# EXTRAPOLATION OF COMPACTNESS ON WEIGHTED MIXED LEBESGUE SPACES

MARÍA J. CARRO, CARLOS PÉREZ, AND RODOLFO H. TORRES

ABSTRACT. A version of the Fréchet-Kolmogorov theorem for the compactness of operators in weighted mixed Lebesgue spaces is proved and a corresponding compact extrapolation theory a la Rubio de Francia is developed. Several applications are presented too.

## 1. INTRODUCTION

The extrapolation result by Rubio de Francia has become a powerful tool for extending the weighted boundedness of an operator from $L^p(w)$ for every $w \in A_p$ to boundedness from $L^q(w)$ for every $w \in A_q$ ([22, 11, 9]).

More recently, several authors have investigated extending Rubio de Francia's extrapolation result to the category of compact operators on weighted Lebesgue spaces. Contributions from Cao, Olivo, and Yabuta [4]) along with Hytönen and Lappas [16] (see also [23]), have established linear, multilinear, and off-diagonal results. The two groups of authors adopt different approaches: while both use a combination of interpolation and extrapolation techniques, the former relies on the classical Fréchet-Kolmogorov theorem to characterize precompactness, whereas the latter employs more abstract arguments concerning compact operators. Generally speaking, these works show that if an operator is compact on a specific initial weighted Lebesgue space, then it is also compact on all corresponding weighted Lebesgue spaces with Muckenhoupt weights.

---

2010 *Mathematics Subject Classification.* 42B20, 42B25, 35S05, 46E30, 47B07.

*Key words and phrases.* Mixed Lebesgue spaces, Fréchet-Kolmogorov, Muckenhoupt weights, compact operators, extrapolation.

The first author has been partially supported by grants PID2020-113048GB-I00 funded by MCIN/AEI/10.13039/501100011033, CEX2019-000904-S funded by MCIN/AEI/ 10.13039/501100011033 and Grupo UCM-970966 (Spain). The second author is supported by grant PID2020-113156GB-I00, Spanish Government; by the Basque Government through grant IT1615-22 and the BERC 2014-2017 program and by BCAM Severo Ochoa accreditation SEV-2013-0323, Spanish Government.

In this article, we build upon the results in [5], where a notion of uniform compactness was introduced and applied via extrapolation to derive new compactness results for pseudodifferential operators. While this assumption of uniform compactness is stronger than the assumptions used in other places in the literature, it provides a very straightforward way to extrapolate compactness via an application of a modern form of the Rubio de Francia extrapolation of continuity result and it is easy to verify in applications. We will extend these ideas to the context of weighted mixed Lebesgue spaces, defined as follows

**Definition 1.1.** *Given $1 \leq p, q < \infty$ and two weights $u$ in $\mathbb{R}^n$ and $v$ in $\mathbb{R}^m$, we define the space*

$$L^p_u L^q_v(\mathbb{R}^{n+m}) = \{f : \mathbb{R}^{n+m} \to \mathbb{C} \text{ measurable} : \|f\|_{L^p_u L^q_v(\mathbb{R}^{n+m})} < \infty\},$$

*where*

$$\|f\|_{L^p_u L^q_v(\mathbb{R}^{n+m})} = \left( \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^m} f(x,y)^q v(y) dy \right]^{p/q} u(x) dx \right)^{1/p}.$$

The unweighted version of these spaces was introduced and thoroughly studied by A. Benedek and R. Panzone in [1]. See [18] and the reference therein for further properties in the weighted case.

When the dimensions are clear from the context or do not need to be specified, we will simply write $L^p_u L^q_v$. We observe that if $p = q$, then $L^p_u L^q_v = L^p(w)$ is the standard Lebesgue space on $\mathbb{R}^{n+m}$ with weight $w(x,y) = u(x)v(y)$.

## 2. Preliminaries

2.1. **Weights.** By a weight we mean a non-negative, measurable, and locally integrable function. We recall that a weight $w$ belongs to the class $A_p$, with $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$ , if

$$[w]_{A_p} = \sup_Q \left( \fint_Q w(y)\, dy \right) \left( \fint_Q w(y)^{1-p'}\, dy \right)^{p-1} < \infty.$$

As it is usually done, we have used here the notation $\fint_E f\, dx = f_E = \frac{1}{|E|} \int_E f\, dx$ for the average of $f$ over $E$ with respect to the Lebesgue measure. Likewise, for a given measure $\mu$ defined for every cube $Q$, we will write $f_{Q,\mu} = \fint_Q f d\mu := \frac{1}{\mu(Q)} \int_Q f d\mu$. In the particular case of densities given by a weight $w$, we will write $f_{Q,w} = \frac{1}{w(Q)} \int_Q f w dx$.

The number $[w]_{A_p}$ above is called the $A_p$ constant of $w$. A weight $w$ belongs to the class $A_1$ if there is a constant $C$ such that

$$\fint_Q w(y)\,dy \leq C \inf_Q w,$$

and the infimum of these constants $C$ is called the $A_1$ constant of $w$.

We will need the following well known property of the $A_p$ weights. Using Hölder's inequality with $p$ and its conjugate $p'$, we have that for every cube $Q$ and every $g \geq 0$,

$$\fint_Q g\ dx \leq [w]_{A_p}^{\frac{1}{p}} \left( \frac{1}{w(Q)} \int_Q g^p\, w dx \right)^{\frac{1}{p}}. \tag{1}$$

Specializing inequality (1) for $g \equiv \chi_E$ we obtain that, for any measurable set $E \subset Q$, that

$$w(Q) \leq [w]_{A_p} \left( \frac{|Q|}{|E|} \right)^p w(E) \tag{2}$$

Since the $A_p$ classes are increasing with respect to $p$, the $A_\infty$ class of weights is defined in a natural way by

$$A_\infty = \cup_{p>1} A_p,$$

and it is characterized by means of this constant

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q)\ dx,$$

as considered in [14] (see also [15]). In fact we will use the following "precise openness property" in the proof of Theorem 4.1 whose proof can be found in [15].

Let $1 < p < \infty$ and let $w \in A_p$. Then $w \in A_{p-\varepsilon}$ where

$$\varepsilon := \frac{p-1}{(r_\sigma)'} = \frac{p-1}{2^{n+1}[\sigma]_{A_\infty}}, \tag{3}$$

and, as usual, $\sigma = w^{1-p'}$. Furthermore,

$$[w]_{A_{p-\varepsilon}} \leq 2^{p-1}[w]_{A_p} \tag{4}$$

The proof of this result is based in the following "precise reverse Hölder property": Let

$$r_w := 1 + \frac{1}{2^{n+1}[w]_{A_\infty} - 1}. \tag{5}$$

Then for any cube $Q$, we have that

$$\left( \fint_Q w^{r_w}\, dx \right)^{\frac{1}{r_w}} \leq 2 \fint_Q w\, dx. \tag{6}$$

In the proof of Theorem 4.1 we will be considering weights $w$ in the class $A_2$ and then in this case we can replace the $\varepsilon$ from (4) by

$$\varepsilon := \frac{1}{2^{n+1}[w]_2}$$

since $\sigma = w^{-1}$ and, in this case, $[\sigma]_{A_\infty} \leq [\sigma]_{A_2} = [w]_{A_2}$.

