GENERALISED HAJŁASZ-BESOV SPACES ON RD-SPACES

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Abstract. An RD space is a doubling measure metric space Ω with the additional property that it has a reverse doubling property. In this paper we introduce a new class of Hajłasz-Besov spaces on Ω and extend several results from classical theory, such as embeddings and Sobolev-type embeddings.

1. INTRODUCTION

Recently, Besov type spaces defined on metric spaces have been rapidly introduced and developed, and several results from classical theory have been extended into this new context (see for example [61], [31], [47], [21], [22], [32], [13], [38], [56], [2], [37] and references therein). In this paper, we continue our research on this topic initiated in our previous papers [42], [43] and [44].

Among the various equivalent expressions for defining Besov spaces in metric measure spaces, we use the approach based on a generalisation of the classical L^p modulus of smoothness introduced in [21]. More specifically (precise definitions and properties used in this introductory section can be found in section 2), we assume that (Ω, d, μ) is a metric measure space equipped with a metric d and a Borel regular outer measure μ, for which the measure of each ball is positive and finite and $\mu(\{x\}) = 0$ for all $x \in \Omega$. Given $t > 0$, $1 \leq p < \infty$ and $f \in L^p_{loc}(\Omega)$, the L^p-modulus of smoothness is defined by

$$\mathcal{E}_p(f,t) = \left(\int_\Omega \left(\fint_{B(x,t)} |f(x) - f(y)|^p \, d\mu(y) \right) d\mu(x) \right)^{1/p},$$

where, as usual, $\fint_A g(x) d\mu(y) = \frac{1}{\mu(A)} \int_A g(x) d\mu(y)$ is the integral average of the function g over a measurable set A with $0 < \mu(A) < \infty$.

For $0 < \theta < \infty$, the homogeneous Besov space $\dot{\mathcal{B}}^\theta_{p,q}(\Omega)$ consists of functions $f \in L^p_{loc}(\Omega)$ for which the seminorm

$$\|f\|_{\dot{\mathcal{B}}^\theta_{p,q}(\Omega)} := \begin{cases} \left(\int_0^\infty \left(t^{-\theta} \mathcal{E}_p(f,t) \right)^q \frac{dt}{t} \right)^{1/q}, & 1 \leq q < \infty, \\ \sup_{t>0} t^{-\theta} \mathcal{E}_p(f,t), & q = \infty, \end{cases}$$

is finite.

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If $0<\theta<1$, this definition gives the usual Besov space in the Euclidean setting, since as was observed in [21], $\mathcal{E}_p(f,t)$ is equivalent to the standard L^p-modulus of smoothness $\omega_p(f,t)=\sup_{|h|\leq t}\|f(x+h)-f(x)\|_{L^p(\mathbb{R}^n)}$.

Classical theory generalises Besov spaces in the following way:

(i) Replacing the regularity index $t^{-\theta}$ by $t^{-\theta}b(t)$, where b is a slowly varying function ($b\in SV$). This change allows to modify the smoothness properties of functions; the resulting spaces are often called generalised smooth Besov spaces, see e.g. [17], [24], [25], [58], [59], [60], the literature on this subject is extensive.

(ii) Replacing the basis space L^p in the definition of the L^p modulus of smoothness by a Lorentz space, an Orlicz space or a rearrangement invariant (r.i) function space[1] (see e.g. [18], [54], [10], [11], [16] and the references given there).

The combination of these two ideas has been a frequently used means of obtaining the complete solution of some natural questions that require a fine control of the behaviour of the functions involved (for example in variational problems and PDE's), and numerous works have been devoted to proving embedding theorems and fundamental topological properties for Besov spaces obtained in this way, which are crucial in applications[2].

In this paper we will introduce a new type of Besov spaces defined in metric measure spaces, which incorporates and generalises the ideas described in items (i) and (ii) above. More specifically,

Definition 1. *Let (Ω,d,μ) be an $RD-$ space. Let X be an r.i. space on Ω, E an r.i. space on $(0,\infty)$, $b\in SV$ and $0<\theta<\infty$. The* ***Generalised Homogeneous Hajłasz-Besov Space*** *$\mathring{B}^{\theta,b}_{X,E}$, is defined as the set of functions $f\in L^1+L^\infty$ such that the quasi-norm*

$$\|f\|_{\mathring{B}^{\theta,b}_{X,E}}=\left\|t^{-\theta}b(t)E_X(f,t)\right\|_{\tilde{E}} \tag{1}$$

is finite. Here $E_X(f,r)$ is the the $X-$modulus of continuity defined by

$$E_X(f,t)=\left\|\fint_{B(x,t)}|f(x)-f(y)|\,d\mu(y)\right\|_X,$$

and $\tilde{E}$ is the corresponding r.i. space with respect to homogeneous measure dt/t.[3]

Similarly, the ***Generalised Hajłasz-Besov space*** *$B^{\theta,b}_{X,E}$ is $\mathring{B}^{\theta,b}_{X,E}\cap X$ with the norm*

$$\|f\|_{B^{\theta,b}_{X,E}}=\left\|t^{-\theta}b(t)E_X(f,t)\right\|_{\tilde{E}(0,1)}+\|f\|_X.$$

Example 2. *If $0<\theta<1$ and $E=L^q(0,\infty)$ then $\tilde{E}=L^q((0,\infty),\frac{dt}{t})$ and $\mathring{B}^{\theta,1}_{L^p,L^q}=\dot{\mathcal{B}}^{\theta}_{p,q}(\Omega)$ (see Remark 15 below), while $\mathring{B}^{\theta,b}_{L^p,L^q}(\mathbb{R}^n)$ is the classical Besov space of generalised smoothness.*

[1] i.e. a function space which satisfies the following condition: if f and g have the same distribution function, then $\|f\|_X=\|g\|_X$ (see Section 2.1 below).

[2] We refer the interested reader, for instance, to [40], [23], [48], [16], [6], [27],[26], [1], [3], [5], [55], [4] and the references quoted therein.

[3] See Section 2.2.2 below.

The main point in (1) is the replacement of the r.i. space $L^q(0,\infty)$ by the richer family of r.i. spaces E, which we believe is the natural framework for presenting this theory. For example, any ultrasymmetric space (see [50]) can be derived in this way, in fact the theory of utrasymmetric spaces developed by E. Pustylnik in [50], and a variant of the real interpolation method introduced and studied in [19] and [20] have been both a motivation and a model for the Besov type spaces introduced above.

The aim of this paper is to extend several results from classical theory, such as embeddings, Sobolev-type embeddings, embedding in BMO, essential continuity and Morrey type embeddings, for the generalised Hajłasz-Besov spaces introduced in the definition above. Notice that Besov spaces defined in this way are also new in the Euclidean setting; moreover, as we will see in sections 3 and 4 below, this general context will be of particular interest since, rather than complicating things, it often makes the results more transparent.

The structure of this paper is as follows. In section 2 we review well-known results and give some further notation and background information, as well as more details on metric spaces, function spaces, SV-functions and parameters. In section 3 we prove some embedding results for generalised Hajłasz-Besov spaces. In section 4 we will see that for $0<\theta<1$ generalised Hajłasz-Besov spaces can be realised as real interpolation spaces, this description will allow us to obtain Sobolev-type embedding theorems. Criteria for essential continuity, embedding in BMO and Morrey type results are also obtained and some examples are given.

Throughout the paper we will write $f \preceq g$ instead of $f \le Cg$ for some constant $C>0$. The functions f and g are equivalent, $f \simeq g$, if $f \preceq g$ and $g \preceq f$. We also say that a function f is almost increasing (almost decreasing) if it is equivalent to an increasing (decreasing) function.

2. Notation and preliminaries

A metric measure space (Ω, d, μ) is a metric space (Ω, d) endowed with a Borel measure μ such that $0<\mu(B)<\infty$, for every ball B in Ω. We will also assume that (Ω, d, μ) is atom free, i.e. $\mu(\{x\})=0$ for all $x \in \Omega$. For any $x\in\Omega$ and $r>0$, set $B(x,r) := \{y \in \Omega : d(x,y) < r\}$.

A metric measure space (Ω, d, μ) is called an $RD-$**space** if there are positive constants $0<C_1\le 1\le C_2$ and $0<k\le n$ such that for all $x\in\Omega$ and $\lambda\ge 1$, we have

$$C_1\lambda^k\mu(B(x,r)) \le \mu(B(x,\lambda r)) \le C_2\lambda^n\mu(B(x,r)). \tag{2}$$

We will call the pair (k,n) the **indices** of the RD space (see [34] and [63] for more information and equivalent characterisations of RD-spaces).

We note that (2) implies the **doubling property**, there exists a constant $C_D \ge 1$ (called the doubling constant) such that

$$\mu(B(x,2r)) \le C_D\mu(B(x,r)) \tag{3}$$

and the **reverse doubling property**: there exists a constant $a\in(1,\infty)$ such that for all $x\in\Omega, \mu(B(x,ar)) \ge 2\mu(B(x,r))$.

It follows from (2) that there exist positive constants c_0, C_0 such that

$$c_0\min(r^k,r^n)\mu(B(x,1)) \le \mu(B(x,r)) \le C_0\max(r^k,r^n)\mu(B(x,1)),$$

for all $x\in\Omega$ and $0<r<\infty$.

We will say that a metric measure space (Ω, d, μ) satisfies the **non-collapsing property** if

$$\kappa = \inf_x \mu(B(x,1)) > 0.$$

Therefore, $RD-$spaces satisfying the non-collapsing property, verify that

$$c_0 \kappa \min(r^k, r^n) \leq \mu(B(x,r)). \tag{4}$$

See [44] for more information on metric measure spaces satisfying the non-collapsing property.

2.1. **Rearrangements of functions and rearrangement invariant spaces.** In this part we will give some basic background from the theory of rearrangements and r.i. spaces, which will be used in what follows (for an exhaustive treatment of these topics, we refer the reader to [8] and [35]).

Let (Ω, d, μ) be metric measure space. For measurable functions $f : \Omega \to \mathbb{R}$, the **decreasing rearrangement** f_μ^* of f is given by

$$f_\mu^*(s) = \inf\{t \geq 0 : \mu\{x \in \Omega : |f(x)| > t\} \leq s\}, \quad s > 0.$$

In the following, we will omit the indices μ whenever it is clear which measure we are working with.

A basic property of rearrangements states that

$$\sup_{\mu(E)=t} \int_E |f(x)|\, d\mu = \int_0^t f^*(s) ds.$$

Since f^* is decreasing, the maximal function f^{**} of f^*, defined by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds,$$

is also decreasing, and

$$f^* \leq f^{**}.$$

We single out two subadditivity properties, if f and g are two $\mu-$measurable functions on Ω, then for $t > 0$

$$(f+g)^* (2t) \leq f^*(t) + g^*(t)$$

and

$$(f+g)^{**} (t) \leq f^{**}(t) + g^{**}(t).$$

The oscillation $O(f, \cdot)$ of f is defined by

$$O(f,t) := f^{**}(t) - f^*(t), \quad 0 < t < \infty.$$

Notice that

$$\frac{\partial}{\partial t} f^{**}(t) = -\frac{O(f,t)}{t} \tag{5}$$

and that the function $t \to tO(f,t)$ in increasing.

A Banach function space $X_\mu(\Omega)$ on (Ω, d, μ) that satisfies the Fatou property is called a rearrangement invariant (r.i.) space if, for any two measurable functions f and g the condition $g \in X_\mu(\Omega)$, $f^*(t) \leq g^*(t)$ for almost all $t > 0$ implies $f \in X_\mu(\Omega)$ and $\|f\|_{X_\mu(\Omega)} \leq \|g\|_{X_\mu(\Omega)}$. Typical examples of r.i. spaces are $L^p_\mu(\Omega)$-spaces, Lorentz spaces, Marcinkiewicz spaces, Lorentz-Zygmund spaces, Orlicz spaces and Lorentz-Orlicz spaces.

To condense the notation throughout the paper, we will omit the measure and the set when referring to a function space defined in (Ω, d, μ), so we write X instead of $X_\mu(\Omega)$.

For any r.i. space X we have

$$L^\infty \cap L^1 \subset X \subset L^1 + L^\infty,$$

with continuous embedding.

A useful property of r.i. spaces states that

$$\int_0^r f^*(s)ds \leq \int_0^r g^*(s)ds, \ \forall r > 0 \Rightarrow \|f\|_X \leq \|g\|_X \tag{6}$$

for any r.i. space X.

Given an r.i. space X, the set

$$X' = \left\{ f : \int_\Omega |f(x)g(x)|\, d\mu(x) < \infty, \quad g \in X \right\}$$

equipped with the norm

$$\|f\|_{X'} = \sup_{\|g\|_X \leq 1} \int_\Omega |f(x)g(x)|\, d\mu(x) \tag{7}$$

is called the associate space of X. It turns out that X' endowed with the norm given by (7) is an r.i. space. Furthermore, the Hölder inequality

$$\int_\Omega |f(x)g(x)|\, d\mu(x) \leq \|f\|_X \|g\|_{X'} \tag{8}$$

holds for every $f \in X$ and $g \in X'$.

