Analytic families of operators in extrapolation theory with application to average operators

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Abstract Our goal is to prove weak type $(1,1)$ boundedness for an operator T_A which is given as an average of a family of operators $\{T_j\}_j$ satisfying certain estimates in the context of weighted Lebesgue spaces. In particular, we shall prove that, if there exists $\alpha > 0$, $s > 0$ and $C > 0$ such that, for every every weight v in the Muckenhoupt class A_p and every measurable set E,

$$\sup_j \|T_j \chi_E\|_{L^{p,\infty}(v)} \leq \frac{C}{(p-1)^s} \|v\|_{A_p}^{\alpha} v(E).$$

and, for some $u_0 \in A_1$ fixed,

$$\sup_j \|T_j \chi_E\|_{L^{1,\infty}(u_0)} \leq C_{u_0} u_0(E),$$

then, for every $\beta > 0$, there exists a constant $C_\beta > 0$, so that

$$\sup_{y>0} \frac{y}{\left(1 + \log^+(1 + \log^+ \frac{1}{y})\right)^\beta} \int_{\{|T_A \chi_E| > y\}} u_0(x)dx \leq C_\beta\, u_0(E).$$

Our main technique is inspired on the theory of analytic families of operators and it is closely related to the Rubio de Francia's extrapolation theory.

With love to Guido

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1 Introduction

Let $\{T_j\}_{j\in\mathbb{N}}$ be a collection of operators so that

$$T_j : L^1 \longrightarrow L^{1,\infty}, \qquad \sup_j ||T_j||_{L^1\longrightarrow L^{1,\infty}} < \infty.$$

Let $(c_j)_j \in \ell^1$ and set

$$S = \sum_j c_j T_j.$$

Since $L^{1,\infty}$ is not a Banach space, the weak type $(1,1)$ boundedness of S may fail. However, (we refer to Section 2 for the definition of the Muckenhoupt class of weights A_p, $p \geq 1$) it has been very recently proved ([1]) that if, for every $u \in A_1$,

$$T_j : L^1(u) \longrightarrow L^{1,\infty}(u), \qquad \sup_j ||T_j||_{L^1(u)\longrightarrow L^{1,\infty}(u)} < \infty, \tag{1}$$

then the restricted weak type $(1,1)$ boundedness of S holds, for every $u \in A_1$; that is, for every measurable set and every $u \in A_1$,

$$||S\chi_E||_{L^{1,\infty}(u)} \leq C_u ||c||_{\ell^1} u(E).$$

We observe that, by Rubio de Francia extrapolation theory ([18]), condition (1) implies that, for every $p > 1$ and every $v \in A_p$,

$$T_j : L^p(v) \longrightarrow L^{p,\infty}(v), \qquad \sup_j ||T_j||_{L^p(v)\longrightarrow L^{p,\infty}(v)} < \infty;$$

that is, condition (1) hides a weighted estimate at level $p > 1$. On the other hand, and this is the main motivation of this paper, there are examples of operators for which the hypothesis (1) is only known for the case $u = 1$ and not for every $u \in A_1$, but some weighted estimate at the level $p > 1$ is also true. The main goal of this paper concerns with this situation.

Now, the Rubio de Francia extrapolation theorem (see [18, 11, 12, 13, 7]), can be formulated, nowadays, as follows (see [9, 10]): Let T be an operator so that, for some $1 \leq p_0 < \infty$, and for every $w \in A_{p_0}$, we have

$$T : L^{p_0}(w) \longrightarrow L^{p_0}(w), \qquad ||T|| \leq N(\|w\|_{A_{p_0}}), \tag{2}$$

where $N(t)$, $t > 0$, is an increasing function. Then, for every $1 < p < \infty$ and every $w \in A_p$,

$$T : L^p(w) \longrightarrow L^p(w), \qquad ||T|| \leq K(w) \tag{3}$$

where

$$K(w) \leq C_1 N\Big(\frac{1}{p-1}\|w\|_{A_p}^{\max\left(1,\frac{p_0-1}{p-1}\right)}\Big).$$

In particular, the above estimate (3) shows that if, T satisfies (2) with $N(t) = t^{\alpha}$, then

$$T : L^p \longrightarrow L^p, \qquad \frac{1}{(p-1)^{\alpha}}, \quad 1 < p \leq p_0,$$

and using Yano's extrapolation theorem [19, 15], we get that

$$T : L(\log L)^{\alpha} \longrightarrow L^1 + L^{\infty}.$$

Hence, if (2) holds for our family of operators $\{T_j\}_j$ uniformly in j, same estimate can be deduced for the sum operator S defined before, and consequently S would satisfy the above boundedness on the space $L(\log L)^{\alpha}$. Our goal is to see, that under some extra condition on the operators $\{T_j\}_j$, this estimate can be improved. We should also mention here paper [16] where it is shown that for a given operator the optimality of the weighted L^p bounds in term of the A_p constant of the weight is related to the unweighted behavior of the operator when $p \to 1$. Hence, the relation between Rubio de Francia's extrapolation and Yano's extrapolation was already in the literature (see also [5]).

