ON WEIGHTED ESTIMATES FOR FRACTIONAL OPERATORS AND APPLICATIONS TO HARDY-TYPE INEQUALITIES

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Abstract. New necessary and sufficient conditions for two-weight estimates for fractional maximal and integral operators are established by means of Harnack-type estimates for Muckenhoupt weights. Applications include weighted Hardy inequalities with weights based on the Hardy-Littlewood maximal operator.

1. Introduction and main results

For $0 < \alpha < n$ the fractional maximal operator $\mathcal{M}_\alpha$ is defined as

$$\mathcal{M}_\alpha(f)(x) := \sup_{Q \ni x} |Q|^{\alpha/n} \fint_Q |f(y)|\, dy,$$

where $Q \subset \mathbb{R}^n$ is a cube (always with sides parallel to the coordinate axes) and the notation $\fint_E u$ stands for $\frac{1}{|E|}\int_E u(x)\,dx$; that is, the average of u, with respect to Lebesgue measure, over a set E with Lebesgue measure $|E| > 0$ (while the notation $u(E)$ will stand for $\int_E u(x)\,dx$). Similarly, I_α denotes the fractional integral operator

$$I_\alpha(f)(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

When $\alpha = 0$, $\mathcal{M}_0 = \mathcal{M}$ is the Hardy-Littlewood maximal operator. The literature on two-weight inequalities for $\mathcal{M}_\alpha$ and I_α is vast, but we will base the statements of our main results on the following classical theorem by E. Sawyer and R. Wheeden (see [13, Theorem 1]): For $1 < p \le q < \infty$ and $0 < \alpha < n$ define

$$\Theta(\alpha) := \alpha + n\left(\frac{1}{q} - \frac{1}{p}\right), \tag{1.1}$$

then, given nonnegative measurable functions u and v and $r > 1$, the condition

$$[(u,v)]_{A^{r,\alpha}_{p,q}} := \sup_Q |Q|^{\Theta(\alpha)/n} \left(\fint_Q u^r\right)^{\frac{1}{rq}} \left(\fint_Q v^{-rp'/p}\right)^{\frac{1}{rp'}} < \infty \tag{1.2}$$

implies the two-weight strong-type (p,q) for I_α

$$\left(\int_{\mathbb{R}^n} |I_\alpha(f)(x)|^q u(x)\,dx\right)^{1/q} \le C_{n,p,q,\alpha}[(u,v)]_{A^{r,\alpha}_{p,q}} \left(\int_{\mathbb{R}^n} |f(x)|^p v(x)\,dx\right)^{1/p} \tag{1.3}$$

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for every measurable function f. Recall that for $1 < p < \infty$ a weight w in $\mathbb{R}^n$ (that is, $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with $w \geq 0$ a.e. in $\mathbb{R}^n$) is said to belong to the *Muckenhoupt class* $A_p(\mathbb{R}^n)$ if

$$[w]_{A_p} := \sup_Q \left(\fint_Q w\right)\left(\fint_Q w^{\frac{-1}{(p-1)}}\right)^{p-1} < \infty \tag{A_p}$$

where $Q \subset \mathbb{R}^n$ is a cube. The endpoint classes for $p = 1$ and $p = \infty$ are defined by

$$[w]_{A_1} := \sup_Q \left(\fint_Q w\right)\left(\operatorname*{ess\,inf}_Q w\right)^{-1} < \infty \tag{A_1}$$

and

$$[w]_{A_\infty} := \sup_Q \left(\fint_Q w\right)\exp\left(-\fint_Q \ln w\right) < \infty. \tag{A_∞}$$

As it turns out, $A_\infty(\mathbb{R}^n) = \bigcup_{p\geq 1} A_p(\mathbb{R}^n)$ (see for instance [5, Section 9.3]).

Throughout this article, our basic standing assumption on the pair (u, v) will be that both u and $v^{-p'/p}$ belong to $A_\infty(\mathbb{R}^n)$. In this case, there are constants $N \geq 1$ and $r > 1$, depending only on $[u]_{A_\infty(\mathbb{R}^n)}$, $[v^{-p'/p}]_{A_\infty(\mathbb{R}^n)}$, and n, such that

$$\left(\fint_Q u^r\right)^{1/r} \leq N\left(\fint_Q u\right) \quad \text{and} \quad \left(\fint_Q v^{-rp'/p}\right)^{1/r} \leq N\left(\fint_Q v^{-p'/p}\right), \quad \forall Q,$$

(see for instance [5, Theorem 7.2.5]) and then (1.2) becomes equivalent to

$$[(u,v)]_{A^\alpha_{p,q}} := \sup_Q |Q|^{\Theta(\alpha)/n}\left(\fint_Q u\right)^{1/q}\left(\fint_Q v^{-p'/p}\right)^{1/p'} < \infty \tag{1.4}$$

with the estimate $[(u,v)]_{A^\alpha_{p,q}} \leq [(u,v)]_{A^{r,\alpha}_{p,q}} \leq N^{1/q+1/p'}[(u,v)]_{A^\alpha_{p,q}}$. Since $1 < p \leq q < \infty$ and $0 < \alpha < n$ we always have $\Theta(\alpha) \leq \alpha < n$. Moreover, we may (and do) assume that $\Theta(\alpha) \geq 0$. Otherwise, Lebesgue's Differentiation Theorem applied in (1.4) would imply $u \equiv 0$ a.e.

To compare $\mathcal{M}_\alpha$ and I_α when $0 < \alpha < n$, a simple computation yields $\mathcal{M}_\alpha(f)(x) \leq 2^{n-\alpha} I_\alpha(|f|)(x)$ for every measurable f and $x \in \mathbb{R}^n$. On the other hand, as B. Muckenhoupt and R. Wheeden proved in [10, Theorem 1], for every $0 < p < \infty$, $0 < \alpha < n$, and $u \in A_\infty(\mathbb{R}^n)$, there exists $C > 0$, depending only on p, α, n, and $[u]_{A_\infty(\mathbb{R}^n)}$ such that

$$\int_{\mathbb{R}^n} |I_\alpha(f)(x)|^p u(x)\,dx \leq C \int_{\mathbb{R}^n} |\mathcal{M}_\alpha(f)(x)|^p u(x)\,dx$$

for every measurable f. Moreover, by [13, Theorem 1] the inequality (1.4) turns out to be necessary for (1.3) to hold (and this does not require $u, v^{-p'/p} \in A_\infty(\mathbb{R}^n)$). Consequently, under the assumption $u, v^{-p'/p} \in A_\infty(\mathbb{R}^n)$ the condition (1.4) is both necessary and sufficient for the inequality (1.3) to hold true for either I_α or $\mathcal{M}_\alpha$. This is also true in the case $\alpha = 0$ (which implies $\Theta(\alpha) = 0$ and hence $p = q$), since the necessity of (1.4) (with $\Theta(\alpha) = 0$ and $p = q$) for the (u, v)-weighted strong (p, p)-type of the Hardy-Littlewood maximal function has been established by B. Muckenhoupt in [9, Theorem 1] and its sufficiency, when $v^{-p'/p} \in A_\infty(\mathbb{R}^n)$, by C. J. Neugebauer in [11, Theorem 5].

The purpose of this note is to provide new, simpler necessary and sufficient conditions for (1.4) (and hence for (1.3)) to hold under the hypothesis $u, v^{-p'/p} \in A_\infty(\mathbb{R}^n)$. Based on such sufficient conditions, examples of pairs (u, v) satisfying (1.4) will be constructed and then used in several applications to Hardy-type inequalities.