A well-known result obtained by Muckenhoupt [20] is that the Hardy-Littlewood maximal function,

$$Mf(x) = \sup_{Q \ni x} \frac{1}{Q} \int_Q |f(y)| dy,$$

satisfies $M : L^p(w) \to L^p(w)$ if and only if $w$ is in $A_p$.

2.2. **Sharp maximal operators.** Let $M^\#$ be the usual sharp maximal function of Fefferman and Stein [12],

$$M^\# f(x) = \sup_{Q \ni x} \inf_c \fint_Q |f(y) - c|\, dy \approx \sup_{Q \ni x} \fint_Q |f(y) - f_Q|\, dy.$$

For a given $\delta \in (0, \infty)$ we will use the notation

$$M_\delta f(x) = \Big( M(|f|^\delta)(x) \Big)^{1/\delta}.$$

and

$$M_\delta^\# f(x) = \Big( M^\#(|f|^\delta)(x) \Big)^{1/\delta} = \Big( \sup_{x \in Q} \fint_Q \big||f|^\delta - |f|_Q^\delta\big| dx \Big)^{1/\delta}$$

The following form of the classical result of Fefferman and Stein [12] (see also [17]) will be useful for our purposes.

Let $0 < p, \delta < \infty$ and let $w$ be a weight in $A_\infty$. Then, there exists $C > 0$ (depending on the $A_\infty$ constant of $w$), such that

$$\int_{\mathbb{R}^n} (M_\delta f(x))^p\, w(x) dx \leq C \int_{\mathbb{R}^n} (M_\delta^\# f(x))^p\, w(x) dx, \tag{7}$$

for all function $f$ for which the left hand side is finite.

2.3. **Extrapolation.** We begin this section by recalling a modern version of the Rubio de Francia extrapolation theorem.

**Theorem 2.1.** *Let $\mathcal{F}$ be a family of pairs of functions $(f, g)$ such that, for some $1 \leq p_0 < \infty$, every $w \in A_{p_0}$, every $(f, g) \in \mathcal{F}$, and an increasing function $\varphi$,*

$$\|g\|_{L^{p_0}(w)} \leq \varphi([w]_{A_{p_0}}) \|f\|_{L^{p_0}(w)}.$$

*Then, for every $1 < p < \infty$ there exists and increasing function $\phi_p$ depending only $\varphi$ such that for every $w \in A_p$ and every $(f,g) \in \mathcal{F}$,*

$$\|g\|_{L^p(w)} \leq \phi_p([w]_{A_p})\|f\|_{L^p(w)}.$$

The dependence of $\phi_p$ on $\varphi$ and $p$ can be tracked explicitly; see [11, Theorem 3.1]. In fact, we shall require the following more specific version which can be found in [5].

**Theorem 2.2.** *Let $W > 0$ and let $\mathcal{F}$ be a family of pairs of measurable functions so that, for some $p_0 > 1$*

$$\varphi(W) := \sup_{\|w\|_{A_{p_0}} \leq W} \sup_{(f,g)\in\mathcal{F}} \frac{\|g\|_{L^{p_0}(w)}}{\|f\|_{L^{p_0}(w)}} < \infty,$$

*then, for every $p > 1$ there exist constants $C_1$ and $C_2$ so that if $M > 0$ satisfies that $C_2 M^{\max(1,\frac{p_0-1}{p-1})} = W$, then*

$$\sup_{\|w\|_{A_p \leq M}} \sup_{(f,g)\in\mathcal{F}} \frac{\|g\|_{L^p(w)}}{\|f\|_{L^p(w)}} \leq C_1\varphi(W) < \infty.$$

We will also use an extrapolation theorem based on the $A_\infty$ class in contrast to the usual $A_{p_0}$ condition, $p_0 \in (1,\infty)$. This approach originated in [8, 19] and was subsequently expanded in [10].

**Theorem 2.3.** *Let $\mathcal{F}$ be a family of pairs of functions $(f,g)$ such that, for some $0 < p_0 < \infty$, every $w \in A_\infty$ and every $(f,g) \in \mathcal{F}$,*

$$\|g\|_{L^{p_0}(w)} \leq \varphi([w]_{A_\infty})\|f\|_{L^{p_0}(w)}$$

*for some increasing function $\varphi$. Then for every $0 < p < \infty$ there exists an increasing function $\phi_p$ depending only $\varphi$ such that for every $w \in A_\infty$ and every $(f,g) \in \mathcal{F}$,*

$$\|g\|_{L^p(w)} \leq \phi_p([w]_{A_\infty})\|f\|_{L^p(w)}.$$

A theory of extrapolation in weighted mixed Lebesgue spaces has been previously developed by Kurtz. In fact, the following theorem was proved in [18] but we include a simpler proof as a direct application of Theorem 2.1.

**Theorem 2.4.** *Let $\mathcal{F}$ be a family of pair of functions $(f,g)$ defined on $\mathbb{R}^{n+m}$ such that, for some $p_0 \geq 1$, every $w \in A_{p_0}(\mathbb{R}^{n+m})$ and every $(f,g) \in \mathcal{F}$,*

$$\|g\|_{L^{p_0}_w(\mathbb{R}^{n+m})} \leq \varphi([w]_{A_{p_0}})\|f\|_{L^{p_0}_w(\mathbb{R}^{n+m})}, \tag{8}$$

*for some increasing function $\varphi$. Then, for every $1 < p,q < \infty$, there exist a function $\psi_{p,q}$ such that for every $u \in A_p(\mathbb{R}^n)$ and every $v \in A_q(\mathbb{R}^m)$*

$$\|g\|_{L^p_u L^q_v} \leq \psi_{p,q}([u]_{A_p},[v]_{A_q})\|f\|_{L^p_u L^q_v},$$

*for all* $(f,g)\in\mathcal{F}$.