Our general context will be the following: let $\{T_\theta\}_\theta$ be a family of operators indexed in a probability measure space $(\mathcal{M}, P)$ such that

$$T_\theta : L^1 \longrightarrow L^{1,\infty}, \qquad \sup_\theta ||T_\theta||_{L^1 \longrightarrow L^{1,\infty}} < \infty.$$

What can we say about the boundedness, near L^1, of the average operator

$$T_A f(x) = \int_{\mathcal{M}} T_\theta f(x) dP(\theta),$$

whenever it is well defined? Our main result is the following:

Theorem 1 *Let $\{T_\theta\}_\theta$ be a family of operators indexed in a probability measure space satisfying the following conditions:*

- *There exist $\alpha > 0$ and $s > 0$ so that, for every $1 < p < 2$ with $\alpha p(p-1) < 1$ and every $v \in A_p$,*

$$T_\theta : L^p(v) \longrightarrow L^{p,\infty}(v), \qquad \sup_\theta ||T_\theta|| \leq \frac{C}{(p-1)^s} \|v\|_{A_p}^{\alpha},$$

with C a universal constant.

- *For some $u_0 \in A_1$ and every measurable set E,*

$$\sup_\theta \|T_\theta \chi_E\|_{L^{1,\infty}(u_0)} \leq C_{u_0} u_0(E).$$

Then, for every $\beta > 0$, the averaging operator T_A satisfies the following estimate

$$\sup_{y>0} \frac{y}{\left(1+\log^+(1+\log^+\frac{1}{y})\right)^\beta} \int_{\{|T_A\chi_E|>y\}} u_0(x)dx \leq C_{u_0,\beta}\, u_0(E).$$

In particular:

Corollary 1 *If T_j satisfies Condition-A uniformly in j and*

$$\sup_j \|T_j\chi_E\|_{L^{1,\infty}} \lesssim |E|,$$

then, for every $\beta > 0$ and every $(c_j)_j \in \ell^1$, the operator $S = \sum_j c_j T_j$, satisfies that

$$\sup_{y>0} \frac{y}{(1+\log^+(1+\log^+\frac{1}{y}))^\beta} |\{x : |S\chi_E(x)| > y\}| \lesssim C_\beta ||c||_{\ell^1} |E|.$$

Remark 1 We have not succeeded in proving the restricted weak type $(1,1)$ of T_A for the weight u_0 neither to find a counterexample and hence this remains as an open question.

As usual, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C, independent of all relevant parameters, such that $A \leq CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$. Moreover, even though C may depend on $p > 1$, we shall only be concerned about the dependence on p if this blows up when $p \to 1$.

2 Preliminary results and some technical lemmas

For our purposes, let us recall that the Lorentz spaces $L^{p,q}(u)$ is defined as the set of measurable functions such that

$$\|f\|_{L^{p,q}(u)} = \left(\frac{q}{p}\int_0^\infty f_u^*(t)^q t^{\frac{q}{p}-1}\, dt\right)^{1/q} = \left(q\int_0^\infty y^{q-1}\lambda_f^u(y)^{q/p} dy\right)^{1/q} < \infty,$$

and $L^{p,\infty}(u)$ is defined by the condition

$$\|f\|_{L^{p,\infty}}(u) = \sup_{t>0} t^{1/p} f_u^*(t) = \sup_{y>0} y\lambda_f^u(y)^{1/p} < \infty,$$

where f_u^* is the decreasing rearrangement of f, with respect to the weight u, defined by

$$f_u^*(t) = \inf\left\{y > 0 : \lambda_f^u(y) \leq t\right\},$$

and $\lambda_f^u(y) = u\big(\{x : |f(x)| > y\}\big)$ is the distribution function of f with respect to u. We shall also use the standard notation $u(E) = \int_E u(x)\, dx$ and, if $u = 1$, we shall write $\lambda_f(y)$, f^* and $|E|$ (see [2]). With the above definition, it holds that, for every $1 < p$ and $1 \le q \le \infty$,

$$\left|\int f(x)g(x)u(x)dx\right| \le \left(\frac{p}{q}\right)^{1/q}\left(\frac{p'}{q'}\right)^{1/q'} ||f||_{L^{p,q}(u)}||g||_{L^{p',q'}(u)},$$

and, for every q, $||\chi_E||_{L^{p,q}(u)} = u(E)^{1/p}$.

