THE ZAREMBA PROBLEM IN TWO-DIMENSIONAL LIPSCHITZ GRAPH DOMAINS

MARÍA JESÚS CARRO, TERESA LUQUE, AND VIRGINIA NAIBO

ABSTRACT. We study the Zaremba problem, or mixed problem associated to the Laplace operator, in two-dimensional Lipschitz graph domains with mixed Dirichlet and Neumann boundary data in Lebesgue and Lorentz spaces. We obtain an explicit value r such that the Zaremba problem is solvable in L^p for $1 < p < r$ and in the Lorentz space $L^{r,1}$. Applications include those where the domain is a cone with vertex at the origin and, more generally, a Schwarz–Christoffel domain. The techniques employed are based on results of the Zaremba problem in the upper-half plane, the use of conformal maps and the theory of solutions to the Neumann problem. For the case when the domain is the upper-half plane, we work in the weighted setting, establishing conditions on the weights for the existence of solutions and estimates for the non-tangential maximal function of the gradient of the solution. In particular, in the L^2-unweighted case, where known examples show that the gradient of the solution may fail to be squared-integrable, we prove restricted weak-type estimates.

1. INTRODUCTION AND MAIN RESULTS

Given a Lipschitz domain $\Omega \subset \mathbb{R}^n$, consider the following mixed problem for Laplace's equation, or Zaremba problem in Ω:

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = f_D & \text{on } D, \\ \nabla v \cdot \mathbf{n} = f_N & \text{on } N, \end{cases} \tag{1.1}$$

where Δ is the Laplace operator in $\mathbb{R}^n$, $\mathbf{n}$ is the outward unit normal vector to $\partial\Omega$, and D and N are disjoint open subsets of $\partial\Omega$ such that $\partial N = \partial D$ and $\overline{N} \cup \overline{D} = \partial\Omega$.

Boundary value problems satisfying mixed Dirichlet-Neumann boundary conditions appear naturally associated with different partial differential equations. They were studied for the first time by Zaremba [26, 27] for Laplace's equation. Other physically interesting mixed boundary value problems extensively studied in many different settings include those associated with Maxwell equations, Lamé systems or certain elliptic equations; see, for instance, [2, 5, 11, 16, 17, 19, 22] and references therein.

In the context of Lipschitz domains, the study of the regularity of solutions of (1.1) was motivated by a question of C. Kenig in [14, p.120, problem 3.2.15] that was first partially solved by R. Brown and J. Sykes [3, 23, 24] for a restricted class of Lipschitz graph domains known as creased domains (roughly speaking, D and N meet at an angle which is strictly less than π). In particular, they obtained existence and uniqueness of solution and non-tangential maximal function estimates for the gradient of the solution in $L^p(\partial\Omega)$ for $1 < p \leq 2$ when f_N

Date: July 20, 2023.

2010 *Mathematics Subject Classification.* Primary: 35J25; secondary: 35J05, 46E30.

Key words and phrases. Mixed problem, Zaremba problem, Lipschitz graph domain, Lebesgue and Lorentz spaces, Muckenhoupt weights, Hilbert transform.

First and second authors partially supported by grants PID2020-113048GB-I00 funded by MCIN/AEI/10.13039/501100011033, CEX2019-000904-S funded by MCIN/AEI/ 10.13039/501100011033 and Grupo UCM-970966 (Spain). Third author partially supported by the NSF under grant DMS 2154113 and the Simons Foundation under grant 705953 (USA).

and the derivatives of f_D are in $L^p(\partial\Omega)$. Subsequenly, I. Mitrea and M. Mitrea [18] investigated the mixed problem for the Laplacian in bounded creased domains in $\mathbb{R}^n$ for $n \geq 3$ when the size and smoothness of the data and the solutions are measured on Sobolev-Besov scales. Under the same assumptions on the boundary data as in [24], Brown, Capogna and Lanzani [15] studied the Zaremba problem in two-dimensional Lipschitz graph domains (i.e. Ω is the upper part of the graph of a real-valued Lipschitz function) with Lipschitz constant less than one, establishing existence and uniqueness of solutions and estimates in $L^p(\partial\Omega)$ for the non-tangential maximal function of the gradient of the solution in the range $1 < p < p_0$, for some $p_0 > 1$. In order to achieve such results, the work [15] also includes the study of the mixed problem with data in weighted L^2 and weighted Hardy space H^1 with power weights. Analogous results for bounded Lipschitz domains in $\mathbb{R}^n$ with $n \geq 2$ were obtained by Brown and Ott [20, 21] under the assumption that the boundary between D and N (relative to $\partial\Omega$) is locally given by a Lipschitz graph. This work was extended under more general conditions on the decomposition of the boundary by Brown, Ott and Taylor [25]. By considering a more restrictive class of bounded domains than those in [20], Croyle and Brown [4] were able to give an explicit range of exponents p for which (1.1) can be solved with data in $L^p(\partial\Omega)$; this range turns out to be sharp in two dimensions. When the domain Ω is an infinite sector in two dimensions with N and D corresponding to the rays of the sector, Awala, Mitrea and Ott [1] established well-posedness of the Zaremba problem by showing sharp invertibility properties for an associated singular integral operator.