*Proof.* Fix $q>1$. By Theorem 2.1, we have that (8) implies that, for every $w\in A_q(\mathbb{R}^{n+m})$,

$$\|g\|_{L^q(w)}\le \phi_q([w]_{A_q})\|f\|_{L^q(w)}.$$

Let now fix $v\in A_q(\mathbb{R}^m)$ and let us consider the function

$$G(x)=\left(\int_{\mathbb{R}^m}|g(x,y)|^q v(y)dy\right)^{1/q},$$

and similarly define $F$ by replacing $g$ with $f$. Then, for every $u\in A_q(\mathbb{R}^n)$, we clearly have that $u\otimes v\in A_q(\mathbb{R}^{n+m})$ with $[u\otimes v]_{A_q}\le [u]_{A_q}[v]_{A_q}$, and hence

$$\begin{aligned}\|G\|_{L^q_u(\mathbb{R}^m)}=\|g\|_{L^q_{u\otimes v}(\mathbb{R}^{n+m})}&\le \phi_q([u\otimes v]_{A_q})\|f\|_{L^q_{u\otimes v}(\mathbb{R}^{n+m})}\\&\le \phi_q([u]_{A_q}[v]_{A_q})\|F\|_{L^q_u(\mathbb{R}^m)}\\&\le \Phi_{q,[v]_{A_q}}([u]_{A_q})\|F\|_{L^q_u(\mathbb{R}^m)},\end{aligned}$$

where $\Phi_{q,[v]_{A_q}}([u]_{A_q})=\phi_q([u]_{A_q}[v]_{A_q})$ is an increasing function of $[u]_{A_q}$. Using Theorem 2.1, we conclude that for every $1<p<\infty$ there exists an increasing function $\Psi_p$ depending on $\Phi_{q,[v]_{A_q}}$ such that for every $u\in A_p(\mathbb{R}^n)$,

$$\|g\|_{L^p_uL^q_v(\mathbb{R}^{n+m})}=\|G\|_{L^p_u}\le\Psi_p([u]_{A_p})\|F\|_{L^p_u}=\|f\|_{L^p_uL^q_v(\mathbb{R}^{n+m})}.$$

Observe that $\Psi_p([u]_{A_p})$ is really a function $\psi_{p,q}([u]_{A_p},[v]_{A_q})$ and the result follows. □

We note that using the precise estimates in [11, Theorem 3.1] one can check that $\psi_{p,q}$ is actually increasing in each variable.

The following result (also proved in [18]) follow as an easy corollary of Theorem 2.4.

**Corollary 2.5.** *Let $M$ be the Hardy-Littlewood maximal operator in $\mathbb{R}^{n+m}$ then, for every $1<p,q<\infty$, $u\in A_p(\mathbb{R}^n)$ and $v\in A_q(\mathbb{R}^m)$,*

$$M: L^p_uL^q_v\longrightarrow L^p_uL^q_v,$$

*with norm controlled by $\varphi_{p,q}([u]_{A_p},[v]_{A_q})$, with $\varphi_{p,q}$ an increasing function.*

Another corollary of Theorem 2.4 is an extension of the Fefferman-Stein inequality (7) within the context of weighted mixed Lebesgue spaces.

**Corollary 2.6.** *Let $M$ and $M^{\#}$ be the Hardy-Littlewood maximal operator and Fefferman and Stein sharp maximal function in $\mathbb{R}^{n+m}$. Let $0 < p, q < \infty$, $\delta > 0$ and let $u \in A_\infty(\mathbb{R}^n)$ and $v \in A_\infty(\mathbb{R}^m)$. Then there exists a function $\psi$ increasing in each variable such that*

$$\|M_\delta f\|_{L^p_u L^q_v} \leq \psi_{p,q}([u]_{A_\infty}, [v]_{A_\infty}) \|M^{\#}_\delta f\|_{L^p_u L^q_v}.$$

*Proof.* By the classical result of Fefferman-Stein (7) for any $0 < p, \delta < \infty$, any $w \in A_\infty(\mathbb{R}^{n+m})$

$$\|M_\delta f\|_{L^q(w)} \leq c_{q,n}\, [w]_{A_\infty} \|M^{\#}_\delta f\|_{L^q(w)}.$$

Now, let $v \in A_\infty(\mathbb{R}^m)$. As before, define the function

$$G(x) = \left( \int_{\mathbb{R}^m} |M_\delta f(x,y)|^q v(y) dy \right)^{1/q},$$

and similarly define $F$ by

$$F(x) = \left( \int_{\mathbb{R}^m} |M^{\#}_\delta f(x,y)|^q v(y) dy \right)^{1/q}.$$

Observing that for every $u \in A_\infty(\mathbb{R}^n)$ and $v \in A_\infty(\mathbb{R}^m)$ we clearly have that $u \otimes v \in A_\infty(\mathbb{R}^{n+m})$. Moreover, if we denote by $Q_n$ cubes in $\mathbb{R}^n$ and similarly $Q_m$, then

$$\begin{aligned}
[u \otimes v]_{A_\infty} &\leq \sup_{Q_n \times Q_m} \frac{1}{u(Q_n)v(Q_m)} \int_{Q_n \times Q_m} M\Big(\chi_{Q_n \times Q_m} u \otimes v\Big)(x,y) dx \\
&\leq \sup_{Q_n \times Q_m} \frac{1}{u(Q_n)v(Q_m)} \int_{Q_n \times Q_m} M(\chi_{Q_n} u)(x) M(\chi_{Q_m} v)(y) dx \\
&\leq [u]_{A_\infty} [v]_{A_\infty},
\end{aligned}$$

and hence, we have that

$$\begin{aligned}
\|G\|_{L^q_u(\mathbb{R}^m)} = \|M_\delta f\|_{L^q_{u\otimes v}(\mathbb{R}^{n+m})} &\leq c_{n,m,q}\, [u \otimes v]_{A_\infty} \|M^{\#}_\delta f\|_{L^q_{u\otimes v}(\mathbb{R}^{n+m})} \\
&\leq c_{n,m,q}\, [u]_{A_\infty} [v]_{A_\infty} \|F\|_{L^q_u(\mathbb{R}^m)}.
\end{aligned}$$

Using Theorem 2.3, we conclude that, for every $v \in A_\infty(\mathbb{R}^m)$ and every $u \in A_\infty(\mathbb{R}^n)$,

$$\begin{aligned}
\|M_\delta f\|_{L^p_u L^q_v(\mathbb{R}^{n+m})} &= \|G\|_{L^p_u} \leq \psi_{p,q}([u]_{A_\infty}, [v]_{A_\infty}) \|F\|_{L^p_u} \\
&= \|M^{\#}_\delta f\|_{L^p_u L^q_v(\mathbb{R}^{n+m})},
\end{aligned}$$

concluding the proof of the corollary. □

## 3. Compactness in weighted mixed Lebesgue spaces

We begin by extending the Fréchet-Kolmogorov compactness theorem to the setting of weighted mixed spaces. The following result, concerning weighted Lebesgue spaces, was proved by Clop and Cruz [6].

