

WEIGHTED ESTIMATES FOR FRACTIONAL INTEGRALS WITH DISTANCES TO BOUNDED MEDIAN POROUS SETS AND APPLICATIONS TO HARDY–SOBOLEV INEQUALITIES

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ABSTRACT. Weighted estimates for the fractional integral operator I_α , governed by distance functions to bounded median porous sets are established and then applied to obtain corresponding Hardy-Sobolev inequalities.

1. INTRODUCTION AND MAIN RESULTS

For $1 < p < n$ the classical Hardy or weighted Hardy-Sobolev inequality establishes the existence of $C_{p,n} > 0$ such that

$$\int_{\mathbb{R}^n} |f(x)|^p |x|^{-p} dx \leq C_{p,n} \int_{\mathbb{R}^n} |\nabla f(x)|^p dx, \quad \forall f \in C_0^1(\mathbb{R}^n), \quad (1.1)$$

where $C_0^1(\mathbb{R}^n)$ denotes the class of continuously differentiable functions vanishing at ∞ . More generally, for $1 < p \leq q \leq np/(n-p) < \infty$, H. Egnell [5, Lemma 7] (see also V. G. Maz'ja's [14, Section 1.3.1]) proved the inequality

$$\left(\int_{\mathbb{R}^n} |f(x)|^q |x|^{\frac{q}{p}(n-p)-n} dx \right)^{1/q} \leq C_{n,p,q} \left(\int_{\mathbb{R}^n} |\nabla f(x)|^p dx \right)^{1/p}, \quad \forall f \in C_0^1(\mathbb{R}^n). \quad (1.2)$$

If $q = p$ then (1.2) reduces to the Hardy inequality (1.1). Given a set $E \subset \mathbb{R}^n$, the distance function to E is defined for $x \in \mathbb{R}^n$ as $d(x, E) := \inf\{|x - y| : y \in E\}$. When $|x| = d(x, \{0\})$ is replaced with the distance to a (nonempty) closed set $E \subset \mathbb{R}^n$, it has been proved by J. Lehrbäck and A. Vähäkangas in [12, Theorem 1.1] that given $1 \leq p \leq q < np/(n-p) < \infty$, the Hardy-type inequality

$$\left(\int_{\mathbb{R}^n} |f(x)|^q d(x, E)^{\frac{q}{p}(n-p)-n} dx \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |\nabla f(x)|^p dx \right)^{1/p}, \quad \forall f \in C_0^1(\mathbb{R}^n), \quad (1.3)$$

holds true if and only if $\dim_A(E) < q(n-p)/p$ (here $\dim_A(E)$ denotes the Assouad dimension of E , see Section 2.4.2).

Back to powers of $|x|$, extensions of (1.2) include the classical Caffarelli-Kohn-Nirenberg inequalities proved in [2]: given $1 < p \leq q \leq np/(n-p) < \infty$ and $\beta > -(n-p)$ there exists $C = C(p, q, \beta, n) > 0$ such that

$$\left(\int_{\mathbb{R}^n} |f(x)|^q |x|^{q(n-p+\beta)/p-n} dx \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |\nabla f(x)|^p |x|^\beta dx \right)^{1/p}, \quad \forall f \in C_0^1(\mathbb{R}^n). \quad (1.4)$$

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Now, by considering $d(x, E)$ instead of $|x|$ in (1.4) for a closed set $E \subset \mathbb{R}^n$, $1 \leq p, q < \infty$, and $\beta \geq 0$ with $q(n - p + \beta)/p \neq n$, from [12, Theorem 6.1] the inequality

$$\left(\int_{\mathbb{R}^n} |f(x)|^q d(x, E)^{q(n-p+\beta)/p-n} dx \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |\nabla f(x)|^p d(x, E)^\beta dx \right)^{1/p}, \quad (1.5)$$

for some constant $C > 0$ and every $f \in C_0^1(\mathbb{R}^n)$, implies $\dim_A(E) < q(n - p + \beta)/p$. In particular, if $q(n - p + \beta)/p < n$, then $\dim_A(E) < n$.

Let us mention that in [4, Theorem 4.1], B. Dyda, L. Ichnatsyeva, J. Lehrbäck, H. Tuominen, and A. Vähäkangas proved that for $1 < p \leq q \leq pn/(n - p) < \infty$ and $\beta \in \mathbb{R}$, the condition

$$\dim_A(E) < \min\{q(n - p + \beta)/p, n - \beta/(p - 1)\} \quad (1.6)$$

is sufficient for (1.5) to hold. By setting $\text{codim}_A(E) := n - \dim_A(E)$, the inequality (1.6) can be recast as

$$\text{codim}_A(E) > \min\{n - q(n - p + \beta)/p, \beta/(p - 1)\}. \quad (1.7)$$

A set $E \subset \mathbb{R}^n$ with $\dim_A(E) < n$ (that is, $\text{codim}_A(E) > 0$) is called a *porous set* (see Section 2.5.1 for equivalent definitions). The class of porous sets includes that of λ -regular sets for $0 < \lambda < n$, where a closed set $E \subset \mathbb{R}^n$ is called λ -regular if there exists $C \geq 1$ such that

$$C^{-1}r^\lambda \leq \mathcal{H}^\lambda(E \cap B(x, r)) \leq Cr^\lambda, \quad \forall x \in E, 0 < r < \text{diam}(E),$$

where $\text{diam}(E) := \sup\{|x - y| : x, y \in E\}$ indicates the diameter of E and \mathcal{H}^λ stands for the λ -dimensional Hausdorff measure. If E is λ -regular, $\dim_A(E) = \lambda$ (see for instance [11, Theorem 10.21]) which makes E porous when $\lambda < n$.

The porosity of E being then a necessary condition for (1.3) and (1.5) leads to the question: Are there corresponding inequalities to (1.3) or (1.5) in the case of nonporous sets? In order to address this question, let us recall the notions of weak and median porosity introduced in [1] and [15] (see Section 2.5 for details). Following T. Anderson, J. Lehrbäck, C. Mudarra, and A. Vähäkangas in [1] a nonempty set $E \subset \mathbb{R}^n$ is *weakly porous* if there exists $\alpha > 0$ such that $d(\cdot, E)^{-\alpha}$ belongs to the Muckenhoupt class $A_1(\mathbb{R}^n)$ (the $A_p(\mathbb{R}^n)$ classes are reviewed in Section 2.1), while after M. Pasquariello and I. Uriarte-Tuero in [15], the set E is *median porous* if there exists $\alpha > 0$ such that $d(\cdot, E)^{-\alpha}$ belongs to $A_\infty(\mathbb{R}^n)$. As it turns out,

$$\text{porosity} \Rightarrow \text{weak porosity} \Rightarrow \text{median porosity}$$

with the converses to the above implications being false in general.

As further described in Section 2.5, the properties of weak and median porosity for E are quantified by a family of nonnegative indices $\{\text{Mu}_p(E)\}_{1 \leq p \leq \infty}$ (with Mu after Muckenhoupt) which come to refine the notion of the codimension codim_A in the sense that E is weakly porous if and only if $\text{Mu}_1(E) > 0$ (with $\text{Mu}_1(E) = \text{codim}_A(E)$ whenever E is porous); whereas E is median porous if and only if $\text{Mu}_\infty(E) > 0$ (with $\text{Mu}_p(E) = \text{codim}_A(E)$ for every $1 \leq p \leq \infty$ whenever E is weakly porous). For example, in [1, Section 7] it is proved that for $n \in \mathbb{N}$ and $\gamma > 0$, the bounded set

$$E := \bigcup_{j \in \mathbb{N}} \partial B(0, j^{-\gamma}) \cup \{0\} \subset \mathbb{R}^n$$

is weakly porous, but not porous, and it satisfies

$$\dim_A(E) = n, \quad \overline{\dim}_M(E) = \max \left\{ n - 1, \frac{n}{1 + \gamma} \right\}, \quad \text{Mu}_1 = n - \overline{\dim}_M(E).$$

Here $\overline{\dim}_M(E)$ stands for the upper Minkowski dimension of $E \subset \mathbb{R}^n$ (see Section 2.4.1). In addition, in [15, Section 9], the authors showed that for $0 < \gamma < 1$, the subset of the real line $E_\gamma := \{\pm m^\gamma : m \in \mathbb{N}\}$ is a median porous set which is not weakly porous.

In the context of the weighted Hardy inequality (1.5), the first result to “break the porosity barrier” comes from [15, Theorem 10.8(a)] where for $n \geq 2$, $1 < p < n$, and $q = p^* := np/(n - p)$ (the critical index), the conditions

$$\text{Mu}_\infty(E) > n - q(n - p + \beta)/p \quad \text{and} \quad -\text{Mu}_p(E) < \beta < 0$$

are shown to imply (1.5); thus, providing a sufficient condition, counterpart to the sufficient condition (1.7), within the median porous case.