A useful tool, in the study of an r.i. space X is the **fundamental function** of X defined by

$$\phi_X(t) = \|\chi_F\|_X\,,$$

where F is any measurable subset of Ω with $\mu(F) = t$. The fundamental function ϕ_X is quasi-concave. Moreover, one has that

$$\phi_X(t)\phi_{X'}(t) = t. \tag{9}$$

Let $1 \leq p < \infty$ and let X be an r.i. space on Ω; the $p-$**convexification** $X^{(r)}$ of X, (see [39]) is the r.i space defined by

$$X^{(p)} = \{f : |f|^p \in X\}, \quad \|f\|_{X^{(p)}} = \||f|^p\|_X^{1/p}.$$

For example $(L^1)^{(p)} = L^p$.

2.2. **Parameters spaces, extension indices and slowly varying functions.**

2.2.1. *Parameters Spaces.* We consider two different measures on $(0, \infty)$, the usual Lebesgue measure dt and the homogeneous measure dt/t. We use characters with a tilde for spaces with the measure dt/t. For example,

$$\|f\|_{\tilde{L}^1} = \int_0^\infty |f(t)| \frac{dt}{t}$$

while $\tilde{L}^\infty(0, \infty) = L^\infty(0, \infty)$.

A **parameter** space $E=E(0,\infty)$ will be an r.i. space on $(0,\infty)$ with respect to the Lebesgue measure. Given a parameter E, we may always assume that E is an exact interpolation space between L^1 and L^∞ i.e. $E:=\mathcal{F}(L^1,L^\infty)$ for some real interpolation functor $\mathcal{F}$. Together with E, we consider the space obtained by the same functor $\mathcal{F}$ from the couple $(\tilde{L}^1,L^\infty)$, i.e. $\tilde{E}:=\mathcal{F}(\tilde{L}^1,L^\infty)$. The space $\tilde{E}$ is an r.i. space respect to the measure dt/t. The spaces $E=E(0,\infty)$ and $\tilde{E}=\tilde{E}(0,\infty)$ can also be connected directly without using interpolation functors, since $f(t)\in\tilde{E}(0,1)$ if and only if $f(e^{-u})\in E$ and $f(t)\in\tilde{E}(1,\infty)$ if and only if $f(e^u)\in E$. Moreover, $\|f\|_{\tilde{E}(0,1)}=\|f(e^{-u})\|_{E(0,\infty)}$ and $\|f\|_{\tilde{E}(1,\infty)}=\|f(e^u)\|_{E(0,\infty)}$ (see [50]).

Sometimes it is useful to divide the interval $(0,\infty)$ into two subintervals $(0,t)$ and (t,∞), considering the spaces $E(0,t)$ and $E(t,\infty)$ separately, using this notation, we can write

$$\|f\|_E\simeq\|f\|_{E(0,t)}+\|f\|_{E(t,\infty).}$$

The same applies to $\tilde{E}$.

A useful property of $\tilde{E}$, (see [51]), is that for any positive numbers $a,b>0$ and any function $g\in E$, the following equivalence holds

$$\left\|g(at^b)\right\|_{\tilde{E}}\leq\min(1,b)\left\|g\right\|_{\tilde{E}}. \tag{10}$$

Associated with each parameter E is a pair of indices, the upper and lower and lower Boyd indices (see [8]), defined by

$$\overline{\alpha}_E=\inf_{s>1}\frac{\ln h_E(s)}{\ln s}\quad\text{and}\quad\underline{\alpha}_E=\sup_{s<1}\frac{\ln h_E(s)}{\ln s}.$$

where $h_E(s)$ is the norm of the dilation operator $D_{\frac{1}{s}}f(t)=f\left(\frac{t}{s}\right),s>0$. For example $\underline{\alpha}_{L^q}=\overline{\alpha}_{L^q}=1/q$.

For more information on this topic see [15], [50], [51], [19] and the references quoted therein.

2.2.2. *Slowly varying functions.* We say that a positive Lebesgue measurable function b, is **slowly varying** on $(0,\infty)$ ($b\in SV$) if, for any $\varepsilon>0$, the function $t^\varepsilon b(t)$ is almost increasing on $(0,\infty)$ while the function $t^{-\varepsilon}b(t)$ is almost decreasing on $(0,\infty)$. In fact, we have that for every $\varepsilon>0$, there is a positive constant c_ε such that

$$c_\varepsilon^{-1}\min(t^{-\varepsilon},t^\varepsilon)b(s)\leq b(st)\leq c_\varepsilon\max(t^{-\varepsilon},t^\varepsilon)b(s),\ t,s>0. \tag{11}$$

The family of SV functions includes powers of logarithms, $\ell(t)=(1+|\ln t|)^\alpha$, $t>0$ and $\alpha\in\mathbb{R}$, broken logarithmic functions

$$\ell^{\mathbb{A}}(t)=\begin{cases}(1+|\ln t|)^\alpha, & 0<t<1,\\ (1+|\ln t|)^\beta, & t\geq 1,\end{cases}$$

$\mathbb{A}=(\alpha,\beta)\in\mathbb{R}^2$ and also functions such as $b(t)=\exp\left(|\ln t|^\alpha\right)$, $\alpha\in(0,1)$. The latter family of functions has the interesting property that each of its members tends to infinity faster than any positive power of the logarithmic function.

The following result (see [19]) will be useful in what follows.

Lemma 3. *Let E be a parameter and $b\in SV$. For all $\alpha>0$ we get*

(i)

$$\|s^\alpha b(s)\|_{\tilde{E}(0,t)}\simeq t^\alpha b(t)\quad\textit{and}\quad\left\|s^{-\alpha}b(s)\right\|_{\tilde{E}(t,\infty)}\simeq t^{-\alpha}b(t).$$

(ii) *When $\alpha > 0$, the Hardy-type inequality*

$$\left\| s^{-\alpha} b(s) \int_0^t f(s) ds \right\|_{\tilde{E}} \leq \left\| s^{1-\alpha} b(s) f(s) ds \right\|_{\tilde{E}}$$

holds for each measurable positive function f on $(0, \infty)$.

For more information and further examples of slowly varying functions see [12] or [45].

2.2.3. *Extension indices.* Let φ be a positive finite function on $(0, \infty)$, we define its associated dilation function as

$$M_\varphi(t) = \sup_{s>0} \frac{\varphi(st)}{\varphi(s)}$$

If $M_\varphi(t)$ is finite everywhere, then the **lower and upper extension indices** of φ exist and they are defined by

$$\underline{\beta}_\varphi = \lim_{t \to 0} \frac{\log M_\varphi(t)}{\log t}, \quad \overline{\beta}_\varphi = \lim_{t \to \infty} \frac{\log M_\varphi(t)}{\log t}.$$

In general, $-\infty < \underline{\beta}_\varphi \leq \overline{\beta}_\varphi < \infty$, but if it is increasing, then $0 \leq \underline{\beta}_\varphi \leq \overline{\beta}_\varphi < \infty$, and if it is quasi-concave, we have $0 \leq \underline{\beta}_\varphi \leq \overline{\beta}_\varphi \leq 1$. Also

$$\varphi(t) \simeq \int_0^t \varphi(s) \frac{ds}{s} \text{ if } 0 < \underline{\beta}_\varphi \text{ and } \varphi(t) \simeq \int_t^\infty \varphi(s) \frac{ds}{s} \text{ if } \overline{\beta}_\varphi < 0. \tag{12}$$

Note also that both indices remain the same after replacing φ by any equivalent function.

The following properties of extension indices are easy to prove:

Proposition 4. *Let $b \in SV$ and f, g positive finite function on $(0, \infty)$. Then*

(i) $\overline{\beta}_b = \underline{\beta}_b = 0$.

(ii) *If $\varphi(t) = t^a b(t) f(t)$, $a \in \mathbb{R}$, then $\underline{\beta}_\varphi = a + \underline{\beta}_f$ and $\overline{\beta}_\varphi = a + \overline{\beta}_f$.*

(iii) *If $\varphi(t) = f(1/t)$ or $\varphi(t) = 1/f(t)$, then $\underline{\beta}_\varphi = -\overline{\beta}_f$ and $\overline{\beta}_\varphi = -\underline{\beta}_f$.*

(iv) *If $\varphi(t) = f(t)g(t)$, then $\underline{\beta}_\varphi \geq \underline{\beta}_f + \underline{\beta}_g$ and $\overline{\beta}_\varphi \leq \overline{\beta}_f + \overline{\beta}_g$.*

It follows from (ii) that $\varphi(t)/t^a$ is almost increasing for any $a < \underline{\beta}_\varphi$ and almost decreasing for any $a > \overline{\beta}_\varphi$.

Given X an r.i. space on Ω, the **fundamental indices** $\underline{\beta}_X$ and $\bar{\beta}_X$ of X are defined as the lower and upper extension indices of its fundamental function φ_X.

3. Embeddings of Generalised Hajłasz-Besov spaces

In this section we first analyse the effect of replacing the basis space X and/or the parameter E by convexifications of them, and we prove several embedding theorems.

Proposition 5. *Let (Ω, d, μ) be an RD−space, X an r.i. space, E a parameter, $b \in SV$ and $0 < \theta < \infty$, then*

$$\overset{\circ}{B}{}^{\theta,b}_{X,L^1} \subset \overset{\circ}{B}{}^{\theta,b}_{X,E} \subset \overset{\circ}{B}{}^{\theta,b}_{X,L^\infty}$$

and

$$B^{\theta,b}_{X,L^1} \subset B^{\theta,b}_{X,E} \subset B^{\theta,b}_{X,L^\infty}.$$

Proof. It follows from (3) that there is a positive constant c which depends only on the doubling constant, such that

$$c^{-1}E_X(f,s) \leq E_X(f,R) \leq cE_X(f,r) \tag{13}$$

for all $0 < R/2 \leq s \leq R \leq r \leq 2R$, thus

$$\begin{aligned} b(r)r^{-\theta}E_X(f,r) &\preceq \int_r^{2r} b(t)t^{-\theta}E_X(f,t)\frac{dt}{t} \leq \left\| b(t)t^{-\theta}E_X(f,t)\right\|_{\tilde{E}} \left\|\chi_{(r,2r)}\right\|_{\tilde{E}'} \\ &= \|f\|_{\mathring{\mathrm{B}}^{\theta,b}_{X,E}} \varphi_{\tilde{E}'}(\ln 2). \end{aligned} \tag{14}$$

Therefore

$$\mathring{\mathrm{B}}^{\theta,b}_{X,E} \subset \mathring{\mathrm{B}}^{\theta,b}_{X,L^\infty}.$$

To see the first embedding, we will see that for any $t > 0$ we have

$$\left\| b(r)r^{-\theta}E_X(f,r)\right\|_{\tilde{E}(0,t)} \preceq \int_0^{2t} b(s)s^{-\theta}E_X(f,s)\frac{ds}{s}.$$

In virtue of interpolation properties of any parameter E, it suffices to prove this inequality for the extreme spaces $\tilde{L}^1$ and L^∞. The above inequality is obvious for $\tilde{E} = \tilde{L}^1$. Let us check it for $\tilde{E} = L^\infty$.

By (14) we get

$$b(r)r^{-\theta}E_X(f,r) \preceq \int_0^{2r} b(s)t^{-\theta}E_X(f,s)\frac{ds}{s},$$

thus

$$\left\| b(r)r^{-\theta}E_X(f,r)\right\|_{L^\infty(0,t)} \preceq \int_0^{2t} b(s)s^{-\theta}E_X(f,s)\frac{ds}{s}.$$

□

The next result is an extension of the well-known embedding $B^\theta_{p,q_1} \subset B^\theta_{p,q_2}$ when $1 \leq q_1 < q_2$.

Proposition 6. *Let (Ω, d, μ) be an RD−space, X an r.i. space on Ω, E a parameter, $b \in SV$ and $0 < \theta < \infty$. Suppose that $1 \leq p < q < \infty$, then*

$$\|f\|_{\mathring{B}^{\theta,b}_{X,E^{(q)}}} \preceq \|f\|_{\mathring{B}^{\theta,b}_{X,E^{(p)}}}.$$

The same embedding holds for the inhomogeneous case.

Proof. By the previous result, $\mathring{\mathrm{B}}^{\theta,b}_{X,E^{(p)}} \subset \mathring{\mathrm{B}}^{\theta,b}_{X,L^\infty}$, hence

$$\begin{aligned} \|f\|^q_{\mathring{\mathrm{B}}^{\theta,b}_{X,E^{(q)}}} &= \left\| \left(t^{-\theta}b(t)E_X(f,t)\right)^q \right\|_{\tilde{E}} = \left\| \left(t^{-\theta}b(t)E_X(f,t)\right)^{q-p} \left(t^{-\theta}b(t)E_X(f,t)\right)^p \right\|_{\tilde{E}} \\ &\leq \sup_t \left(t^{-\theta}b(t)E_X(f,t)\right)^{q-p} \left\| \left(t^{-\theta}b(t)E_X(f,t)\right)^p \right\|_{\tilde{E}} \\ &= \|f\|^{q-p}_{B^{\theta,b}_{X,L^\infty}} \left\| t^{-\theta}b(t)E_X(f,t)\right\|^p_{\tilde{E}^{(p)}} \\ &\preceq \left\| t^{-\theta}b(t)E_X(f,t)\right\|^q_{\tilde{E}^{(p)}}, \end{aligned}$$

that implies the desired result. □

We will now consider the effect of replacing the space X by $X^{(q)}$ and the relation between the spaces $\mathring{\mathrm{B}}^{\theta,b}_{X,E}$ and $\mathring{\mathrm{B}}^{\theta,b}_{X^{(q)},E}$, for which we need the following technical result obtained by Ranjbar-Motlagh in [53] for the special case $X = L^q$. We will give its proof in the Appendix section at the end of this paper.