Let us now describe our main results according to the cases $\Theta(\alpha) = 0$ and $\Theta(\alpha) > 0$. When $\Theta(\alpha) = 0$ the Lebesgue Differentiation Theorem again, now applied to (1.4), yields the pointwise inequality $u(x)^{1/q} \leq [(u, v)]_{A^{\alpha}_{p,q}} v(x)^{1/p}$ for a.e. $x \in \mathbb{R}^n$. Our first result shows that, conversely, if $u, v^{-p'/p} \in A_\infty(\mathbb{R}^n)$ then the pointwise condition $u^{1/q} \leq C v^{1/p}$ implies (1.4). More precisely,

Theorem 1.1. *Fix $1 < p \leq q < \infty$ and $0 < \alpha < n$ (with $\alpha = 0$ allowed for $\mathcal{M}_0$) such that $\Theta(\alpha) = 0$ (that is, $1/q = 1/p - \alpha/n$). Given (u, v) with $u, v^{-p'/p} \in A_\infty(\mathbb{R}^n)$ suppose that there exists $C_0 > 0$ with*

$$u(x)^{1/q} \leq C_0\, v(x)^{1/p}, \quad a.e.\ x \in \mathbb{R}^n. \tag{1.5}$$

Then (1.4) holds true and consequently

$$\left(\int_{\mathbb{R}^n} |I_\alpha(f)(x)|^q u(x)\, dx\right)^{1/q} \leq C_0 C_1 \left(\int_{\mathbb{R}^n} |f(x)|^p v(x)\, dx\right)^{1/p},$$

for every measurable f (with I_α replaced by $\mathcal{M}_0$ if $\alpha = 0$), where $C_1 > 0$ depends only on n, p, q, α, $[u]_{A_\infty(\mathbb{R}^n)}$, and $[v^{-p'/p}]_{A_\infty(\mathbb{R}^n)}$.

For the case $\Theta(\alpha) > 0$ (that is, $1/q > 1/p - \alpha/n$) we prove the following

Theorem 1.2. *Fix $1 < p \leq q < \infty$ and $0 < \alpha < n$ such that $\Theta(\alpha) > 0$. Given (u, v) with $u, v^{-p'/p} \in A_\infty(\mathbb{R}^n)$ the condition $u^{1/q}/v^{1/p} \in L^{n/\Theta(\alpha),\infty}(\mathbb{R}^n)$ implies (1.4) with the estimate*

$$[(u, v)]_{A^{\alpha}_{p,q}} \leq C_2 \|u^{1/q}/v^{1/p}\|_{L^{n/\Theta(\alpha),\infty}(\mathbb{R}^n)}, \tag{1.6}$$

where $C_2 > 0$ depends only on n, α, p, q, $[u]_{A_\infty(\mathbb{R}^n)}$, and $[v^{-p'/p}]_{A_\infty(\mathbb{R}^n)}$. In particular,

$$\left(\int_{\mathbb{R}^n} |I_\alpha(f)(x)|^q u(x)\, dx\right)^{1/q} \leq C_2 \|u^{1/q}/v^{1/p}\|_{L^{n/\Theta(\alpha),\infty}(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |f(x)|^p v(x)\, dx\right)^{1/p},$$

for every measurable f.

If, in addition, $u^{1/q}/v^{1/p} \in A_\infty(\mathbb{R}^n)$, then (1.4) is equivalent to

$$\mathcal{M}_{\Theta(\alpha)}(u^{1/q}/v^{1/p}) \in L^\infty(\mathbb{R}^n)$$

with the estimates

$$\theta \|\mathcal{M}_{\Theta(\alpha)}(u^{1/q}/v^{1/p})\|_{L^\infty(\mathbb{R}^n)} \leq [(u, v)]_{A^{\alpha}_{p,q}} \leq \theta^{-1} \|\mathcal{M}_{\Theta(\alpha)}(u^{1/q}/v^{1/p})\|_{L^\infty(\mathbb{R}^n)},$$

where $\theta \in (0, 1)$ depends only on n, α, p, $[u]_{A_\infty(\mathbb{R}^n)}$, $[v^{-p'/p}]_{A_\infty(\mathbb{R}^n)}$, and $[u^{1/q}/v^{1/p}]_{A_\infty(\mathbb{R}^n)}$.

Remark 1.3. Notice that given $1 < p \leq q < \infty$ and $0 < \alpha < n$ with $\Theta(\alpha) = 0$, by setting $t := q/p' + 1$ the condition (1.2) follows from (1.5) together with the assumption $u \in A_t(\mathbb{R}^n)$. Indeed, given $u \in A_t(\mathbb{R}^n)$ there are constants $r > 1$ and $N \geq 1$, depending only on $[u]_{A_t(\mathbb{R}^n)}$, t, and n, such that $u^r \in A_t(\mathbb{R}^n)$ with $[u^r]_{A_t(\mathbb{R}^n)} \leq N[u]^r_{A_t(\mathbb{R}^n)}$ (see for instance [5, Theorem 9.2.5]). Hence, by using that $v^{-1/p} \leq Cu^{-1/q}$ a.e. from (1.5), we can write

$$\begin{aligned}\left(\fint_Q u^r\right)^{\frac{1}{rq}}\left(\fint_Q v^{-rp'/p}\right)^{\frac{1}{rp'}} &\leq C_0\left(\fint_Q u^r\right)^{\frac{1}{rq}}\left(\fint_Q u^{-rp'/q}\right)^{\frac{1}{rp'}}\\ &= C_0\left[\left(\fint_Q u^r\right)\left(\fint_Q u^{-rp'/q}\right)^{\frac{q}{p'}}\right]^{\frac{1}{rq}} \leq C_0[u^r]_{A_t(\mathbb{R}^n)}^{\frac{1}{rq}}\end{aligned}$$

and (1.2) follows with $[(u,v)]_{A_{p,q}^{r,\alpha}} \leq C_0 N^{\frac{1}{rq}}[u]_{A_t(\mathbb{R}^n)}^{1/q}$. Similarly, the inequality (1.5) now together with the condition $v^{-p'/p} \in A_{t'}(\mathbb{R}^n)$ (where $(t'-1)(t-1)=1$) also quantitatively implies (1.2). Thus, Theorem 1.1 allows the passage from the assumption $u \in A_t(\mathbb{R}^n)$ or $v^{-p'/p} \in A_{t'}(\mathbb{R}^n)$, for the precise index $t := q/p' + 1$, to $u, v^{-p'/p} \in A_\infty(\mathbb{R}^n)$.

Remark 1.4. To the best of our knowledge, Theorem 1.2 seems to be the first to establish sufficient conditions for the (u,v)-based estimates (1.3) phrased in terms of the quotient $u^{1/q}/v^{1/p}$.

The rest of the article is organized as follows: Section 2 includes an account on basic properties of Muckenhoupt weights. In Section 3 we identify the largest rearrangement-invariant space D_α so that the fractional maximal operator $\mathcal{M}_\alpha : D_\alpha \to L^\infty(\mathbb{R}^n)$ is bounded. This boundedness property is crucial to the proof of Theorem 1.2. In Section 4 we prove our main results Theorems 1.1 and 1.2. In Section 5 we construct examples of weight pairs (u,v) satisfying (1.4) where u takes the form $u = \mathcal{M}(g)^{\Theta(\alpha)q/n} v^{q/p}$ with $g \in L^1(\mathbb{R}^n)$. Finally, in Section 6 we apply Theorems 1.1 and 1.2 with the classes of weights constructed in Section 5 to produce new families of weighted Hardy inequalities, including some extensions of the classical Caffarelli-Kohn-Nirenberg inequalities (see Theorem 6.5 and Corollary 6.6).