**Theorem 3.1.** *Let $1 < p < \infty$ and $u \in A_p$ Then $\mathcal{F} \subset L^p_u$ is relatively compact in $L^p_u$ if the following conditions hold.*

1. *There exists $K > 0$ such that, for every $f \in \mathcal{F}$, $\|f\|_{L^p_u} \leq K$.*
2. *For every $\varepsilon > 0$, there exists $R > 0$ such that*
$$\|\chi_{B(0,R)^c} f\|_{L^p_u} < \varepsilon,$$
*for all $f \in \mathcal{F}$.*
3. *For every $\varepsilon > 0$, there exists $\delta > 0$ so that, for every $h$, $|h| \leq \delta$,*
$$\|f - f(\cdot + h)\|_{L^p_u} \leq \varepsilon,$$
*for all $f \in \mathcal{F}$.*

**Definition 3.2.** *A subset $A$ of a metric space $(X, d)$ is called totally bounded if, for every $\epsilon > 0$, there exists a finite collection of points $x_1, x_2, \ldots, x_n \in X$ such that*

$$A \subseteq \bigcup_{i=1}^{n} B(x_i, \epsilon),$$

*where $B(x_i, \epsilon) = \{y \in X : d(x_i, y) < \epsilon\}$.*

This means that the set $A$ can be covered by a finite number of open balls of arbitrarily small radius, reflecting a certain compact-like behavior.

We will use the following lemma, which appears in [13] and was also employed in [6].

**Lemma 3.3.** *Let $X$ be a metric space. Suppose that for every $\varepsilon > 0$, there exists $\delta > 0$, a metric space $W$ and a mapping $\Phi : X \to W$, such that $\Phi(X)$ is totally bounded, and the implication*

$$d(\Phi(x), \Phi(y)) < \delta \implies d(x, y) < \varepsilon$$

*holds for any $x, y \in X$. Then $X$ is totally bounded.*

Following the approach in [6], we prove the following extension of the Fréchet-Kolmogorov compactness theorem to weighted mixed spaces.

**Theorem 3.4.** *Let $1 < p, q < \infty$, $u \in A_p$, and $v \in A_q$. Then $\mathcal{F} \subset L^p_u L^q_v$ is relatively compact in $L^p_u L^q_v$ if and only if the following conditions hold.*

1. *There exists $K > 0$ such that, for every $f \in \mathcal{F}$, $\|f\|_{L^p_u L^q_v} \leq K$.*
2. *For every $\varepsilon > 0$, there exists $R > 0$ such that*

   $$\|\chi_{B(0,R)^c} f\|_{L^p_u L^q_v} < \varepsilon,$$

   *for all $f \in \mathcal{F}$.*
3. *For every $\varepsilon > 0$, there exists $\delta > 0$ so that, for every $h = (h_1, h_2)$, $|h| \leq \delta$,*

   $$\|f - f(\cdot + h_1, \cdot + h_2)\|_{L^p_u L^q_v} \leq \varepsilon,$$

   *for all $f \in \mathcal{F}$.*

*Proof.* For notational simplicity, we will denote cubes in $\mathbb{R}^n$ by $Q$, cubes in $\mathbb{R}^m$ by $Q'$ and cubes in $\mathbb{R}^{n+m}$ by $Q''$.

We begin by proving the sufficient conditions. Let $\varepsilon > 0$ be given and choose $R > 0$ such that, by (2)

$$\sup_{f \in \mathcal{F}} \|f - f\chi_{Q''(0,R)}\|_{L^p_u L^q_v} \leq \frac{\varepsilon}{8}. \tag{9}$$

Condition (3) guarantees the existence of a $\delta > 0$ such that

$$\sup_{|h| \leq \delta} \sup_{f \in \mathcal{F}} \|f - f(\cdot + h)\|_{L^p_u L^q_v} \leq \frac{\varepsilon}{8}. \tag{10}$$

Now, given $R > 0$ and $\delta > 0$, let us consider $\{Q_i \times Q'_i\}_{i=1}^N$ pairwise disjoint such that, $l(Q_i) = l(Q'_i) = \delta$, and

$$Q''(0,R) \subset \bigcup_i Q_i \times Q'_i.$$

Now, let us define the linear operator

$$\Phi(f)(x,y) = \sum_i \left( \frac{1}{|Q_i \times Q'_i|} \int_{Q_i \times Q'_i} f(z,\xi)\, dz d\xi \right) \chi_{Q_i \times Q'_i}(x,y).$$

Notice that $\Phi(f)(x,y) \leq Mf(x,y)$, where $M$ is the Hardy-Littlewood maximal operator in $\mathbb{R}^{n+m}$. Therefore, by Corollary 2.5,

$$\Phi : L^p_u L^q_v \to L^p_u L^q_v$$

is bounded. Moreover since $\mathcal{F}$ is bounded and $\Phi(f)$ takes only a finite number of values, for each $f \in \mathcal{F}$, $\Phi(\mathcal{F})$ is a totally bounded set in a finite dimensional space.

On the other hand, if $Q_\delta = Q(0,\delta)$ is the cube centered at 0 and side length $\delta$, (resp. $Q'_\delta = Q'(0,\delta)$) in $\mathbb{R}^n$ (resp. $\mathbb{R}^m$), then

$$\begin{aligned}
&\|f\chi_{\cup_i Q_i\times Q'_i} - \Phi(f)\|_{L^p_uL^q_v} \\
= &\left\|\sum_i \left(\frac{1}{|Q_i\times Q'_i|}\int_{Q_i\times Q'_i} f(x,y)-f(z,\xi)dzd\xi\right)\chi_{Q_i\times Q'_i}(x,y)\right\| \\
\le &\left\|\sum_i \frac{\chi_{Q_i\times Q'_i}(x,y)}{|Q_i\times Q'_i|}\int_{Q_i\times Q'_i}|f(x,y)-f(z,\xi)|dzd\xi\right\|_{L^p_uL^q_v} \\
\le &\left\|\sum_i \frac{\chi_{Q_i\times Q'_i}(x,y)}{|Q_\delta\times Q'_\delta|}\int_{Q_{2\delta}\times Q'_{2\delta}}|f(x,y)-f(x+h_1,y+h_2)|dh_1dh_2\right\|,
\end{aligned}$$

where the last inequality follows since, for every $x\in Q_i$, we have that $Q_i \subset x + Q_{2\delta}$. Using now, Minkowski integral inequality and the fact that $\{Q_i\times Q'_i\}_{i=1}^N$ are pairwise disjoint sets, we have that

$$\begin{aligned}
&\|f\chi_{\cup_i Q_i\times Q'_i} - \Phi(f)\|_{L^p_uL^q_v} \\
\lesssim\ &\frac{1}{|Q_\delta\times Q'_\delta|}\int_{Q_{3\delta}\times Q'_{3\delta}}\left\|\sum_i \chi_{Q_i\times Q'_i}|f-f(\cdot+h_1,\cdot+h_2)|\right\| dh_2 \\
\le\ &\frac{1}{|Q_\delta\times Q'_\delta|}\int_{Q_{3\delta}\times Q'_{3\delta}}||f-f(\cdot+h_1,\cdot+h_2)||_{L^p_uL^q_v}\,dh_1dh_2 \\
\lesssim\ &\sup_{|h|\le\delta}\|f-f(\cdot+h\|_{L^p_uL^q_v}\le\frac{\varepsilon}{8},
\end{aligned}$$

for all $f\in\mathcal{F}$. From here, it follows that

$$\|f-\Phi(f)\|_{L^p_uL^q_v}\le$$
$$\|f-f\chi_{\cup_i Q_i\times Q'_i}\|_{L^p_uL^q_v}+\|f\chi_{\cup_i Q_i\times Q'_i}-\Phi(f)\|_{L^p_uL^q_v}<\frac{\varepsilon}{4}.$$