In this paper, we study the mixed problem (1.1) when Ω is a Lipschitz graph domain in the plane with data associated to Lebesgue and Lorentz spaces. Before presenting our results, we describe with more details the settings in which we work.

Let Λ be a curve in the complex plane given parametrically by $x + i\gamma(x)$ for $x \in \mathbb{R}$, where γ is a real-valued Lipschitz function with constant L, and consider the Lipschitz graph domain

$$\Omega = \{z \in \mathbb{C} : \mathrm{Im}(z) > \gamma(\mathrm{Re}(z))\}; \tag{1.2}$$

note that $\Lambda = \partial\Omega$. Given $0 < \alpha < \arctan(1/L)$, define the non-tangential maximal operator $\mathcal{M}_\alpha$ as

$$\mathcal{M}_\alpha(F)(\xi) = \sup_{z \in \Gamma_\alpha(\xi)} |F(z)|, \quad \xi \in \partial\Omega,$$

where F is a complex-valued function defined in Ω and

$$\Gamma_\alpha(\xi) = \{z \in \mathbb{C} : \mathrm{Im}(z) > \mathrm{Im}(\xi) \text{ and } |\mathrm{Re}(\xi) - \mathrm{Re}(z)| < \tan(\alpha)|\mathrm{Im}(z) - \mathrm{Im}(\xi)|\}.$$

We will study the mixed problem (1.1) with Ω as in (1.2) and with the sets D and N given by

$$D = \{(x, \gamma(x)) \in \partial\Omega : x > 0\} \text{ and } N = \{(x, \gamma(x)) \in \partial\Omega : x < 0\}. \tag{1.3}$$

The notations f_D and f_N will be used for functions defined on $\partial\Omega$ that are zero outside D and N, respectively. We have the following definitions regarding solutions of (1.1):

Definition 1.1. For $f_D \in L^1_{\mathrm{loc}}(D)$ and $f_N \in L^1_{\mathrm{loc}}(N)$, a function v defined on Ω is *a solution of the mixed problem* (1.1) *in* Ω *with data* f_D *and* f_N if v is harmonic in Ω and the equalities $v = f_D$ on D and $\nabla v \cdot \mathbf{n} = f_N$ on N hold almost everywhere in the sense of non-tangential convergence. The latter means that there exists $0 < \alpha < \arctan(1/L)$ such that $\lim_{z\in\Gamma_\alpha(\xi), z\to\xi} v(z) = f_D(\xi)$ for almost every $\xi \in D$ and $\lim_{z\in\Gamma_\alpha(\xi), z\to\xi} \nabla v(z) \cdot \mathbf{n}(\xi) = f_N(\xi)$ for almost every $\xi \in N$, with respect to arc-length.

Definition 1.2. Given $f : \partial\Omega \longrightarrow \mathbb{R}$ and $\xi = \nu(x) = x + i\gamma(x) \in \partial\Omega$, we define $f'(\xi)$ by the condition

$$f'(\xi)(1 + i\gamma'(x)) = (f \circ \nu)'(x),$$

whenever $f \circ \nu$ and γ are differentiable at x. We say that f is differentiable (in a weak sense) over the curve $\partial\Omega$ if this holds for almost every $x\in\mathbb{R}$.

Definition 1.3. If X is a Banach space of measurable functions defined on $\partial\Omega$, we say that *the mixed problem* (1.1) *in* Ω *is solvable in* X if there exist a Banach space Y of measurable functions defined on $\partial\Omega$ and $0 < \alpha < \arctan(1/L)$ such that for every $f_D \in L^1_{\text{loc}}(D)$ with $f_D' \in X$ and every $f_N \in L^1_{\text{loc}}(N)$ with $f_N \in X$ there exists a solution v of the Neumann problem in Ω with data f_D and f_N and

$$\|\mathcal{M}_\alpha(\nabla v)\|_Y \lesssim \|f_D'\|_X + \|f_N\|_X,$$

where the implicit constant is independent of f_D and f_N and, $\|\cdot\|_X$ and $\|\cdot\|_Y$ denote the norms in X and Y, respectively.

Our first main result, Theorem 1.4 below, deals with the mixed problem (1.1) in a general Lipschitz graph domain Ω in the plane. It is motivated by the results in [15], where, as mentioned in the introduction, it is proved that there exists $p_0 > 1$ such that the mixed problem in a two-dimensional Lipschitz graph domain with Lipschitz constant less than one is solvable in $L^p(\partial\Omega)$ for $1 < p < p_0$.