2. PRELIMINARIES

Given two nonnegative quantities or functions A and B, we will frequently use the notation $A \lesssim B$ to indicate the presence of an implied constant $C > 0$ such that $A \leq CB$ and the nature of the constant $C > 0$ will be described in each case. We write $A \simeq B$ if $A \lesssim B$ and $B \lesssim A$.

2.1. **On Muckenhoupt weights.**

Typical examples of A_p-weights are the locally integrable powers of $|x|$. More precisely, for $1 < p < \infty$, the weight $|x|^a \in A_p(\mathbb{R}^n)$ if and only if $-n < a < n(p-1)$ and $|x|^a \in A_1(\mathbb{R}^n)$ if and only if $-n < a \leq 0$, see for instance [5, p. 286]. In the construction of $A_\infty(\mathbb{R}^n)$ weights in Section 5, we will use the facts that if $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ with $\mathcal{M}(g)(x) < \infty$ for a.e. $x \in \mathbb{R}^n$ and $\theta \in (0,1)$, then $\mathcal{M}(g)^\theta \in A_1(\mathbb{R}^n)$ with the estimate

$$[\mathcal{M}(g)^\theta]_{A_1(\mathbb{R}^n)} \leq \frac{C_n}{1-\theta}, \tag{2.1}$$

for a dimensional constant $C_n > 0$, see for instance [5, Theorem 9.2.7]. Also, if $w_1, w_2 \in A_1(\mathbb{R}^n)$ and $\tau > 1$, then

$$w_1 w_2^{-(\tau-1)} \in A_\tau(\mathbb{R}^n), \tag{2.2}$$

see for instance [5, Exercise 7.1.2].

During the proofs of Theorems 1.1 and 1.2 (in Section 4) we will repeatedly make use of the following characterization of $A_\infty(\mathbb{R}^n)$ established in [8, Theorem 1.1].

Theorem 2.1. *Fix a weight w in $\mathbb{R}^n$ with $w(Q) > 0$ for every cube $Q \subset \mathbb{R}^n$. Then the following are quantitatively equivalent:*

(i) $w \in A_\infty(\mathbb{R}^n)$,

(ii) for every $\varepsilon \in (0,1)$ there exists $C_\varepsilon \geq 1$ such that for every cube $Q \subset \mathbb{R}^n$ there is a set $H_\varepsilon \subset Q$ satisfying

$$\min\left\{\frac{|H_\varepsilon|}{|Q|}, \frac{w(H_\varepsilon)}{w(Q)}\right\} \geq (1-\varepsilon) \quad \textit{and} \quad \operatorname*{ess\,sup}_{H_\varepsilon} w \leq C_\varepsilon \operatorname*{ess\,inf}_{H_\varepsilon} w.$$

Remark 2.2. Given $w \in A_\infty(\mathbb{R}^n)$, $\varepsilon \in (0,1)$, a cube $Q \subset \mathbb{R}^n$, and $H_\varepsilon \subset Q$ as in Theorem 2.1 (ii), for a.e. $x \in H_\varepsilon$ we can write

$$\begin{aligned} w(x) &\leq \operatorname*{ess\,sup}_{H_\varepsilon} w \leq C_\varepsilon \operatorname*{ess\,inf}_{H_\varepsilon} w \leq C_\varepsilon \fint_{H_\varepsilon} w \leq \frac{C_\varepsilon}{(1-\varepsilon)} \fint_Q w \\ &\leq \frac{C_\varepsilon}{(1-\varepsilon)^2} \fint_{H_\varepsilon} w \leq \frac{C_\varepsilon}{(1-\varepsilon)^2} \operatorname*{ess\,sup}_{H_\varepsilon} w \leq \frac{C_\varepsilon^2}{(1-\varepsilon)^2} \operatorname*{ess\,inf}_{H_\varepsilon} w \leq \frac{C_\varepsilon^2}{(1-\varepsilon)^2} w(x). \end{aligned}$$

That is,

$$w(x) \simeq \fint_{H_\varepsilon} w \simeq \fint_Q w \quad \text{a.e. } x \in H_\varepsilon, \tag{2.3}$$

where the implied constants depend only on ε, $[w]_{A_\infty}$, and n.

Remark 2.3. Fix N weights $w_1, \dots, w_N \in A_\infty(\mathbb{R}^n)$, $\varepsilon \in (0,1)$, and a cube $Q \subset \mathbb{R}^n$. If H_ε^j denotes the set $H_\varepsilon \subset Q$ from Theorem 2.1 (ii), corresponding to each $w_j \in A_\infty(\mathbb{R}^n)$, $j = 1, \dots, N$, then

$$|Q \setminus \cap_{j=1}^N H_\varepsilon^j| = |\cup_{j=1}^N (Q \setminus H_\varepsilon^j)| \leq \sum_{j=1}^N |Q \setminus H_\varepsilon^j| \leq \varepsilon N |Q|,$$

since $|Q \setminus H_\varepsilon^j| \leq \varepsilon |Q|$ for every $j = 1, \dots, N$. Thus, by choosing $\varepsilon < 1/N$,

$$\left|\bigcap_{j=1}^N H_\varepsilon^j\right| \geq (1-\varepsilon N)|Q|. \tag{2.4}$$

2.2. **Reverse Hölder classes.** For $1 < s < \infty$ we write $w \in RH_s(\mathbb{R}^n)$ if

$$[w]_{RH_s} := \sup_Q \left(\fint_Q w(x)^s \, dx\right)^{1/s} \left(\fint_Q w(x) \, dx\right)^{-1} < \infty.$$

As it turns out (see for instance [5, Section 9.3]), $\bigcup_{s>1} RH_s(\mathbb{R}^n) = A_\infty(\mathbb{R}^n)$.

For $s = \infty$ we write $w \in RH_\infty(\mathbb{R}^n)$ if

$$[w]_{RH_\infty} := \sup_Q \left(\operatorname*{ess\,sup}_Q w\right) \left(\fint_Q w(x) \, dx\right)^{-1} < \infty.$$

It is a known fact that the class $RH_\infty(\mathbb{R}^n)$ is invariant under positive powers, that is, the implication

$$w \in RH_\infty(\mathbb{R}^n), \ell > 0 \Rightarrow w^\ell \in RH_\infty(\mathbb{R}^n) \tag{2.5}$$

holds true with $[w^\ell]_{RH_\infty(\mathbb{R}^n)}$ depending only on $[w]_{RH_\infty(\mathbb{R}^n)}$, ℓ, and n (see [4, Theorem 4.2]) and that the inclusion $RH_\infty(\mathbb{R}^n) \subset \bigcap_{s>1} RH_s(\mathbb{R}^n)$ is proper (see [4, p. 2948]). In addition, the class $\bigcap_{s>1} RH_s(\mathbb{R}^n)$ constitutes the multipliers of $A_\infty(\mathbb{R}^n)$. In particular, the implication

$$w \in A_\infty(\mathbb{R}^n), u \in \bigcap_{s>1} RH_s(\mathbb{R}^n) \Rightarrow wu \in A_\infty(\mathbb{R}^n) \tag{2.6}$$

holds true with $[wu]_{A_\infty(\mathbb{R}^n)} \leq [w^\tau]^{1/\tau}_{A_\infty(\mathbb{R}^n)} [u^{\tau'}]^{1/\tau'}_{A_\infty(\mathbb{R}^n)}$ for some $\tau > 1$ depending only on $[w]_{A_\infty(\mathbb{R}^n)}$ and n, see for instance [7, Theorem 3.5].