Let us also recall some well known facts of the class A_p. Let M be the Hardy-Littlewood maximal operator, defined by

$$Mf(x) = \sup_{x\in Q}\frac{1}{|Q|}\int_Q |f(y)|\,dy,$$

where Q denotes a cube in $\mathbb{R}^n$ and let v be a positive locally integrable function w (called weight) such that

$$\|v\|_{A_r} = \sup_Q\left(\frac{1}{|Q|}\int_Q v(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q v^{-1/(r-1)}(x)\,dx\right)^{r-1} < \infty,$$

with $r > 1$. This class of weights is known as the Muckenhoupt class A_r $(r > 1)$. If $r = 1$, we say that $u \in A_1$, if $Mu(x) \le Cu(x)$, at almost every point $x \in \mathbb{R}^n$ and $\|w\|_{A_1}$ will be the least constant C satisfying such inequality.

In [17, 3], it was proved that, if $p > 1$, then

$$M : L^p(v) \longrightarrow L^p(v)$$

is bounded if, and only if, $v \in A_p$ and also, for every $1 \le p < \infty$, M is of weak-type (p, p) if and only if $v \in A_p$ and, in this case,

$$\|M\|_{L^p(v)\to L^{p,\infty}(v)} \le C\|v\|_{A_p}^{\frac{1}{p}}.$$

These classes of weights satisfy the following properties:

1. $u \in A_1$ if and only if there exists $h \in L^1_{loc}(\mathbb{R}^n)$ and k such that $k, k^{-1} \in L^\infty$ satisfying that, for some $0 < \delta < 1$,

$$u(x) = k(x)(Mh)^\delta.$$

2. Factorization: $v \in A_p$ if and only if there exist $u_0, u_1 \in A_1$ such that

$$v = u_0u_1^{1-p}, \qquad \|u_0\|_{A_1} \le \|v\|_{A_p}, \|u_1\|_{A_1} \le \|v\|_{A_p}^{\frac{1}{p-1}}.$$

   Moreover, if $v = u_0u_1^{1-p}$, then $\|v\|_{A_p} \lesssim \|u_0\|_{A_1}\|u_1\|_{A_1}^{p-1}$.

3. For every $g \in L^1_{loc}$ with $Mg > 0$, every $u \in A_1$, and $0 < \delta < 1$, it holds that $v(x) = u(x)(Mg)^{\delta(1-p)} \in A_p$ with

$$\|v\|_{A_p} \lesssim \frac{\|u\|_{A_1}}{(1-\delta)^{(p-1)}}. \tag{4}$$

4. If $u \in A_1$ and $0 < \theta < 1$, then $(Mf)^{1-\theta}u^\theta \in A_1$ with

$$\left\|(Mf)^{1-\theta}u^\theta\right\|_{A_1} \lesssim \frac{\|u\|_{A_1}}{\theta}.$$

The following lemma in complex variable will be fundamental in our theory.

Lemma 1 *Let $S = \{0 < Re\, z < 1\}$ and set $H(S)$ the space of analytic functions in S and continuous in $\bar{S}$. If $F \in H(S)$ and $\sup_{t\in\mathbb{R}} |F(j+it)| \leq M_j$, with $j = 0, 1$, then for every $0 < \theta < 1$,*

$$|F^{N)}(\theta)| \lesssim \left(\frac{1}{\theta(1-\theta)}\right)^N M_0^{1-\theta} M_1^\theta \left(1 + \left|\log \frac{M_1}{M_0}\right|\right)^N.$$

Using the dual operator of M as in [8], it was also proved in [6] the following result.

Lemma 2 *[6] For every $1 \leq p < \infty$ and every $u \in A_1$, there exists $C_{p,u}$ such that*

$$\left\|\frac{M(fu(Mg)^{1-p})}{u(Mg)^{1-p}}\right\|_{L^{p',\infty}(u(Mg)^{1-p})} \leq C_{p,u}\left\|f\right\|_{L^{p',1}(u(Mg)^{1-p})}, \tag{5}$$

where if $p = 1$ $L^{p',\infty}(u(Mg)^{1-p}) = L^{p',1}(u(Mg)^{1-p}) = L^\infty$

The constant $C_{p,u}$ was not explicitly computed in [6], but it can be proved that

$$C_{p,u} \lesssim \|u\|_{A_1}^2.$$

3 Main results

In this paper, we shall start with an operator T satisfying the following condition:

Condition-A: There exists $\alpha > 0$ and $s > 0$, so that, for every $1 < p < 2$ with $\alpha p(p-1) < 1$ and every $v \in A_p$,

$$\|T\chi_E\|_{L^{p,\infty}(v)} \lesssim \frac{\|v\|_{A_p}^\alpha}{(p-1)^s} v(E)^{1/p}.$$

This is the case of important operators in Harmonic Analysis such as the Hardy-Littlewood maximal operator or the maximal Calderón-Zygmund operators ([14]), among many others.