Lemma 7. *Let (Ω, d, μ) be an RD−space that satisfies the non-collapsing condition, X an r.i. space on Ω, $q \geq p \geq 1$. Then, for any $f \in L^1 + L^\infty$ and any $r > 0$, there exists a constant $c = c(c_0, \kappa, C_D)$ such that*

(i)

$$\left\| \fint_{B(x,r)} |f(y)|\, d\mu(y) \right\|_{X^{(q)}} \leq \frac{c}{\varphi_X \left(\min(r^k, r^n)\right)^{\frac{1}{p}-\frac{1}{q}}} \|f\|_{X^{(p)}}$$

(ii)

$$E_{X^{(q)}}(f,r) \leq c \int_0^{4r} \frac{\mathcal{E}_{X^{(p)}}(f,s)}{\varphi_X \left(\min(s^k, s^n)\right)^{\frac{1}{p}-\frac{1}{q}}} \frac{ds}{s},$$

where

$$\mathcal{E}_{X^{(p)}}(f,t) := \left\| \left(\fint_{B(x,t)} |f(x) - f(y)|^p \, d\mu(y) \right)^{1/p} \right\|_{X^{(p)}}.$$

Theorem 8. *Let (Ω, d, μ) be an RD−space, X an r.i. space, E a parameter, $b \in SV$, $0 < \theta < \infty$, $1 \leq p < \infty$ and $1 < \sigma < \infty$. Let $\bar{\beta}_X \leq \gamma \leq 1$ be such that the function $\frac{\phi_X(t)}{t^\gamma}$ is quasi decreasing*[4] *and $\alpha = \theta + \frac{n\gamma}{p}\left(1 - \frac{1}{\sigma}\right)$. Then*

(i)

$$\|f\|_{\mathring{B}^{\theta,b}_{X^{(\sigma p)},E}} \preceq \|f\|_{B^{\alpha,b}_{X^{(p)},E}},$$

(ii)

$$\|f\|_{B^{\theta,b}_{X^{(\sigma p)},E}} \preceq \|f\|_{B^{\alpha,b}_{X^{(p)},E}},$$

(iii) *and*

$$\|f\|_{X^{(p\sigma)}} \preceq \|f\|_{X^{(p)}}^{\frac{\theta}{\alpha}} \|f\|_{B^{\alpha,b}_{X^{(p)},E}}^{1-\frac{\theta}{\alpha}} \left(1 + \frac{1}{b\left(\left(\|f\|_{X^{(p)}} / \|f\|_{B^{\alpha,b}_{X^{(p)},E}} \right)^{\frac{\theta}{\alpha}} \right)} \right). \tag{15}$$

[4]By proposition 4, this is always satisfied if $\bar{\beta}_X < \gamma \leq 1$.

Proof. (i) By the previous Lemma, and since $\phi_X(t)/t^\gamma$ is quasi decreasing, we get

$$\begin{aligned}
\|f\|_{\mathring{\mathrm{B}}^{\theta,b}_{X^{(\sigma p)},E}} &= \left\|b(t)t^{-\theta}E_{X^{(\sigma p)}}(f,t)\right\|_{\tilde{E}} \\
&\preceq \left\|b(t)t^{-\theta}\int_0^{4t}\frac{\mathcal{E}_{X^{(p)}}(f,s)}{\varphi_X\left(\min(s^k,s^n)\right)^{\frac{1}{p}-\frac{1}{\sigma p}}}\frac{ds}{s}\right\|_{\tilde{E}} \\
&\simeq \left\|b(4t)\,(4t)^{-\theta}\int_0^{4t}\frac{\mathcal{E}_{X^{(p)}}(f,s)}{\varphi_X\left(\min(s^k,s^n)\right)^{\frac{1}{p}-\frac{1}{\sigma p}}}\frac{ds}{s}\right\|_{\tilde{E}} \\
&\simeq \left\|b(t)t^{-\theta}\int_0^{t}\frac{\mathcal{E}_{X^{(p)}}(f,s)}{\varphi_X\left(\min(s^k,s^n)\right)^{\frac{1}{p}-\frac{1}{\sigma p}}}\frac{ds}{s}\right\|_{\tilde{E}} \quad \text{(by (10))} \\
&\preceq \left\|b(t)t^{-\theta}\frac{\mathcal{E}_{X^{(p)}}(f,t)}{\varphi_X\left(\min(t^k,t^n)\right)^{\frac{1}{p}-\frac{1}{\sigma p}}}\right\|_{\tilde{E}} \quad \text{(by Lemma 3)} \\
&\preceq \left\|b(t)t^{-\theta}\left(\frac{t^{n\gamma}}{\varphi_X\left(t^n\right)}\right)^{\frac{1}{p}-\frac{1}{\sigma p}}\frac{\mathcal{E}_{X^{(p)}}(f,t)}{t^{n\gamma\left(\frac{1}{p}-\frac{1}{\sigma p}\right)}}\right\|_{\tilde{E}(0,1)} + \left\|b(t)t^{-\theta}\frac{\mathcal{E}_{X^{(p)}}(f,t)}{\varphi_X\left(t^k\right)^{\frac{1}{p}-\frac{1}{\sigma p}}}\right\|_{\tilde{E}(1,\infty)} \\
&\preceq \left\|b(t)t^{-\theta}\frac{\mathcal{E}_{X^{(p)}}(f,t)}{t^{n\gamma\left(\frac{1}{p}-\frac{1}{\sigma p}\right)}}\right\|_{\tilde{E}(0,1)} + \left\|b(t)t^{-\theta}\mathcal{E}_{X^{(p)}}(f,t)\right\|_{\tilde{E}(1,\infty)} \\
&\preceq \|f\|_{\mathring{\mathrm{B}}^{\alpha,b}_{X^{(p)},E}} + \left\|b(t)t^{-\theta}\mathcal{E}_{X^{(p)}}(f,t)\right\|_{\tilde{E}(1,\infty)}.
\end{aligned}$$

Finally, since

$$\begin{aligned}
\mathcal{E}_{X^{(p)}}(f,t) &= \left\|\left(\fint_{B(x,t)}|f(x)-f(y)|^p\,d\mu(y)\right)^{1/p}\right\|_{X^{(p)}} \\
&\leq \|f\|_{X^{(p)}} + \left\|\left(\fint_{B(x,t)}|f(y)|^p\,d\mu(y)\right)^{1/p}\right\|_{X^{(p)}} \\
&\leq \|f\|_{X^{(p)}} + \frac{c}{\varphi_X(t^k,t^n)}\|f\|_{X^{(p)}} \quad \text{(by Lemma 7)}
\end{aligned}$$

it follows that

$$\begin{aligned}
(16)\quad \left\|b(t)t^{-\theta}\mathcal{E}_{X^{(p)}}(f,t)\right\|_{\tilde{E}(1,\infty)} &\leq \left\|b(t)t^{-\theta}\left(1+\frac{c}{\varphi_X(t^k,t^n)}\right)\|f\|_{X^{(p)}}\right\|_{\tilde{E}(1,\infty)} \\
&\leq (1+c)\,\|f\|_{X^{(p)}}\left\|b(t)t^{-\theta}\right\|_{\tilde{E}(1,\infty)} \\
&\preceq \|f\|_{X^{(p)}} \quad \text{(by Lemma 3).}
\end{aligned}$$

(ii)We know from part (i)

$$\|f\|_{\mathring{\mathrm{B}}^{\theta,b}_{X^{(\sigma p)},E}} \preceq \|f\|_{B^{\theta,b}_{X^{(p)},E}}.$$

On the other hand

$$\|f\|_{X^{(\sigma p)}} \leq \left| \|f\|_{X^{(\sigma p)}} - \left\| \fint_{B(x,1/4)} |f(x)|\, d\mu(x) \right\|_{X^{(\sigma p)}} \right| + \left\| \fint_{B(x,1/4)} |f(x)|\, d\mu(x) \right\|_{X^{(\sigma p)}} = I + II. \tag{17}$$

From Lemma 7 (i) with $r = 1/4$ we get

$$II \preceq \|f\|_{X^{(p)}} . \tag{18}$$

By Hölder's inequality, the concavity of φ_X and the fact that $\phi_X(t)/t^{\gamma}$ is quasi decreasing, we have

$$\begin{aligned} I &\leq \left\| \left| f(x) - \fint_{B(x,1/4)} f(y) d\mu(y) \right|^{\sigma p} \right\|_X^{1/\sigma p} \\ &\leq \left\| \left(\fint_{B(x,1/4)} |f(x) - f(y)|\, d\mu(y) \right)^{\sigma p} \right\|_X^{1/\sigma p} \\ &= E_{X^{(\sigma p)}}(f, 1/4) \preceq \int_0^1 \frac{\mathcal{E}_{X^{(p)}}(f,s)}{\left(\varphi_X\left(s^n\right)\right)^{\frac{1}{p} - \frac{1}{\sigma p}}} \frac{ds}{s} \text{ (by Lemma 7 (ii))} \\ &= \int_0^1 \frac{s^{-\theta} b(s) \mathcal{E}_{X^{(p)}}(f,s)}{\left(\varphi_X\left(s^n\right)\right)^{\frac{1}{p} - \frac{1}{\sigma p}}} \frac{s^{\theta}}{b(s)} \frac{ds}{s} \\ &\preceq \left\| b(t) t^{-\theta} \frac{\mathcal{E}_{X^{(p)}}(f,t)}{t^{n\gamma\left(\frac{1}{p} - \frac{1}{\sigma p}\right)}} \right\|_{\tilde{E}(0,1)} \left\| \frac{s^{\theta} s^{n\gamma\left(\frac{1}{p} - \frac{1}{\sigma p}\right)}}{b(s) \left(\varphi_X\left(s^n\right)\right)^{\frac{1}{p} - \frac{1}{\sigma p}}} \right\|_{\tilde{E}'(0,1)} \\ &\preceq \left\| b(t) t^{-\theta} \frac{\mathcal{E}_{X^{(p)}}(f,t)}{t^{n\gamma\left(\frac{1}{p} - \frac{1}{\sigma p}\right)}} \right\|_{\tilde{E}(0,1)} \left\| \frac{s^{\theta}}{b(s)} \right\|_{\tilde{E}'(0,1)} \\ &\preceq \left\| b(t) t^{-\theta} \frac{\mathcal{E}_{X^{(p)}}(f,t)}{t^{n\gamma\left(\frac{1}{p} - \frac{1}{\sigma p}\right)}} \right\|_{\tilde{E}(0,1)} \text{ (by Lemma 3).} \end{aligned} \tag{19}$$

Inserting (18) and (19) into (17), we find that

$$\|f\|_{X^{(\sigma p)}} \leq \left\| b(t) t^{-\theta} \frac{\mathcal{E}_{X^{(p)}}(f,t)}{t^{\gamma n\left(\frac{1}{p} - \frac{1}{\sigma p}\right)}} \right\|_{\tilde{E}(0,1)} + \|f\|_{X^{(p)}} = \|f\|_{B^{\alpha,b}_{X^{(p)},E}} .$$

Thus

$$\begin{aligned} \|f\|_{B^{\theta,b}_{X^{(\sigma p)},E}} &= \|f\|_{X^{(\sigma p)}} + \left\| b(t) t^{-\theta} E_{X^{(\sigma p)}}(f,t) \right\|_{\tilde{E}(0,1)} \\ &\preceq \|f\|_{B^{\alpha,b}_{X^{(p)},E}} \end{aligned}$$

as we wanted to show.