We are now in position to state our main results on weighted estimates for fractional integrals which, in turn, will produce a number of Hardy-Sobolev inequalities, including counterparts to (1.3) and (1.5), for bounded, median porous sets. Given one such set E let $I_\infty(E)$ denote the collection of all the β 's with $d(\cdot, E)^{-\beta} \in A_\infty(\mathbb{R}^n)$ and for $1 < p \leq q < \infty$ and $0 < \alpha < n$ let us define

$$\Theta(\alpha) := \alpha + n \left(\frac{1}{q} - \frac{1}{p} \right). \quad (1.8)$$

Theorem 1.1. *Fix $1 < p \leq q < \infty$ and $0 < \alpha < n$ with $\Theta(\alpha) > 0$. Let $E \subset \mathbb{R}^n$ with $0 < \text{diam}(E) < \infty$ be a median porous set. Then, for any exponents $\gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$ verifying*

$$-\gamma_j \in I_\infty(E) \quad \text{for } j = 1, 2, \quad (1.9)$$

$$\delta_j p'/p \in I_\infty(E) \quad \text{for } j = 1, 2, \quad (1.10)$$

$$\overline{\dim}_M(E) < \frac{n}{\Theta(\alpha)} \left(\frac{\gamma_1}{q} - \frac{\delta_1}{p} \right) + n < n, \quad (1.11)$$

and

$$\left(\frac{\delta_2}{p} - \frac{\gamma_2}{q} \right) = \Theta(\alpha), \quad (1.12)$$

there exists $C > 0$, depending only on $\gamma_1, \gamma_2, \delta_1, \delta_2, p, q, \alpha, n$, and E , such that

$$\left(\int_{\mathbb{R}^n} |I_\alpha(f)(x)|^q u_{\gamma_1, \gamma_2}(x) dx \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^p v_{\delta_1, \delta_2}(x) dx \right)^{1/p}, \quad (1.13)$$

for every measurable f , where u_{γ_1, γ_2} and v_{δ_1, δ_2} are defined for $x \in \mathbb{R}^n \setminus E$ as

$$u_{\gamma_1, \gamma_2}(x) := \begin{cases} \max \left\{ \left(\frac{d(x, E)}{\text{diam}(E)} \right)^{\gamma_1}, \left(\frac{d(x, E)}{\text{diam}(E)} \right)^{\gamma_2} \right\} & \text{if } \gamma_1 < \gamma_2 \\ \min \left\{ \left(\frac{d(x, E)}{\text{diam}(E)} \right)^{\gamma_1}, \left(\frac{d(x, E)}{\text{diam}(E)} \right)^{\gamma_2} \right\} & \text{if } \gamma_1 \geq \gamma_2 \end{cases} \quad (1.14)$$

and

$$v_{\delta_1, \delta_2}(x) := \begin{cases} \max \left\{ \left(\frac{d(x, E)}{\text{diam}(E)} \right)^{\delta_1}, \left(\frac{d(x, E)}{\text{diam}(E)} \right)^{\delta_2} \right\} & \text{if } \delta_1 < \delta_2 \\ \min \left\{ \left(\frac{d(x, E)}{\text{diam}(E)} \right)^{\delta_1}, \left(\frac{d(x, E)}{\text{diam}(E)} \right)^{\delta_2} \right\} & \text{if } \delta_1 \geq \delta_2. \end{cases} \quad (1.15)$$

Remark 1.2. By Lemma 2.7 below we have $\text{Mu}_\infty(E) \leq n - \overline{\dim}_M(E)$ for every nonempty bounded set $E \subset \mathbb{R}^n$. If, in addition, E is median porous (i.e., $\text{Mu}_\infty(E) > 0$) it follows that $\overline{\dim}_M(E) < n$.

The case $\alpha = 1$ in Theorem 1.1 yields the following two-weight Hardy-Sobolev inequality.

Theorem 1.3. *Fix $n > 1$ and $1 < p \leq q < \infty$ with*

$$\Theta(1) := 1 + n(1/q - 1/p) > 0,$$

that is, the subcritical case $q < np/(n-p)$. Let $E \subset \mathbb{R}^n$ with $0 < \text{diam}(E) < \infty$ be a median porous set. Then, given exponents $\gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$ satisfying

$$\begin{aligned} -\gamma_j, \delta_j p'/p &\in I_\infty(E) \quad \text{for } j = 1, 2, \\ \overline{\dim}_M(E) - n &< \frac{n}{\Theta(1)} \left(\frac{\gamma_1}{q} - \frac{\delta_1}{p} \right) < 0, \\ \left(\frac{\delta_2}{p} - \frac{\gamma_2}{q} \right) &= \Theta(1), \end{aligned}$$

there exists $C > 0$, depending only on $p, q, n, \gamma_1, \gamma_2, \delta_1, \delta_2$, and E such that

$$\left(\int_{\mathbb{R}^n} |f(x)|^q u_{\gamma_1, \gamma_2}(x) dx \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |\nabla f(x)|^p v_{\delta_1, \delta_2}(x) dx \right)^{1/p}, \quad \forall f \in C_0^1(\mathbb{R}^n),$$

with u_{γ_1, γ_2} and v_{δ_1, δ_2} as in (1.14) and (1.15).

The case $\delta_1 = \delta_2 = 0$ in Theorem 1.3 provides a counterpart to (1.3) when E is bounded and median porous. More precisely,

Theorem 1.4. *Fix $1 < p < n$ and $p \leq q < np/(n-p)$. Let $E \subset \mathbb{R}^n$ be a median porous set with $0 < \text{diam}(E) < \infty$. Then, for any exponents γ_1, γ_2 satisfying*

$$0 > \gamma_1 > \left(\frac{q}{p}(n-p) - n \right) \left(1 - \frac{\overline{\dim}_M(E)}{n} \right) \quad (1.16)$$

$$0 > \gamma_2 = \frac{q}{p}(n-p) - n > -\text{Mu}_\infty(E), \quad (1.17)$$

there exists $C > 0$ depending only on $\gamma_1, \gamma_2, p, q, n$, and E such that

$$\left(\int_{\mathbb{R}^n} |I_1(f)(x)|^q \min \{d(x, E)^{\gamma_1}, d(x, E)^{\gamma_2}\} dx \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p}, \quad (1.18)$$

for every $f \in L^p(\mathbb{R}^n)$. Consequently, the weighted Hardy-Sobolev inequality

$$\left(\int_{\mathbb{R}^n} |f(x)|^q \min \{d(x, E)^{\gamma_1}, d(x, E)^{\gamma_2}\} dx \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |\nabla f(x)|^p dx \right)^{1/p} \quad (1.19)$$

holds true for every $f \in C_0^1(\mathbb{R}^n)$.

Remark 1.5. The Hardy-type inequality (1.19), with its mixed (γ_1, γ_2) -homogeneity, comes then as a substitute for (1.3) in the absence of porosity. Indeed, for γ_1, γ_2 as in (1.16) and (1.17) we have $\gamma_2 < \gamma_1 < 0$ and the inequality $d(x, E) \leq 1$ is equivalent to $d(x, E)^{\gamma_1} \leq d(x, E)^{\gamma_2}$. Hence if $\text{supp}(f) \subset \{x \in \mathbb{R}^n : d(x, E) > 1\}$, then (1.19) reduces to (1.3) (since

γ_2 in (1.17) equals the corresponding exponent in (1.3)), whereas if $\text{supp}(f) \subset \{x \in \mathbb{R}^n : d(x, E) \leq 1\}$, then (1.19) means

$$\left(\int_{\mathbb{R}^n} |f(x)|^q d(x, E)^{\gamma_1} dx \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |\nabla f(x)|^p dx \right)^{1/p},$$

with $\gamma_1 = \left(1 - \frac{\overline{\dim_{\mathbb{M}}(E)}}{n}\right) \gamma_2$.

From the case $p = q$ and $\alpha = 1$ (hence, $\Theta(1) = 1$) in Theorem 1.1 we obtain

Theorem 1.6. *Fix $n > 1$ and $1 < p < \infty$. Let $E \subset \mathbb{R}^n$ be a median porous set with $0 < \text{diam}(E) < \infty$. Then, for any exponents $\gamma_1, \gamma_2, \delta_1, \delta_2$ satisfying*

$$\begin{cases} -\gamma_j, \delta_j p'/p \in I_\infty(E) & \text{for } j = 1, 2 \\ \overline{\dim_{\mathbb{M}}(E)} - n < \frac{n}{p}(\gamma_1 - \delta_1) < 0 \\ \gamma_2 - \delta_2 = -p, \end{cases} \quad (1.20)$$

there exists $C > 0$, depending only on $\gamma_1, \gamma_2, \delta_1, \delta_2, p, n$, and E , such that

$$\int_{\mathbb{R}^n} |I_1(f)(x)|^p u_{\gamma_1, \gamma_2}(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v_{\delta_1, \delta_2}(x) dx, \quad (1.21)$$

for every measurable function f , with u_{γ_1, γ_2} and v_{δ_1, δ_2} as in (1.14) and (1.15). Consequently,

$$\int_{\mathbb{R}^n} |f(x)|^p u_{\gamma_1, \gamma_2}(x) dx \leq C \int_{\mathbb{R}^n} |\nabla f(x)|^p v_{\delta_1, \delta_2}(x) dx, \quad \forall f \in C_0^1(\mathbb{R}^n). \quad (1.22)$$

The next theorem establishes two-weight inequalities for I_α in the critical case $\Theta(\alpha) = 0$, that is, $q = np/(n - \alpha p)$.

Theorem 1.7. *Fix $0 \leq \alpha < n$, $1 < p < n/\alpha$, and set $q := np/(n - \alpha p)$. Let $E \subset \mathbb{R}^n$ be a median porous set with $0 < \text{diam}(E) < \infty$. Then, for any exponents $\gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$ verifying (1.9), (1.10), and*

$$\gamma_1 \geq \frac{q}{p} \delta_1, \quad (1.23)$$

$$\gamma_2 \leq \frac{q}{p} \delta_2, \quad (1.24)$$

there exists $C > 0$, depending only on $\gamma_1, \gamma_2, \delta_1, \delta_2, p, q, \alpha, n$, and E , such that

$$\left(\int_{\mathbb{R}^n} |I_\alpha(f)(x)|^q u_{\gamma_1, \gamma_2}(x) dx \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^p v_{\delta_1, \delta_2}(x) dx \right)^{1/p}, \quad (1.25)$$

for every measurable f and with u_{γ_1, γ_2} and v_{δ_1, δ_2} as in (1.14) and (1.15).