Then

$$\|f\|_{L^p_uL^q_v}\le\frac{\varepsilon}{4}+\|\Phi(f)\|_{L^p_uL^q_v}. \tag{11}$$

Notice that if $f,g\in\mathcal{F}$, (9) and (10) hold for $f-g$ if we replace $\frac{\varepsilon}{8}$ by $\frac{\varepsilon}{4}$ on the right hand side. Moreover since $\Phi$ is linear, we also obtain as in (11)

$$\|f-g\|_{L^p_uL^q_v}\le\frac{\varepsilon}{2}+\|\Phi(f-g)\|_{L^p_uL^q_v}=\frac{\varepsilon}{2}+\|\Phi(f)-\Phi(g)\|_{L^p_uL^q_v}.$$

Therefore, if $f,g\in\mathcal{F}$ and $\|\Phi(f)-\Phi(g)\|_{L^p_uL^q_v}\le\frac{\varepsilon}{2}$, we must have $\|f-g\|_{L^p_uL^q_v}\le\varepsilon$, and by Lemma 3.3, $\mathcal{F}$ is totally bounded which means in this case precompact.

Finally, the necessary condition follows as in Theorem 2 of [1]. □

We now extend to the mixed Lebesgue setting the notion of uniformly compactness introduced in [5].

**Definition 3.5.** *An operator $T$ is uniformly compact in $L^p_uL^q_v$ with respect to $(A_p, A_q)$ if there exist an increasing function $\varphi : (0,\infty) \to (0,\infty)$ such that the following hold.*

*(1) For every every $K \geq 1$, and every $u \in A_p$ and $v \in A_q$ with $\max([u]_{A_p}\,[v]_{A_q}) \leq K$*

$$\sup_{|f\|_{L^p_uL^q_v}\leq 1} \|Tf\|_{L^p_uL^q_v} \leq \varphi(K),$$

*(2) For every $\varepsilon > 0$, every $K \geq 1$, there exists $R = R(\varepsilon, K) > 0$ such that*

$$\sup_{\max([u]_{A_p}\,[v]_{A_q})\leq K}\ \sup_{\|f\|_{L^p_uL^q_v}\leq 1} \|\chi_{B(0,R)^c}Tf\|_{L^p_uL^q_v} \leq \varphi(K)\varepsilon,$$

*(3) For every $\varepsilon > 0$ and every $K \geq 1$, there exists $\delta = \delta(\varepsilon, K) > 0$ so that, for every $h = (h_1, h_2)$ with $|h| \leq \delta$,*

$$\sup_{\max([u]_{A_p}\,[v]_{A_q})\leq K}\ \sup_{\|f\|_{L^p_uL^q_v}\leq 1} \|Tf - Tf(\cdot + h_1, \cdot + h_2)\|_{L^p_uL^q_v} \leq \varphi(K)\varepsilon.$$

Clearly if an operator is uniformly compact it is also compact, since by Theorem 3.4 the image of the unit ball under $T$ is precompact.

As in [5], we will also say that $T$ is uniformly compact in $L^p_w(\mathbb{R}^{n+m})$ if the conditions in Definition 3.5 hold with the norm in $L^p_w(\mathbb{R}^{n+m})$ and all $w \in A_p(\mathbb{R}^{n+m})$.

**Remark 3.6.** We also say that an operator $T$ is uniformly approximated with respect to $(A_p, A_q)$ by a sequence of operators $\{T_j\}_j$, if there exist a function $\varphi : (0,\infty) \to (0,\infty)$ such that given $\varepsilon > 0$ and $K \geq 1$, there exists $j_0 = j_0(\varepsilon, K)$ such that for all $j > j_0$

$$\sup_{\max([u]_{A_p},[v]_{A_q})\leq K}\ \sup_{|f\|_{L^p_uL^q_v}\leq 1} \|Tf - T_jf\|_{L^p_uL^q_v} \leq \varphi(K)\varepsilon,$$

It is easy to see that if a sequence of operators $\{T_j\}_j$ satisfies that each $T_j$ is uniformly compact in $L^p_uL^q_v$ with respect to $(A_p, A_q)$ and uniformly approximates $T$ with respect to $(A_p, A_q)$, then $T$ is not only compact (as the limit of compact operators) but also uniformly compact in $L^p_uL^q_v$ with respect to $(A_p, A_q)$. In fact (1) and (2) in Definition 3.5 follow at once, while for (3) we simply observe that, for any $g \in L^p_uL^q_v$,

$$\|g(\cdot + h_1, \cdot + h_2)\|_{L^p_uL^q_v} = \|g\|_{L^p_{\tau_{h_1}u}L^q_{\tau_{h_2}v}},$$

where $\tau_hu(x) = u(x-h)$ and that for any weight in $A_q$, $[\tau_hu]_{A_q} = [u]_{A_q}$. We leave the details to the reader.

The following compact extrapolation result was proved in [5].

**Theorem 3.7.** *If $T$ is an operator such that, for some $p_0 > 1$*

$$T : L^{p_0}(w) \longrightarrow L^{p_0}(w)$$

*is uniformly compact with respect to $A_{p_0}$, then for every $1 < p < \infty$,*

$$T : L^p_u \longrightarrow L^p_u$$

*is uniformly compact with respect to $A_p$.*

Our main theorem now reads as follows.

**Theorem 3.8.** *If $T$ is an operator such that, for some $p_0 > 1$*

$$T : L^{p_0}(w) \longrightarrow L^{p_0}(w)$$

*is uniformly compact with respect to $A_{p_0}(\mathbb{R}^{n+m})$, then for every $p, q > 1$,*

$$T : L^p_u L^q_v \longrightarrow L^p_u L^q_v$$

*is uniformly compact with respect to $(A_p, A_q)$.*

*Proof.* By the compact extrapolation result in [5] we immediately get that for all $1 < p < \infty$,

$$T : L^p(w) \longrightarrow L^p(w)$$

is uniformly compact with respect to $A_p(\mathbb{R}^{n+m})$.