(iii) Let $0 < R < 1/4$, then

$$\begin{aligned}\|f\|_{X^{(p\sigma)}} &\leq \left| \|f\|_{X^{(p\sigma)}} - \left\| \fint_{B(x,R)} |f(x)|\, d\mu(x) \right\|_{X^{(p\sigma)}} \right| \\ &+ \left\| \fint_{B(x,R)} |f(x)|\, d\mu(x) \right\|_{X^{(p\sigma)}} \\ &= I + II.\end{aligned}$$

From Lemma 7, we get

$$II \preceq \frac{1}{\left(\varphi_X(R^n)\right)^{\frac{1}{p}-\frac{1}{p\sigma}}} \|f\|_{X^{(p)}},$$

and with the same argument that was used to get (19), we have

$$I \preceq \int_0^{4R} \frac{\mathcal{E}_{X^{(p)}}(f,s)}{\left(\varphi_X(s^n)\right)^{\frac{1}{p}-\frac{1}{p\sigma}}} \frac{ds}{s}.$$

On the other hand, using the fact that $\varphi_X(t)/t^{\gamma}$ is quasi decreasing and Lemma 3, we have

$$\begin{aligned}\int_0^{4R} \frac{\mathcal{E}_{X^{(p)}}(f,s)}{\left(\varphi_X(s^n)\right)^{\frac{1}{p}-\frac{1}{p\sigma}}} \frac{ds}{s} &= \int_0^{4R} \frac{\mathcal{E}_{X^{(p)}}(f,s)}{s^{n\gamma\left(\frac{1}{p}-\frac{1}{p\sigma}\right)}} \frac{s^{n\gamma\left(\frac{1}{p}-\frac{1}{p\sigma}\right)}}{\left(\varphi_X(s^n)\right)^{\frac{1}{p}-\frac{1}{p\sigma}}} \frac{ds}{s} \\ &\preceq \left(\frac{R^{\gamma n}}{\varphi_X(R^n)}\right)^{\frac{1}{p}-\frac{1}{p\sigma}} \int_0^{4R} \frac{\mathcal{E}_{X^{(p)}}(f,s)}{s^{n\gamma\left(\frac{1}{p}-\frac{1}{p\sigma}\right)}} \frac{ds}{s} \\ &\leq \left(\frac{R^{\gamma n}}{\varphi_X(R^n)}\right)^{\frac{1}{p}-\frac{1}{p\sigma}} \left\| \frac{b(s)s^{-\theta}\mathcal{E}_{X^{(p)}}(f,s)}{\min(s^k,s^n)^{\gamma\left(\frac{1}{p}-\frac{1}{p\sigma}\right)}} \right\|_{\tilde{E}(0,4R)} \left\| \frac{s^{\theta}}{b(s)} \right\|_{\tilde{E}'(0,4R)} \\ &\preceq \left(\frac{R^{\gamma n}}{\varphi_X(R^n)}\right)^{\frac{1}{p}-\frac{1}{p\sigma}} \frac{R^{\theta}}{b(R)} \left\| \frac{b(s)s^{-\theta}\mathcal{E}_{X^{(p)}}(f,s)}{\min(s^k,s^n)^{\gamma\left(\frac{1}{p}-\frac{1}{p\sigma}\right)}} \right\|_{\tilde{E}}.\end{aligned}$$

In summary

$$\|f\|_{X^{(p\sigma)}} \preceq \left(\frac{R^{\gamma n}}{\varphi_X(R^n)}\right)^{\frac{1}{p}-\frac{1}{p\sigma}} \left(\frac{1}{R^{n\gamma\left(\frac{1}{p}-\frac{1}{p\sigma}\right)}} \|f\|_{X^{(p)}} + \frac{R^{\theta}}{b(R)} \|f\|_{B^{\alpha,b}_{X^{(p)},E}} \right).$$

So for $0 < R < 1/4$, one gets

$$\|f\|_{X^{(p\sigma)}} \preceq \inf_{0<R<1} \left(\frac{1}{R^{n\gamma\left(\frac{1}{p}-\frac{1}{p\sigma}\right)}} \|f\|_{X^{(p)}} + \frac{R^{\theta}}{b(R)} \|f\|_{B^{\alpha,b}_{X^{(p)},E}} \right).$$

If we choose

$$R = \left(\frac{\|f\|_{X^{(p)}}}{4\|f\|_{B^{\alpha,b}_{X^{(p)},E}}} \right)^{\frac{1}{\alpha}},$$

then the submultiplicative inequality (15) holds. □

Remark 9. *For the special case* $X = L^1(\Omega)$ *and* $E = L^q(0,\infty)$*, the previous theorem was proved in* [52, Theorem 3.3] *and* [52, Theorem 3.6]*. See* [36, Theorem 7.1] *in the Euclidean case.*

Example 10. *If* $r > p \geq 1$ *and* $\alpha = \theta + \frac{n}{p}\left(1 - \frac{p}{r}\right)$, *then the Theorem 8 gives the following submultiplicative inequality for classical Besov spaces of logarithmic smoothness* $B^{\theta,\ell}_{p,q}(\mathbb{R}^n)$*:*

$$\|f\|_{L^r(\mathbb{R}^n)} \preceq \|f\|_{L^p(\mathbb{R}^n)}^{\frac{\theta}{\alpha}} \|f\|_{B^{\theta,\ell}_{p,q}(\mathbb{R}^n)}^{1-\frac{\theta}{\alpha}} \left(1 + \frac{1}{\left(1 + \ln\left(\frac{\|f\|_{B^{\theta,\ell}_{p,q}(\mathbb{R}^n)}}{\|f\|_{L^p(\mathbb{R}^n)}}\right)\right)^{\frac{\theta}{\alpha}}}\right).$$

4. Generalized Hajłasz-Besov spaces as interpolation spaces

In this section we prove that the spaces $\overset{\circ}{\mathrm{B}}{}^{\theta,b}_{X^{(\sigma p)},E}$ and $B^{\theta,b}_{X,E}$, $0 < \theta < 1$, can be constructed using the real interpolation method described in [19]; this description will allow us to obtain Sobolev-type embedding theorems in the subsection 4.1.

Let (Ω, d, μ) be an $RD-$space, let f be a $\mu-$measurable function, a non-negative measurable function g that satisfies

$$|f(x) - f(y)| \leq d(x,y)\,(g(x) + g(y)) \quad \mu - a.e. \;\; x, y \in \Omega.$$

is called a $1-$gradient of f. We denote by $D(f)$ the collection of all $1-$gradients of f.

The **homogeneous Hajłasz-Sobolev** space $\overset{\circ}{\mathrm{M}}{}^{1,X}$ consists of measurable functions $f \in L^1 + L^\infty$ for which

$$\|f\|_{\overset{\circ}{\mathrm{M}}{}^{1,X}} = \inf_{g \in D(f)} \|g\|_X$$

is finite.

The **Hajłasz-Sobolev** space $M^{1,X}$ is $\overset{\circ}{\mathrm{M}}{}^{1,X} \cap X$ equipped with the quasi-norm

$$\|f\|_{M^{1,X}} = \|f\|_X + \|f\|_{\overset{\circ}{\mathrm{M}}{}^{1,X}}.$$

Remark 11. *The definition formulated above is motivated by the Hajłasz approach to the definition of Sobolev spaces on a metric measure space (see* [29] *and* [30]*), where* $M^{1,p}$ *was defined as the set of measurable functions* f *for which*

$$\|f\|_{\overset{\circ}{M}{}^{1,p}} = \inf_{g \in D(f)} \|g\|_{L^p}$$

is finite[5]*. Based on this definition* $\overset{\circ}{M}{}^{1,X}$ *appears naturally when the Lebesgue norm is replaced by the norm* $\|\cdot\|_X$*, for example, if* X *is an Orlicz space, we get the Orlicz-Hajłasz-Sobolev space (see* [62]*), and if* X *is a Lorentz space, we get the Lorentz-Hajłasz-Sobolev space (see* [14]*).*

Recall that given (X_0, X_1) a compatible couple of (quasi-semi)normed spaces (i.e. there is a topological vector space $\mathcal{X}$ such that X_0 and X_1 are continuously embedded into $\mathcal{X}$). For every $f \in X_0 + X_1$ and $t > 0$, the Peetre $K-$functional is

$$K(f, t; X_0, X_1) := \inf_{f = f_0 + f_1} \left\{ \|f_0\|_{X_0} + t\,\|f_1\|_{X_1} \right\}.$$

The following interpolation method has been introduced in [19] and is an extension of the well-known real interpolation method with a functional parameter (see [28] and [46]).

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[5]For $p > 1$, see [29], [30], $M^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$, while for $n/(n+1) < p \leq 1$, $M^{1,p}(\mathbb{R}^n)$ coincides with the Hardy-Sobolev space $H^{1,p}(\mathbb{R}^n)$ [33, Theorem 1].

Definition 12. *Let (X_0, X_1) be a compatible couple of (quasi-semi)normed spaces, E a parameter, $b \in SV$ and $0 < \theta < 1$. The real interpolation space $(X_0, X_1)_{\theta,b,E}$ consists of all f in $X_0 + X_1$ for which the norm*

$$\|f\|_{(X_0,X_1)_{\theta,b,E}} = \left\|t^{-\theta} b(t) K(f,t;X_0,X_1)\right\|_{\tilde{E}}$$

is finite.

If $E = L^q$ and $b = 1$, then the space $(X_0, X_1)_{\theta,b,E}$ coincides with the classical real interpolation space $(X_0, X_1)_{\theta,q}$. For more information about the spaces $(X_0, X_1)_{\theta,b,E}$ and their applications, see for example [19], [46] and [49].

Theorem 13. *Let (Ω, d, μ) be an RD space, X be an r.i. space, and $p \geq 1$, then for all $f \in X^{(p)} + \mathring{M}^{1,X^{(p)}}$ and $t > 0$, we have that*

$$E_{X^{(p)}}(f,t) \leq \mathcal{E}_{X^{(p)}}(f,t) \preceq K(f,t,X^{(p)}, \mathring{M}^{1,X^{(p)}}) \preceq \sum_{j=0}^{\infty} 2^{-j} E_{X^{(p)}}(f, 2^j t) \leq \sum_{j=0}^{\infty} 2^{-j} \mathcal{E}_{X^{(p)}}(f, 2^j t) \tag{20}$$

and

$$K(f,t,X^{(p)}, M^{1,X^{(p)}}) \simeq K(f,t,X^{(p)}, \mathring{M}^{1,X^{(p)}}) + \min(1,t) \|f\|_{X^{(p)}}.$$

Where

$$\mathcal{E}_{X^{(p)}}(f,t) := \left\| \left(\fint_{B(x,t)} |f(x) - f(y)|^p \, d\mu(y) \right)^{1/p} \right\|_{X^{(p)}}.$$

Proof. In [44, Theorem 11] we proved that

$$E_{X^{(p)}}(f,t) \preceq K(f,t,X^{(p)}, \mathring{\mathrm{M}}^{1,X^{(p)}}) \preceq \sum_{j=0}^{\infty} 2^{-j} E_{X^{(p)}}(f, 2^j t).$$

By Hölder's inequality we get $E_{X^{(p)}}(f,t) \leq \mathcal{E}_{X^{(p)}}(f,t)$, so the right hand side inequality of (20) follows. To see the left hand side inequality, given $f \in L^1 + L^\infty$, $1 \leq p < \infty$ and $t > 0$ we define define

$$T_r^p f(x) = \left(\fint_{B(x,r)} |f(y)|^p \, d\mu(y) \right)^{1/p}$$

we claim that the family of operators $\{T_r^p\}_{r>0}$ is uniformly bounded on $X^{(p)}$.

Obviously

$$\|T_r^p f\|_{L^\infty} \leq \|f\|_{L^\infty}. \tag{21}$$

On the other hand, by Fubini's theorem, we get

$$\begin{aligned} \|T_r^p f\|_{L^p}^p &= \int_\Omega \left(\fint_{B(x,r)} |f(y)|^p \, d\mu(y) \right) d\mu(x) \\ &= \int_\Omega |f(y)|^p \left(\int_{B(y,r)} \frac{1}{\mu(B(x,r))} d\mu(x) \right) d\mu(y). \end{aligned}$$

Using the doubling property and the fact that $B(x,r) \subset B(y,2r)$ whenever $y \in B(x,r)$, we conclude that

$$\begin{aligned}\int_{B(y,r)} \frac{1}{\mu(B(x,r))} d\mu(x) &\leq C_\mu \int_{B(y,r)} \frac{1}{\mu(B(x,2r))} d\mu(x)\\ &\leq C_\mu \int_{B(y,r)} \frac{1}{\mu(B(y,r))} d\mu(x)\\ &= C_\mu.\end{aligned}$$

Thus, for all $r > 0$

$$\|T_r^p f\|_{L^p}^p \leq C_\mu \int_\Omega |f(y)|^p \, d\mu(y) = C_\mu \, \|f\|_{L^p}^p \,. \tag{22}$$

By combining (21) and (22) with the definition of the K functional, we obtain

$$K(T_r^p f, t^{1/p}, L^p, L^\infty) \preceq K(f, t^{1/p}, L^p, L^\infty), \quad (t > 0).$$

Since (see [9, Theorem 5.1])

$$K(g, t^{1/p}, L^p, L^\infty) \simeq \left(\int_0^t (g^*(s))^p \, ds \right)^{1/p}$$

we have

$$\int_0^t \left((T_r^p f)^* (s) \right)^p ds \preceq \int_0^t (f^*(s))^p \, ds$$

which by (6) implies that

$$\|(T_r^p f)^p\|_X \preceq \||f|^p\|_X \,,$$

which is equivalent to

$$\|T_r^p f\|_{X^{(p)}} \preceq \|f\|_{X^{(p)}} \,. \tag{23}$$

Let $f = g + h$, where $g \in X^{(p)}$, $h \in \dot{M}^{1,X^{(p)}}$ and let $t > 0$. Then

$$\left(\fint_{B(x,t)} |g(x) - g(y)|^p \, d\mu(y) \right)^{1/p} \preceq |g(x)| + T_r^p g(x),$$

consequently, by the previous claim,

$$E_{X^{(p)}}(g,t) \preceq \|g\|_{X^{(p)}} + \|T_r^p g\|_{X^{(p)}} \preceq \|g\|_{X^{(p)}} \,.$$

On the other hand, since $h \in \dot{M}^{1,X^{(p)}}$, by the definition of the $1-$gradient, if $\varrho \in D(h) \cap X^{(p)}$, then

$$\begin{aligned}\left(\fint_{B(x,t)} |h(x) - h(y)|^p d\mu(y) \right)^{1/p} &\leq \left(\fint_{B(x,t)} d(x,y)^p |\varrho(x) + \varrho(y)|^p d\mu(y) \right)^{1/p}\\ &\leq \left(\fint_{B(x,t)} d(x,y)^p \left(\varrho(x)^p + \varrho(y)^p \right) d\mu(y) \right)^{1/p}\\ &\leq t \left(\varrho(x) + T_t^p \varrho(x) \right),\end{aligned}$$

hence

$$\begin{aligned}\mathcal{E}_{X^{(p)}}(h,t) &\leq t \, \|\varrho\|_{X^{(p)}} + t \, \|T_t^p \varrho \,\|_{X^{(p)}}\\ &\preceq t \, \|\varrho\|_{X^{(p)}} \quad \text{(by (23))}.\end{aligned}$$

In consequence

$$\mathcal{E}_{X^{(p)}}(h,t) \preceq t \inf_{\varrho \in D(u)} \|\varrho\|_{X^{(p)}} = t \|h\|_{\dot{M}^{1,X^{(p)}}(\Omega)},$$

and taking the infimum over all representations of f in $X^{(p)} + \dot{M}^{1,X^{(p)}}(\Omega)$, we obtain

$$\mathcal{E}_{X^{(p)}}(f,t) \preceq K(f,t,X^{(p)},\dot{M}^{1,X^{(p)}}).$$

□

Having determined the $K-$functional between X and $\mathring{\mathrm{M}}^{1,X}$ in terms of the $X-$modulus of smoothness, it is a known routine to determine the corresponding interpolation spaces.