As a particular case of Theorem 1.7 we recover [15, Theorem 10.4] in the case of bounded sets. Namely,

Corollary 1.8. *Fix $0 \leq \alpha < n$, $1 < p < n/\alpha$, and set $q =: np/(n - \alpha p)$. Let $E \subset \mathbb{R}^n$ be a median porous set with $0 < \text{diam}(E) < \infty$. Consider exponents $\beta_1, \beta_2 \in \mathbb{R}$ satisfying*

$$-\beta_2 = -\frac{q}{p} \beta_1 \in I_\infty(E) \quad (1.26)$$

and

$$\frac{\beta_1}{p-1} \in I_\infty(E). \quad (1.27)$$

Then, there exists $C > 0$ depending only on $\beta_1, \beta_2, \alpha, p, n$, and E , such that

$$\left(\int_{\mathbb{R}^n} |I_\alpha(f)(x)|^q d(x, E)^{\beta_2} dx \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^p d(x, E)^{\beta_1} dx \right)^{1/p},$$

for every measurable function f .

As another consequence of Theorem 1.7 we obtain the following two-weight Hardy-Sobolev inequality in the critical case $q = np/(n-p)$ (that is, $\Theta(1) = 0$).

Theorem 1.9. *Fix $1 < p < n$ and set $q := np/(n-p)$. Let $E \subset \mathbb{R}^n$ be a median porous set with $0 < \text{diam}(E) < \infty$. Then, for any exponents $\gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$ verifying*

$$\begin{aligned} -\gamma_j, \delta_j p'/p &\in I_\infty(E) \quad \text{for } j = 1, 2, \\ \gamma_1 &\geq \frac{n}{(n-p)} \delta_1 \\ \gamma_2 &\leq \frac{n}{(n-p)} \delta_2, \end{aligned}$$

then there exists $C > 0$ depending only on $p, n, \gamma_1, \gamma_2, \delta_1, \delta_2$, and E , such that

$$\left(\int_{\mathbb{R}^n} |f(x)|^q u_{\gamma_1, \gamma_2}(x) dx \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |\nabla f(x)|^p v_{\delta_1, \delta_2}(x) dx \right)^{1/p}, \quad \forall f \in C_0^1(\mathbb{R}^n),$$

with u_{γ_1, γ_2} and v_{δ_1, δ_2} as in (1.14) and (1.15).

1.1. A blow-up rate for the constant C in (1.5) under weak porosity. Regarding the blow-up rate of the constant $C > 0$ from (1.5), in the case of a bounded, weakly porous set and as the left integral in (1.5) is performed increasingly closer to E we have

Theorem 1.10. *Let $n > 1$ and $1 < p \leq q < n/\Theta(1)$ with $\Theta(1) := 1 + n(1/q - 1/p) > 0$. Fix a weakly porous set $E \subset \mathbb{R}^n$ with $0 < \text{diam}(E) < \infty$ and $\beta > 0$ with $\beta p'/p < \text{Mu}_1(E)$. Then, there exists $C > 0$ such that*

$$\left(\int_{D(\kappa)} |f(x)|^q d_E(x)^{q(n-p+\beta)/p-n} dx \right)^{1/q} \leq C \kappa^{-\Theta(1)} \left(\int_{\mathbb{R}^n} |\nabla f(x)|^p d_E(x)^\beta dx \right)^{1/p}$$

for every $f \in C_0^1(\mathbb{R}^n)$ and $0 < \kappa < 1$, with $D(\kappa) := \{x \in \mathbb{R}^n : d(x, E) \geq \kappa \text{diam}(E)\}$.

The rest of the article is organized as follows: Section 2 contains preliminary material on Muckenhoupt weights as well as on various notions of dimension and porosity of sets in \mathbb{R}^n . Section 3 introduces a function g_E associated to a bounded set $E \subset \mathbb{R}^n$ and characterizes the condition $g_E \in L^{1,\infty}(\mathbb{R}^n)$ in terms of measure-theoretic properties of E . Section 4 contains the proofs of Theorems 1.1, 1.3, 1.4, 1.6, 1.7, 1.9 and that of Corollary 1.8. These proofs heavily rely on the aforementioned function g_E to identify pairs $(u_{\gamma_1, \gamma_2}, v_{\delta_1, \delta_2})$ satisfying either the condition (2.2) from Theorem 2.2 or the condition (2.3) from Theorem 2.3, when $E \subset \mathbb{R}^n$ is a bounded, median porous set. Finally, Section 5 is devoted to the proof of Theorem 1.10.

2. PRELIMINARIES

2.1. Muckenhoupt weights. Recall that for $1 < p < \infty$ a weight w in \mathbb{R}^n (that is, $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ with $w \geq 0$ a.e. in \mathbb{R}^n) is said to belong to the *Muckenhoupt class* $A_p(\mathbb{R}^n)$ if

$$[w]_{A_p} := \sup_Q \left(\int_Q w(x) dx \right) \left(\int_Q w(x)^{\frac{-1}{p-1}} dx \right)^{p-1} < \infty \quad (A_p)$$

where $Q \subset \mathbb{R}^n$ is a cube and $\int_Q u$ denotes the average $\frac{1}{|Q|} \int_Q u$. The endpoint classes for $p = 1$ and $p = \infty$ are defined by

$$[w]_{A_1} := \sup_Q \left(\int_Q w(x) dx \right) \left(\text{ess inf}_Q w \right)^{-1} < \infty \quad (A_1)$$

and

$$[w]_{A_\infty} := \sup_Q \left(\int_Q w(x) dx \right) \exp \left(- \int_Q \ln w(x) dx \right) < \infty. \quad (A_\infty)$$

As it turns out, $A_\infty(\mathbb{R}^n) = \bigcup_{p \geq 1} A_p(\mathbb{R}^n)$ (see for instance [8, Section 9.3]). Typical examples of A_p -weights are the locally integrable powers of $|x|$. More precisely, for $1 < p < \infty$, the weight $|x|^a \in A_p(\mathbb{R}^n)$ if and only if $-n < a < n(p-1)$ and $|x|^a \in A_1(\mathbb{R}^n)$ if and only if $-n < a \leq 0$, see for instance [8, p. 286]. When the locally integrable powers of $|x|$ are replaced by those of a distance to a set $E \subset \mathbb{R}^n$, their membership to $A_\infty(\mathbb{R}^n)$, or to a particular $A_p(\mathbb{R}^n)$ class, will be determined by corresponding notions of porosity applied to E , as described in Section 2.5. In Section 4 we will construct examples of $A_\infty(\mathbb{R}^n)$ -weights based on the following fact: For $1 < p < \infty$, the class $A_p(\mathbb{R}^n)$ forms a lattice in the sense that if w_1 and w_2 belong to $A_p(\mathbb{R}^n)$, then $\max\{w_1, w_2\}$ and $\min\{w_1, w_2\}$ belong to $A_p(\mathbb{R}^n)$ as well, see [10, Proposition 4.3]. Since $A_\infty(\mathbb{R}^n) = \bigcup_{1 \leq p < \infty} A_p(\mathbb{R}^n)$ and $A_{p_1}(\mathbb{R}^n) \subset A_{p_2}(\mathbb{R}^n)$ whenever $1 \leq p_1 \leq p_2 < \infty$, it follows that if w_1 and w_2 belong to $A_\infty(\mathbb{R}^n)$, then so do $\max\{w_1, w_2\}$ and $\min\{w_1, w_2\}$.

2.2. Reverse Hölder classes. For $1 < s < \infty$ we write $w \in RH_s(\mathbb{R}^n)$ if

$$[w]_{RH_s} := \sup_Q \left(\int_Q w(x)^s dx \right)^{1/s} \left(\int_Q w(x) dx \right)^{-1} < \infty.$$

As it turns out (see for instance [8, Section 9.3]), $\bigcup_{s > 1} RH_s(\mathbb{R}^n) = A_\infty(\mathbb{R}^n)$. For $s = \infty$ we write $w \in RH_\infty(\mathbb{R}^n)$ if

$$[w]_{RH_\infty} := \sup_Q \left(\text{ess sup}_Q w \right) \left(\int_Q w(x) dx \right)^{-1} < \infty.$$

It is a known fact that the class $RH_\infty(\mathbb{R}^n)$ is invariant under positive powers, that is, the implication

$$w \in RH_\infty(\mathbb{R}^n), \ell > 0 \Rightarrow w^\ell \in RH_\infty(\mathbb{R}^n) \quad (2.1)$$

holds true with $[w^\ell]_{RH_\infty(\mathbb{R}^n)}$ depending only on $[w]_{RH_\infty(\mathbb{R}^n)}$, ℓ , and n , see [3, Theorem 4.2].

Remark 2.1. If $w \in A_1(\mathbb{R}^n)$ then there exists $\theta_0 > 0$ (depending only on $[w]_{A_1(\mathbb{R}^n)}$ and n) such that $w^{-\theta_0} \in RH_\infty(\mathbb{R}^n)$. Conversely, if $w \in RH_\infty(\mathbb{R}^n)$, then there exists $\theta_1 > 0$ (depending only on $[w]_{RH_\infty(\mathbb{R}^n)}$ and n) such that $w^{-\theta_1} \in A_1(\mathbb{R}^n)$, see [3, Corollary 4.5].