Proposition 1 *Let T be an operator satisfying Condition-A. Then, for every $\alpha p(p-1) < \beta \leq 1$, every measurable set E and every $u \in A_1$,*

$$\sup_{y>0} y^p \int_{\{|T\chi_E(x)|>y\}} u(x) \frac{(M\chi_E(x))^{(1-p)}}{\left(\log \frac{e}{M\chi_E(x)}\right)^{\beta}} dx \lesssim C(p,\beta,\alpha)\|u\|_{A_1}^{\alpha p} u(E), \tag{6}$$

with

$$C(p,\beta,\alpha) = \frac{(p-1)^{\beta-s}}{(\beta - \alpha p(p-1))}.$$

Proof Set $C_p = (p-1)^{-s}$. Let $0 < \mu < 1$, and let $K \subset \{|T\chi_E(x)| > y\}$ be a compact set. Set the analytic function on the unit strip $S = \{0 < Re\, z < 1\}$,

$$F(z) = \int_K u(x) \left(\frac{M\chi_E(x)}{e}\right)^{\left(\frac{1-\mu}{2}z+\mu\right)(1-p)} dx.$$

Then, by (4), we have that, for $j = 0, 1$,

$$|F(j+it)| \lesssim \frac{C_p \|u\|_{A_1}^{\alpha p} u(E)}{(1-\mu)^{\alpha p(p-1)} y^p}$$

uniform in $t \in \mathbb{R}$. Thus,

$$|F(1/2)| = \int_K u(x) \left(\frac{M\chi_E(x)}{e}\right)^{\frac{1+3\mu}{4}(1-p)} dx \lesssim \frac{C_p \|u\|_{A_1}^{\alpha p} u(E)}{(1-\mu)^{\alpha p(p-1)} y^p}, \tag{7}$$

and, by Lemma 1,

$$\begin{aligned}
&(1-\mu)(p-1) \int_K u(x)(M\chi_E(x))^{\frac{1+3\mu}{4}(1-p)} \log \frac{e}{M\chi_E(x)} dx \lesssim |F'(1/2)| \\
&\lesssim \frac{C_p \|u\|_{A_1}^{\alpha p} u(E)}{y^p (1-\mu)^{\alpha p(p-1)}},
\end{aligned}$$

and therefore, letting K tend to $\{|T\chi_E(x)| > y\}$, we obtain that

$$\begin{aligned}
y^p \int_{\{|T\chi_E(x)|>y\}} & u(x)(M\chi_E(x))^{\frac{1+3\mu}{4}(1-p)} \log \frac{e}{M\chi_E(x)} dx \\
&\lesssim \frac{C_p \|u\|_{A_1}^{\alpha p} u(E)}{(1-\mu)^{\alpha p(p-1)+1}(p-1)}.
\end{aligned} \tag{8}$$

By (7) and (8), we get that, for every $t > 0$,

$$
\begin{aligned}
&y^p \int_{\{|T\chi_E(x)|>y\}} u(x)(M\chi_E(x))^{\frac{1+3\mu}{4}(1-p)} \min\Big(1, t\log\frac{e}{M\chi_E(x)}\Big)dx \\
&\lesssim \frac{C_p\|u\|_{A_1}^{\alpha p}}{(1-\mu)^{\alpha p(p-1)}} \min\Big(1, \frac{t}{(1-\mu)(p-1)}\Big)u(E),
\end{aligned}
$$

and integrating in $t \in (0, \infty)$ against $\frac{1}{t^{\gamma+1}}$, we obtain that, for every $0 \leq \gamma < 1$,

$$
\begin{aligned}
&\int_{\{|T\chi_E(x)|>y\}} u(x) \left(\frac{e}{M\chi_E(x)}\right)^{\frac{1+3\mu}{4}(p-1)} \Big(\log\frac{e}{M\chi_E(x)}\Big)^{\gamma} dx \\
&\lesssim \frac{C_p\|u\|_{A_1}^{\alpha p} u(E)}{(1-\mu)^{\alpha p(p-1)+\gamma}(p-1)^{\gamma}}.
\end{aligned}
$$

Hence, if $\alpha p(p-1) + \gamma < 1$, we can integrate in the variable $\mu \in (0, 1)$ and we obtain that if $\beta = 1 - \gamma$,

$$
\begin{aligned}
&y^p \int_{\{|T\chi_E(x)|>y\}} u(x)(M\chi_E(x))^{(1-p)} \frac{1}{\Big(\log\frac{e}{M\chi_E(x)}\Big)^{\beta}} dx \\
&\lesssim y^p \int_{\{|T\chi_E(x)|>y\}} u(x)(M\chi_E(x))^{\frac{1-p}{4}} dx + C_p \frac{(p-1)^{\beta}\|u\|_{A_1}^{\alpha p}}{(\beta - \alpha p(p-1))} u(E) \\
&\lesssim C_p\Big(\frac{(p-1)^{\beta}}{(\beta-\alpha p(p-1))} + 1\Big)\|u\|_{A_1}^{\alpha p} u(E),
\end{aligned}
$$

as we wanted to see.

Our next step is to prove that an estimate of the form (6) is also true for other values of p. To this end, the following lemma is needed.