Before stating Theorem 1.4, we briefly describe the notation used in the statement, directing the reader to Sections 2 and 4 for more details and references. Consider a conformal map $\Phi : \mathbb{R}^2_+ \to \Omega$; we then have that Φ extends as a homeomorphism from $\overline{\mathbb{R}^2_+}$ onto $\overline{\Omega}$, and $\Phi'(x)$ exists for almost every $x \in \mathbb{R}$ and is locally integrable. For $1 < p < \infty$, the notation A_p stands for the family of weights w in $\mathbb{R}$ that characterizes the boundedness of the Hardy-Littlewood maximal operator on the weighted Lebesgue spaces $L^p(w)$ (Muckenhoupt weights). The class $A_p^{\mathcal{R}}$ is the set of weights w in $\mathbb{R}$ that characterizes the boundedness of the Hardy-Littlewood maximal operator from the weighted Lorentz space $L^{p,1}(w)$ to the weighted Lorentz space $L^{p,\infty}(w)$; we have that $A_p \subsetneq A_p^{\mathcal{R}}$. As usual, p' denotes the conjugate exponent of p, i.e. $1/p + 1/p' = 1$. We define $1 \le r_\Phi \le \infty$ such that its conjugate exponent is given by

$$r_\Phi' = \inf\{q \in (1,\infty) : |x\,\Phi'(x^2)| \in A_q \text{ and } |\Phi'(x)| \in A_q\}.$$

Denote by $L^{p,1}(\partial\Omega)$ and $L^{p,\infty}(\partial\Omega)$ the Lorentz spaces with respect to arc-length in $\partial\Omega$.

Theorem 1.4 (Solvability of the mixed problem (1.1) in a Lipschitz graph domain in the plane)**.** *Let* Ω *be as in* (1.2)*,* D *and* N *be as in* (1.3) *and* $\Phi : \mathbb{R}^2_+ \to \Omega$ *be a conformal map such that* $\Phi((-\infty,0)) = N$ *and* $\Phi((0,\infty)) = D$*. Then the mixed problem* (1.1) *is solvable in the following settings:*

(a) $X = Y = L^p(\partial\Omega)$ *with* $1 < p < r_\Phi$.

(b) If $r_\Phi < \infty$, $X = L^{r_\Phi,1}(\partial\Omega)$, $Y = L^{r_\Phi,\infty}(\partial\Omega)$ *and*

$$|x\,\Phi'(x^2)| \in A^{\mathcal{R}}_{r_\Phi'} \text{ and } |\Phi'(x)| \in A_{r_\Phi'}. \tag{1.4}$$

The techniques used in [15] are completely different from ours and do not give an explicit expression of p_0 as Theorem 1.4 does along with the solvability of the mixed problem at such endpoint in the setting of Lorentz spaces; as opposed to [15], we do not assumed that the Lipschitz constant of the boundary is less than one. The arguments in [15] are based on weighted L^2 estimates established through Rellich indentity, techniques by Dahlberg and Kenig [9] to obtain weighted estimates in Hardy spaces and interpolation. On the other hand, the methods of proof we use for Theorem 1.4 rely on the application of the solvability of the mixed problem (1.1) in the upper half-plane with data in weighted spaces (Theorem 1.5 below), flattening the boundary via conformal maps (see Figure 1), and results obtained in [6] on the solvability of the Neumann problem in Lipschitz graph domains in the plane.

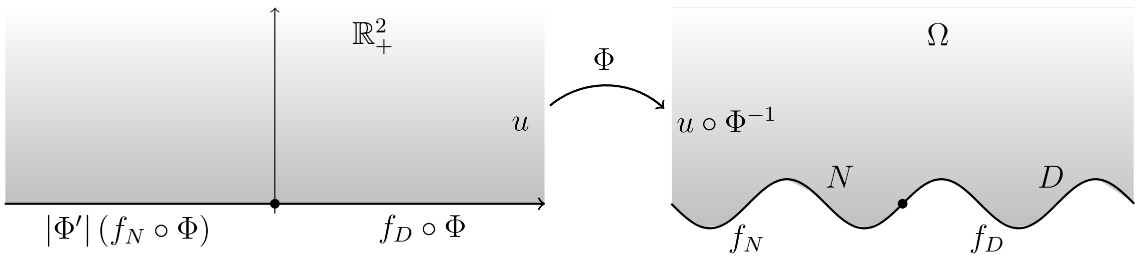

FIGURE 1. Flattening the boundary via a conformal map

For the statement of Theorem 1.5, given a weight w in $\mathbb{R}$, the space $\mathcal{L}^{p,1}(w)$ (respectively, $\mathcal{L}^{p,\infty}(w)$) consists of all measurable functions f defined on $\mathbb{R}$ such that $fw^{-1} \in L^{p,1}(w)$ (respectively, $L^{p,\infty}(w)$).