2.3. Weighted inequalities for the fractional integral operator. For $0 < \alpha < n$ the fractional integral operator I_α are defined as

$$I_\alpha(f)(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

Let us record the following two results from [13] regarding weighted estimates for I_α in the cases $\Theta(\alpha) = 0$ and $\Theta(\alpha) > 0$, with $\Theta(\alpha)$ as in (1.8). In the statements below, the notation (u, v) denotes a pair of nonnegative functions u, v defined on \mathbb{R}^n .

Theorem 2.2 (Theorem 1.1 from [13]). *Fix $1 < p \leq q < \infty$ and $0 < \alpha < n$ such that $\Theta(\alpha) = 0$. Given (u, v) with $u, v^{-p'/p} \in A_\infty(\mathbb{R}^n)$ suppose that there exists $C_0 > 0$ with*

$$u(x)^{1/q} \leq C_0 v(x)^{1/p}, \quad \text{a.e. } x \in \mathbb{R}^n. \quad (2.2)$$

Then

$$\left(\int_{\mathbb{R}^n} |I_\alpha(f)(x)|^q u(x) dx \right)^{1/q} \leq C_1 C_0 \left(\int_{\mathbb{R}^n} |f(x)|^p v(x) dx \right)^{1/p},$$

for every measurable f , where $C_1 > 0$ depends only on $n, p, q, \alpha, [u]_{A_\infty(\mathbb{R}^n)}$, and $[v^{-p'/p}]_{A_\infty(\mathbb{R}^n)}$.

For the case $\Theta(\alpha) > 0$ we have

Theorem 2.3 (Theorem 1.2 from [13]). *Fix $1 < p \leq q < \infty$ and $0 < \alpha < n$ such that $\Theta(\alpha) > 0$ and (u, v) with $u, v^{-p'/p} \in A_\infty(\mathbb{R}^n)$ verifying*

$$\frac{u^{1/q}}{v^{1/p}} \in L^{n/\Theta(\alpha), \infty}(\mathbb{R}^n). \quad (2.3)$$

Then

$$\left(\int_{\mathbb{R}^n} |I_\alpha(f)(x)|^q u(x) dx \right)^{1/q} \leq C_2 \|u^{1/q}/v^{1/p}\|_{L^{n/\Theta(\alpha), \infty}(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |f(x)|^p v(x) dx \right)^{1/p},$$

for every measurable f , where $C_2 > 0$ depends only on $n, \alpha, p, q, [u]_{A_\infty(\mathbb{R}^n)}$, and $[v^{-p'/p}]_{A_\infty(\mathbb{R}^n)}$.

Remark 2.4. The proofs of Theorems 1.1, 1.3, 1.4, 1.6, 1.7, and 1.9 will rely on Theorems 2.2 and 2.3 as well as on the pointwise estimate (see [7, Lemma 7.14])

$$|f(x)| \leq \frac{1}{n\omega_n} I_1(|\nabla f|)(x), \quad \forall f \in C_0^1(\mathbb{R}^n), x \in \mathbb{R}^n, \quad (2.4)$$

where ω_n denotes the Euclidean measure of a unit ball in \mathbb{R}^n .

Although we focus on Hardy-Sobolev inequalities, weighted Poincaré-type inequalities can be obtained as well by using that for a convex set $S \subset \mathbb{R}^n$ and $f \in C^1(S)$ we have

$$\left| f(x) - \int_S f \right| \leq \frac{\text{diam}(S)^n}{n|S|} I_1(|\nabla f| \chi_S)(x), \quad \forall x \in S, \quad (2.5)$$

see for instance [7, Lemma 7.16]. From (2.5) and Theorems 2.2 and 2.3 (always in the case $\alpha = 1$), given a cube $Q \subset \mathbb{R}^n$ weighted inequality

$$\left(\int_Q \left| f(x) - \int_Q f \right|^q u(x) dx \right)^{1/q} \leq C \left(\int_Q |\nabla f(x)|^p v(x) dx \right)^{1/p}$$

holds true for every $f \in C^1(\mathbb{R}^n)$ whenever a pair (u, v) with $u, v^{-p'/p} \in A_\infty(\mathbb{R}^n)$ satisfies (2.2) or (2.3).

Moreover, for $0 < \alpha < n$, the pointwise inequality (2.4) can be replaced with the fractional-derivative identity

$$f = I_\alpha((-\Delta)^{\alpha/2}(f))$$

for every f in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ (see for instance [16, p.117]) and then Theorems 2.2 and 2.3 yield

$$\left(\int_{\mathbb{R}^n} |f(x)|^q u(x) dx \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |(-\Delta)^{\alpha/2} f(x)|^p v(x) dx \right)^{1/p}, \quad \forall f \in \mathcal{S}(\mathbb{R}^n),$$

whenever a pair (u, v) with $u, v^{-p'/p} \in A_\infty(\mathbb{R}^n)$ satisfies (2.2) or (2.3)

2.4. Dimensions of sets. For $\delta > 0$, E_δ stands for the δ -neighborhood of E , that is,

$$E_\delta := \{x \in \mathbb{R}^n : d(x, E) < \delta\}.$$

Notice that, given $\delta > 0$, we have $|E_\delta| < \infty$ if and only if E is bounded.

2.4.1. The upper Minkowski dimension of $E \subset \mathbb{R}^n$. Given a set $E \subset \mathbb{R}^n$ and $\delta > 0$, let $N(E, \delta)$ denote the smallest number of closed balls of radius δ required to cover E . In particular, given $\delta > 0$, $N(E, \delta) < \infty$ if and only if E is bounded. The *upper box-counting* or *upper Minkowski* dimension of E is defined as

$$\overline{\dim}_M(E) := \limsup_{\delta \rightarrow 0^+} \frac{\log N(E, \delta)}{\log(1/\delta)}, \quad (2.6)$$

see for instance [6, p. 43]. Equivalently, $\overline{\dim}_M(E)$ can be regarded as the infimum of the nonnegative λ 's such that there exists $C > 0$ (which may depend on λ and E) satisfying

$$|E_\delta| \leq C\delta^{n-\lambda}, \quad \forall 0 < \delta < \text{diam}(E), \quad (2.7)$$

see for instance [1, Remark 6.8]. Notice that $\overline{\dim}_M(E) < \infty$ if and only if E is bounded. In addition, we have $\overline{\dim}_M(E) = \overline{\dim}_M(\overline{E})$ for every $E \subset \mathbb{R}^n$ by [6, Proposition 3.4]. Let us record here a simple result concerning $\overline{\dim}_M(E)$ that will be useful later on.

Lemma 2.5. *Let $\emptyset \neq E \subset \mathbb{R}^n$ be a bounded set and fix $0 < s < n$. If $\overline{\dim}_M(E) < s$, then there exists $C > 0$ such that*

$$|E_\delta| \leq C\delta^{n-s}, \quad \forall 0 < \delta < \text{diam}(E). \quad (2.8)$$

Proof. Set $d_B := \overline{\dim}_M(E)$ and $f(\delta) := \frac{\log N(E, \delta)}{\log(1/\delta)}$, then (2.6) means $d_B = \inf_{a>0} \sup f((0, a))$.

Put $\varepsilon_0 := s - \overline{\dim}_M(E) > 0$, then there exists $a_0 > 0$ (depending on ε_0 and E) such that

$$\sup f((0, a_0)) < d_B + \varepsilon_0,$$

that is,

$$f(\delta) < d_B + \varepsilon_0, \quad \forall \delta \in (0, a_0). \quad (2.9)$$

Now, given $\delta \in (0, \text{diam}(E))$, let us cover E with $N(E, \delta) < \infty$ closed balls of radius δ , in particular, the set $\{x \in \mathbb{R}^n : d(x, E) < \delta\}$ can be covered with $N(E, \delta)$ closed balls concentric with the first balls but with radius 2δ . Hence, $|E_\delta| \leq \omega_n N(E, \delta)(2\delta)^n$, with $\omega_n := |B(0, 1)|$.

Now, if $\delta \in (0, a_0)$, then (2.9) implies

$$\frac{\log N(E, \delta)}{\log(1/\delta)} < d_B + \varepsilon_0 = s,$$

which means $N(E, \delta) < \delta^{-s}$ and then $|E_\delta| \leq \omega_n 2^n \delta^{n-s}$. If $\delta \geq a_0$, fix $x_0 \in E$ and use the inclusion $E_\delta \subset B(x_0, 2\text{diam}(E))$, so that

$$|E_\delta| \leq \omega_n 2^n \text{diam}(E)^n \leq \omega_n 2^n \text{diam}(E)^n \left(\frac{\delta}{a_0}\right)^{n-s}$$

and (2.8) follows with $C = \omega_n 2^n \max\{1, \text{diam}(E)^n / a_0^{n-s}\}$. \square

2.4.2. The Assouad dimension of $E \subset \mathbb{R}^n$. The *Assouad dimension* of a set $E \subset \mathbb{R}^n$, denoted by $\dim_A(E)$, is defined as the infimum of $\lambda \geq 0$ such that there exists $C > 0$ with

$$N(E \cap B(x, R), r) \leq C \left(\frac{R}{r}\right)^\lambda, \quad \forall x \in E, 0 < r < R. \quad (2.10)$$