Lemma 3 *Let* $\alpha > 0$ *and* $0 < \varepsilon \leq 1$. *Let* $q > 1$ *and* $1 < p \leq \min(2, q)$ *such that*

$$
(\alpha q + 1)(p - 1) = \frac{\varepsilon}{2}.
$$

Let us define $\gamma > 0$ *and* $\nu > 0$ *such that*

$$
\frac{\varepsilon\gamma}{q-1} = \alpha p + \frac{1}{q}, \qquad \gamma\frac{p-1}{q-1} + \nu\frac{q-p}{q-1} = 1 \tag{9}
$$

Then, it holds that:

$$
\frac{1}{q'} < \nu, \qquad \frac{1}{1-\frac{1}{\nu q'}} \leq 2\Big(1 + \frac{1}{\alpha p}\Big)q.
$$

Proof We observe that,

$$\nu = \frac{q-1}{q-p}\left(1-\gamma\frac{p-1}{q-1}\right) = \frac{q-1-\gamma(p-1)}{q-p} = \frac{q-1-\frac{(q-1)(\alpha p+\frac{1}{q})}{\varepsilon}(p-1)}{q-p}$$
$$= (q-1)\frac{\varepsilon-(\alpha p+\frac{1}{q})(p-1)}{\varepsilon(q-p)} > \frac{q-1}{q}$$

if and only if

$$\frac{\varepsilon-(\alpha p+\frac{1}{q})(p-1)}{\varepsilon(q-p)} > \frac{1}{q}.$$

That is,

$$\varepsilon-\left(\alpha p+\frac{1}{q}\right)(p-1) > \varepsilon\left(1-\frac{p}{q}\right),$$

or equivalently

$$\varepsilon\frac{p}{q} > \left(\alpha p+\frac{1}{q}\right)(p-1),$$
$$\varepsilon > (\alpha pq+1)\frac{p-1}{p} = \left(\alpha q+\frac{1}{p}\right)(p-1).$$

On the other hand,

$$\frac{\nu q'}{\nu q'-1} = \frac{q\frac{\varepsilon-(\alpha p+\frac{1}{q})(p-1)}{\varepsilon(q-p)}}{q\frac{\varepsilon-(\alpha p+\frac{1}{q})(p-1)}{\varepsilon(q-p)}-1} = \frac{q(\varepsilon-(\alpha p+\frac{1}{q})(p-1))}{q(\varepsilon-(\alpha p+\frac{1}{q})(p-1))-\varepsilon(q-p)}$$
$$= \frac{q(2(\alpha q+1)(p-1)-(\alpha p+\frac{1}{q})(p-1))}{2p(\alpha q+1)(p-1)-q(\alpha p+\frac{1}{q})(p-1))} = \frac{q(2(\alpha q+1)-(\alpha p+\frac{1}{q}))}{2p(\alpha q+1)-q(\alpha p+\frac{1}{q})}$$
$$= \frac{2\alpha q^2+2q-\alpha pq-1}{\alpha pq+2p-1} \leq \frac{2\alpha q^2+2q}{\alpha pq} \leq 2\left(1+\frac{1}{\alpha p}\right)q,$$

and the result is proved. □

Proposition 2 *Let T be an operator satisfying Condition-A. Then, for every $0<\varepsilon\leq 1$, every $q>1$, every measurable set E and every $u\in A_1$,*

$$\sup_{y>0} y^q \int_{\{|T\chi_E(x)|>y\}} u(x)\frac{(M\chi_E(x))^{(1-q)}}{\left(\log\frac{e}{M\chi_E(x)}\right)^{\varepsilon}}dx \lesssim \frac{1}{\varepsilon^{q(s+1+\alpha)}}\|u\|_{A_1}^{(\alpha+2)q}u(E). \quad (10)$$

Proof Let $0\leq\varepsilon<1$ be fixed. Let K be a compact set such that $K\subset\{|T\chi_E(x)|>y\}$ and let us define

$$A_q := \int_K u(x)\frac{(M\chi_E(x))^{(1-q)}}{\left(\log\frac{e}{M\chi_E(x)}\right)^{\varepsilon}}dx.$$

Set p, γ and ν as in Lemma 3 and set $C_p=(p-1)^{-s}$. Then, we have that

$$A_q \leq \int_K (M\chi_E)^{(1-p)} \frac{u^{\frac{p-1}{q-1}} \left[M\Big(u\chi_{\chi_K} \frac{(M\chi_E)^{1-q}}{\left(\log \frac{e}{M\chi_E}\right)^{\varepsilon\gamma}} \Big) \right]^{\frac{q-p}{q-1}}}{(\log \frac{e}{M\chi_E})^{\varepsilon\gamma \frac{p-1}{q-1}}} dx,$$