Theorem 1.5 (Solvability of the mixed problem (1.1) in the upper half-plane)**.** *The mixed problem (1.1) is solvable in the following settings for $\Omega = \mathbb{R}^2_+$, $N = (-\infty, 0)$ and $D = (0, \infty)$:*

(a) $X = Y = L^p(w)$, $1 < p < \infty$, $w \in A_p$ *and* $|x|w(x^2) \in A_p$,
(b) $X = Y = L^p(w)$, $1 < p < \infty$, $w \in A_p$ *and* $|x|^{1-p}w(x^2) \in A_p$,
(c) $X = L^{p,1}(w)$, $Y = L^{p,\infty}(w)$, $1 < p < \infty$, $w \in A_p$ *and* $|x|w(x^2) \in A_p^{\mathcal{R}}$,
(d) $X = \mathcal{L}^{p,1}(w)$, $Y = \mathcal{L}^{p,\infty}(w)$, $1 < p < \infty$, $w \in A_{p'}$ *and* $|x|w(x^2) \in A_{p'}^{\mathcal{R}}$.

The proof of Theorem 1.5 employs results from [6] on the solvability of the Neumann problem in $\mathbb{R}^2_+$, which allows to find explicit solutions for the mixed problem in the upper half-plane.

Simple examples (see, for instance, [15, p. 93] or [20, p. 1334]) show that the mixed problem (1.1) with $\Omega = \mathbb{R}^2_+$, $N = (-\infty, 0)$ and $D = (0, \infty)$ is not solvable in $X = L^2(\mathbb{R})$ with $Y = L^2(\mathbb{R})$. However, the following corollary of Theorem 1.5 gives that this problem is solvable in the Lorentz space setting with $X = L^{2,1}(\mathbb{R}) \subset L^2(\mathbb{R})$ and $Y = L^{2,\infty}(\mathbb{R}) \supset L^2(\mathbb{R})$.

Corollary 1.6. *The mixed problem (1.1) is solvable in the following settings for $\Omega = \mathbb{R}^2_+$, $N = (-\infty, 0)$ and $D = (0, \infty)$:*

(a) $X = Y = L^p(\mathbb{R})$ *if and only if* $1 < p < \infty$ *and* $p \neq 2$.
(b) $X = L^{2,1}(\mathbb{R})$ *and* $Y = L^{2,\infty}(\mathbb{R})$.

A different proof for the solvability stated in Part (a) of Corollary 1.6 is given in [1, Theorem 5] (see more details in Section 5.1) while the result in Part (b) of Corollary 1.6 is new.

The organization of the article is as follows. In Section 2, we present notation, definitions and preliminaries related to our results on the solvability of the Zaremba problem in the upper half-plane. The proofs of Theorem 1.5 and Corollary 1.6 are then presented in Section 3. The main content of Section 4 is the proof of Theorem 1.4 on the solvability of the Zaremba problem in a general two-dimensional Lipschitz graph domain. In Section 5 we present applications of Theorem 1.4 when the domain is a cone with vertex at the origin (Corollary 5.1) and, more generally, for Schwarz–Christoffel domains (Corollary 5.4).

2. Notation and preliminaries

In this section, we introduce notation and definitions associated to the results on the solvability of the Zaremba problem in the upper half-plane; we also state some lemmas that will be used in the proofs.

2.1. **Functions spaces and weights.** Consider a weight w on $\mathbb{R}$ (i.e. a non-negative locally integrable function defined in $\mathbb{R}$). For $1 \le p \le \infty$, the space $L^p(w)$ is the class of measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}} |f(x)|^p w(x)\, dx \right)^{\frac{1}{p}} < \infty,$$

with the corresponding changes for $p = \infty$. For $1 \le p < \infty$, we denote by $L^{p,1}(w)$ and $L^{p,\infty}(w)$ the Lorentz spaces defined as the classes of measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$\|f\|_{L^{p,1}(w)} = \int_0^\infty \left(\lambda_f^w(t)\right)^{\frac{1}{p}} dt < \infty \quad \text{and} \quad \|f\|_{L^{p,\infty}(w)} = \sup_{t>0} t \left(\lambda_f^w(t)\right)^{\frac{1}{p}} < \infty,$$

respectively, where $\lambda_f^w(t) = w(\{x \in \mathbb{R} : |f(x)| > t\})$ and $w(A) = \int_A w(x)dx$ for A a measurable subset of $\mathbb{R}$. For $1 < p < \infty$, we define the spaces $\mathcal{L}^{p,1}(w)$ and $\mathcal{L}^{p,\infty}(w)$ as the classes of measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that, respectively,

$$\|f\|_{\mathcal{L}^{p,1}(w)} = \|fw^{-1}\|_{L^{p,1}(w)} < \infty \quad \text{and} \quad \|f\|_{\mathcal{L}^{p,\infty}(w)} = \|fw^{-1}\|_{L^{p,\infty}(w)} < \infty.$$

When $w \equiv 1$, we use the notation $L^p(\mathbb{R})$ instead of $L^p(w)$, and similarly for the other spaces introduced above.