Notice that if $0 < \text{diam}(E) < \infty$ and if $\lambda \geq 0$ satisfies (2.10), then by fixing an $x \in E$ and $R := \text{diam}(E)$ we get $E \cap B(x, R) = E$ as well as

$$N(E, r) \leq C \text{diam}(E)^\lambda r^{-\lambda}, \quad \forall 0 < r < \text{diam}(E),$$

which together with (2.6) implies $\overline{\dim}_M(E) \leq \lambda$ and thus

$$\overline{\dim}_M(E) \leq \dim_A(E) \quad (2.11)$$

whenever $E \subset \mathbb{R}^n$ is bounded. For every $E \subset \mathbb{R}^n$ we have $\dim_A(E) \leq n$ (see for instance [11, Remark 10.18]) and the *Assouad co-dimension* of E is defined as $\underline{\text{codim}}_A(E) := n - \dim_A(E)$. Equivalently, (see [9, Lemma 3.4]) $\underline{\text{codim}}_A(E)$ equals the supremum of $\nu \geq 0$ such that there exists $C > 0$ with

$$\frac{|E_r \cap B(x, R)|}{|B(x, R)|} \leq C \left(\frac{R}{r}\right)^{-\nu}, \quad \forall x \in E, 0 < r < R.$$

2.5. Distance weights and notions of porosity. This section summarizes certain geometric and measure-theoretic conditions on a set $E \subset \mathbb{R}^n$ that can be characterized by regularity properties of negative powers of the distance function $d(\cdot, E)$ considered as a weight. For a cube $Q_0 \subset \mathbb{R}^n$, $\mathcal{D}(Q_0)$ stands for the class of all dyadic subcubes of Q_0 . Given sets $A, E \subset \mathbb{R}^n$, A is called *E-free* if $A \cap E = \emptyset$. Given a nonempty set $E \subset \mathbb{R}^n$, a cube Q is called *maximal E-free* if Q is free but $2Q$ is not. For the next definitions we follow [15, Section 7]. Fix a cube Q_0 with $Q_0 \cap E \neq \emptyset$, then

$$\mathcal{V}_1(Q_0) := \max\{|Q| : Q \in \mathcal{D}(Q_0), Q \cap E = \emptyset\},$$

for $\delta > 0$ let $\mathcal{S}_\delta(Q_0)$ be the maximal subfamily of $\{Q \in \mathcal{D}(Q_0) : Q \cap E = \emptyset, |Q| \geq \delta\}$, and for $0 < s < 1$, let

$$\mathcal{V}_s(Q_0) = \sup \left\{ \delta > 0 : \sum_{Q \in \mathcal{S}_\delta(Q_0)} |Q| \geq (1-s)|Q_0| \right\}.$$

Finally, for $0 < s \leq 1$, let $\mathcal{L}_s(Q_0) := \mathcal{V}_s(Q_0)^{1/n}$.

2.5.1. *Porous sets.* Following [11, Definition 10.11], a set $E \subset \mathbb{R}^n$ is *porous* if there exists $C > 0$ such that for every $x \in \mathbb{R}^n$ and $r > 0$ there exists $y \in \mathbb{R}^n$ satisfying $B(y, Cr) \subset B(x, r) \setminus E$. Equivalently, E is porous if there exists $c > 0$ such that for every cube $Q_0 \subset \mathbb{R}^n$ there exists an E -free subcube $Q \in \mathcal{D}(Q_0)$ with $|Q| \geq c|Q_0|$.

Moreover, the following are equivalent (see for instance [11, Theorem 10.25]):

- (i) a nonempty set $E \subset \mathbb{R}^n$ is porous,
- (ii) $\dim_A(E) < n$.

2.5.2. *Weakly porous sets.* In [1], Anderson, Lehrbäck, Mudarra, and Vähäkangas introduced and developed the notion of weak porosity for a nonempty set $E \subset \mathbb{R}^n$. Given a cube $P \subset \mathbb{R}^n$, let $\mathcal{M}(P)$ indicate an E -free subcube in $\mathcal{D}(P)$ of largest side-length. Then E is a *weakly porous set* if there exist $c, \delta \in (0, 1)$ such that for every cube $P \subset \mathbb{R}^n$, there is a family of pairwise disjoint E -free subcubes $\{Q_k\}_{k=1}^N \subset \mathcal{D}(P)$ satisfying $|Q_k| \geq \delta|\mathcal{M}(P)|$ for $k = 1, \dots, N$ as well as $\sum_{k=1}^N |Q_k| \geq c|P|$, see [1, Section 3]. It then quickly follows that a porous set is weakly porous. Moreover, by [1, Theorem 1.1] the following are equivalent:

- (i) a nonempty set $E \subset \mathbb{R}^n$ is weakly porous,
- (ii) there exists $\alpha > 0$ such that $d(\cdot, E)^{-\alpha} \in A_1(\mathbb{R}^n)$,
- (iii) $d(\cdot, E) \in RH_\infty(\mathbb{R}^n)$,

where the equivalence between the last two statements is due to Remark 2.1. The precise range of α 's that make $d(\cdot, E)^{-\alpha} \in A_1(\mathbb{R}^n)$ or $d(\cdot, E)^{-\alpha} \in A_p(\mathbb{R}^n)$ for $1 < p < \infty$ for a weakly porous set $E \subset \mathbb{R}^n$ relies on the notion of Muckenhoupt exponent Mu_1 . Given a nonempty set $E \subset \mathbb{R}^n$, its *Muckenhoupt exponent* $\text{Mu}_1(E)$ is defined as the supremum of $\alpha \geq 0$ such that there exists $C > 0$ satisfying

$$\frac{|E_r \cap Q|}{|Q|} \leq C \left(\frac{\mathcal{L}_1(Q)}{r} \right)^{-\alpha}$$

for every cube Q centered at a point of E and every $0 < r < \mathcal{L}_1(Q)$, and if $\mathcal{L}_1(Q) = 0$ for some such cube Q , then $\text{Mu}_1(E) := 0$, see [1, Definition 6.1] and [15, Definition 8.1 and Remark 8.2]. Then, by [1, Theorem 1.2], given a weakly porous set $E \subset \mathbb{R}^n$ we have

- (i) $d(\cdot, E)^{-\alpha} \in A_1(\mathbb{R}^n)$ if and only if $0 \leq \alpha < \text{Mu}_1(E)$, and
- (ii) for $1 < p < \infty$, $d(\cdot, E)^{-\alpha} \in A_p(\mathbb{R}^n)$ if and only if $(1 - p)\text{Mu}_1(E) < \alpha < \text{Mu}_1(E)$.

In addition, by [1, Theorem 1.3], given a nonempty set $E \subset \mathbb{R}^n$ and $\alpha \in \mathbb{R} \setminus \{0\}$, then $d(\cdot, E)^{-\alpha} \in A_1(\mathbb{R}^n)$ if and only if $0 < \alpha < \text{Mu}_1(E)$. Consequently, a nonempty set $E \subset \mathbb{R}^n$ is weakly porous if and only if $\text{Mu}_1(E) > 0$.

Remark 2.6. Notice that given a weakly porous set $E \subset \mathbb{R}^n$, $1 < p < \infty$, and $\beta > 0$ with $\beta p'/p < \text{Mu}_1(E)$, by setting $v(x) := d(x, E)^\beta$ we have $v^{-p'/p} = d(\cdot, E)^{-\beta p'/p} \in A_1(\mathbb{R}^n)$ and consequently $v \in RH_\infty(\mathbb{R}^n)$ due to (2.1) and Remark 2.1.

2.5.3. *Median porous sets.* In [15], Pasquariello and Uriarte-Tuero extended the notion of weak porosity by introducing that of median porosity. A set $\emptyset \subsetneq E \subsetneq \mathbb{R}^n$ is a *median porous set* if there are constants $0 < \delta < 1$ and $0 < s < t \leq 1$ such that

$$\sum_{Q \in \mathcal{S}_\delta^t} |Q| \geq (1 - s)|Q_0|$$

for every cube Q_0 with $Q_0 \cap E \neq \emptyset$, where $\mathcal{S}_\delta^t = \mathcal{S}_{\delta \mathcal{V}_t(Q_0)}(Q_0)$, see [15, Definition 7.3]. As noted in [15, Remark 7.5] the definition of weak porosity corresponds to the case $t = 1$. By

[15, Theorem 1.5] a nonempty $E \subset \mathbb{R}^n$ is median porous if and only if there exists $\alpha > 0$ such that $d(\cdot, E)^{-\alpha} \in A_\infty(\mathbb{R}^n)$. The precise range for the α 's with $d(\cdot, E)^{-\alpha} \in A_\infty(\mathbb{R}^n)$ requires the notion of the Muckenhoupt exponents Mu_p for $1 < p < \infty$ and $\text{Mu}_\infty(E)$ of a nonempty set E . Let $E \subset \mathbb{R}^n$ be a nonempty set. For $1 < p < \infty$, let $\text{Mu}_p(E)$ denote the supremum of $\alpha \geq 0$ so that there are constants $C > 0$ and $0 < s < 1$ such that for every cube Q centered at a point of E

$$\frac{|E_r \cap Q|}{|Q|} \leq C \left(\frac{\mathcal{L}_s(Q)}{r} \right)^{-\alpha}, \quad \forall 0 < r < \mathcal{L}_s(Q)$$

as well as

$$\frac{|(E_r)^c \cap Q|}{|Q|} \leq C \left(\frac{\mathcal{L}_s(Q)}{r} \right)^{\frac{\alpha}{p-1}}, \quad \forall \mathcal{L}_s(Q) < r < l(Q).$$

Let $\text{Mu}_\infty(E)$ be the supremum of $\alpha \geq 0$ so that there are constants $C > 0$ and $0 < s < 1$ such that for every cube Q centered at a point of E

$$\frac{|E_r \cap Q|}{|Q|} \leq C \left(\frac{\mathcal{L}_s(Q)}{r} \right)^{-\alpha}, \quad \forall 0 < r < \mathcal{L}_s(Q).$$

If $\mathcal{L}_s(Q) = 0$ for some $0 < s < 1$ and some cube Q , set $\text{Mu}_p(E) = \text{Mu}_\infty(E) = 0$, see [15, Definition 8.6]. Now, by [15, Theorem 8.7 and Corollary 8.10], given a nonempty set $E \subset \mathbb{R}^n$ we have

$$d(\cdot, E)^{-\alpha} \in A_\infty(\mathbb{R}^n) \Leftrightarrow -\sup_{p>1} (p-1)\text{Mu}_p(E) < \alpha < \text{Mu}_\infty(E)$$

and the following are equivalent:

- (i) E is median porous,
- (ii) there exists $\alpha > 0$ such that $d(\cdot, E)^{-\alpha} \in A_\infty(\mathbb{R}^n)$,
- (iii) $\text{Mu}_\infty(E) > 0$,
- (iv) $\text{Mu}_p > 0$ for some $1 < p < \infty$,
- (v) $\text{Mu}_p > 0$ for every $1 < p < \infty$.