Corollary 14. *Suppose X is an r.i. space on Ω and $p \geq 1$. If $0 < \theta < 1$, E is a parameter and $b \in SV$, then*

$$\mathring{B}^{\theta,b}_{X^{(p)},E} = (X^{(p)}, \mathring{M}^{1,X^{(p)}})_{\theta,b,E} \tag{24}$$

and

$$B^{\theta,b}_{X^{(p)},E} = (X^{(p)}, M^{1,X^{(p)}})_{\theta,b,E}, \tag{25}$$

with equivalent norms.

Proof. Let us write $K(f,t,X^{(p)},\mathring{\mathrm{M}}^{1,X^{(p)}}) = K(f,t)$. Obviously

$$\|f\|_{\mathring{\mathrm{B}}^{\theta,b}_{X^{(p)},E}} = \left\|t^{-\theta}b(t)E_{X^{(p)}}(f,t)\right\|_{\tilde{E}} \preceq \left\|t^{-\theta}b(t)K(f,t)\right\|_{\tilde{E}} \leq \|f\|_{(X^{(p)},\mathring{\mathrm{M}}^{1,X^{(p)}})_{\theta,b,E}}.$$

Conversely,

$$\begin{aligned}
\|f\|_{(X^{(p)},\mathring{\mathrm{M}}^{1,X^{(p)}})_{\theta,b,E}} &= \left\|t^{-\theta}b(t)K(f,t)\right\|_{\tilde{E}} \preceq \left\|t^{-\theta}b(t)\sum_{j=0}^{\infty} 2^{-j}E_{X^{(p)}}(f,2^jt)\right\|_{\tilde{E}} \\
&\leq \sum_{j=0}^{\infty} 2^{j(\theta-1)} \sup_{t>0}\frac{b(t)}{b(2^jt)} \left\|\left(2^jt\right)^{-\theta} b(2^jt)E_{X^{(p)}}(f,2^jt)\right\|_{\tilde{E}} \\
&\preceq \left(\sum_{j=0}^{\infty} 2^{j(\theta-1)} \sup_{t>0}\frac{b(t)}{b(2^jt)}\right) \left\|t^{-\theta}b(t)E_{X^{(p)}}(f,t)\right\|_{\tilde{E}} \text{ (by (10)).}
\end{aligned}$$

To complete the proof, we need to see that the above series is convergent. Consider $\varepsilon > 0$ such that $\theta - 1 + \varepsilon < 0$, then it follows from (11) that

$$\sup_{t>0}\frac{b(t)}{b(2^jt)} \preceq 2^{j\varepsilon}$$

which implies convergence of the series.

To see (25), we observe that from Lemma 3, $\left\|t^{-\theta}b(t)\min(1,t)\right\|_{\tilde{L}^1}$ and $\left\|t^{-\theta}b(t)\min(1,t)\right\|_{L^\infty}$ are both finite, so by interpolation $\left\|t^{-\theta}b(t)\min(1,t)\right\|_{\tilde{E}} < \infty$, hence

$$(X^{(p)}, M^{1,X^{(p)}})_{\theta,b,E} \simeq (X^{(p)}, \mathring{\mathrm{M}}^{1,X^{(p)}})_{\theta,b,E} + \|f\|_{X^{(p)}}.$$

Therefore

$$\begin{aligned}\|f\|_{B^{\theta,b}_{X^{(p)},E}} &= \left\|t^{-\theta}b(t)E_{X^{(p)}}(f,t)\right\|_{\tilde{E}(0,1)} + \|f\|_{X^{(p)}} \\ &\leq \left\|t^{-\theta}b(t)K(f,t)\right\|_{\tilde{E}} + \|f\|_{X^{(p)}} \\ &\simeq \|f\|_{(X^{(p)},M^{1,X^{(p)}})_{\theta,b,E}}.\end{aligned}$$

Conversely, $E_{X^{(p)}}(f,t) \leq \|f\|_{X^{(p)}}$ gives

$$\begin{aligned}\left\|t^{-\theta}b(t)K(f,t)\right\|_{\tilde{E}} &\simeq \left\|t^{-\theta}b(t)E_{X^{(p)}}(f,t)\right\|_{\tilde{E}} \\ &\simeq \left\|t^{-\theta}b(t)E_{X^{(p)}}(f,t)\right\|_{\tilde{E}(0,1)} + \left\|t^{-\theta}b(t)E_{X^{(p)}}(f,t)\right\|_{\tilde{E}(1,\infty)} \\ &\preceq \left\|t^{-\theta}b(t)E_{X^{(p)}}(f,t)\right\|_{\tilde{E}(0,1)} + \|f\|_{X^{(p)}} \left\|t^{-\theta}b(t)\right\|_{\tilde{E}(1,\infty)} \\ &\preceq \left\|t^{-\theta}b(t)E_{X^{(p)}}(f,t)\right\|_{\tilde{E}(0,1)} + \|f\|_{X^{(p)}} \ \text{(by Lemma 3)} \\ &= \|f\|_{B^{\theta,b}_{X^{(p)},E}}.\end{aligned}$$

□

Remark 15. *It is clear, that if in the previous corollary we use the functional $\mathcal{E}_{X^{(p)}}(f,t)$ instead of $E_{X^{(p)}}(f,t)$, then the same conclusion holds. In particular, since for $1 < p < \infty$, $L^p = \left(L^1\right)^{(p)}$, we get*

$$\overset{\circ}{B}{}^{\theta}_{L^p,q} = \left(L^p, \dot{M}^{1,p}\right)_{\theta,q} = \left(\left(L^1\right)^{(p)}, \dot{M}^{1,p}\right)_{\theta,q} = \dot{\mathcal{B}}^s_{p,q},$$

with equivalent norms. Similarly $B^{\theta}_{L^p,q} = \mathcal{B}^s_{p,q}$.

A direct application of Theorem 5.1 and Corollary 5.2 of [19] gives us

Theorem 16. *Suppose X is an r.i. space on Ω, E_0, E_1 and E parameters, and b, b_0 and b_1 $SV-$ functions. Then, for $0 < \theta_0 < \theta_1 < 1$ and $0 < \theta < 1$, we get*

$$\left(\overset{\circ}{B}{}^{\theta_0,b_0}_{X,E_0}, \overset{\circ}{B}{}^{\theta_1,b_1}_{X,E_1}\right)_{\theta,b,E} = \overset{\circ}{B}{}^{\tilde{\theta},\tilde{b}}_{X,E}$$

where $\tilde{\theta} = (1-\theta)\theta_0 + \theta\theta_1$ and $\tilde{b}(t) = b_0^{1-\theta}(t)b_1^{\theta}(t)b\left(t^{\theta_1-\theta_0}b_0(t)/b_1(t)\right)$, and

$$\left(\overset{\circ}{M}{}^{1,X}, \overset{\circ}{B}{}^{\theta_1,b_1}_{X,E_1}\right)_{\theta,b,E} = \overset{\circ}{B}{}^{\hat{\theta},\hat{b}}_{X,E}$$

where $\hat{\theta} = \theta\theta_1$ and $\hat{b}(t) = b_1^{\theta}(t)b\left(t^{\theta_1}/b_1(t)\right)$.

A similar result holds in the inhomogeneous case.

4.1. **Sobolev embedding theorems for Hajłasz-Besov spaces.** Throughout what follows we will write $K(f,t)$ instead of $K(f,t,X,\overset{\circ}{M}{}^{1,X})$.

Theorem 17. *Let (Ω, d, μ) be an $RD-$space with indices (k,n) that satisfy the non-collapsing condition. Let $R(t) = \max(t^{1/n}, t^{1/k})$ and $f \in \overset{\circ}{M}{}^{1,L^1+L^\infty}$. Then for all $t > 0$, we have*

$$O(f,t) \preceq \frac{K(f,R(t))}{\phi_X(t)}. \tag{26}$$

Proof. Fix $t>0$ and a decomposition $f=g+h$ with $g\in X$ and $h\in \dot{M}^{1,X}$, since $f^{*}(t)\leq g^{*}(t/2)+h^{*}(t/2)$ and $f^{*}(t)\geq h^{*}(2t)-g^{*}(t)$ we have that

$$\begin{aligned} f^{**}(t)-f^{*}(t) &\leq g^{**}(t)+h^{**}(t)+g^{*}(t)-h^{*}(2t) \\ &\leq \left(g^{**}(t)+g^{*}(t)\right)+\left(h^{**}(t)-h^{**}(2t)\right)+\left(h^{**}(2t)-h^{*}(2t)\right) \\ &\leq 2g^{**}(t)+2\left(h^{**}(2t)-h^{*}(2t)\right). \end{aligned} \tag{27}$$

On the other hand, by Hölder's inequality

$$g^{**}(t)=\frac{1}{t}\int_{0}^{\infty}g^{*}(s)\,\chi_{[0,t]}(s)ds\leq \|g\|_{X}\,\frac{\phi_{X'}(t)}{t}=\frac{\|g\|_{X}}{\phi_{X}(t)}. \tag{28}$$

By [43, Theorem 1], it follows that for all $\rho\in D(h)$ and for all $t>0$, we have

$$h^{**}(2t)-h^{*}(2t)\preceq R(t)g^{**}(t).$$

So, using the same argument as in (28)

$$\begin{aligned} h^{**}(2t)-h^{*}(2t) &\preceq R(2t)\rho^{**}(2t) \\ &\preceq R(t)\rho^{**}(t) \\ &\leq R(t)\frac{\|\rho\|_{X}}{\phi_{X}(t)} \\ &= R(t)\frac{\|h\|_{\mathring{M}^{1,X}}}{\phi_{X}(t)}. \end{aligned} \tag{29}$$

Putting (29) and (28) back into (27), we find that

$$f^{**}(t)-f^{*}(t)\preceq \frac{1}{\phi_{X}(t)}\left(\|g\|_{X}+R(t)\,\|h\|_{\dot{M}^{1,X}}\right).$$

Taking the infimum over all decompositions of $f=g+h$ as shown above gives the required estimate (26). □

The Theorem 17 is the key to prove the following Sobolev-type embedding results for generalised Hajłasz-Besov spaces.

Theorem 18. *Let (Ω,d,μ) be an $RD-$space with indices (k,n) which satisfies the non-collapsing condition. Let E be a parameter, $b\in SV$, $0<\theta<1$ and X be an r.i. space on Ω. Then for all $f\in L^{1}+L^{\infty}$ such that[6] $f^{*}(\infty)=0$, we have*

$$\left\|\frac{b(t^{1/n})\phi_{X}(t)}{t^{\theta/n}}O(f,t)\right\|_{\tilde{E}(0,1)}+\left\|\frac{b(t^{1/k})\phi_{X}(t)}{t^{\theta/k}}O(f,t)\right\|_{\tilde{E}(1,\infty)}\preceq \|f\|_{\mathring{B}^{\theta,b}_{X,E}}.$$

Moreover:

(i) *If $\underline{\beta}_{X}>\frac{\theta}{k}$, then*

$$\left\|\frac{b(t^{1/n})\phi_{X}(t)}{t^{\theta/n}}f^{**}(t)\right\|_{\tilde{E}(0,1)}+\left\|\frac{b(t^{1/k})\phi_{X}(t)}{t^{\theta/k}}f^{**}(t)\right\|_{\tilde{E}(1,\infty)}\preceq \|f\|_{\mathring{B}^{\theta,b}_{X,E}}. \tag{30}$$

(ii) *If $\frac{\theta}{n}<\underline{\beta}_{X}\leq\frac{\theta}{k}$, then*

$$\left\|\frac{b(t^{1/n})\phi_{X}(t)}{t^{\theta/n}}f^{**}(t)\right\|_{\tilde{E}(0,1)}+\left\|\frac{b(t^{1/k})\phi_{X}(t)}{t^{\theta/k}}O(f,t)\right\|_{\tilde{E}(1,\infty)}\preceq \|f\|_{\mathring{B}^{\theta,b}_{X,E}}+\|f\|_{L^{1}+L^{\infty}}.$$

(iii) *If $\underline{\beta}_{X}=\frac{\theta}{n}$, then*

[6] Condition $f^{*}(\infty)=0$ is equivalent to $\mu\{|f|>t\}<\infty$ for all $t>0$.