The Hilbert transform $\mathcal{H}$ is defined by

$$\mathcal{H}f(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|t|>\varepsilon} \frac{f(x-t)}{t}\, dt.$$

We will work with two classes of weights in $\mathbb{R}$: the Muckenhoupt class A_p and the larger class $A_p^{\mathcal{R}}$. If $1 < p < \infty$,

$$w \in A_p \quad \text{iff} \quad \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \|\chi_I\|_{L^p(w)} \|\chi_I w^{-1}\|_{L^{p'}(w)} < \infty, \tag{2.1}$$

$$w \in A_p^{\mathcal{R}} \quad \text{iff} \quad \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \|\chi_I\|_{L^{p,1}(w)} \|\chi_I w^{-1}\|_{L^{p',\infty}(w)} < \infty, \tag{2.2}$$

where the supremum is taken over all intervals contained in $\mathbb{R}$. We recall that, for $1 < p < \infty$, $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$, $A_p \subset A_q$ if $p < q$ and, if $w \in A_p$ with $p > 1$ then $w \in A_{p-\varepsilon}$ for some $\varepsilon > 0$. Moreover, it holds that

$$A_p \subsetneq A_p^{\mathcal{R}} \subsetneq \cap_{\varepsilon>0} A_{p+\varepsilon}. \tag{2.3}$$

2.2. **Boundedness results and identities for $\mathcal{H}$.** The following two statements summarize results regarding boundedness properties and identities for $\mathcal{H}$ in weighted spaces that will be used throughout the article.

Theorem 2.1. *Let* $1 < p < \infty$.
(a) $\mathcal{H}$ *is bounded from* $L^p(w)$ *to* $L^p(w)$ *if and only if* $w \in A_p$.
(b) $\mathcal{H}$ *is bounded from* $L^{p,\infty}(w)$ *to* $L^{p,\infty}(w)$ *and from* $L^{p,1}(w)$ *to* $L^{p,1}(w)$ *if* $w \in A_p$.
(c) $\mathcal{H}$ *is bounded from* $L^{p,1}(w)$ *to* $L^{p,\infty}(w)$ *if* $w \in A_p^{\mathcal{R}}$.

For Part (a), see Hunt–Muckenhoupt–Wheeden [12]; Part (b) follows from Part (a) by interpolation; for (c), see [8].

Lemma 2.2. *Let* $1 < p < \infty$.
(a) If $w \in A_{p'}^{\mathcal{R}}$, *then*

$$\left\| \frac{\mathcal{H}(wf)}{w} \right\|_{L^{p,\infty}(w)} \lesssim \|f\|_{L^{p,1}(w)}.$$

(b) If $w \in A_{p'}$, then

$$\left\| \frac{\mathcal{H}(wf)}{w} \right\|_{L^{p,1}(w)} \lesssim \|f\|_{L^{p,1}(w)}.$$

Proof. Proof of Part (a): Let $g \in L^{p',1}(w)$ be such that $\|g\|_{L^{p',1}(w)} \leq 1$. Using duality, the fact that the adjoint of $\mathcal{H}$ is $-\mathcal{H}$, and that $\mathcal{H}$ is bounded from $L^{p',1}(w)$ to $L^{p',\infty}(w)$ since $w \in A_{p'}^{\mathcal{R}}$, we have

$$\begin{aligned}\left| \int_{\mathbb{R}} \frac{\mathcal{H}(fw)(x)}{w(x)} g(x) w(x)\, dx \right| &= \left| \int_{\mathbb{R}} f(x) \mathcal{H}g(x) w(x)\, dx \right| \leq \|f\|_{L^{p,1}(w)} \|\mathcal{H}g\|_{L^{p',\infty}(w)} \\ &\lesssim \|f\|_{L^{p,1}(w)} \|g\|_{L^{p',1}(w)} \leq \|f\|_{L^{p,1}(w)}.\end{aligned}$$

Taking supremum over all such g gives the desired result.

Proof of Part (b): The proof is analogous to that of Part (a) using the fact that $\mathcal{H}$ is bounded from $L^{p',\infty}(w)$ to $L^{p',\infty}(w)$ for $w \in A_{p'}$. □

Lemma 2.3. *If $1 < p < \infty$ and f belongs to $L^{p,\infty}(w)$ with $w \in A_p$ or to $\mathcal{L}^{p,\infty}(w)$ with $w \in A_{p'}$ then*

$$-\int_{\mathbb{R}} \mathcal{H}\varphi(t)\, f(t)\, dt = \int_{\mathbb{R}} \mathcal{H}f(t)\, \varphi(t)\, dt$$

for all functions φ infinitely differentiable in $\mathbb{R}$ and with compact support.

Proof. The identity holds for functions in $L^2(\mathbb{R})$ and is obtained by density arguments for functions in $L^p(w)$ with $w \in A_p$. Since $L^{p,\infty}(w) \subset L^{p-\varepsilon}(w) + L^{p+\varepsilon}(w)$ for $\varepsilon > 0$ such that $w \in A_{p-\varepsilon} \cap A_{p+\varepsilon}$, the result then follows for functions in $L^{p,\infty}(w)$. If $w \in A_{p'}$, we have $\mathcal{L}^{p,\infty}(w) \subset L^{p-\varepsilon}(w^{1-(p-\varepsilon)}) + L^{p+\varepsilon}(w^{1-(p+\varepsilon)})$ for $\varepsilon > 0$ such that $w^{1-(p-\varepsilon)} \in A_{p-\varepsilon}$ and $w^{1-(p+\varepsilon)} \in A_{p+\varepsilon}$, from which the result holds for $\mathcal{L}^{p,\infty}(w)$. □