Let us set $\ell_\infty(E) := -\sup_{p>1} (p-1)\text{Mu}_p(E)$ and

$$I_\infty(E) := (\ell_\infty(E), \text{Mu}_\infty(E)). \quad (2.12)$$

That is, $d(\cdot, E)^{-\alpha} \in A_\infty(\mathbb{R}^n)$ if and only if $\alpha \in I_\infty(E)$. Also, by [15, Corollary 8.10.3] it follows that $\ell_\infty(E) < 0$ whenever $\text{Mu}_\infty(E) > 0$. Also, by [15, Corollary 8.10], if $E \subset \mathbb{R}^n$ then $\text{Mu}_\infty(E) \leq n$ and $\text{Mu}_\infty(E) := \sup_{1 < q < \infty} \text{Mu}_q(E)$. Moreover, by [15, Corollary 8.10], if

E is weakly porous, then it is median porous with $\text{Mu}_\infty(E) = \text{Mu}_p(E) > 0$ for every $1 \leq p < \infty$. The next lemma guarantees that $\overline{\dim}_M(E) < n$ for every bounded, median porous set $E \subset \mathbb{R}^n$.

Lemma 2.7. *Given a nonempty bounded set $E \subset \mathbb{R}^n$, we have*

$$\text{Mu}_\infty(E) \leq n - \overline{\dim}_M(E). \quad (2.13)$$

Proof. If $\text{Mu}_\infty(E) = 0$ then there is nothing to prove due to (2.11) and the aforementioned fact that $\dim_A(E) \leq n$ for every $E \subset \mathbb{R}^n$. If $\text{Mu}_\infty(E) > 0$, let us take $\alpha \in (0, \text{Mu}_\infty(E))$ which makes for $d(\cdot, E)^{-\alpha} \in A_\infty(\mathbb{R}^n) \subset L^1_{\text{loc}}(\mathbb{R}^n)$. Next, by [17, Lemma 3], given a set $A \subset \mathbb{R}^n$, an open set $\Omega \subset \mathbb{R}^n$, and $\gamma > 0$ we have

$$\int_{A_r \cap \Omega} d(x, A)^{-\gamma} dx = r^{-\gamma} |A_r \cap \Omega| + \gamma \int_0^r |A_t \cap \Omega| t^{-\gamma-1} dt, \quad \forall r > 0. \quad (2.14)$$

Now, for a bounded set $E \subset \mathbb{R}^n$ and $0 < r < 1$ let us choose $A := E$, $\Omega := E_1$ so that $E_r \cap \Omega = E_r$, and $\gamma := \alpha$ so that $d(\cdot, E)^{-\alpha} \in L^1_{\text{loc}}(\mathbb{R}^n)$. The identity (2.14) then yields

$$r^{-\alpha}|E_r| \leq \int_{E_1} d(x, A)^{-\alpha} dx =: \eta < \infty.$$

Define $\lambda := n - \alpha$ (which is positive since $\alpha < \text{Mu}_\infty(E) \leq n$) so that $|E_r| \leq \eta r^{n-\lambda}$ for every $0 < r < 1$, and from the definition of $\overline{\dim}_M(E)$ in (2.7) it follows that $\overline{\dim}_M(E) \leq \lambda$, that is, $\alpha \leq n - \overline{\dim}_M(E)$. Finally, letting $\alpha \rightarrow \text{Mu}_\infty(E)$ proves (2.13). \square

3. ON A FUNCTION $g_E \in L^{1,\infty}(\mathbb{R}^n)$

In this section we introduce and study a function g_E associated to a bounded set $E \subset \mathbb{R}^n$ and characterize the condition $g_E \in L^{1,\infty}(\mathbb{R}^n)$ in terms of measure-theoretic properties of E , this characterization plays a central role in the construction of examples from Section 4.

Lemma 3.1. *Fix $0 \leq s < n$ and a set $E \subset \mathbb{R}^n$ with $0 < \text{diam}(E) < \infty$ and $|\overline{E}| = 0$. For a.e. $x \in \mathbb{R}^n$ define*

$$g_E(x) := \begin{cases} \left(\frac{d(x,E)}{\text{diam}(E)} \right)^{-(n-s)} & \text{if } d(x, E) < \text{diam}(E) \\ \left(\frac{d(x,E)}{\text{diam}(E)} \right)^{-n} & \text{if } d(x, E) \geq \text{diam}(E). \end{cases}$$

Then, if there exists $K > 0$ such that

$$|E_r| \leq K r^{n-s}, \quad \forall 0 < r < \text{diam}(E), \quad (3.1)$$

it follows that $g_E \in L^{1,\infty}(\mathbb{R}^n)$ with

$$\|g_E\|_{L^{1,\infty}(\mathbb{R}^n)} \leq K \text{diam}(E)^{n-s} + \omega_n 2^n \text{diam}(E)^n. \quad (3.2)$$

Conversely, if $g_E \in L^{1,\infty}(\mathbb{R}^n)$, then (3.1) holds with $K := \|g_E\|_{L^{1,\infty}(\mathbb{R}^n)} / \text{diam}(E)^{n-s}$.

Proof. For $t > 0$ let us write

$$\begin{aligned} |\{x \in \mathbb{R}^n : g_E(x) > t\}| &= |\{x \in \mathbb{R}^n : d(x, E) < \text{diam}(E) \text{ and } g_E(x) > t\}| \\ &\quad + |\{x \in \mathbb{R}^n : d(x, E) \geq \text{diam}(E) \text{ and } g_E(x) > t\}| =: I + II. \end{aligned}$$

By setting $r := \text{diam}(E) \min\{t^{-\frac{1}{n-s}}, 1\}$ and using it in (3.1) we get

$$I = |\{x \in \mathbb{R}^n : d(x, E) < r\}| \leq K \text{diam}(E)^{n-s} t^{-1}.$$

On the other hand, by defining $R := \text{diam}(E)/t^{1/n}$, we can write

$$II = |\{x \in \mathbb{R}^n : \text{diam}(E) \leq d(x, E) < R\}|$$

and notice that, for any fixed $x_E \in E$, the inclusion

$$\{x \in \mathbb{R}^n : \text{diam}(E) \leq d(x, E) < R\} \subset \overline{B(x_E, 2R)} \quad (3.3)$$

holds true, since given $x \in \mathbb{R}^n$ with $\text{diam}(E) \leq d(x, E) < R$ and $\varepsilon > 0$ let $y_E \in E$ satisfy $|x - y_E| \leq d(x, E) + \varepsilon$ and then

$$|x - x_E| \leq |x - y_E| + |y_E - x_E| \leq d(x, E) + \text{diam}(E) + \varepsilon < R + \text{diam}(E) + \varepsilon < 2R + \varepsilon,$$

and we obtain (3.3) by letting $\varepsilon \rightarrow 0$. Consequently,

$$II \leq |B(0, 2)|R^n = |B(0, 2)|\text{diam}(E)^n/t$$

and (3.2) follows. Conversely, $g_E \in L^{1,\infty}(\mathbb{R}^n)$ means

$$t|\{x \in \mathbb{R}^n : g_E(x) > t\}| \leq \|g_E\|_{L^{1,\infty}(\mathbb{R}^n)}, \quad \forall t > 0 \quad (3.4)$$

and, given $0 < r \leq \text{diam}(E)$, for $x \in \mathbb{R}^n$ with $d(x, E) < r$ we finally get $d(x, E) < \text{diam}(E)$ and then $g_E(x) = \left(\frac{d(x, E)}{\text{diam}(E)}\right)^{-(n-s)}$. By using (3.4) with $t := (r/\text{diam}(E))^{-(n-s)}$,

$$\begin{aligned} |\{x \in \mathbb{R}^n : d(x, E) < r\}| &\leq |\{x \in \mathbb{R}^n : \text{diam}(E)g_E(x)^{-\frac{1}{n-s}} < r\}| \\ &= |\{x \in \mathbb{R}^n : g_E(x) > (r/\text{diam}(E))^{-(n-s)}\}| \\ &\leq \|g_E\|_{L^{1,\infty}(\mathbb{R}^n)} \left(\frac{r}{\text{diam}(E)}\right)^{n-s} \end{aligned}$$

which proves (3.1) with $K := \|g_E\|_{L^{1,\infty}(\mathbb{R}^n)}/\text{diam}(E)^{n-s}$. \square