and since,

$$v = u^{\frac{p-1}{q-1}} \left[M\Big(u\chi_K \frac{(M\chi_E)^{1-q}}{\left(\log \frac{e}{M\chi_E}\right)^{\varepsilon\gamma}} \Big) \right]^{\frac{q-p}{q-1}} \in A_1,$$

with $\|v\|_{A_1} \lesssim \frac{q-1}{p-1} \|u\|_{A_1}$ and, by (9), $\varepsilon\gamma \frac{p-1}{q-1} > \alpha p(p-1)$, we have by Proposition 1,

$$\begin{aligned}
A_q &\leq \frac{B}{y^p} \int_E u^{\frac{p-1}{q-1}}(x) \left[M\Big(u\chi_K \frac{(M\chi_E)^{1-q}}{(\log \frac{e}{M\chi_E(x)})^{\varepsilon\gamma}} \Big) \right]^{\frac{q-p}{q-1}} dx \\
&= \frac{B}{y^p} \int_E \left[\frac{M\Big(u\chi_K \frac{(M\chi_E)^{1-q}}{(\log \frac{e}{M\chi_E(x)})^{\varepsilon\gamma}} \Big)}{u(x)} \right]^{\frac{q-p}{q-1}} u(x) dx \\
&= \frac{B}{y^p} \int_E \left[\frac{M\Big(u\chi_K \frac{(M\chi_E)^{1-q}}{(\log \frac{e}{M\chi_E(x)})^{\varepsilon\gamma}} \Big)}{u(x)(M\chi_E(x))^{1-q}} \right]^{\frac{q-p}{q-1}} u(x)(M\chi_E(x))^{1-q} dx
\end{aligned}$$

with

$$\begin{aligned}
B &\lesssim C_p \frac{\|v\|_{A_1}^{\alpha p} (p-1)^{(\alpha p + \frac{1}{q})(p-1)}}{(\varepsilon\gamma \frac{p-1}{q-1} - \alpha p(p-1))} \lesssim C_p \frac{q\|v\|_{A_1}^{\alpha p}}{(p-1)} \\
&\lesssim C_p \frac{q^{1+\alpha p}}{\varepsilon^{1+\alpha p}} \|u\|_{A_1}^{\alpha p} \approx \frac{q^{2+\alpha p}}{\varepsilon^{s+1+\alpha p}} \|u\|_{A_1}^{\alpha p}.
\end{aligned}$$

Then, by duality,

$$A_q \lesssim \frac{Bu(E)^{p/q}}{y^p} \left\| \frac{M\Big(u\chi_K \frac{(M\chi_E)^{1-q}}{(\log \frac{e}{M\chi_E})^{\varepsilon\gamma}} \Big)}{u(M\chi_E)^{1-q}} \right\|_{L^{q',\infty}(u(M\chi_E)^{1-q})}^{\frac{q-p}{q-1}}.$$

Now, using (5) with the behavior of the constant $C_{p,u}$ we have that

$$
\begin{aligned}
&\left\| \frac{M\left(u\chi_K \frac{(M\chi_E)^{1-q}}{(\log \frac{e}{M\chi_E})^{\varepsilon\nu}}\right)}{u(M\chi_E)^{1-q}} \right\|_{L^{q',\infty}(u(M\chi_E)^{1-q})} \\
&\lesssim \|u\|_{A_1}^2 \left\| \frac{\chi_K}{(\log \frac{e}{M\chi_E})^{\varepsilon\nu}} \right\|_{L^{q',1}(u(M\chi_E)^{1-q})} \\
&= \|u\|_{A_1}^2 \int_0^\infty \left(\int_{\left\{x\in K: \frac{1}{(\log \frac{e}{M\chi_E})^{\varepsilon\nu}} > z\right\}} u(x)(M\chi_E(x))^{1-q} dx \right)^{1/q'} dz \\
&= t\|u\|_{A_1}^2 \int_0^1 \left(\int_{\left\{x\in K: \frac{1}{(\log \frac{e}{M\chi_E})^{\varepsilon\nu}} > z\right\}} u(x)(M\chi_E(x))^{1-q} dx \right)^{1/q'} dz \\
&\lesssim \|u\|_{A_1}^2 \left(\int_K u(x) \frac{(M\chi_E)^{1-q}(x)}{\left(\log \frac{e}{M\chi_E(x)}\right)^{\varepsilon}} dx \right)^{1/q'} \int_0^1 z^{-\frac{1}{\nu q'}} dz \\
&\lesssim \frac{\|u\|_{A_1}^2}{1-\frac{1}{\nu q'}} A_q^{1/q'} \lesssim \|u\|_{A_1}^2 A_q^{1/q'}
\end{aligned}
$$