Let us now fix $K \geq 1$, $p, q > 1$ and $u \in A_p$ and $v \in A_q$ such that $\max([u]_{A_p}, [v]_{A_q}) \leq K$. We need to verify the three conditions in Definition 3.5. We will see that they follow by applying three times Theorem 2.4.

For *(1)*, let $\|f\|_{L^p(L^q w)} \leq 1$. Then,

$$\|Tf\|_{L^p_u L^q_v} \leq \phi([u]_{A_p}, [v[_{A_q}) \leq \varphi(K),$$

if we use $(f, g) = (f, T(f))$ in Theorem 2.4.

For *(2)*, since for every $K > 0$ and every $\varepsilon > 0$ there exists $R > 0$ such that

$$\left(\int_{|x| \geq R} |Tf(x)|^{p_0} w(x) dx\right)^{1/p_0} \lesssim \varphi(K) \varepsilon \left(\int_{\mathbb{R}^n} |f(x)|^{p_0} w(x) dx\right)^{1/p_0},$$

whenever $[w]_{A_{p_0}} \leq K$, then by Theorem 2.4 with

$$(f, g) = (f, \chi_{B(0,R)^c} Tf),$$

for every $u$ and $v$ so that $\max([u]_{A_p}, [v]_{A_q}) \leq K$, there exists $R > 0$ so that

$$\|\chi_{B(0,R)^c} Tf\|_{L^p_u L^q_v} \lesssim \psi(K, K) \varepsilon \|f\|_{L^p_u L^q_v},$$

and the result follows.

Similarly, for *(3)* we use the extrapolation theorem with

$$(f, g) = (f, (Tf(x+t) - Tf(x))).$$

$\square$

## 4. Applications

### 4.1. **Commutators of singular integrals and point-wise multiplication.**

There exist a very extensive and ever growing literature about the boundedness and compactness of commutators of various singular integrals and point-wise multiplication by function in a variety of contexts, including linear, multilinear, and weighted settings. In general commutation with function in $BMO$ produces bounded operators, while commutation with functions in $CMO$ produces compact ones. We refer the, reader for example, to [2] and [3] for comprehensive approaches and references. Here $BMO$ is the usual space of functions of bounded mean oscillations, while $CMO$ is defined as the closure in the topology of $BMO$ of the space $C_c^\infty$ of smooth functions with compact support.

The first to prove the compactness of the commutators of Calderón-Zygmund operators with $CMO$-functions in a weighted $L^p$ setting were Clopp and Cruz in the already cited work [6]. In order to apply our theory to extrapolate this result to weighted mixed Lebesgue spaces we need to verify that those authors proved that the operators in question are actually uniformly compact in $L^p$ with respect to $A_p$. This can be done by inspection of their proof. We will not repeat their computations here but, for the benefit of the reader, we shall summarize some key points where estimates depending on the classes of weights are obtained.

For simplicity and the purposes of this article, we define a Calderón-Zygmund operator $T$ to be a bounded linear operator on $L^2(\mathbb{R}^n)$ that can be written as

$$Tf(x) = \int_{\mathbb{R}^n} \mathcal{K}(x,y) f(y)\, dy$$

for $x \notin \operatorname{supp} f$, where the kernel $\mathcal{K}$ satisfies

$$|\partial^\beta \mathcal{K}(x,y)| \leq C_{\mathcal{K}} |x-y|^{-n-|\beta|}, \quad |\beta| \leq 1. \tag{12}$$

We will often write in such a case $T \in CZO$.

**Theorem 4.1.** *Let $T \in CZO$ and $b \in CMO$ and consider the commutator*

$$C_b f = [T, b] f = T(bf) - bTf.$$

*Then, for every $u \in A_p(\mathbb{R}^n)$ and every $v \in A_q(\mathbb{R}^m)$*

$$C_b : L^p_u L^q_v \longrightarrow L^p_u L^q_v$$

*is uniformily compact.*

*Proof.* Let $d := n + m$. To prove that $C_b : L^p_u L^q_v \longrightarrow L^p_u L^q_v$ is compact whenever $u \in A_p(\mathbb{R}^n)$ and $v \in A_q(\mathbb{R}^m)$, we will use the extrapolation Theorem 3.8 with exponent $p_0 = 2$, namely we have to prove that

$$C_b : L^2(w) \longrightarrow L^2(w)$$

is uniformly compact with respect to $w \in A_2(\mathbb{R}^d)$.

We shall follow the proof of [6] checking all the details that remains open in our setting.

**Step 1:** Let us approximate $b \in CMO$ by functions $b_j \in C^\infty_c$. By a result in [21], we have that, for $w \in A_2(\mathbb{R}^d)$,

$$\|C_b f\|_{L^2(w)} \lesssim \|b\|_* \|M^2 f\|_{L^2(w)},$$

where $|b\|_*$ is the $BMO$ norm. Therefore,

$$\|C_b f - C_{b_j}\|_{L^2(w)} \lesssim \|b - b_j\|_* [w]^2_{A_2} \|f\|_{L^2(w)},$$

and $C_{b_j} \to C_b$ uniformly on $[w]_{A_2(\mathbb{R}^d)} \leq K$, with $K$ a fixed and arbitrary constant. So it is enough to verify that the $C_{b_j}$ are uniformly compact.

**Step 2:** Next, for each $j$ the operators $C_{b_j}$ are then approximated by $C^\eta_{b_j}$ which is the commutator obtained by replacing the kernel $\mathcal{K}$ of $T$ by a smooth truncated version $\mathcal{K}^\eta$, where $\eta$ is an appropriate parameter. The authors of [6] show that

$$|Cb_j f(x) - Cb^\eta_j(x)| \lesssim \eta \|\nabla b_j\|_\infty Mf(x),$$

from which it follows that $Cb^\eta_j \to Cb_j$ in the uniform sense as $\eta \to 0$.

**Step 3:** For a fixed $b$ and $\eta$, it is then needed to prove the compactness of $C^\eta_b$. To this end, we will use the version of the Fréchet-Kolmogorov result in Theorem 3.1. Condition *(1)* is uniformly satisfied by the already mentioned estimates on the commutator. To verify *(2)* and *(3)*, Clop and Cruz separate the quantities in questions into several terms. They obtained several pointwise estimates ultimately controlled by $M$ and $T_*$ (the maximal truncated singular integral), see [6, p.98]. These terms clearly produce uniform estimates with respect to $A_2(\mathbb{R}^d)$.