Remark 2.4. $RH_\infty(\mathbb{R}^n)$ and $A_1(\mathbb{R}^n)$. The classes $RH_\infty(\mathbb{R}^n)$ and $A_1(\mathbb{R}^n)$ are related as follows: if $w \in A_1(\mathbb{R}^n)$ then there exists $\theta_0 > 0$ (depending only on $[w]_{A_1(\mathbb{R}^n)}$ and n) such that $w^{-\theta_0} \in RH_\infty(\mathbb{R}^n)$. Conversely, if $w \in RH_\infty(\mathbb{R}^n)$, then there exists $\theta_1 > 0$ (depending only on $[w]_{RH_\infty(\mathbb{R}^n)}$ and n) such that $w^{-\theta_1} \in A_1(\mathbb{R}^n)$, see [4, Corollary 4.5].

In particular, one has that $|x|^\delta \in RH_\infty(\mathbb{R}^n)$ for every $\delta \geq 0$ since by fixing $0 < q < n$ the weight $|x|^{-q}$ belongs to $A_1(\mathbb{R}^n)$, then $|x|^\theta \in RH_\infty(\mathbb{R}^n)$ for some $\theta > 0$. Finally, (2.5) applied with $\ell = \delta/\theta$ yields $|x|^\delta \in RH_\infty(\mathbb{R}^n)$.

3. On $L^\infty(\mathbb{R}^n)$ as the range for $\mathcal{M}_\alpha$

Given $0 < \alpha < n$, in this section we will characterize the optimal domain D_α, for the fractional maximal operator $\mathcal{M}_\alpha$, so that the boundedness $\mathcal{M}_\alpha : D_\alpha \to L^\infty(\mathbb{R}^n)$ holds. For this purpose, we recall the definition of the nonincreasing rearragment of $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, denoted as f^* (see [1]):

$$f^*(t) = \inf\{s > 0 : \big|\{x \in \mathbb{R}^n : |f(x)| > s\}\big| < t\}, \quad \forall t > 0.$$

We will also need the following result [3, Theorem 1.1], where the authors show that if $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then

$$(\mathcal{M}_\alpha f)^*(t) \lesssim \sup_{t<s<\infty} s^{\alpha/n} f^{**}(s), \quad \forall t > 0, \tag{3.1}$$

and, there exists a function $g \in L^1_{\text{loc}}(\mathbb{R}^n)$, with $g^* = f^*$, such that

$$\sup_{t<s<\infty} s^{\alpha/n} f^{**}(s) \lesssim (\mathcal{M}_\alpha g)^*(t), \tag{3.2}$$

where $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds$ and the implied constants depend only on α and n.

In view of this, it is reasonable to introduce the following rearrangement invariant structure: Given a set X, satisfying that $\big(L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big) \subset X \subset L^1_{\text{loc}}(\mathbb{R}^n)$, we define

$$\widetilde{X} = \bigcup_{Y \subset X \text{ is an r.i. space}} Y \qquad ,$$

where $(Y, \|\cdot\|)$ is a rearrangement invariant space (r.i.) if it satisfies the following conditions (see [1, Definitions I.1.1 and II.4.1]):

$Y = \{f \text{ is measurable and } \|f\| < \infty\}$ with a norm $\|\cdot\|$ (called Banach function norm) for which, given $f, g, f_n \in Y$, $A \subset \mathbb{R}^n$ measurable:

(i) $\|f\| \leq \|g\|$, if $|f| \leq |g|$,

(ii) $0 \leq f_n \leq f_{n+1} \to f$, a.e. $\Rightarrow \|f_n\| \to \|f\|$,

(iii) $\chi_E \in Y$, if $|E| < \infty$,

(iv) $\displaystyle\int_E |f(x)|\,dx \leq C_E \|f\|$, if $|E| < \infty$, and

(v) $\|f\| = \|g\|$, if $f^* = g^*$.

Since, for every r.i. space Y, we have that [1, II.Theorem 6.6],

$$\big(L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big) \subset Y \subset \big(L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)\big),$$

it is then clear that

$$\big(L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big) \subset \widetilde{X} \subset X \cap \big(L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)\big).$$

Also, if $f \in \widetilde{X}$ and $f^* = g^*$, then $g \in \widetilde{X}$. Moreover, if X is already an r.i. space, then $\widetilde{X} = X$. Let us define

$$X_\alpha := \Big\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \mathcal{M}_\alpha f \in L^\infty(\mathbb{R}^n) \Big\}. \tag{3.3}$$

The main result of this section is the following:

Theorem 3.1. *For $0 < \alpha < n$, let X_α be as in* (3.3)*. Then, $\widetilde{X_\alpha} = L^{n/\alpha,\infty}(\mathbb{R}^n)$. Hence, $L^{n/\alpha,\infty}(\mathbb{R}^n)$ is the largest r.i. space Y for which $\mathcal{M}_\alpha : Y \to L^\infty(\mathbb{R}^n)$ is bounded.*

Proof. Let us first observe that $\widetilde{X_\alpha}$ is well defined; that is, $\big(L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big) \subset X_\alpha$. In fact, if $f \in \big(L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$, then for every cube Q:

$$|Q|^{\alpha/n-1} \int_Q |f(x)|\,dx \leq \|f\|_\infty, \text{ if } |Q| < 1,$$

and

$$|Q|^{\alpha/n-1} \int_Q |f(x)|\,dx \leq \|f\|_1, \text{ if } |Q| \geq 1.$$

Thus, $\|\mathcal{M}_\alpha f\|_\infty \leq \|f\|_{L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)} := \max\{\|f\|_{L^1(\mathbb{R}^n)}, \|f\|_{L^\infty(\mathbb{R}^n)}\}$.