We next present an additional lemma for functions f on $\mathbb{R}$ related to the condition

$$\int_{\mathbb{R}} \frac{|f(x)|}{1+|x|}\, dx < \infty. \tag{2.4}$$

Lemma 2.4. *The condition (2.4) is satisfied by any f in the following function spaces:*

(a) $L^{p,1}(w)$ with $w \in A_p^{\mathcal{R}}$ and $1 \leq p < \infty$,
(b) $\mathcal{L}^{p,1}(w)$ with $w \in A_{p'}^{\mathcal{R}}$ and $1 < p < \infty$,
(c) $L^{p,\infty}(w)$ with $w \in A_p$ and $1 < p < \infty$,
(d) $\mathcal{L}^{p,\infty}(w)$ with $w \in A_{p'}$ and $1 < p < \infty$.

Proof.

Proofs of (a) and (b): See [6, Proof of Lemma 3.1].

Proofs of (c) and (d): The case $f \in L^p(w)$ with $w \in A_p$ can be seen in [6, Proof of Lemma 3.1]. For $f \in L^{p,\infty}(w)$ with $w \in A_p$ or $f \in \mathcal{L}^{p,\infty}(w)$ with $w \in A_{p'}$, the result follows by considering the same space decompositions as in the proof of Lemma 2.3. □

3. Solvability of the Zaremba problem in the upper half-plane

The main goal of this section is to present the poofs of Theorem 1.5 about the solvability of the Zaremba problem in the upper half-plane and its Corollary 1.6. Among other tools, we will use results regarding the solution of the Neumann problem in the upper half-plane, which we review in Section 3.1. The proofs of Theorem 1.5 and Corollary 1.6 are contained in Sections 3.2 and 3.3, respectively.

3.1. **The Neumann problem in the upper half-plane.** Consider the classical Neumann boundary value problem in $\mathbb{R}^2_+$:

$$\Delta u = 0 \text{ on } \mathbb{R}^2_+ \qquad \text{and} \qquad \nabla u \cdot (0,-1) = f \text{ on } \mathbb{R}, \tag{3.1}$$

where the equality $\nabla u \cdot (0,-1) = f$ is in the sense of non-tangential convergence.

For $f : \mathbb{R} \to \mathbb{C}$ satisfying (2.4) define

$$u_f(x,y) := -\frac{1}{\pi}\int_{\mathbb{R}} \log\left(\frac{\sqrt{(x-t)^2+y^2}}{1+|t|}\right) f(t)\,dt, \quad (x,y)\in\mathbb{R}^2_+. \tag{3.2}$$

We note that the integral on the right-hand side of (3.2) is absolutely convergent for all f satisfying (2.4); in particular, u_f is well defined and absolutely convergent for all f in any of the function spaces of Lemma 2.4.

As proved in [6, Theorem 1.3], u_f is a solution of the Neumann problem (3.1) in the upper half-plane; more precisely, the following result holds.

Theorem 3.1 (Solvability of the Neumann problem in the upper half-plane; Theorem 1.3 in [6])**.** *Given $1 < p < \infty$, consider $X = Y = L^p(w)$ with $w \in A_p$ or $X = L^{p,1}(w)$ and $Y = L^{p,\infty}(w)$ with $w \in A_p^{\mathcal{R}}$ or $X = \mathcal{L}^{p,1}(w)$ and $Y = \mathcal{L}^{p,\infty}(w)$ with $w \in A_{p'}^{\mathcal{R}}$. If $f \in X$, then u_f is harmonic in $\mathbb{R}^2_+$, $\nabla u \cdot (0,-1) = f$ on $\mathbb{R}$ in the sense of non-tangential convergence and*

$$\|\mathcal{M}_\alpha(\nabla u_f)\|_Y \lesssim \|f\|_X,$$

where the implicit constant is independent of f and $0 < \alpha < \pi/2$.

Remark 3.2. We note that Theorem 3.1 and interpolation give that the results of Theorem 3.1 also hold for $X = Y = L^{p,\infty}(w)$ with $w \in A_p$ and $X = Y = \mathcal{L}^{p,\infty}(w)$ with $w \in A_{p'}$, where $1 < p < \infty$.

The next lemma deals with the boundary values of u_f in terms of non-tangential convergence.

Lemma 3.3. *Let $f : \mathbb{R} \to \mathbb{C}$ satisfy* (2.4)*.*

(a) The function $x \to \int_{\mathbb{R}} \left|\log\left(\frac{|x-t|}{1+|t|}\right) f(t)\right| dt$ is locally integrable in $\mathbb{R}$. Moreover, the function given by

$$\mathcal{B}f(x) = -\frac{1}{\pi}\int_{\mathbb{R}} \log\left(\frac{|x-t|}{1+|t|}\right) f(t)\,dt \tag{3.3}$$

satisfies $(\mathcal{B}f)' = \mathcal{H}f$ in the sense of distributions if, for some $1 < p < \infty$, f belongs to $L^{p,\infty}(w)$ with $w \in A_p$ or to $\mathcal{L}^{p,\infty}(w)$ with $w \in A_{p'}$.