Finally there are some terms which are dominated by expressions that can be controlled (up to constants) by

$$\int_{3B_0}\left(\int_{B_x\cap 4B_0} w(y)^{-1}\,dy\right) w(x)\,dx \tag{13}$$
$$+\int_{{}^c(3B_0)}\left(\int_{B_x\cap 4B_0} w(y)^{-1}\,dy\right)\frac{w(x)}{|x|^{2d}}\,dx,$$

where $B_0$ is a fixed ball centered at the origin of radius $R_0$, which we can take bigger than 1, and $B_x$ is a set depending on $x$ but with size $|B_x| \approx h$ where $|h| \to 0$; and

$$\left(\int_{|x|<R_0} w(x)^{-1}dx\right)^{1/2}\left(\int_{|x|>R}\frac{w(x)}{|x|^{2d}}dx\right)^{1/2}, \tag{14}$$

with $R > R_0$ fixed, $R \to \infty$; see [6, pp.99-100]. We need to show that these terms, which obviously tend to zero when $h \to 0$ in (13) and $R \to \infty$ in (14), do so uniformly on $[w]_{A_2}$, whenever $[w]_{A_2} \le K$ with $K$ any fixed number.

**Estimates for** (13): Note that by Hölder's inequality we get that, for any $r > 1$,

$$\left(\int_{B_x\cap 4B_0} w(y)^{-1}\,dy\right) \le \left(\int_{4B_0} w(y)^{-r}dy\right)^{1/r}|B_x|^{1/r'}.$$

To control (13), we shall choose $r_1 > 1$ and $r_2 > 1$ appropriately and estimate

$$I := \left(\int_{4B_0} w(x)^{-r_1}dx\right)^{1/r_1}\left(\int_{3B_0} w(x)dx\right)|h|^{1/r_1'} \tag{15}$$

and

$$II := \left(\int_{4B_0} w(x)^{-r_2}dx\right)^{1/r_2}\int_{(3B_0)^c}\frac{w(x)}{|x|^{2d}}dx\;|h|^{1/r_2'}. \tag{16}$$

To estimate $I$ in (15), and using that $w \in A_2$ implies $w^{-1} \in A_2$ with same $A_2$ constant, we can choose $q = 2-\varepsilon$ with

$$\varepsilon = \frac{1}{c_d[w]_{A_2}} = \frac{1}{c_d[w^{-1}]_{A_2}} \le \frac{1}{c_d[w^{-1}]_{A_\infty}},$$
$$[w]_{A_q} \le 2[w]_{A_2}$$

by the precise openness property of the $A_2$ class of weights (see (3) and (4)). It follows that with $r_1 = \frac{q'}{q} > 1$,

$$I \le \left(\fint_{4B_0} w(x)^{-r_1}dx\right)^{1/r_1}\fint_{4B_0} w(x)dx\;|4B_0|^{1+1/r_1}|h|^{1/r_1'},$$

$$\leq [w]_{A_q}|4B_0|^{1+1/r_1}|h|^{1/r_1'} \leq 2[w]_{A_2}|4B_0|^{1+1/r}|h|^{1/r_1'}.$$

Now, fixed $K$ and $[w|_{A_2} \leq K$, it can be check that

$$r_1 = \frac{1}{q-1} = \frac{1}{1-\varepsilon} \implies r_1' = \frac{1}{\varepsilon} = c_d[w]_{A_2} \leq c_d K,$$

and hence

$$I \leq K|4B_0|^2|h|^{1/(c_dK)},$$

which tends to zero as $|h| \to 0$, uniformly on $[w]_{A_2} \leq K$.

We now estimate $II$ in (16). Setting $R_1$ equal to the radius of $3B_0$,

$$\int_{|x|>R_1} \frac{w(x)}{|x|^{2d}}dx = \sum_{j=0}^{\infty} \int_{2^jR_1<|x|\leq 2^{j+1}R_1} \frac{w(x)}{|x|^{2d}}dx$$

$$\leq \sum_{j=0}^{\infty} (2^jR_1)^{-2d} w(B(0,2^{j+1}R_1)).$$

Let $q$ be the exponent from the precise openness property as above. Then, by standard properties of the $A_p$ class of weights (see (2)) we have that for any pair of balls $B_1 \subset B_2$, that

$$w(B_2) \leq [w]_{A_q}\left(\frac{|B_2|}{|B_1|}\right)^q w(B_1) \tag{17}$$

and we can continue with

$$\lesssim [w]_{A_q}\sum_{j=0}^{\infty}(2^jR_1)^{-2d}(2^{j+1}R_1)^{qd}w(B(0,1)) \lesssim [w]_{A_2}w(B(0,1))R_1^{d(q-2)},$$

since $q < 2$.

Now, taking $r_2 = 1 + \frac{1}{2^{d+1}[w]_{A_2}-1} \leq r_{w^{-1}}$ from the precise reverse Hölder property (5), we have as in (6)

$$\left(\fint_{4B_0} w(x)^{-r_2}dx\right)^{1/r_2} \leq 2\fint_{4B_0} w(x)^{-1}dx,$$

and we can use again (17) and $[w^{-1}]_{A_2} = [w]_{A_2}$ to obtain

$$\fint_{4B_0} w(x)^{-1}dx \lesssim [w]_{A_2}w^{-1}(B(0,1))|4B_0|.$$

So all together we obtain that, for some constant $C_K$,

$$\begin{aligned} II &\lesssim [w]_{A_2}^2|4B_0|^{1+\frac{1}{r}2}R_1^{n(q-2)}w(B(0,1))w^{-1}(B(0,1))|h|^{1/r_2'} \\ &\lesssim [w]_{A_2}^3|4B_0|^2R_1^{d(q-2)}|h|^{C_K}, \end{aligned}$$

which again tends to zero uniformly in $[w]_{A_2} \leq K$ as $|h| \to 0$.

**Estimate for** (14):

We have, as before, that since $q < 2$,

$$\int_{|x|>R} \frac{w(x)}{|x|^{2d}} dx = \sum_{j=0}^{\infty} \int_{2^j R<|x|\le 2^{j+1}R} \frac{w(x)}{|x|^{2d}} dx \lesssim [w]_{A_2} w(B(0,1)) R^{d(q-2)}.$$

Also, for the first factor in (14) we use again (17),

$$\int_{|x|<R_0} w(x)^{-1} dx \lesssim [w]_{A_2} R_0^{qd} w^{-1}(B(0,1)).$$

So combining all the estimates, (14) is controlled up to a constant by

$$[w]_{A_2}^3 R^{d(q-2)} R_0^{qd}$$

which tends to zero uniformly on $[w]_{A_2} \le K$ as $R \to \infty$. □

4.2. **Pseudodifferential operators.** The first example of non-trivial compact pseudodifferential operators on weighted Lebesgue spaces that we are aware of was obtained in [5]. The authors combined conditions on the symbol considered by Cordes [7] to produce compact operators on $L^2$ with additional conditions that provide a priori boundedness in $L^p$ for $p \neq 2$. More precisely the following was obtained in [5].