Let $h \in L^1_{\text{loc}}(\mathbb{R}^+)$ and define the Hardy operator $Ah(t) = \frac{1}{t}\int_0^t h(s)\,ds$. It is well known that (see [6, p. 240]), for $1 < p \leq \infty$, $A : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ and hence, for every $1 < p < \infty$, $A : L^{p,\infty}(\mathbb{R}^+) \to L^{p,\infty}(\mathbb{R}^+)$. Lets us now prove the following remark:

$$f \in L^{n/\alpha,\infty}(\mathbb{R}^n) \iff \sup_{t<s<\infty} s^{\alpha/n} f^{**}(s) \in L^\infty(\mathbb{R}^+). \tag{3.4}$$

In fact, since $f^* \leq f^{**} = A(f^*)$,

$$\begin{aligned} \sup_{t<s<\infty} s^{\alpha/n} f^{**}(s) \in L^\infty(\mathbb{R}^+) &\iff A(f^*) \in L^{n/\alpha,\infty}(\mathbb{R}^+) \\ &\Rightarrow f^* \in L^{n/\alpha,\infty}(\mathbb{R}^+) \iff f \in L^{n/\alpha,\infty}(\mathbb{R}^n). \end{aligned}$$

Conversely, if $f \in L^{n/\alpha,\infty}(\mathbb{R}^n)$, since $1 < n/\alpha < \infty$,

$$\begin{aligned} \Big\| \sup_{t<s<\infty} s^{\alpha/n} f^{**}(s) \Big\|_{L^\infty(\mathbb{R}^+)} &= \sup_{0<s<\infty} s^{\alpha/n} f^{**}(s) = \|A(f^*)\|_{L^{n/\alpha,\infty}(\mathbb{R}^+)} \\ &\lesssim \|f^*\|_{L^{n/\alpha,\infty}(\mathbb{R}^+)} = \|f\|_{L^{n/\alpha,\infty}(\mathbb{R}^n)} < \infty. \end{aligned}$$

To finish, if $f \in L^{n/\alpha,\infty}(\mathbb{R}^n)$, then using (3.1) and (3.4), we have that

$$\|\mathcal{M}_\alpha f\|_{L^\infty(\mathbb{R}^n)} \lesssim \|f\|_{L^{n/\alpha,\infty}(\mathbb{R}^n)},$$

and hence $f \in X_\alpha$. Thus, $L^{n/\alpha,\infty}(\mathbb{R}^n)$ is an r.i. space contained in X_α and, therefore, $L^{n/\alpha,\infty}(\mathbb{R}^n) \subset \widetilde{X_\alpha}$.

Conversely, if $f \in \widetilde{X_\alpha}$, consider g as in (3.2). Then $g^* = f^*$, and hence $g \in \widetilde{X_\alpha}$. Thus, $\mathcal{M}_\alpha g \in L^\infty(\mathbb{R}^n)$ and, using both (3.1) and (3.2),

$$\Big\| \sup_{t<s<\infty} s^{\alpha/n} f^{**}(s) \Big\|_{L^\infty(\mathbb{R}^+)} \simeq \|\mathcal{M}_\alpha g\|_{L^\infty(\mathbb{R}^n)} < \infty.$$

Finally, (3.4) gives us that $f \in L^{n/\alpha,\infty}(\mathbb{R}^n)$. □

4. Proofs of Theorems 1.1 and 1.2

4.1. **Proof of Theorem 1.1.** Fix $u, v^{-p'/p} \in A_\infty(\mathbb{R}^n)$. By Remark 2.3 with $N = 2$, fix $\varepsilon \in (0, 1/2)$ and given a cube Q, let H_ε^1 and H_ε^2 denote the subset $H_\varepsilon \subset Q$ from Theorem 2.1 (ii), corresponding to u and $v^{-p'/p}$, respectively. Set $H := H_\varepsilon^1 \cap H_\varepsilon^2$, so that (2.4) yields $|H| \geq (1 - 2\varepsilon)|Q| > 0$, and then (2.3) applied to u and $v^{-p'/p}$ gives

$$\left(\fint_Q u \right)^{1/q} \simeq u(x)^{1/q}, \quad \text{a.e. } x \in H \tag{4.1}$$

as well as

$$\left(\fint_Q v^{-p'/p} \right)^{1/p'} \simeq v(x)^{-1/p}, \quad \text{a.e. } x \in H. \tag{4.2}$$

Then, by fixing $x \in H$, (4.1) and (4.2) along with the hypothesis $u^{1/q} \leq C v^{1/p}$ a.e. $\mathbb{R}^n$ yield

$$\left(\fint_Q u \right)^{1/q} \left(\fint_Q v^{-p'/p} \right)^{1/p'} \simeq u(x)^{1/q} v(x)^{-1/p} \lesssim C,$$

where the implied constants depend only on $[u]_{A_\infty(\mathbb{R}^n)}$, $[v^{-p'/p}]_{A_\infty(\mathbb{R}^n)}$, ε, p, and n. □

4.2. **Proof of Theorem 1.2.** In keeping with the notation from the proof of Theorem 1.1, from (4.1), (4.2), and the fact that $|H| \geq (1 - 2\varepsilon)|Q| > 0$, given $1 < p \leq q < \infty$ we have

$$\left(\fint_Q u \right)^{1/q} \left(\fint_Q v^{-p'/p} \right)^{1/p'} \simeq \frac{u(x)^{1/q}}{v(x)^{1/p}}.$$

Now, since (4.1) and (4.2) imply $u(x) \simeq u(y)$ as well as $v(x) \simeq v(y)$ for a.e. $x, y \in H$, it follows that

$$\frac{u(x)^{1/q}}{v(x)^{1/p}} \simeq \fint_H \frac{u^{1/q}}{v^{1/p}}, \quad \text{a.e. } x \in H.$$

Consequently,

$$\left(\fint_Q u \right)^{1/q} \left(\fint_Q v^{-p'/p} \right)^{1/p'} \simeq \fint_H \frac{u^{1/q}}{v^{1/p}} \lesssim \fint_Q \frac{u^{1/q}}{v^{1/p}} \tag{4.3}$$

which, from the definition of $\Theta(\alpha)$ in (1.1), yields

$$|Q|^{\alpha/n + 1/q - 1/p} \left(\fint_Q u \right)^{1/q} \left(\fint_Q v^{-p'/p} \right)^{1/p'} \lesssim |Q|^{\Theta(\alpha)/n} \fint_Q \frac{u^{1/q}}{v^{1/p}}. \tag{4.4}$$

Finally, the hypothesis $u^{1/q}/v^{1/p} \in L^{\frac{n}{\Theta(\alpha)},\infty}(\mathbb{R}^n)$ and Theorem 3.1 imply

$$|Q|^{\Theta(\alpha)/n} \fint_Q \frac{u^{1/q}}{v^{1/p}} \leq \|\mathcal{M}_{\Theta(\alpha)}(u^{1/q}/v^{1/p})\|_{L^\infty(\mathbb{R}^n)} \leq C\|u^{1/q}/v^{1/p}\|_{L^{n/\Theta(\alpha),\infty}(\mathbb{R}^n)}$$

and (1.4) follows. If, in addition, $u^{1/q}/v^{1/p} \in A_\infty(\mathbb{R}^n)$, then fix $\varepsilon \in (0, 1/3)$ and given a cube Q, let H_ε^1, H_ε^2, and H_ε^3 denote the subset $H_\varepsilon \subset Q$ from Theorem 2.1 (ii), corresponding to u, $v^{-p'/p}$, and $u^{1/q}/v^{1/p}$, respectively, and put $H := H_\varepsilon^1 \cap H_\varepsilon^2 \cap H_\varepsilon^3$, so that (2.4) yields $|H| \geq (1-3\varepsilon)|Q| > 0$. Then, the inequality $\fint_H \frac{u^{1/q}}{v^{1/p}} \lesssim \fint_Q \frac{u^{1/q}}{v^{1/p}}$ from (4.3) becomes the equivalence $\fint_H \frac{u^{1/q}}{v^{1/p}} \simeq \fint_Q \frac{u^{1/q}}{v^{1/p}}$, which then turns (4.4) into

$$|Q|^{\alpha/n+1/q-1/p} \left(\fint_Q u\right)^{1/q} \left(\fint_Q v^{-p'/p}\right)^{1/p'} \simeq |Q|^{\Theta(\alpha)/n} \fint_Q \frac{u^{1/q}}{v^{1/p}}$$

and therefore (1.4) is equivalent to $\mathcal{M}_{\Theta(\alpha)}(u^{1/q}/v^{1/p}) \in L^\infty(\mathbb{R}^n)$. □