(a) *If* $\left\|\frac{t^{\theta/n}}{b(t^{1/n})\phi_X(t)}\right\|_{\tilde{E}'(0,1)} < \infty$, *then*

$$\|f\|_{L^\infty} \preceq \|f\|_{\mathring{B}^{\theta,b}_{X,E}} + \|f\|_{L^1+L^\infty}.$$

(b) *If the function* $b(t^{1/n})\phi_X(t)/t^{\theta/n}$ *is quasi-increasing and Boyd's indices* $\underline{\alpha}_E, \bar{\alpha}_E \in (0,1)$, *then*

$$\left\|\frac{b(t^{1/n})\phi_X(t)}{t^{\theta/n}\ell(t)} f^{**}(t)\right\|_{\tilde{E}(0,1)} \preceq \|f\|_{\mathring{B}^{\theta,b}_{X,E}} + \|f\|_{L^1+L^\infty}.$$

(c) *If*

$$\sup_{t\in(0,1)} \frac{t^{\theta/n}}{b(t^{1/n})\phi_X(t)} < \infty,$$

then[7]

$$\|f\|_{bmo} \preceq \|f\|_{\mathring{B}^{\theta,b}_{X,L^\infty}} + \|f\|_{L^1+L^\infty}.$$

(iv) *If* $\bar{\beta}_X < \frac{\theta}{n}$, *then*

$$\|f\|_{L^\infty} \preceq \|f\|_{\mathring{B}^{\theta,b}_{X,E}} + \|f\|_{L^1+L^\infty}.$$

Proof. It follows from (26) that

$$O(f,t)\phi_X(t) \preceq K(f,R(t))$$

thus

$$\frac{b(R(t))\phi_X(t)}{R(t)^\theta} O(f,t) \preceq b(R(t))\frac{K(f,R(t))}{R(t)^\theta}.$$

Whence

(31)

$$\begin{aligned}\left\|\frac{b(t^{1/n})\phi_X(t)}{t^{\theta/n}} O(f,t)\right\|_{\tilde{E}(0,1)} + \left\|\frac{b(t^{1/k})\phi_X(t)}{t^{\theta/k}} O(f,t)\right\|_{\tilde{E}(1,\infty)} &\simeq \left\|\frac{b(R(t))\phi_X(t)}{R(t)^\theta} O(f,t)\right\|_{\tilde{E}} \\ &\preceq \left\|b(R(t))\frac{K(f,R(t))}{R(t)^\theta}\right\|_{\tilde{E}} \\ &\preceq \left\|b(t)\frac{K(f,t)}{t^\theta}\right\|_{\tilde{E}} \quad \text{(by (10))} \\ &= \|f\|_{\mathring{\mathrm{B}}^{\theta,b}_{X,E}}.\end{aligned}$$

and we have the inequality (30).

(i) Let $\varphi(t) = \frac{b(R(t))\phi_X(t)}{R(t)^\theta}$, by (31) we know that

$$\|\varphi(t)O(f,t)\|_{\tilde{E}} \preceq \|f\|_{\mathring{\mathrm{B}}^{\theta,b}_{X,E}}.$$

Taking into account (5) and $f^*(\infty) = 0$, from the fundamental theorem of calculus we deduce that

$$f^{**}(t) = \int_t^\infty O(f,s)\frac{ds}{s},$$

[7]The local spaces of functions of bounded mean oscillation, *bmo*, consists of all locally integrable functions satisfying the condition

$$bmo(\Omega) = \left\{f : \sup_{\mu(B)\le 1} \fint_B \left|f(y) - \fint_B f(s)d\mu(s)\right| d\mu(y) + \sup_{\mu(B)>1} \fint_B f(s)d\mu(s)\right\}$$

where B is a ball in Ω. This definition coincides with [57, 2.2.2 (viii), p. 37].

thus we need to see that

$$\|\varphi(t)f^{**}(t)\|_{\tilde{E}} = \left\|\varphi(t)\int_t^\infty O(f,s)\frac{ds}{s}\right\|_{\tilde{E}} \preceq \|\varphi(t)O(f,t)\|_{\tilde{E}}. \tag{32}$$

To do this, we look at the operator

$$(Ug)(t) = \varphi(t)\int_t^\infty g(z)\frac{1}{\varphi(z)}\frac{dz}{z}, \quad (t>0),$$

and we will prove that it is bounded for any parameter $\tilde{E}$. Due to the interpolation properties of the space $\tilde{E}$, it suffices to prove that U is bounded in $\tilde{L}^1$ and in L^∞.

Since $b(R(t)) \in SV$, it follows from the proposition 4 that $\underline{\beta}_\varphi = -\frac{\theta}{k} + \underline{\beta}_X > 0$ and $\overline{\beta}_{1/\varphi} = -\underline{\beta}_\varphi < 0$, hence by (12)

$$\int_0^t \varphi(z)\frac{dz}{z} \preceq \varphi(t) \text{ and } \int_t^\infty \frac{1}{\varphi(z)}\frac{dz}{z} \preceq \frac{1}{\varphi(t)}.$$

Consequently

$$\begin{aligned}\|Ug\|_{\tilde{L}^1} &\leq \int_0^\infty \left(\varphi(t)\int_t^\infty |g(z)|\frac{1}{\varphi(z)}\frac{dz}{z}\right)\frac{dt}{t} \\ &= \int_0^\infty |g(t)|\,\varphi(t)\left(\int_0^t \frac{1}{\varphi(z)}\frac{dz}{z}\right)\frac{dt}{t} \preceq \|g\|_{\tilde{L}^1};\end{aligned}$$

and

$$\|Ug\|_{L^\infty} = \sup_{t>0}\int_t^\infty |g(z)|\,\varphi(t)\frac{dz}{z} \leq \|Ug\|_{L^\infty}\sup_{t>0}\int_t^\infty \frac{1}{\varphi(z)}\frac{dz}{z} \preceq \|g\|_{L^\infty}.$$

If we take $g(t) = O(f,t)\varphi(t)$, we have the desired inequality (32).

(ii) Using an argument analogous to the previous case, it follows that the operator

$$(Ug)(t) = \frac{b(t^{1/n})\phi_X(t)}{t^{\theta/n}}\int_t^1 g(z)\frac{z^{\theta/n}}{b(z^{1/n})\phi_X(z)}\frac{dz}{z}$$

is bounded in $\tilde{E}(0,1)$.

Taking $g(t) = \frac{b(t^{1/n})\phi_X(t)}{t^{\theta/n}}\left(f^{**}(t) - f^*(t)\right)$, we have that

$$\begin{aligned}Uh(t) &= \frac{b(t^{1/n})\phi_X(t)}{t^{\theta/n}}\int_t^1 \left(f^{**}(z) - f^*(z)\right)\frac{dz}{z} \\ &= \frac{b(t^{1/n})\phi_X(t)}{t^{\theta/n}}\left(f^{**}(t) - f^{**}(1)\right) \\ &= \frac{b(t^{1/n})\phi_X(t)}{t^{\theta/n}}\left(f^{**}(t) - \|f\|_{L^1+L^\infty}\right)\end{aligned} \tag{33}$$

thus

$$\begin{aligned}\left\|\frac{b(t^{1/n})\phi_X(t)}{t^{\theta/n}}f^{**}(t)\right\|_{\tilde{E}(0,1)} &\preceq \|Uh\|_{\tilde{E}(0,1)} \\ &\preceq \left\|\frac{b(t^{1/n})\phi_X(t)}{t^{\theta/n}}O(f,t)\right\|_{\tilde{E}(0,1)} + \|f\|_{L^1+L^\infty} \\ &\preceq \|f\|_{\mathring{B}^{\theta,b}_{X,E}} + \|f\|_{L^1+L^\infty}.\end{aligned}$$

(iii) case (a) From the fundamental theorem of calculus and the Hölder inequality, we obtain

$$\begin{aligned} f^{**}(t)-\|f\|_{L^1+L^\infty} &= \int_t^1 O(f,t)\frac{b(t^{1/n})\phi_X(t)}{t^{\theta/n}}\frac{t^{\theta/n}}{b(t^{1/n})\phi_X(t)}\frac{dt}{t} \\ &\leq \left\|O(f,t)\frac{b(t^{1/n})\phi_X(t)}{t^{\theta/n}}\right\|_{\tilde{E}(0,1)}\left\|\frac{s^{\theta/n}}{b(s^{1/n})\phi_X(s)}\chi_{[t,1]}(s)\right\|_{\tilde{E}'(0,1)} \text{ (by (8))} \end{aligned}$$

thus

$$\|f\|_{L^\infty}=\sup_{0<t<1} f^{**}(t)\preceq \|f\|_{\mathring{\mathrm{B}}^{\theta,b}_{X,E}}+\|f\|_{L^1+L^\infty},$$

as we want to show.

(iii) case (b) Consider the operator

$$(Ug)(t)=\frac{b(t^{1/n})\phi_X(t)}{t^{\theta/n}\ell(t)}\int_t^1 g(z)\frac{dz}{z}.$$

If $1<p<\infty$, then by [7, Theorem 6.5] we have

$$\begin{aligned} \|Ug\|_{\tilde{L}^p(0,1)} &\preceq \left(\int_0^1\left(\frac{1}{\ell(t)}\int_t^1|g(z)|\frac{b(z^{1/n})\phi_X(z)}{z^{\theta/n}}\frac{dz}{z}\right)^p\frac{dt}{t}\right)^{1/p} \\ &\leq \left(\int_0^1\left(g(t)\frac{b(t^{1/n})\phi_X(t)}{t^{\theta/n}}\right)^p\frac{dt}{t}\right)^{1/p} \\ &= \left\|g(t)\frac{b(t^{1/n})\phi_X(t)}{t^{\theta/n}}\right\|_{\tilde{L}^p(0,1)}, \end{aligned}$$

and

$$\begin{aligned} \|Ug\|_{L^\infty[0,1]} &\preceq \sup_{0<t<1}\left(\frac{1}{\ell(t)}\int_t^1|g(z)|\frac{b(z^{1/n})\phi_X(z)}{z^{\theta/n}}\frac{dz}{z}\right) \\ &\leq \sup_{0<t<1}\left(\sup_{t<s<1}\left|g(t)\frac{b(t^{1/n})\phi_X(t)}{t^{\theta/n}}\right|\left(\frac{1}{\ell(t)}\int_t^1\frac{dz}{z}\right)\right) \\ &\preceq \left\|g(t)\frac{b(t^{1/n})\phi_X(t)}{t^{\theta/n}}\right\|_{L^\infty[0,1]}. \end{aligned}$$

Choose $1<p<\infty$ such $\overline{\alpha}_E<1/p$, since also $\underline{\alpha}_E>0$, E is an interpolation space between $L^p[0,1]$ and $L^\infty[0,1]$ (see [39, Theorem 2.b.11]), therefore

$$\|Ug\|_{\tilde{E}(0,1)}\preceq\left\|\frac{b(t^{1/n})\phi_X(t)}{t^{\theta/n}}g(t)\right\|_{\tilde{E}(0,1)}.$$

Taking $g(t)=O(f,t)$, we get

$$\begin{aligned} \left\|\frac{b(t^{1/n})\phi_X(t)}{\ell(t)t^{\theta/n}}f^{**}(t)\right\|_{\tilde{E}(0,1)} &\preceq \left\|\frac{b(t^{1/n})\phi_X(t)}{t^{\theta/n}}O(f,t)\right\|_{\tilde{E}(0,1)}+\|f\|_{L^1+L^\infty} \\ &\preceq \|f\|_{\mathring{\mathrm{B}}^{\theta,b}_{X,E}}+\|f\|_{L^1+L^\infty}. \end{aligned}$$

(iii) case (c) Given $B=B(x,r)$ a ball centered at x with radius r, we have that

$$\begin{aligned} I(B) &:= \fint_B \left| f(y) - \fint_B f(s)d\mu(s) \right| d\mu(y) \\ &\leq \fint_B \left(\fint_B |f(y)-f(s)|\, d\mu(s) \right) d\mu(y) \\ &\leq \frac{E_X(f,r)}{\mu(B)} \|\chi_B\|_{X'} \text{ (by (8))} \\ &= \frac{E_X(f,r)}{\phi_X(\mu(B))} \\ &\preceq \frac{K(f,r)}{\phi_X(\mu(B))} \text{ (by Theorem 13).} \end{aligned}$$

We claim that given $t>0$ and $x\in\Omega$, there exists a ball $B(x)$ centered at x such that

$$t/C_D \leq \mu(B(x)) \leq t \tag{34}$$

furthermore, for any ball $B(x,r)$ satisfying (34), there exists a constant $c=c(c_0,\kappa,n,k)$, such that

$$r \leq cR(t). \tag{35}$$

Assuming for the moment the validity of (34) and (35), and considering that that $K(f,\cdot)$ is increasing, we get

$$\begin{aligned} \sup_{\mu(B)\leq 1} I(B) &\preceq \sup_{\mu(B)\leq 1} \frac{K(f,r)}{\phi_X(\mu(B))} \leq \sup_{0<t\leq 1} \sup_{t/C_D\leq\mu(B)\leq t\leq 1} \frac{K(f,r)}{\phi_X(\mu(B))} \\ &\leq \sup_{0<t\leq 1} \sup_{t/C_D\leq\mu(B)\leq t\leq 1} \frac{K(f,ct^{1/n})}{\phi_X(\mu(B))} \leq \sup_{0<t\leq 1} \frac{K(f,ct^{1/n})}{\phi_X(t/C_D)} \\ &\preceq \sup_{0<t\leq 1} \frac{K(f,t^{1/n})}{\phi_X(t)} \\ &\leq \sup_{0<t\leq 1} \frac{K(f,t^{1/n})b(t^{1/n})}{t^{\theta/n}} \left(\sup_{0<t\leq 1} \frac{t^{\theta/n}}{\phi_X(t)b(t^{1/n})} \right) \\ &\preceq \|f\|_{\mathring{\mathrm{B}}^{\theta,b}_{X,L^\infty}}, \end{aligned} \tag{36}$$

where in the third line of the above inequality we have used the concavity of the functions $K(f,\cdot)$ and $\phi_X(\cdot)$.