**Theorem 4.2.** *Let*

$$T_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\xi) \widehat{f}(\xi) e^{ix\xi}\, d\xi$$

*be a pseudodifferential operator with symbol $\sigma$ satisfying the conditions*

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \lesssim C_{\alpha,\beta}(x,\xi)(1+|\xi|)^{-|\beta|}, \tag{18}$$

*for all $|\alpha|, |\beta|$, where $C_{\alpha,\beta}$ is a bounded function which tends to zero as $|x|^2 + |\xi|^2 \to \infty$. Then $T_\sigma$ is compact in $L^p(w)$ for all $1 < p < \infty$ and all $w \in A_p$.*

Although the authors stated the conclusion as $T_\sigma$ being compact, it is actually uniformly compact with respect to $A_2$. In fact, the proof of the theorem involves the construction of a sequence of compact Calderón-Zygmund operators $\{T_j\}_j$ converging to $T_\sigma$ in the "Calderón-Zygmund norm" defined by

$$\|T\|_{CZO} = \|T\|_{L^2} + C_K,$$

where $C_K$ is the best constant in (12). Moreover the $T_j$ were proved to be uniformly compact with respect to $A_2$ and hence by compact extrapolation uniformly compact with respect to all $A_p$, $1 < p < \infty$.

However, it was also shown using the Fefferman-Stein inequality, see [5, Remark 2.2 ], that

$$\int_{\mathbb{R}^d} |(T_\sigma - T_j)(f)(x)|^p w(x)\,dx \lesssim \|T_\sigma - T_j\|_{CZO} \int_{\mathbb{R}^d} M(f)(x)^p w(x)\,dx.$$

It follows then than $T_j \to T_\sigma$ uniformly with respect to $A_p$. By Remark 3.6 (analogously applied to $L^p$ instead of $L^pL^q$) it follows that $T_\sigma$ is uniformly compact in $L^p(w)$ with respect to $A_p$ for all $1 < p < \infty$.

As a corollary of Theorem 3.8 we then obtain from Theorem 4.2 the following result.

**Theorem 4.3.** *$T_\sigma$ is uniformly in compact on $L^p_uL^q_v$ for all $1 < p, q < \infty$, $u \in A_p$, and $v \in A_q$.*

## References

[1] A. Benedek and R. Panzone *The space $L^p$ with mixed norm*, Duke Math. J. **28** (1961), 301–324.

[2] Á. Bényi, J. M. Martell, K. Moen E. Stachura, and R.H. Torres, *Boundedness results for commutators with BMO functions via weighted estimates: a comprehensive approach*, Math. Ann. **376** (2020), 853–871.

[3] Á. Bényi and R.H. Torres, *An update on the compactness of bilinear commutators*, The Mathematical Heritage of Guido Weiss, E. Hernández et al. (eds.) Applied and Numerical Harmonic Analysis, Springer Nature Switzerland (2025), 83–100.

[4] M. Cao, A. Olivo, and K. Yabuta *Extrapolation for multilinear compact operators and applications*, Trans. Amer. Math. Soc. **375** (2022), no. 7, 5011–5070.

[5] M.J. Carro, J. Soria, and R.H. Torres *Extrapolation of compact operators with applications to pseudodifferential operators*, Revista Mat. Argentina **66** (2023), no. 1, 177–186.

[6] A. Clop and V. Cruz, *Weighted estimates for Beltrami equations*, Ann. Acad. Sc. Fennica Mat. **38** (2013), 91–113.

[7] H.O. Cordes, *On compactness of commutators of multiplications and convolutions, and boundedness of pseudodifferential operators*, J. Funct. Anal. **18** (1975), 115–131.

[8] D. Cruz-Uribe, J.M. Martell y C. Pérez, *Extrapolation results for $A_\infty$ weights and applications*, J. Funct. Anal. , **213** (2004), 412–439.

[9] D. Cruz-Uribe, J.M. Martell and C. Pérez, *Weights, Extrapolation and the Theory of Rubio de Francia,* Series: Operator Theory: Advances and Applications, Vol. 215, Birkauser, Basel (2011).

[10] G.P. Curbera, J. García-Cuerva, J.M. Martell and C. Pérez, *Extrapolation with weights, Rearrangement Invariant Function Spaces, modular inequalities and applications to Singular Integrals,* Adv. in Math. **203** (2006), 256-318.

[11] J. Duoandikoetxea, *Extrapolation of weights revisited: new proofs and sharp bounds*, J. Funct. Anal. **260** (2011), no. 6, 1886–1901.

[12] C. Fefferman and E.M. Stein, *$H^p$ spaces of several variables*, Acta Math. **129** (1972), no. 3-4, 137–193.

[13] H. Hanche-Olsen, H., and H. Holden, *The Kolmogorov–Riesz compactness theorem*, Expo. Math. **28:4** (2010), 385–394.

[14] T. Hytönen and C. Pérez, *Sharp weighted bounds involving $A_\infty$*, Anal. PDE **6** (2013), no. 4, 777–818.

[15] T. Hytönen, C. Pérez and E. Rela, *Sharp Reverse Hölder property for $A_\infty$ weights on spaces of homogeneous type*, J. Funct. Anal. **263** (2012), 3883–3899.

[16] T. Hytönen and S. Lappas, *Extrapolation of compactness on weighted spaces*, Rev. Mat. Iberoam. **39** (2023), no. 1, 91–122.

[17] J. L. Journé, *Calderón–Zygmund operators, pseudo–differential operators and the Cauchy integral of Calderón*, Lecture Notes in Math. **994** (1983), Springer Verlag.

[18] D.S. Kurtz, *Classical operators on mixed-normed spaces with product weights*, Rocky Mountain J. **37** (2007), no. 1, 269–283.

[19] J.M. Martell, C. Pérez and R. Trujillo-Gonzalez, *Lack of natural weighted estimates for some singular integral operators*, Trans. Amer. Math. Soc. **357** (2005), 385-396.

[20] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.

[21] C. Pérez, *Sharp estimates for commutators of singular integrals via iterations of the Hardy-Littlewood maximal function*, J. Fourier Anal. Appl. **3**, (1997), no. 6, 743–756.

[22] J. L. Rubio de Francia, *Factorization theory and $A_p$ weights*, Amer. J. Math. **106** (1984), no. 3, 533–547.

[23] Q. Xue, K. Yabuta, and J. Yan, *Weighted Fréchet-Kolmogorov theorem and compactness of vector-valued multilinear operators*, J. Geom. Anal. **31** (2021), no. 10, 9891–9914.

María J. Carro, Department of Analysis and Applied Mathematics, Complutense University of Madrid, 28040 Madrid, Spain.
*Email address*: `mjcarro@ucm.es`

Carlos Pérez, Department of Mathematics University of the Basque Country/Euskal Herriko Unibertsitatea, 48080 Bilbao Spain.
*Email address*: `cperez@bcamath.org`

Rodolfo H. Torres, Department of Mathematics, University of California, Riverside, CA 92521, USA.
*Email address*: `rodolfo.h.torres@ucr.edu`