5. Examples of (u, v) with $u = \mathcal{M}(g)^{\Theta(\alpha)q/n} v^{q/p}$ and $g \in L^1(\mathbb{R}^n)$

For $1 < p < \infty$ and $0 < \alpha < n$, in [12, Section 3] C. Pérez introduced the class $A_{p,\alpha}$ of pairs (u, v) satisfying

$$[(u,v)]_{A_{p,\alpha}} := \sup_Q |Q|^{\alpha/n} \left(\fint_Q u\right)^{1/p} \left(\fint_Q v^{-p'/p}\right)^{1/p'} < \infty$$

as well as the classes

$$\begin{aligned} B_{p,\alpha} :=& \{(u,v) \in A_{p,\alpha} : v^{-p'/p} \in A_\infty(\mathbb{R}^n)\}, \\ C_{p,\alpha} :=& \{(u,v) \in A_{p,\alpha} : u \in A_\infty(\mathbb{R}^n)\}, \\ D_\alpha :=& \left\{u \in A_\infty(\mathbb{R}^n) : \sup_Q |Q|^{\alpha/n} \fint_Q u < \infty\right\}. \end{aligned}$$

Notice that weight pairs in the intersection $B_{p,\alpha} \cap C_{p,\alpha}$ satisfy (1.4) (with $p = q$). As proved in [12, Section 3], if $0 < \alpha < n$, $1 < p < \infty$, and $v^{-p'/p} \in A_\infty(\mathbb{R}^n)$, then $(\mathcal{M}_{\alpha p'}(v^{-p'/p}), v) \in B_{p,\alpha} \cap C_{p,\alpha}$ as long as $\mathcal{M}_{\alpha p'}(v^{-p'/p})$ is finite a.e. Similarly, if $\alpha p < n$, then $(u, \mathcal{M}_{\alpha p}(u)) \in A_{p,\alpha}$, for a general weight u with $\mathcal{M}_{\alpha p}(u)$ finite a.e. Notice however that if $w \equiv 1$, then $\mathcal{M}_\alpha(w) \equiv \infty$ for $0 < \alpha < n$. In this section we will construct examples of weight pairs $(u, v) \in B_{p,\alpha} \cap C_{p,\alpha}$ based on the Hardy-Littlewood maximal operator $\mathcal{M}$.

Given $1 < p \leq q < \infty$, $u, v^{-p'/p} \in A_\infty(\mathbb{R}^n)$, and $0 < \alpha < n$ with $\Theta(\alpha) > 0$, by Theorem 1.2 the condition $u^{1/q}/v^{1/p} \in L^{n/\Theta(\alpha),\infty}(\mathbb{R}^n)$ implies (1.4). In this section we seek a pair (u, v) such that

$$u := \mathcal{M}(g)^{\Theta(\alpha)q/n} v^{q/p}, \tag{5.1}$$

which immediately yields $u^{1/q}/v^{1/p} = \mathcal{M}(g)^{\Theta(\alpha)/n} \in L^{n/\Theta(\alpha),\infty}(\mathbb{R}^n)$ for any $g \in L^1(\mathbb{R}^n)$ with the estimate

$$\|u^{1/q}/v^{1/p}\|_{L^{n/\Theta(\alpha),\infty}(\mathbb{R}^n)} = \|\mathcal{M}(g)^{\Theta(\alpha)/n}\|_{L^{n/\Theta(\alpha),\infty}(\mathbb{R}^n)} \lesssim \|g\|_{L^1(\mathbb{R}^n)}^{\Theta(\alpha)/n}. \tag{5.2}$$

In what follows we are then only to find sufficient conditions for $u, v^{-p'/p} \in A_\infty(\mathbb{R}^n)$.

Theorem 5.1. *Fix $1 < p \leq q < \infty$ and $0 < \alpha < n$ with $\Theta(\alpha) := \alpha + n(1/q - 1/p) > 0$ and $p'\Theta(\alpha) < n$. Then, given $u \in A_1(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$ the pair*

$$(u, v) = (u, \mathcal{M}(g)^{-\Theta(\alpha)p/n} u^{p/q})$$

satisfies (1.4) *with the estimate*

$$[(u, v)]_{A^{\alpha}_{p,q}} \lesssim \|g\|^{\Theta(\alpha)/n}_{L^1(\mathbb{R}^n)}, \tag{5.3}$$

where the implied constants depend only on n, α, p, q, and $[u]_{A_1(\mathbb{R}^n)}$. In particular, from (1.3) *we obtain the weighted inequality*

$$\left(\int_{\mathbb{R}^n} |I_\alpha(f)(x)|^q u(x)\, dx\right)^{\frac{1}{q}} \leq C\|g\|^{\Theta(\alpha)/n}_{L^1(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |f(x)|^p \mathcal{M}(g)(x)^{-\Theta(\alpha)p/n} u(x)^{p/q}\, dx\right)^{\frac{1}{p}},$$

for every $g \in L^1(\mathbb{R}^n)$ and measurable f, where $C > 0$ depends only on n, α, p, q, and $[u]_{A_1(\mathbb{R}^n)}$.

Proof. Let us write (5.1) as

$$v^{-p'/p} = \mathcal{M}(g)^{\Theta(\alpha)p'/n} u^{-p'/q} = \mathcal{M}(g)^{\Theta(\alpha)p'/n} u^{-(\tau-1)}$$

with $\tau := p'/q + 1$. Since we have $\mathcal{M}(g)^{\Theta(\alpha)p'/n} \in A_1(\mathbb{R}^n)$, due to (2.1) with $\theta := \Theta(\alpha)p'/n \in (0, 1)$, as well as $u \in A_1(\mathbb{R}^n)$ by hypothesis, it follows from (2.2) that $v^{-p'/p} \in A_\tau(\mathbb{R}^n)$. Thus, by Theorem 1.2, the pair (u, v) satisfies (1.4) also with the estimate (5.3) due to (1.6) and (5.2). □

Theorem 5.2. *Fix $0 < \alpha < n$ and $1 < p \leq q < n/\Theta(\alpha)$ with $\Theta(\alpha) := \alpha + n(1/q - 1/p) > 0$. Fix $v \in RH_\infty(\mathbb{R}^n)$ such that $v^{-p'/p} \in A_\infty(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$. Then the pair*

$$(u, v) = (\mathcal{M}(g)^{\Theta(\alpha)q/n} v^{q/p}, v)$$

satisfies (1.4) *with the estimate*

$$[(u, v)]_{A^{\alpha}_{p,q}} \leq C_{\alpha,\delta,p,q,n} \|g\|^{\Theta(\alpha)/n}_{L^1(\mathbb{R}^n)}. \tag{5.4}$$

In particular, from (1.3) *we obtain the weighted inequality*

$$\left(\int_{\mathbb{R}^n} |I_\alpha(f)(x)|^q \mathcal{M}(g)(x)^{\Theta(\alpha)q/n} v(x)^{q/p}\, dx\right)^{\frac{1}{q}} \leq C\|g\|^{\Theta(\alpha)/n}_{L^1(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |f(x)|^p v(x)\, dx\right)^{\frac{1}{p}},$$

for every $g \in L^1(\mathbb{R}^n)$ and measurable f, where $C > 0$ depends only on n, α, p, q, $[v^{-p/p}]_{A_\infty(\mathbb{R}^n)}$, and $[v]_{RH_\infty(\mathbb{R}^n)}$.