4. PROOFS OF THEOREMS 1.1, 1.3, 1.4, 1.6, 1.7, 1.9 AND OF COROLLARY 1.8

The proofs are based on the function g_E from Section 3 and on the right choice of indices $\gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$ so that the pair $(u_{\gamma_1, \gamma_2}, v_{\delta_1, \delta_2})$ satisfies either the condition (2.3) from Theorem 2.3 (when $\Theta(\alpha) > 0$) or (2.2) from Theorem 2.2 (when $\Theta(\alpha) = 0$), whenever $E \subset \mathbb{R}^n$ is a bounded, median porous set. Recall that u_{γ_1, γ_2} and v_{δ_1, δ_2} are defined for $x \in \mathbb{R}^n \setminus E$ as

$$u_{\gamma_1, \gamma_2}(x) := \begin{cases} \max \left\{ \left(\frac{d(x, E)}{\text{diam}(E)}\right)^{\gamma_1}, \left(\frac{d(x, E)}{\text{diam}(E)}\right)^{\gamma_2} \right\} & \text{if } \gamma_1 < \gamma_2 \\ \min \left\{ \left(\frac{d(x, E)}{\text{diam}(E)}\right)^{\gamma_1}, \left(\frac{d(x, E)}{\text{diam}(E)}\right)^{\gamma_2} \right\} & \text{if } \gamma_1 \geq \gamma_2 \end{cases} \quad (4.1)$$

and

$$v_{\delta_1, \delta_2}(x) := \begin{cases} \max \left\{ \left(\frac{d(x, E)}{\text{diam}(E)}\right)^{\delta_1}, \left(\frac{d(x, E)}{\text{diam}(E)}\right)^{\delta_2} \right\} & \text{if } \delta_1 < \delta_2 \\ \min \left\{ \left(\frac{d(x, E)}{\text{diam}(E)}\right)^{\delta_1}, \left(\frac{d(x, E)}{\text{diam}(E)}\right)^{\delta_2} \right\} & \text{if } \delta_1 \geq \delta_2 \end{cases} \quad (4.2)$$

and notice that they can be rewritten as

$$u_{\gamma_1, \gamma_2}(x) := \begin{cases} \left(\frac{d(x, E)}{\text{diam}(E)}\right)^{\gamma_1} & \text{if } d(x, E) < \text{diam}(E) \\ \left(\frac{d(x, E)}{\text{diam}(E)}\right)^{\gamma_2} & \text{if } d(x, E) \geq \text{diam}(E) \end{cases} \quad (4.3)$$

as well as

$$v_{\delta_1, \delta_2}(x) := \begin{cases} \left(\frac{d(x, E)}{\text{diam}(E)}\right)^{\delta_1} & \text{if } d(x, E) < \text{diam}(E) \\ \left(\frac{d(x, E)}{\text{diam}(E)}\right)^{\delta_2} & \text{if } d(x, E) \geq \text{diam}(E). \end{cases} \quad (4.4)$$

Proof of Theorem 1.1. Fix $1 < p \leq q < \infty$ and $0 < \alpha < n$ such that $\Theta(\alpha) > 0$ (always with $\Theta(\alpha)$ as in (1.8)). Let $E \subset \mathbb{R}^n$ be a median porous set with $0 < \text{diam}(E) < \infty$. In particular, $\overline{\dim}_M(E) < n$ (due to (2.13)) and $d(\cdot, E)^{-\vartheta} \in A_\infty(\mathbb{R}^n)$ if and only if $\vartheta \in I_\infty(E)$ as defined in (2.12). Next consider exponents γ_1, γ_2 as in (1.9). Notice that the local integrability of $d(\cdot, E)^{-\vartheta}$ for $\vartheta \in I_\infty(E)$ along with the fact that $d(\cdot, E) = d(\cdot, \overline{E})$ implies $|\overline{E}| = 0$ and then $u_{\gamma_1, \gamma_2}(x)$ is well defined for a.e. $x \in \mathbb{R}^n$. Now, the hypothesis (1.9) means precisely $d(\cdot, E)^{\gamma_1}, d(\cdot, E)^{\gamma_2} \in A_\infty(\mathbb{R}^n)$ which together with (4.1) and the fact (mentioned

in Section 2.1) that $A_\infty(\mathbb{R}^n)$ forms a lattice yields $u_{\gamma_1, \gamma_2} \in A_\infty(\mathbb{R}^n)$. Similarly, δ_1, δ_2 as in (1.10) and (4.2) imply $v_{\delta_1, \delta_2}^{-p'/p} \in A_\infty(\mathbb{R}^n)$. Next, we use (4.3) and (4.4) to write

$$\left(\frac{u_{\gamma_1, \gamma_2}(x)^{\frac{1}{q}}}{v_{\delta_1, \delta_2}(x)^{\frac{1}{p}}} \right)^{\frac{n}{\Theta(\alpha)}} = \begin{cases} \left(\frac{d(x, E)}{\text{diam}(E)} \right)^{\frac{n}{\Theta(\alpha)} \left(\frac{\gamma_1}{q} - \frac{\delta_1}{p} \right)} & \text{if } d(x, E) < \text{diam}(E) \\ \left(\frac{d(x, E)}{\text{diam}(E)} \right)^{\frac{n}{\Theta(\alpha)} \left(\frac{\gamma_2}{q} - \frac{\delta_2}{p} \right)} & \text{if } d(x, E) \geq \text{diam}(E), \end{cases}$$

and by choosing $\gamma_1, \gamma_2, \delta_1, \delta_2$ such that

$$\begin{cases} \frac{n}{\Theta(\alpha)} \left(\frac{\gamma_1}{q} - \frac{\delta_1}{p} \right) = -(n - s) \\ \frac{n}{\Theta(\alpha)} \left(\frac{\gamma_2}{q} - \frac{\delta_2}{p} \right) = -n, \end{cases} \quad (4.5)$$

for some $s \in (\overline{\dim}_M(E), n)$ it follows that $\left(\frac{u_{\gamma_1, \gamma_2}(x)^{\frac{1}{q}}}{v_{\delta_1, \delta_2}(x)^{\frac{1}{p}}} \right)^{n/\Theta(\alpha)} = g_E(x)$ with g_E as in Lemma 3.1 satisfying $g_E \in L^{1, \infty}(\mathbb{R}^n)$ with the estimate

$$\|g_E\|_{L^{1, \infty}(\mathbb{R}^n)} \leq K \text{diam}(E)^{n-s} + \omega_n 2^n \text{diam}(E)^n$$

for some $K > 0$ depending on s, n and E , due to Lemma 2.5 (since $\overline{\dim}_M(E) < s < n$). Consequently,

$$\|u^{1/q}/v^{1/p}\|_{L^{n/\Theta(\alpha), \infty}(\mathbb{R}^n)} = \|g_E\|_{L^{1, \infty}(\mathbb{R}^n)}^{n/\Theta(\alpha)} \leq (K \text{diam}(E)^{n-s} + \omega_n 2^n \text{diam}(E)^n)^{n/\Theta(\alpha)}.$$

Finally, by solving for s from the first equality in (4.5), the inequalities $\overline{\dim}_M(E) < s < n$ amount to (1.11), while the second equality in (4.5) means (1.12). Thus the conditions $u_{\gamma_1, \gamma_2}, v_{\delta_1, \delta_2}^{-p'/p} \in A_\infty(\mathbb{R}^n)$ and $u_{\gamma_1, \gamma_2}^{1/q}/v_{\delta_1, \delta_2}^{1/p} \in L^{n/\Theta(\alpha), \infty}(\mathbb{R}^n)$ in Theorem 2.3 are met and (1.13) follows from it. \square

Proof of Theorem 1.3. A consequence of Theorem 1.1 applied to I_1 , by noticing that the conditions on $\gamma_1, \gamma_2, \delta_1, \delta_2$ coincide with (1.11) and (1.12) when $\alpha = 1$. \square

Proof of Theorem 1.4. Since $\delta_1 = \delta_2 = 0$, the condition $\delta_j p'/p \in I_\infty(E)$ holds trivially for $j = 1, 2$ (since $\ell_\infty(E) < 0 < \text{Mu}_\infty(E)$), also $v_{\delta_1, \delta_2} \equiv 1$. Then (1.9), (1.10), (1.11), and (1.12) reduce to

$$\begin{cases} -\gamma_j \in I_\infty(E) \quad \text{for } j = 1, 2 \\ \overline{\dim}_M(E) - n < \frac{n}{q\Theta(1)}\gamma_1 < 0 \\ \gamma_2 = -q\Theta(1) = \frac{q}{p}(n-p) - n < 0, \end{cases}$$

which are equivalent to (1.16) and (1.17). Notice that the second inequality for γ_2 in (1.17), along with the fact that $\text{Mu}_\infty(E) \leq n$ by [15, Corollary 8.10.6], imposes $p < n$, while the first one makes $q < np/(n-p)$. Finally, since $\gamma_1 > \gamma_2 \left(1 - \frac{\overline{\dim}_M(E)}{n}\right)$, with $\gamma_2 < 0$, it follows that $\gamma_1 > \gamma_2$ and (1.14) yields

$$u_{\gamma_1, \gamma_2}(x) = \min \left\{ \left(\frac{d(x, E)}{\text{diam}(E)} \right)^{\gamma_1}, \left(\frac{d(x, E)}{\text{diam}(E)} \right)^{\gamma_2} \right\}.$$

Finally, the weighted Hardy-Sobolev inequality (1.19) follows from (1.18) and (2.4). \square

Proof of Theorem 1.6. In this case $\alpha = \Theta(\alpha) = 1$ (in particular, $n > 1$) and (1.9), (1.10), (1.11), and (1.12) reduce to (1.20) and therefore (1.21) follows from Theorem 1.1. In turn, (1.21) and (2.4) yield (1.22) \square