(b) Let $0 < \alpha < \pi/2$. If $\mathcal{M}_\alpha(\nabla u_f)$ is finite almost everywhere in $\mathbb{R}$, then $u_f = \mathcal{B}f$ almost everywhere on $\mathbb{R}$ in the sense of non-tangential convergence. In particular, this holds if, for some $1 < p < \infty$, f belongs to $L^{p,\infty}(w)$ with $w \in A_p$ or to $\mathcal{L}^{p,\infty}(w)$ with $w \in A_{p'}$.

Proof. Proof of Part (a): The fact that the function $x \to \int_{\mathbb{R}} \left|\log\left(\frac{|x-t|}{1+|t|}\right) f(t)\right| dt$ is locally integrable in $\mathbb{R}$ can be proved with standard techniques and is left to the reader.

We next show that $(\mathcal{B}f)' = \mathcal{H}f$ in the sense of distributions if f belongs to one of the function spaces stated in the lemma. Let φ be an infinitely differentiable function defined in $\mathbb{R}$ and with compact support. We will show that $\langle(\mathcal{B}f)',\varphi\rangle = \langle\mathcal{H}f,\varphi\rangle$, where $\langle T,\varphi\rangle$ denotes the action of a distribution T on the test function φ. Assume $\operatorname{supp}(\varphi) \subset (a,b)$ for some $-\infty <$

$a < b < \infty$, then

$$\begin{aligned}\langle(\mathcal{B}f)', \varphi\rangle &= -\int_{\mathbb{R}} \mathcal{B}f(x)\varphi'(x)\,dx = \frac{1}{\pi}\int_a^b \left(\int_{\mathbb{R}} \log\left(\tfrac{|x-t|}{1+|t|}\right) f(t)\,dt\right)\varphi'(x)\,dx \\ &= \frac{1}{\pi}\int_{\mathbb{R}} \left(\int_a^b \log\left(\tfrac{|x-t|}{1+|t|}\right)\varphi'(x)\,dx\right) f(t)\,dt = \frac{1}{\pi}\int_{\mathbb{R}}\left(\int_a^b \log|x-t|\,\varphi'(x)\,dx\right) f(t)\,dt \\ &= -\int_{\mathbb{R}} \mathcal{H}\varphi(t)\,f(t)\,dt = \int_{\mathbb{R}} \mathcal{H}f(t)\,\varphi(t)\,dt = \langle \mathcal{H}f, \varphi\rangle,\end{aligned}$$

where in the third equality we have used Fubini since $\int_{\mathbb{R}} \left|\log\left(\frac{|x-t|}{1+|t|}\right) f(t)\right|\,dt$ is locally integrable and φ' is bounded, in the fourth equality we have used that $\int_a^b \varphi'(x)\,dx = 0$, in the fifth equality we have used that $(\log|x|)' = \text{p.v.}\frac{1}{x}$ and in the second to last equality we have used Lemma 2.3.

Proof of Part (b): Fix $x_0 \in \mathbb{R}$ such that $\mathcal{M}_\alpha(\nabla u_f)(x_0) < \infty$ and the integral defining $\mathcal{B}f(x_0)$ is absolutely convergent; for $(x, y) \in \Gamma_\alpha(x_0)$ we then have

$$\begin{aligned}|u_f(x,y) - \mathcal{B}f(x_0)| &\le |u_f(x,y) - u_f(x_0,y)| + |u_f(x_0,y) - \mathcal{B}f(x_0)| \\ &\le \left|\int_{x_0}^x \left|\frac{\partial u_f}{\partial t}(t,y)\right|\,dt\right| + |u_f(x_0,y) - \mathcal{B}f(x_0)| \\ &\le \mathcal{M}_\alpha(\nabla u_f)(x_0)\,|x - x_0| + |u_f(x_0,y) - \mathcal{B}f(x_0)|\,.\end{aligned}$$

The desired convergence will follow if we prove that $|u_f(x_0,y) - \mathcal{B}f(x_0)|$ as $y \to 0$. The latter is a consequence of monotone convergence since $\log\left(\frac{\sqrt{(x_0-t)^2+y^2}}{1+|t|}\right)$ decreases to $\log\left(\frac{|x_0-t|}{1+|t|}\right)$ as $y \to 0$, and the integrals $\int_{\mathbb{R}}\left|\log\left(\frac{\sqrt{(x_0-t)^2+y^2}}{1+|t|}\right)\right| g(t)\,dt$ and $\int_{\mathbb{R}}\left|\log\left(\frac{|x_0-t|}{1+|t|}\right)\right| g(t)\,dt$ are finite for $y > 0$, $g = f^+$ and $g = f^-$.