Proof. First we show that $u := \mathcal{M}(g)^{\Theta(\alpha)q/n} v^{q/p} \in A_\infty(\mathbb{R}^n)$. By assumption, $\Theta(\alpha)q < n$ and $g \in L^1(\mathbb{R}^n)$, it then follows from (2.1) that $\mathcal{M}(g)(x)^{\Theta(\alpha)q/n} \in A_1(\mathbb{R}^n)$. Next, the fact that $v \in RH_\infty(\mathbb{R}^n)$ along with (2.5) makes $v^{q/p} \in RH_\infty(\mathbb{R}^n)$. In particular, $v^{q/p}$ is a multiplier of $A_\infty(\mathbb{R}^n)$ (recall (2.6)), consequently $u \in A_\infty(\mathbb{R}^n)$ and all the hypotheses of Theorem (1.2) are met. Hence, (u, v) satisfies (1.4) with the estimate (5.4) due to (1.6) and (5.2). □

For the case $v(x) := |x|^\delta$ with $0 \leq \delta < n(p-1)$ Theorem 5.2 yields

Corollary 5.3. *Fix $0 < \alpha < n$ and $1 < p \leq q < n/\Theta(\alpha)$ with $\Theta(\alpha) := \alpha + n(1/q - 1/p) > 0$. Then, given $0 \leq \delta < n(p-1)$ we have*

$$\left(\int_{\mathbb{R}^n} |I_\alpha(f)(x)|^q \mathcal{M}(g)(x)^{\Theta(\alpha)q/n} |x|^{q\delta/p} \, dx \right)^{\frac{1}{q}} \leq C \|g\|_{L^1(\mathbb{R}^n)}^{\Theta(\alpha)/n} \left(\int_{\mathbb{R}^n} |f(x)|^p |x|^\delta \, dx \right)^{\frac{1}{p}},$$

for every $g \in L^1(\mathbb{R}^n)$ and measurable f, and $C > 0$ depends only on n, α, p, q, and, δ.

Proof. Set $v(x) := |x|^\delta$ (which belongs to $RH_\infty(\mathbb{R}^n)$ due to Remark 2.4) so that the condition $\delta < n(p-1)$ makes $|x|^{-\delta p'/p} \in A_\infty(\mathbb{R}^n)$. □

6. Applications to weighted Hardy inequalities

In [2], L. Caffarelli, R. Kohn, and L. Nirenberg proved, among other results, that for $n > 2$, $-\infty < a < \frac{n-2}{2}$, $a \leq b \leq a+1$, and $q = \frac{2n}{n-2+2(b-a)}$, the weighted inequality

$$\left(\int_{\mathbb{R}^n} |f(x)|^q |x|^{-bq} \, dx \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |\nabla f(x)|^2 |x|^{-2a} \, dx \right)^{1/2}, \quad \forall f \in C_c^1(\mathbb{R}^n), \tag{6.1}$$

for some $C = C(a, b, q, n) > 0$. As an illustration of the applications of Theorems 1.1 and 1.2, in this section we will establish new weighted Hardy inequalities, some of which will extend inequalities such as (6.1), by utilizing the pairs (u, v) developed in Section 5. Our starting point will be the pointwise estimate

$$|f(x)| \leq C_n I_1(|\nabla f|)(x), \quad \forall x \in \mathbb{R}^n, \tag{6.2}$$

for every $f \in C_0^1(\mathbb{R}^n)$ (i.e., f continuously differentiable, vanishing at ∞), which is quickly proved, for given $f \in C_0^1(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, and $\omega \in S^{n-1}$ the fundamental theorem of calculus gives

$$f(x) = -\int_0^\infty \frac{d}{dr} f(x + r\omega) \, dr$$

and then, after integration on both sides with respect to $\omega \in S^{n-1}$,

$$f(x) = -\frac{1}{n\omega_n} \int_0^\infty \int_{S^{n-1}} \frac{d}{dr} f(x + r\omega) \, dr d\omega = \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{(x-y) \cdot \nabla f(y)}{|x-y|^n} \, dy$$

and (6.2) follows. By combining (6.2) with $\|I_1(h)\|_{L^q(u)} \leq C[(u,v)]_{A^1_{p,q}} \|h\|_{L^p(v)}$ from (1.3), we obtain the Hardy-Sobolev inequality

$$\|f\|_{L^q(u)} \leq C[(u,v)]_{A^\alpha_{p,q}} \|\nabla f\|_{L^p(v)}, \quad \forall f \in C_0^1(\mathbb{R}^n) \tag{6.3}$$

whenever the pair (u, v) satisfies the Sawyer-Wheeden condition (1.4).

Remark 6.1. Poincaré-type inequalities can be obtained as well by using that given a cube $Q \subset \mathbb{R}^n$ the weighted inequality

$$\left(\int_Q |f(x) - f_Q|^q u(x) \, dx \right)^{1/q} \leq C_S \left(\int_Q |\nabla f(x)|^p v(x) \, dx \right)^{1/p} \tag{6.4}$$

holds true for every $f \in C^1(\mathbb{R}^n)$ whenever the pair (u, v) satisfies (1.4) with $u, v^{-p'/p} \in A_\infty(\mathbb{R}^n)$. Moreover, in this case the constant $C_S > 0$ from (6.4) can be expressed as $C_S = C_{p,q,n}[(u,v)]_{A^\alpha_{p,q}}$, see for instance [13, Theorem 5].

Remark 6.2. Although the applications below will focus on I_1, the method to obtain Hardy-type inequalities can be implemented for a fractional exponent $0 < \alpha < n$ by means of fractional derivatives $(-\Delta)^{\alpha/2}$ by replacing (6.2) with the identity

$$f = I_\alpha((-\Delta)^{\alpha/2}(f))$$

for every f in the Schwartz class (see for instance [14, p.117]) which along with (1.3) yields

$$\left(\int_{\mathbb{R}^n} |f(x)|^q u(x)\,dx\right)^{1/q} \leq C[(u,v)]_{A^\alpha_{p,q}} \left(\int_{\mathbb{R}^n} |(-\Delta)^{\alpha/2} f(x)|^p v(x)\,dx\right)^{1/p}.$$

6.1. **Hardy-type inequalities after Theorems 1.1 and 1.2.** Before passing to specific choices of pairs (u, v), let us state here the general forms of the two-weight Hardy-type inequalities that follow from Theorems 1.1 and 1.2. Recall that, for $1 < p \leq q < \infty$, we have defined

$$\Theta(1) := 1 + n\left(\frac{1}{q} - \frac{1}{p}\right).$$

For the cases $\Theta(1) = 0$ and $\Theta(1) > 0$ (corresponding to the cases $1/q = 1/p - 1/n$ and $1/q > 1/p - 1/n$, respectively) we have the following Theorems 6.3 and 6.4.