Proof of Theorem 1.7. Having $\Theta(\alpha) = 0$ means $q = np/(n - \alpha p)$. In this critical case our guide will be Theorem 2.2. We will choose the indices $\gamma_1, \gamma_2, \delta_1, \delta_2$ such that $u^{1/q} \leq v^{1/p}$, which is quickly done from the definitions of u and v in (1.14) and (4.4), respectively. Indeed, by considering the cases $d(x, E) < \text{diam}(E)$ and $d(x, E) \geq \text{diam}(E)$ the inequality $u^{1/q} \leq v^{1/p}$ is achieved by choosing $\gamma_1, \gamma_2, \delta_1, \delta_2$ with $\gamma_1/q - \delta_1/p \geq 0$ and $\gamma_2/q - \delta_2/p \leq 0$, which amount to (1.23) and (1.24), and together with (1.9) and (1.10) (always to guarantee $u_{\gamma_1, \gamma_2}, v_{\delta_1, \delta_2}^{-p'/p} \in A_\infty(\mathbb{R}^n)$) as well as Theorem 2.2, yield (1.25). \square

Proof of Corollary 1.8. Given β_1, β_2 verifying (1.26), define $\gamma_1 := \beta_2, \gamma_2 := \beta_2, \delta_1 := \beta_1$ and $\delta_2 := \beta_1$. Then, due to (1.26), the exponents $\gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$ satisfy (1.23) and (1.24). Also, by (1.26) since $\gamma_1 = \gamma_2 = \beta_2$ we get $-\gamma_1 = -\gamma_2 \in I_\infty(E)$ and the condition (1.9) is met as is the condition (1.10) since $p'\beta_1/p \in I_\infty(E)$ means precisely (1.27). \square

Proof of Theorem 1.9 It follows from Theorem 1.7 applied with $\alpha = 1$ and by noticing that the conditions on $\gamma_1, \gamma_2, \delta_1, \delta_2$ coincide with (1.9), (1.10), (1.23) and (1.24) when $\alpha = 1$ and $q = np/(n - p)$. \square

5. PROOF OF THEOREM 1.10

We will make use of the following result from [13].

Theorem 5.1 (Theorem 6.8 from [13]). *Fix $n > 1$ and $1 < p \leq q < n/\Theta(1)$ with $\Theta(1) := 1 + n(1/q - 1/p) > 0$. Given $v \in RH_\infty(\mathbb{R}^n)$ such that $v^{-p'/p} \in A_\infty(\mathbb{R}^n)$ there exists $C > 0$ depending only on $n, p, q, [v^{-p'/p}]_{A_\infty(\mathbb{R}^n)}$, and $[v]_{RH_\infty(\mathbb{R}^n)}$ such that*

$$\left(\int_{\mathbb{R}^n} |f(x)|^q \mathcal{M}(g)(x)^{\Theta(1)q/n} v(x)^{q/p} dx \right)^{1/q} \leq C \|g\|_{L^1(\mathbb{R}^n)}^{\Theta(1)/n} \left(\int_{\mathbb{R}^n} |\nabla f(x)|^p v(x) dx \right)^{1/p}$$

for every $f \in C_0^1(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$.

In the case where v is a power of $d(\cdot, E)$ for a weakly porous set $E \subset \mathbb{R}^n$, we obtain the next corollary of Theorem 5.1.

Corollary 5.2. *Fix $n > 1$ and $1 < p \leq q < n/\Theta(1)$ with $\Theta(1) := 1 + n(1/q - 1/p) > 0$. Given a nonempty weakly porous set $E \subset \mathbb{R}^n$ and $\beta > 0$ with $\beta p'/p < \text{Mu}_1(E)$, there exists $C > 0$, depending only on $\beta, p, q, \text{Mu}_1(E)$, and n , such that*

$$\left(\int_{\mathbb{R}^n} |f(x)|^q \mathcal{M}(g)(x)^{\Theta(1)q/n} d_E(x)^{q\beta/p} dx \right)^{\frac{1}{q}} \leq C \|g\|_{L^1(\mathbb{R}^n)}^{\Theta(1)/n} \left(\int_{\mathbb{R}^n} |\nabla f(x)|^p d_E(x)^\beta dx \right)^{\frac{1}{p}}$$

for every $f \in C_0^1(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$.

Proof. A consequence of Theorem 5.1 by setting $v(x) := d(x, E)^\beta$ so that $v^{-p'/p} \in A_1(\mathbb{R}^n)$ due to $0 < \beta p'/p < \text{Mu}_1(E)$ and then $v \in RH_\infty(\mathbb{R}^n)$ by Remark 2.6. \square

The proof of Theorem 1.10 will also rely on the following lemma.

Lemma 5.3. *Fix a set $E \subset \mathbb{R}^n$ with $0 < \text{diam}(E) < \infty$, and for $\kappa > 0$ let us write $d_\kappa := \kappa \text{diam}(E)$. Then, for each $\gamma < n - \overline{\dim}_M(E)$ there exists $K > 0$ depending on $\gamma, \kappa, \text{diam}(E)$, and n such that*

$$\mathcal{M}(d(\cdot, E)^{-\gamma} \chi_{E_{d_\kappa}})(x) \geq \begin{cases} d(x, E)^{-\gamma} & \text{if } d(x, E) < d_\kappa \\ K d(x, E)^{-n} & \text{if } d(x, E) \geq d_\kappa, \end{cases} \quad \text{for a.e. } x \in \mathbb{R}^n. \quad (5.1)$$

Proof. Let us first notice that the condition $\gamma < n - \overline{\dim}_M(E)$ guarantees

$$\int_{E_R} d(z, E)^{-\gamma} dz < \infty \quad (5.2)$$

for every $R > 0$, see for instance [17, Lemma 1]. Now, for a.e. x with $d(x, E) < d_\kappa$, that is, if a.e. $x \in E_{d_\kappa}$, we have $\mathcal{M}(d(\cdot, E)^{-\gamma} \chi_{E_{d_\kappa}})(x) \geq d(x, E)^{-\gamma}$. Let us then consider the case $x \notin E_{d_\kappa}$, that is, $d_\kappa \leq d(x, E)$. In this case we have

$$E_{d_\kappa} \subset B(x, (2 + 1/\kappa) d(x, E)), \quad (5.3)$$

since, given $y \in E_{d_\kappa}$, for $\varepsilon > 0$ there exist $x_0, x_1 \in E$ with $|x - x_0| < d(x, E) + \varepsilon$ and $|y - x_1| < d(y, E) + \varepsilon < d_\kappa + \varepsilon$ and then

$$\begin{aligned} |y - x| &\leq |y - x_1| + |x_1 - x_0| + |x_0 - x| \leq d_\kappa + \text{diam}(E) + d(x, E) + 2\varepsilon \\ &= d_\kappa + \frac{1}{\kappa} d_\kappa + d(x, E) + 2\varepsilon \leq (2 + 1/\kappa) d(x, E) + 2\varepsilon \end{aligned}$$

and (5.3) follows by letting $\varepsilon \rightarrow 0$ (and by recalling that E_{d_κ} is an open set). Next, define

$$C(\gamma, d_\kappa) := \int_{E_{d_\kappa}} d(z, E)^{-\gamma} dz < \infty \quad (5.4)$$

so that by setting $B_x := B(x, (2 + 1/\kappa) d(x, E))$ we get

$$\frac{1}{|B_x|} \int_{B_x} d(z, E)^{-\gamma} \chi_{E_{d_\kappa}}(z) dz = \frac{1}{|B_x|} \int_{E_{d_\kappa}} d(z, E)^{-\gamma} dz = \frac{C(\gamma, d_\kappa)}{|B_x|}$$

and then

$$\mathcal{M}(d(\cdot, E)^{-\gamma} \chi_{E_{d_\kappa}})(x) \geq C(\gamma, d_\kappa) \omega_n^{-1} ((2 + 1/\kappa) d(x, E))^{-n}$$

and (5.1) is proved with $K := (2 + 1/\kappa)^{-n} \omega_n^{-1} C(\gamma, d_\kappa)$. \square

Proof of Theorem 1.10. The proof will be a combination of Corollary 5.2 and Lemma 5.3. Fix $0 < \gamma < n - \overline{\dim}_M(E)$ and for $\kappa > 0$ let us use Corollary 5.2 with the function

$$g_\kappa := d(\cdot, E)^{-\gamma} \chi_{E_{d_\kappa}}$$

(which belongs to $L^1(\mathbb{R}^n)$ due to (5.2)) where $d_\kappa := \kappa \text{diam}(E)$, so that by Lemma 5.3 we have

$$\mathcal{M}(g_\kappa)(x) \geq K d(x, E)^{-n},$$

for every $x \in \mathbb{R}^n$ with $d(x, E) \geq d_\kappa$ where

$$K := (2 + 1/\kappa)^{-n} \omega_n^{-1} C(\gamma, d_\kappa) > \kappa^n \omega_n^{-1} C(\gamma, d_\kappa)$$

with $C(\gamma, d_\kappa)$ being precisely $\|g_\kappa\|_{L^1(\mathbb{R}^n)}$ by (5.4). Therefore, we get

$$\mathcal{M}(g_\kappa)(x)^{q\Theta(1)/n} \geq \kappa^{q\Theta(1)} \omega_n^{-1} C(\gamma, d_\kappa)^{q\Theta(1)/n} d(x, E)^{-q\Theta(1)}$$

for every $x \in \mathbb{R}^n$ with $d(x, E) \geq \kappa \text{diam}(E)$ and the result follows from Corollary 5.2 as the finite quantity $\|g\|_{L^1(\mathbb{R}^n)}^{\Theta(1)/n}$ appearing on the right-hand side of the inequality cancels out with the one on the left-hand side. \square

Statements and Declarations

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