It remains to prove (34) and (35). Consider $r_0=\sup\{r:\mu(B(x,r))<t/C_D\}$, then

$$\mu(B(x,r_0)) \leq t/C_D \leq \mu(B(x,2r_0)) \leq C_D\mu(B(x,r_0)) \leq t.$$

If $B(x,r)$ satisfies (34), then obviously there exists a constant $c=c(c_0,\kappa,n,k)$ such that $R(t/c_0\kappa)\leq cR(t)$, so it will be enough to see that

$$r \leq R(t/c_0\kappa).$$

If $\frac{t}{c_0\kappa}\leq 1$, then $R(t/c_0\kappa)=(t/c_0\kappa)^{1/n}$, therefore, if $r>(t/c_0\kappa)^{1/n}$, then from (4) we will get $\mu(B(x,r))>t$, in contradiction to (34). In the case of $t/c_0\kappa\geq 1$, we proceed in a similar way.

Finally

$$\begin{aligned}
\|f\|_{bmo} &= \sup_{\mu(B)\leq 1} I(B) + \sup_{\mu(B)\geq 1} \fint_B |f(x)|\, d\mu(x) \\
&\preceq \|f\|_{\mathring{\mathrm{B}}^{\theta,b}_{X,L^\infty}} + \sup_{\mu(B)\geq 1} \fint_B |f(x)|\, d\mu(x) \text{ (by (36))} \\
&\leq \|f\|_{\mathring{\mathrm{B}}^{\theta,b}_{X,L^\infty}} + \sup_{\mu(B)\geq 1} \frac{1}{\mu(B)} \int_0^{\mu(B)} f^*(s)ds \\
&= \|f\|_{\mathring{\mathrm{B}}^{\theta,b}_{X,L^\infty}} + f^{**}(1) \\
&= \|f\|_{\mathring{\mathrm{B}}^{\theta,b}_{X,L^\infty}} + \|f\|_{L^1+L^\infty} .
\end{aligned}$$

(iv) Let $\varphi(t) = \frac{b(t^{1/n})\phi_X(t)}{t^{\theta/n}}$. Since $b(t^{1/n}) \in SV$, from the Proposition 4 we get $\overline{\beta}_\varphi = -\frac{\theta}{n} + \overline{\beta}_X < 0$ and $\underline{\beta}_{1/\varphi} = -\overline{\beta}_\varphi > 0$, therefore by (12)

$$\left\| \frac{1}{\varphi} \right\|_{\tilde{L}^1(0,t)} = \int_0^t \frac{1}{\varphi(s)} \frac{ds}{s} \preceq \frac{1}{\varphi(t)}.$$

With the choice of $\underline{\beta}_{1/\varphi} > a > 0$ we have the function $\frac{1}{\varphi(t)t^a}$ is almost increasing, therefore

$$\left\| \frac{1}{\varphi} \right\|_{L^\infty(0,t)} = \sup_{0<s<t} \frac{s^a}{\varphi(s)s^a} \preceq \frac{1}{\varphi(t)}.$$

Using interpolation, for any parameter E we have

$$\left\| \frac{1}{\varphi} \right\|_{\tilde{E}(0,t)} \preceq \frac{1}{\varphi(t)},$$

and, by Hölder inequality

$$\begin{aligned}
\|f\|_{L^\infty} = f^{**}(0) &= \int_0^1 O(f,t)\frac{dt}{t} + \|f\|_{L^1+L^\infty} \\
&\leq \|O(f,t)\varphi(t)\|_{\tilde{E}(0,1)} \left\| \frac{1}{\varphi(t)} \right\|_{\tilde{E}'(0,1)} + \|f\|_{L^1+L^\infty} \\
&\preceq \|f\|_{\mathring{\mathrm{B}}^{\theta,b}_{X,E}} + \|f\|_{L^1+L^\infty}
\end{aligned}$$

as we wanted to see. □

Now we consider the essential continuity problem and obtain Morrey type results for functions in generalised Hajłasz-Besov spaces.

Theorem 19. *Let (Ω, d, μ) be an RD−space with indices (k, n) which satisfies the non-collapsing condition and X an r.i. space on Ω and $f \in X \cap L^\infty$. Then*

$$|f(x) - f(y)| \preceq \int_0^{8d(x,y)} \frac{E_X(f,s)}{\varphi_X\left(\min(s^k, s^n)\right)} \frac{ds}{s}.$$

In particular, if

$$\int_0^1 \frac{E_X(f,s)}{\varphi_X\left(s^n\right)} \frac{ds}{s} < \infty,$$

then, f is essentially continuous.

Proof. Given $r > 0$ and $f \in X \cap L^\infty$, let

$$\nabla_r f(x) := \fint_{B(x,r)} |f(x) - f(y)|\, dy,$$

Since $\nabla_r f \in X \cap L^\infty$, it follows that

$$\lim_{q\to\infty} E_{X^{(q)}}(f,r) = \lim_{q\to\infty} \|(\nabla_r f)^q\|_X^{1/q} = \sup_{x\in\Omega} \nabla_r f(x).$$

On the other hand, from Lemma 7 (ii) we get (notice that $\mathcal{E}_{X^{(1)}}(f,s) = E_X(f,s)$)

$$\lim_{q\to\infty} E_{X^{(q)}}(f,r) \leq c \int_0^{4r} \frac{E_X(f,s)}{\varphi_X\left(\min(s^k,s^n)\right)} \frac{ds}{s}. \tag{37}$$

Let $B(x,r)$ be a ball centered at x with radius r, since if $y \in B(x,r)$, then $B(x,r) \subset B(y,2r)$ we have

$$\begin{aligned}
|f(x) - f(y)| &\leq \fint_{B(x,r)} |f(x) - f(z)|\, d\mu(z) + \fint_{B(x,r)} |f(z) - f(y)|\, d\mu(z) \\
&\preceq \fint_{B(x,2r)} |f(x) - f(z)|\, d\mu(z) + \fint_{B(y,2r)} |f(z) - f(y)|\, d\mu(z) \text{ (since } \mu \text{ is doubling)} \\
&\leq \sup_{x\in\Omega} \fint_{B(x,2r)} |f(x) - f(z)|\, d\mu(z) + \sup_{y\in\Omega} \fint_{B(y,2r)} |f(z) - f(y)|\, d\mu(z) \\
&\preceq \int_0^{8r} \frac{E_X(f,s)}{\varphi_X\left(\min(s^k,s^n)\right)} \frac{ds}{s} \text{ (by (37)).}
\end{aligned}$$

So, if $x, y \in \Omega$ are such that $r/8 = d(x,y) < 1$, then

$$|f(x) - f(y)| \preceq \int_0^{d(x,y)} \frac{E_X(f,s)}{\varphi_X\left(s^n\right)} \frac{ds}{s}$$

and the essential continuity of f follows. □

Corollary 20. *Let (Ω, d, μ) be an RD−space with indices (k,n) which satisfies the non-collapsing condition. Let E be a parameter, $b \in SV$, $0 < \theta < 1$ and X be an r.i. space on Ω. Let $f \in B^{\theta,b}_{X,E}$, then*

(i) *If $\underline{\beta}_X = \frac{\theta}{n}$ and $\left\| \frac{t^{\theta/n}}{b(t^{1/n})\phi_X(t)} \right\|_{\tilde{E}'(0,1)} < \infty$, then f is essentially continuous.*

(ii) *If $\underline{\beta}_X < \frac{\theta}{n}$, then for any $\bar{\beta}_X \leq \gamma \leq 1$ such that the function $\frac{\phi_X(t)}{t^\gamma}$ is quasi decreasing, the following Morrey type embedding result holds:*

$$|f(x) - f(y)| \preceq \|f\|_{B^{\theta,b}_{X,E}} \frac{d(x,y)^{\theta-\gamma n}}{b(d(x,r))}.$$

Proof. (i) Since $\underline{\beta}_X = \frac{\theta}{n}$, by Theorem 18 (iii) we have that $f \in L^\infty \cap X$. Furthermore, by Hölder's inequality, we get

$$\begin{aligned}
\int_0^1 \frac{E_X(f,s)}{\varphi_X(s^n)}\frac{ds}{s} &= \int_0^1 s^{-\theta} b(s)\mathcal{E}_X(f,s)\frac{s^\theta}{b(s)\varphi_X(s^n)}\frac{ds}{s} \\
&\le \left\|s^{-\theta}b(s)E_X(f,s)\right\|_{\tilde{E}(0,1)}\left\|\frac{s^\theta}{b(s)\varphi_X(s^n)}\right\|_{\tilde{E}'(0,1)} \\
&\preceq \left\|s^{-\theta}b(s)E_X(f,s)\right\|_{\tilde{E}(0,1)}\left\|\frac{t^{\theta/n}}{b(t^{1/n})\phi_X(t)}\right\|_{\tilde{E}'(0,1)} \\
&\preceq \|f\|_{B^{\theta,b}_{X,E}}
\end{aligned}$$

and the essential continuity of f follows from the Theorem 19.

(ii) Since $\underline{\beta}_X < \frac{\theta}{n}$, by Theorem 18 (iv) $f \in X \cap L^\infty$ so by Theorem 19.

$$|f(x) - f(y)| \preceq \int_0^{8r} \frac{E_X(f,s)}{\varphi_X(\min(s^k, s^n))}\frac{ds}{s}.$$

Considering that $\gamma n - \theta < 0$ and that the function $\frac{\phi_X(t)}{t^\gamma}$ is quasi decreasing, we get the following

$$\begin{aligned}
r^{\gamma n-\theta}b(r)\,|f(x)-f(y)| &\preceq r^{\gamma n-\theta}b(r)\int_0^{8r}\frac{E_X(f,s)}{\varphi_X(\min(s^k,s^n))}\frac{ds}{s} \\
&\le \sup_{r>0}\left((8r)^{\gamma n-\theta}\,b(8r)\int_0^{8r}\frac{E_X(f,s)}{\varphi_X(\min(s^k,s^n))}\frac{ds}{s}\right) \\
&\preceq \sup_{r>0}\left(r^{\gamma n-\theta}b(r)\frac{E_X(f,r)}{\varphi_X(\min(r^k,r^n))}\right) \text{ (by Lemma 3)} \\
&\le \sup_{0<r<1} b(r)t^{\theta-\gamma n}\left(\frac{r^{\gamma n}}{\varphi_X(r^n)}\right)\frac{E_X(f,r)}{r^{\gamma n}} + \sup_{r\ge 1} b(r)r^{\theta-\gamma n}\frac{E_X(f,r)}{\varphi_X(r^k)} \\
&\preceq \sup_{0<r<1} b(r)t^{-\theta}E_X(f,t) + \sup_{r\ge 1} b(r)r^{\theta-\gamma n}\frac{E_X(f,r)}{\varphi_X(r^k)} \\
&\le \sup_{0<r<1} b(r)t^{-\theta}E_X(f,t) + \sup_{r\ge 1} b(r)r^{\theta-\gamma n}E_X(f,r) \\
&\preceq \|f\|_{\mathring{B}^{\theta,b}_{X,L^\infty}} + \|f\|_X \quad \text{(by (16))} \\
&= \|f\|_{B^{\theta,b}_{X,L^\infty}} \\
&\preceq \|f\|_{B^{\theta,b}_{X,E}} \text{ (by Proposition 5).}
\end{aligned}$$

Finally, if $x, y \in \Omega$ are such that $r = d(x,y)$, then

$$|f(x) - f(y)| \preceq \|f\|_{B^{\theta,b}_{X,E}} \frac{d(x,y)^{\theta-\gamma n}}{b(d(x,r))}.$$

□

Remark 21. *For finite measure metric spaces with convex isoperimetric profile, a related result was obtained in* [41, Chapter 4].

We conclude this section with an example and some comments.