Theorem 6.3. *Fix $n > 1$ and $1 < p \leq q < \infty$ such that $\Theta(1) = 0$. Given weights $u, v^{-p'/p} \in A_\infty(\mathbb{R}^n)$ such that $u(x)^{1/q} \leq C_1 v(x)^{1/p}$ for some $C_1 > 0$ and a.e. $x \in \mathbb{R}^n$, we have*

$$\left(\int_{\mathbb{R}^n} |f(x)|^q u(x)\,dx\right)^{1/q} \leq CC_1 \left(\int_{\mathbb{R}^n} |\nabla f(x)|^p v(x)\,dx\right)^{1/p}, \quad \forall f \in C^1_0(\mathbb{R}^n),$$

where $C > 0$ depends only on n, p, q, $[u]_{A_\infty(\mathbb{R}^n)}$, and $[v^{-p'/p}]_{A_\infty(\mathbb{R}^n)}$.

Proof. It follows from Theorem 1.1 with $\alpha = 1$. ☐

Theorem 6.4. *Fix $n > 1$ and $1 < p \leq q < \infty$ such that $\Theta(1) > 0$. Given weights $u, v^{-p'/p} \in A_\infty(\mathbb{R}^n)$ such that $u^{1/q}/v^{1/p} \in L^{n/\Theta(1),\infty}(\mathbb{R}^n)$, we have, for every $f \in C^1_0(\mathbb{R}^n)$,*

$$\left(\int_{\mathbb{R}^n} |f(x)|^q u(x)\,dx\right)^{1/q} \leq C_2 \|u^{1/q}/v^{1/p}\|_{L^{n/\Theta(\alpha),\infty}(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |f(x)|^p v(x)\,dx\right)^{1/p},$$

where $C_2 > 0$ depends only on n, α, p, q, $[u]_{A_\infty(\mathbb{R}^n)}$, and $[v^{-p'/p}]_{A_\infty(\mathbb{R}^n)}$.

Proof. It follows from Theorem 1.2 with $\alpha = 1$. ☐

6.2. **Hardy-type inequalities with $u = \mathcal{M}(g)^{\Theta(\alpha)q/n} v^{q/p}$ and $g \in L^1(\mathbb{R}^n)$.** In this section we express the inequality (6.3) in terms of the weights developed in Section 5 and $\alpha = 1$.

Theorem 6.5. *Fix $n > 1$, $1 < p \leq q < \infty$ with $\Theta(1) := 1 + n(1/q - 1/p) > 0$ and $\Theta(1)p' < n$. Then, given $u \in A_1(\mathbb{R}^n)$ there exists $C > 0$ depending only on n, p, q, and $[u]_{A_1(\mathbb{R}^n)}$ such that*

$$\left(\int_{\mathbb{R}^n} |f(x)|^q u(x)\,dx\right)^{1/q} \leq C\|g\|^{\Theta(1)/n}_{L^1(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |\nabla f(x)|^p \mathcal{M}(g)(x)^{-\Theta(1)p/n} u(x)^{p/q}\,dx\right)^{1/p}$$

for every $f \in C^1_0(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$.

Proof. It follows from Theorem 5.1 with $\alpha = 1$. $\square$

Corollary 6.6. *Fix* $n > 1$, $1 < p \le q < \infty$ *with* $\Theta(1) := 1 + n(1/q - 1/p) > 0$ *and* $p'\Theta(1) < n$. *Then, given* $u \in A_1(\mathbb{R}^n)$ *there exists* $C > 0$ *depending only on* n, p, q, *and* $[u]_{A_1(\mathbb{R}^n)}$ *such that*

$$\left(\int_{\mathbb{R}^n} |f(x)|^q u(x)\, dx\right)^{1/q} \le C \left(\int_{\mathbb{R}^n} |\nabla f(x)|^p |x|^{\Theta(1)p} u(x)^{p/q}\, dx\right)^{1/p}, \quad \forall f \in C_0^1(\mathbb{R}^n).$$

Proof. For $k \in \mathbb{N}$ apply Theorem 6.5 to $g_k := k^n \chi_{B(0,1/k)}$, which then gives $\|g_k\|_{L^1(\mathbb{R}^n)} = C_n$ as well as $\mathcal{M}(g_k)(x) \ge |x|^{-n}$ (and hence $\mathcal{M}(g_k)(x)^{-\Theta(1)p/n} \le |x|^{\Theta(1)p}$) for every $x \in \mathbb{R}^n$ with $|x| > 1/k$. The corollary then follows by letting $k \to \infty$. $\square$

Remark 6.7. When $n > 2$ one recovers the Caffarelli-Kohn-Nirenberg inequality (6.1) from Corollary 6.6 by choosing $p = 2$ and, given q and a, b, with $a < b$, as in (6.1), by setting $u(x) := |x|^{-bq}$. Then, $u \in A_1(\mathbb{R}^n)$ if and only if $bq < n$, which means

$$bq = \frac{2nb}{n - 2 + 2(b-a)} < n$$

and it amounts to $a < (n-2)/2$. Also, from the definition of $q = \frac{2n}{n-2+2(b-a)}$ we get $\Theta(1) = b - a$ and then $|x|^{p\Theta(1)} u(x)^{p/q} = |x|^{2(\Theta(1)-b)} = |x|^{-2a}$. In particular, $(b-a) = \Theta(1) > 0$ and $\Theta(1) \le 1$ (because $q \ge p = 2$). Finally, $\Theta(1)p' < n$ means $2\Theta(1) < n$ which holds true because $0 < \Theta(1) \le 1$ and $n > 2$.

Theorem 6.8. *Fix* $n > 1$ *and* $1 < p \le q < n/\Theta(1)$ *with* $\Theta(1) := 1 + n(1/q - 1/p) > 0$. *Given* $v \in RH_\infty(\mathbb{R}^n)$ *such that* $v^{-p'/p} \in A_\infty(\mathbb{R}^n)$ *there exists* $C > 0$ *depending only on* n, p, q, $[v^{-p/p}]_{A_\infty(\mathbb{R}^n)}$, *and* $[v]_{RH_\infty(\mathbb{R}^n)}$ *such that*

$$\left(\int_{\mathbb{R}^n} |f(x)|^q \mathcal{M}(g)(x)^{\Theta(1)q/n} v(x)^{q/p}\, dx\right)^{1/q} \le C \|g\|_{L^1(\mathbb{R}^n)}^{\Theta(1)/n} \left(\int_{\mathbb{R}^n} |\nabla f(x)|^p v(x)\, dx\right)^{1/p}$$

for every $f \in C_0^1(\mathbb{R}^n)$ *and* $g \in L^1(\mathbb{R}^n)$.

Proof. It follows from Theorem 5.2 with $\alpha = 1$. $\square$

Corollary 6.9. *Fix* $n > 1$ *and* $1 < p \le q < n/\Theta(1)$ *with* $\Theta(1) := 1 + n(1/q - 1/p) > 0$. *Given* $0 \le \delta < n(p-1)$ *there exists* $C > 0$, *depending only on* δ, p, q, *and* n, *such that*

$$\left(\int_{\mathbb{R}^n} |f(x)|^q \mathcal{M}(g)(x)^{\Theta(1)q/n} |x|^{q\delta/p}\, dx\right)^{1/q} \le C \|g\|_{L^1(\mathbb{R}^n)}^{\Theta(1)/n} \left(\int_{\mathbb{R}^n} |\nabla f(x)|^p |x|^\delta\, dx\right)^{1/p} \tag{6.5}$$

for every $f \in C_0^1(\mathbb{R}^n)$ *and* $g \in L^1(\mathbb{R}^n)$.

Proof. The inequality (6.5) follows from Theorem 6.8 by taking $v(x) = |x|^\delta$. $\square$

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