Consequently, we have that

$$
A_q \lesssim \frac{\|u\|_{A_1}^{\alpha p}}{\varepsilon^{s+1+\alpha p} y^p} u(E)^{p/q} \left(\|u\|_{A_1}^2\right)^{\frac{q-p}{q-1}} A_q^{\frac{q-p}{q'(q-1)}} \lesssim \frac{\|u\|_{A_1}^{(\alpha+2)p}}{\varepsilon^{s+1+\alpha p} y^p} u(E)^{p/q} A_q^{\frac{q-p}{q}},
$$

and hence,

$$
A_q \lesssim \frac{\|u\|_{A_1}^{(\alpha+2)q}}{\varepsilon^{(s+1+\alpha p)\frac{q}{p}} y^q} u(E) \le \frac{\|u\|_{A_1}^{(\alpha+2)q}}{\varepsilon^{q(s+1+\alpha)} y^q} u(E),
$$

as we wanted to see. □

Remark 2 We observe that, in general, the constant has to blow up when $\varepsilon \to 0$, since on the contrary it can be proved that T will be of weak type $(1,1)$ for every weight $u \in A_1$, which is false in general.

We shall now fix a weight $u_0 \in A_1$ and let us consider the condition:

$$
||T\chi_E||_{L^{1,\infty}(u_0)} \lesssim u_0(E), \qquad \forall E. \tag{11}
$$

Corollary 2 *Let T be an operator satisfying Condition-A and* (11). *Then, for every $q > 1$, every measurable set E, and every $\beta > (q-1)(s+1+\alpha)$,*

$$
\sup_{y>0} \frac{y^q}{(1+\log(1+\log_+ \frac{1}{y}))^\beta} \int_{\{|T\chi_E(x)|>y\}} u_0(x)(M\chi_E(x))^{(1-q)} dx \lesssim C_{u_0,q} u_0(E).
$$

Proof First of all we observe that,

$$\frac{y^q}{(1+\log(1+\log_+\frac{1}{y}))^\beta}\int_{\{|T\chi_E(x)|>y, M\chi_E(x)>y\}} u_0(x)(M\chi_E(x))^{(1-q)}dx$$
$$\leq y\int_{\{|T\chi_E(x)|>y\, M\chi_E(x)>y\}} u_0(x)dx \leq C_{u_0}u(E)$$

Now, let $0<\theta<1$ and let $p=1+\frac{q-1}{\theta}$. Let us define, in the case $y<1$, the following analytic function

$$F(z)=\int_K u_0(x)\frac{(\frac{M\chi_E(x)}{e})^{z(1-p)}}{\left(\log\frac{e}{M\chi_E(x)}\right)^\varepsilon}dx,$$

where $K\subset\{|T\chi_E(x)|>y, M\chi_E(x)\leq y\}$ is an arbitrary compact set. Then, by (11),

$$|F(it)|\lesssim\frac{1}{y}u_0(E)$$

and, by (10),

$$|F(1+it)|\lesssim\frac{1}{\varepsilon^{p(s+1+\alpha)}y^p}u_0(E),$$

and thus, since $1-q=\theta(1-p)$, we have by Lemma 1, that

$$|F'(\theta)|\approx\left|(p-1)\int_K u_0(x)\log\left(\frac{e}{M\chi_E(x)}\right)^{1-\varepsilon}\left(\frac{M\chi_E(x)}{y}\right)^{1-q}dx\right|$$
$$\lesssim\frac{C_q}{\varepsilon^{(q-1+\theta)(s+1+\alpha)}y^q}\left(1+\log_+\frac{1}{y}\right)\left(1+\log_+\frac{1}{\varepsilon}\right),$$

from which it follows that, for every $\beta>(q-1)(s+1+\alpha)$, we can take θ so that

$$\frac{1}{\varepsilon^{(q-1+\theta)(s+1+\alpha)}}\left(1+\log_+\frac{1}{\varepsilon}\right)\leq\frac{1}{\varepsilon^\beta},$$

and hence

$$\left(\log\frac{e}{y}\right)^{1-\varepsilon}\left|\int_K u_0(x)\left(\frac{M\chi_E(x)}{y}\right)^{1-q}dx\right|\lesssim\frac{C_q}{\varepsilon^\beta y^q}\left(1+\log_+\frac{1}{y}\right).$$

Therefore,

$$\sup_\varepsilon\frac{\varepsilon^\beta}{\left(\log\frac{e}{y}\right)^\varepsilon}\left|\int_K u_0(x)\left(\frac{M\chi_E(x)}{y}\right)^{1-q}dx\right|\lesssim\frac{C_q}{y^q},$$

from which the result follows in the case $y<1$.

Now, when $y>1$, we consider the analytic function

$$F(z) = \int_K u_0(x) \frac{\left(\frac{M\chi_E(x)}{y}\right)^{z(1-p)}}{\left(\log \frac{ey}{M\chi_E(x)}\right)} dx.$$

Then, by (11) and (10),

$$|F(j+it)| \lesssim \frac{u_0(E),}{y}, \qquad j = 0, 1,$$

and thus,

$$|F'(1/2)| \approx \left|(q-1) \int_K u_0(x) \Big(\frac{M\chi_E(x)}{y}\Big)^{1-q} dx\right| \lesssim \frac{C_q}{y} u_0(E),$$

and the result follows, letting K tends to $\{|T\chi_E(x)| > y, M\chi_E(x) \le y\}$. □

4 Applications to average operators

The following result was proved in [4], but we include the proof for the sake of completeness.