Example 22. *Let (Ω,d,μ) be an RD−space with indices (k,n) that satisfies the non-collapsing condition. Let X be an r.i. space on Ω, $E=L^q$ $(1\leq q\leq\infty)$, $b\in SV$ and $0<\theta<1$. Assume that $\varphi_X(t)=t^{1/p}\psi(t)$ with $1\leq p<\infty$ and $\psi\in SV$. Then*

(i) *If $\frac{1}{p}>\frac{\theta}{k}$, then*

$$\left(\int_0^1\left(b(t^{1/n})\psi(t)f^{**}(t)\,t^{\frac{1}{p}-\frac{\theta}{n}}\right)^q\frac{dt}{t}\right)^{1/q}+\left(\int_1^\infty\left(b(t^{1/n})\psi(t)f^{**}(t)\,t^{\frac{1}{p}-\frac{\theta}{k}}\right)^q\frac{dt}{t}\right)^{1/q}$$
$$\preceq\|f\|_{\mathring{B}^{b,\theta}_{X,L^q}}.$$

(ii) *If $\frac{\theta}{n}<\frac{1}{p}\leq\frac{\theta}{k}$, then*

$$\left(\int_0^1\left(b(t^{1/n})\psi(t)f^{**}(t)\,t^{\frac{1}{p}-\frac{\theta}{n}}\right)^q\frac{dt}{t}\right)^{1/q}+\left(\int_1^\infty\left(b(t^{1/n})\psi(t)O(f,t)t^{\frac{1}{p}-\frac{\theta}{k}}\right)^q\frac{dt}{t}\right)^{1/q}$$
$$\preceq\|f\|_{\mathring{B}^{b,\theta}_{X,L^q}}+\|f\|_{L^1+L^\infty}.$$

(iii) *If $\frac{\theta}{n}=\frac{1}{p}$, then:*

(a) *If $\sup_{t\in(0,1)}\frac{1}{b\left(s^{1/n}\right)\psi(s)}<\infty$, then*

$$\|f\|_{L^\infty}\preceq\|f\|_{\mathring{B}^{b,\theta}_{X,L^1}}+\|f\|_{L^1+L^\infty}.$$

and f is and essentially continuous.

(b) *If $b\left(s^{1/n}\right)\psi(s)$ is quasi-increasing and $1<q<\infty$, then*

$$\left(\int_0^1\left(\frac{b(t^{1/n})\psi(s)}{1+\ln\left(\frac{1}{s}\right)}f^{**}(s)\right)^q\frac{ds}{s}\right)^{1/q}\preceq\|f\|_{\mathring{B}^{b,\theta}_{X,L^q}}+\|f\|_{L^1+L^\infty}.$$

(c) *If $0<\frac{1}{p}<1$, then*

$$\|f\|_{bmo}\leq\|f\|_{\mathring{B}^{b,\theta}_{X,L^\infty}}+\|f\|_{L^1+L^\infty},$$

(iv) *If $\frac{1}{p}<\frac{\theta}{n}$, then*

$$\|f\|_{L^\infty}\preceq\|f\|_{\mathring{B}^{b,\theta}_{X,L^q}}+\|f\|_{L^1+L^\infty}.$$

Moreover

$$|f(x)-f(y)|\preceq\|f\|_{B^{\theta,b}_{X,E}}\frac{d(x,y)^{\theta-\frac{n}{p}}}{b(d(x,r))}$$

in case that $\psi(s)$ is quasi-increasing, otherwise, for any $\frac{1}{p}<\gamma\leq1$

$$|f(x)-f(y)|\preceq\|f\|_{B^{\theta,b}_{X,E}}\frac{d(x,y)^{\theta-\gamma n}}{b(d(x,r))}.$$

Remark 23. *The above example includes the case where X is a Lebesgue space, a Lorentz space, a Lorentz-Zygmund space or a generalized Lorentz-Zygmund spaces.*

Remark 24. *In the particular case $\Omega=\mathbb{R}^n$, $E=L^q$ $(1\leq q\leq\infty)$, $b=1$ and $X=L^p$ $(1\leq p<\infty)$ we obtain the following well-known result:*

(i) *If $\frac{1}{p}>\frac{\theta}{n}$, then*

$$\left(\int_0^\infty\left(f^{**}(t)\,t^{\frac{1}{p}-\frac{\theta}{n}}\right)^q\frac{dt}{t}\right)^{1/q}\preceq\|f\|_{B^{\theta}_{p,q}}.$$

(ii) *If* $\frac{\theta}{n} = \frac{1}{p}$, *then*

$$\|f\|_{L^\infty} \preceq \|f\|_{B^\theta_{p,1}},$$

and f *is and essentially continuous.*

$$\left(\int_0^1 \left(\frac{f^{**}(s)}{1+\ln\frac{1}{s}}\right)^q \frac{ds}{s}\right)^{1/q} \preceq \|f\|_{B^\theta_{p,q}} + \|f\|_{L^1+L^\infty} \quad (\textit{if } 1<q<\infty),$$

and

$$\|f\|_{bmo} \preceq \|f\|_{B^\theta_{p,\infty}}.$$

(iii) *If* $\frac{1}{p} < \frac{\theta}{n}$, *then*

$$\|f\|_{L^\infty} \preceq \|f\|_{B^\theta_{p,q}}$$

and

$$|f(x)-f(y)| \preceq \|f\|_{B^\theta_{p,q}(\Omega)}\, d(x,y)^{\theta-\frac{n}{p}}.$$

5. Appendix

In this section we give the proof of the lemma 7, which follows from a persual of the proofs of Lemma 2.3 and Proposition 2.5 of [52] combined with Hölder's inequality (8).

Proof. Part 1) By Hölder's inequality, we have

$$\begin{aligned}
\left(T^1_R f(x)\right)^q &:= \left(\fint_{B(x,R)} |f(y)|\, d\mu(y)\right)^q \\
&\leq \left(\fint_{B(x,R)} |f(y)|^p\, d\mu(y)\right)^{q/p} \\
&= \mu(B(x,R))^{q/p} \int_{B(x,R)} |f(y)|^p\, d\mu(y) \left(\int_{B(x,R)} |f(y)|^p\, d\mu(y)\right)^{-1+q/p} \\
&\leq \mu(B(x,R))^{q/p} \int_{B(x,R)} |f(y)|^p\, d\mu(y) \left(\||f|^p\|_X^{-1+q/p}\, \varphi_{X'}\left(\mu(B(x,R))\right)^{-1+q/p}\right) \\
&\leq \frac{\||f|^p\|_X^{-1+q/p}}{\inf_{x\in\Omega}\varphi_X(B(x,R))^{q/p-1}} \left(\fint_{B(x,R)} |f(y)|^p\, d\mu(y)\right) \text{ (by (9)).}
\end{aligned}$$

Taking $\|\cdot\|_X$

$$\begin{aligned}
\left\|\left(T^1_R f(x)\right)^q\right\|_X &\leq \frac{\||f|^p\|_X^{-1+q/p}}{\inf_{x\in\Omega}\varphi_X(B(x,R))^{q/p-1}} \left\|\fint_{B(x,R)} |f(y)|^p\, d\mu(y)\right\|_X \\
&\preceq \frac{\||f|^p\|_X^{-1+q/p}}{\inf_{x\in\Omega}\varphi_X(B(x,R))^{q/p-1}} \||f|^p\|_X \text{ (by Lemma 7 (i))} \\
&= \frac{\||f|^p\|_X^{q/p}}{\inf_{x\in\Omega}\varphi_X(B(x,R))^{q/p-1}} \\
&\preceq \frac{\|f\|^q_{X^{(p)}}}{\varphi_X(\min(R^k,R^n))^{q/p-1}} \text{ (by (4)).}
\end{aligned}$$

Part 2) Let us write

$$I_R(f,x)^q = \left(\fint_{B(x,2R)} \fint_{B(x,R)} |f(y)-f(z)|\, d\mu(z) d\mu(y) \right)^q$$

and

$$\nabla_R^p f(x) = \left(\fint_{B(x,R)} |f(y)-f(x)|^p\, d\mu(y) \right)^{1/p}.$$

From Hölder's inequality we can deduce that (see [52, Page 7])

$$I_R(f,x)^q \preceq \frac{1}{\mu(B(x,R))^{q/p-1}} \left(\int_{B(x,3R)} \fint_{B(x,3R)} |f(y)-f(z)|^p\, d\mu(z) d\mu(y) \right)^{-1+q/p} \left(\nabla_{2R}^p f(x)\right)^p.$$

On the other hand by (8),

$$\begin{aligned} &\left(\int_{B(x,3R)} \fint_{B(x,3R)} |f(y)-f(z)|^p\, d\mu(z) d\mu(y) \right)^{-1+q/p} \\ &\leq \left\| \fint_{B(x,3R)} |f(y)-f(z)|^p\, d\mu(z) \right\|_X^{-1+q/p} \varphi_{X'}\left(\mu(B(x,3R))\right)^{-1+q/p} \\ &= \left\| \left(\nabla_{3R}^p f(x)\right)^p \right\|_X^{-1+q/p} \varphi_{X'}\left(\mu(B(x,3R))\right)^{-1+q/p}. \end{aligned}$$

So, using the concavity of φ_X and the fact that μ is a doubling, we get

$$\begin{aligned} I_R(f,x)^q &\preceq \left(\frac{\varphi_{X'}\left(\mu(B(x,3R))\right)}{\mu(B(x,R))} \right)^{q/p-1} \left\| \left(\nabla_{3R}^p f(x)\right)^p \right\|_X^{-1+q/p} \left(\nabla_{2R}^p f(x)\right)^p \\ &\preceq \frac{1}{\inf_{x\in\Omega} \varphi_X(B(x,R))^{q/p-1}} \left\| \left(\nabla_{3R}^p f(x)\right)^p \right\|_X^{-1+q/p} \left(\nabla_{3R}^p f(x)\right)^p, \end{aligned}$$

and, taking the X norm in the above expression we get

$$\|I_R(f,x)^q\|_X \preceq \frac{1}{\inf_{x\in\Omega} \varphi_X(B(x,R))^{q/p-1}} \left\| \left(\nabla_{3R}^p f(x)\right)^p \right\|_X^{q/p},$$

or equivalently

$$\|I_R(f,x)\|_{X^{(q)}} \preceq \frac{1}{\inf_{x\in\Omega} \varphi_X(B(x,R))^{1/p-1/q}} \mathcal{E}_{X^{(p)}}(f,3R). \tag{38}$$

On the other hand, for every integer m, we have

$$\begin{aligned} &\fint_{B(x,R)} \fint_{B(x,R/2^{m+1})} |f(y)-f(z)|\, d\mu(z) d\mu(y) \\ &\leq \sum_{i=0}^{m} \fint_{B(x,R/2^i)} \fint_{B(x,R/2^{i+1}))} |f(y)-f(z)|\, d\mu(z) d\mu(y). \end{aligned} \tag{39}$$

Consequently,

$$\begin{aligned} &\left\| \fint_{B(x,R)} \fint_{B(x,R/2^{m+1})} |f(y)-f(z)|\, d\mu(z) d\mu(y) \right\|_{X^{(q)}} \\ &\leq \sum_{i=0}^{m} \left\| \fint_{B(x,R/2^i)} \fint_{B(x,R/2^{i+1}))} |f(y)-f(z)|\, d\mu(z) d\mu(y) \right\|_{X^{(q)}}. \end{aligned}$$

On the other hand, for every non-negative integer m we have

$$\left\| \fint_{B(x,R)} \left| f(y) - \fint_{B(x,R/2^{m+1})} f(z) d\mu(z) \right| d\mu(y) \right\|_{X^{(q)}} \tag{40}$$
$$\leq \left\| \fint_{B(x,R)} \fint_{B(x,R/2^{m+1})} |f(y) - f(z)|\, d\mu(z) d\mu(y) \right\|_{X^{(q)}}.$$

From Lebesgue's differentiation theorem (for the doubling measures), Fatou's lemma and letting $m \to \infty$, it follows that

$$\begin{aligned}
E_{X^{(q)}}(f,R) &= \left\| \fint_{B(x,R)} |f(y) - f(x)|\, d\mu(z) d\mu(y) \right\|_{X^{(q)}} \\
&= \left\| \fint_{B(x,R)} \lim_{m\to\infty} \left| f(y) - \fint_{B(x,R/2^{m+1})} f(z) d\mu(z) \right| d\mu(y) \right\|_{X^{(q)}} \\
&\leq \left\| \lim_{m\to\infty} \fint_{B(x,R)} \fint_{B(x,R/2^{m+1})} |f(y) - f(z)|\, d\mu(z) d\mu(y) \right\|_{X^{(q)}} \quad \text{(by (40))} \\
&\leq \left\| \sum_{i=0}^{\infty} \fint_{B(x,R/2^i)} \fint_{B(x,R/2^{i+1}))} |f(y) - f(z)|\, d\mu(z) d\mu(y) \right\|_{X^{(q)}} \quad \text{(by (39))} \\
&\leq \sum_{i=0}^{\infty} \frac{1}{\inf_{x\in\Omega} \varphi_X(B(x,R/2^{i+1}))^{1/p-1/q}} \mathcal{E}_{X^{(p)}}(f, 3R/2^{i+1}) \quad \text{(by (38))} \\
&\leq \sum_{i=0}^{\infty} \frac{1}{\inf_{x\in\Omega} \varphi_X(B(x,R/2^{i-2}))^{1/p-1/q}} \mathcal{E}_{X^{(p)}}(f, 3R/2^{i-1}) \\
&\leq \sum_{i=0}^{\infty} \int_{R/2^{i-1}}^{R/2^{i-2}} \frac{\mathcal{E}_{X^{(p)}}(f,r)}{\inf_{x\in\Omega} \varphi_X(B(x,r))^{1/p-1/q}} \frac{dr}{r} \quad \text{(by (13))} \\
&\preceq \int_0^{4R} \frac{\mathcal{E}_{X^{(p)}}(f,r)}{\varphi_X\left(\min(r^k, r^n)\right)^{1/p-1/q}} \frac{dr}{r} \quad \text{(by (4))}.
\end{aligned}$$

□

Conflict of interest

The authors declare that they have no conflict of interest.

Data availability statement

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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