Proposition 3 *If there exists* $C > 0$ *so that* $\sup_{y>0} W(y)\lambda_{T_\theta f}(y) \le C$*, then* $\sup_{y>0} \widetilde{W}(y)\lambda_{T_A f}(y) \le C$*, where*

$$\widetilde{W}(R) = \sup_{x \le R} \frac{R-x}{\int_x^\infty \frac{1}{W(u)}\, du} \cdot$$

Proof Let $\phi(t) = \int_0^t h(s)\, ds$, with h a positive and increasing function. Then ϕ is a convex function and, by Jensen's inequality,

$$\phi(|T_A f(x)|) \le \int_{\mathcal{M}} \phi(T_\theta f(x)|)\, dP(\theta) \cdot$$

Hence, for every $R > 0$, $\phi(R)\chi_{\{x; |T_A f(x)| > R\}}(x) \le \int_{\mathcal{M}} \phi(T_\theta f(x))\, dP(\theta)$ and integrating over $\mathbb{R}^n$, we obtain

$$\phi(R)\lambda_{T_A f}(R) \le \int_{\mathcal{M}} \int_{\mathbb{R}^n} \phi(T_\theta f(x))\, dx\, dP(\theta) \cdot$$

Now,

$$\int_{\mathbb{R}^n} \phi(T_\theta f(x))\, dx = \int_0^\infty \lambda_{T_\theta f}(y) d\phi(y) = \int_0^\infty \lambda_{T_\theta f}(y) h(y)\, dy$$

and hence, we deduce that

$$\phi(R)\lambda_{T_A f}(R) \le \int_{\mathcal{M}} \int_0^\infty \lambda_{T_\theta f}(y) h(y)\, dy\, dP(\theta) \le C \int_0^\infty \frac{h(y)}{W(y)}\, dy,$$

and therefore, if $h \uparrow$ indicates that h is an increasing function, we get that

$$\left(\sup_{h\uparrow} \frac{\int_0^R h}{\int_0^\infty \frac{h(y)}{W(y)}\, dy} \right) \lambda_{T_A f}(R) \le C.$$

The result now follows by computing the exact expression of the function between parenthesis by a simple change of variable and the well known fact that

$$\sup_{f\downarrow} \frac{\int_0^\infty f(t)v(t)dt}{\int_0^\infty f(t)u(t)dt} = \sup_{r>0} \frac{\int_0^r v(t)dt}{\int_0^r u(t)dt}.$$

where $f \downarrow$ indicates that f is a decreasing function. □

Theorem 2 *If T_θ satisfies Condition-A uniformly in θ and, for some u_0,*

$$\sup_\theta \|T_\theta \chi_E\|_{L^{1,\infty}(u_0)} \lesssim u_0(E),$$

then, for every $\beta > 0$,

$$\sup_{y>0} \frac{y}{(1+\log^+(1+\log^+\frac{1}{y}))^\beta} \int_{\{|T_A\chi_E|>y\}} u_0(x)dx \lesssim C_\beta\, u_0(E).$$

Proof Let us fix $q > 1$ so that $\beta > (q-1)(s+1+\alpha)$, and let us choose $W(y) = \frac{y^q}{(1+\log^+(1+\log^+\frac{1}{y}))^\beta}$. Then a simple computation shows that, in this case,

$$(q-1)W(y) \lesssim \widetilde{W}(y),$$

and hence, by Proposition 3,

$$\sup_{y>0} \frac{y^q}{\left(1+\log(1+\log_+\frac{1}{y})\right)^\beta} \int_{\{|T_A\chi_E(x)|>y\}} u_0(x)(M\chi_E(x))^{(1-q)} dx \lesssim C_\beta\, u_0(E).$$

Therefore,

$$\frac{y}{(1+\log^+(1+\log^+\frac{1}{y}))^\beta}\int_{\{|T_A\chi_E|>y\}}u_0(x)dx$$
$$\leq \|u_0\|_{A_1}u_0(E)+\frac{y}{(1+\log^+(1+\log^+\frac{1}{y}))^\beta}\int_{\{|T_A\chi_E|>y, M\chi_E(x)\leq y\}}u_0(x)dx$$
$$\leq \|u_0\|_{A_1}u_0(E)+\frac{y^q}{(1+\log^+(1+\log^+\frac{1}{y}))^\beta}\int_{\{|T_A\chi_E|>y\}}u_0(x)(M\chi_E(x))^{1-q}dx$$
$$\lesssim C_\beta\, u_0(E),$$

and the result follows.

Finally, the proof of Corollary 1 follows immediately.

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