Finally, we note that Remark 3.2 implies that $\mathcal{M}_\alpha(\nabla u_f)$ is finite almost everywhere in $\mathbb{R}$ if, for some $1 < p < \infty$, f belongs to $L^{p,\infty}(w)$ with $w \in A_p$ or $\mathcal{L}^{p,\infty}(w)$ with $w \in A_{p'}$.

□

3.2. **Proof of Theorem 1.5.** Let f_D and f_N be zero in the complements of D and N, respectively.

Proof of Part (a): Let f_D be such that $f_D' \in L^p(w)$ and $f_N \in L^p(w)$. Consider h_1 given by

$$h_1 = f_N + p_D,$$

where p_D is zero outside D and is to be determined so that $h_1 \in L^p(w)$ and $(\mathcal{B}h_1)' = f_D'$ on D in the sense of distributions, with $\mathcal{B}$ defined as in (3.3). Once h_1 is found, we will show that there exists a constant C such that $u_{h_1} + C$, with u_{h_1} as in (3.2), is a solution of the mixed problem (1.1) in the upper half-plane and that

$$\|\mathcal{M}_\alpha(\nabla(u_{h_1} + C))\|_{L^p(w)} \lesssim \|f_D'\|_{L^p(w)} + \|f_N\|_{L^p(w)}, \tag{3.4}$$

where $0 < \alpha < \pi/2$ and the implicit constant is independent of f_D and f_N.

By Lemma 3.3, we must have

$$f_D'(x) = \mathcal{H}h_1(x) \quad \text{for a.e. } x > 0. \tag{3.5}$$

Reasoning heuristically, we then obtain

$$\begin{aligned} f_D'(x^2) - \mathcal{H}(f_N)(x^2) = \mathcal{H}(p_D)(x^2) &= \frac{1}{\pi}\int_0^\infty \frac{p_D(t)}{x^2-t}\,dt \\ &= \frac{1}{\pi}\int_0^\infty p_D(t^2)\frac{2t}{x^2-t^2}\,dt = \frac{1}{\pi}\int_0^\infty p_D(t^2)\left(\frac{1}{x-t}-\frac{1}{x+t}\right)dt \\ &= \frac{1}{\pi}\left(\int_0^\infty \frac{p_D(t^2)}{x-t}\,dt - \int_{-\infty}^0 \frac{p_D(t^2)}{x-t}\,dt\right) = \mathcal{H}q_1(x), \end{aligned}$$

where

$$q_1(x) = \begin{cases} p_D(x^2) & \text{for } x>0, \\ -p_D(x^2) & \text{for } x<0. \end{cases}$$

Since $\mathcal{H}^2 = -\mathrm{Id}$, then $q_1(x) = -\mathcal{H}\left(f_D'(t^2) - \mathcal{H}(f_N)(t^2)\right)(x)$ for $x \in \mathbb{R}$, which leads to

$$p_D(x) = -\mathcal{H}\left(f_D'(t^2) - \mathcal{H}(f_N)(t^2)\right)(\sqrt{x}) \quad \text{for } x>0. \tag{3.6}$$

We note that $f_D'(x^2) - \mathcal{H}(f_N)(x^2) \in L^p(|x|w(x^2))$ (see line (3.9) below); since $|x|w(x^2) \in A_p$, q_1 and $\mathcal{H}q_1$ are well-defined and the above heuristic reasoning is justified. We then obtain that h_1 is given by

$$h_1(x) = \begin{cases} f_N(x) & \text{for } x<0, \\ -\mathcal{H}\left(f_D'(t^2) - \mathcal{H}(f_N)(t^2)\right)(\sqrt{x}) & \text{for } x>0. \end{cases} \tag{3.7}$$

We next prove that, under the assumptions on w, $h_1 \in L^p(w)$ and

$$\|h_1\|_{L^p(w)} \lesssim \|f_D'\|_{L^p(w)} + \|f_N\|_{L^p(w)}. \tag{3.8}$$

Indeed, we have

$$\|h_1\|_{L^p(w)}^p = \|f_N\|_{L^p(w)}^p + \int_0^\infty |h_1(x)|^p w(x)\,dx,$$

and using that $|x|w(x^2) \in A_p$ and $w \in A_p$, it follows that

$$\begin{aligned} \int_0^\infty |h_1(x)|^p w(x)\,dx &= \int_0^\infty |\mathcal{H}\left(f_D'(t^2) - \mathcal{H}(f_N)(t^2)\right)(\sqrt{x})|^p w(x)\,dx \\ &= 2\int_0^\infty |\mathcal{H}\left(f_D'(t^2) - \mathcal{H}(f_N)(t^2)\right)(x)|^p |x| w(x^2)\,dx \\ &\lesssim \int_{-\infty}^\infty |f_D'(x^2) - \mathcal{H}(f_N)(x^2)|^p |x| w(x^2)\,dx \qquad (3.9) \\ &= \int_0^\infty |f_D'(x) - \mathcal{H}(f_N)(x)|^p w(x)\,dx \lesssim \|f_D'\|_{L^p(w)}^p + \|f_N\|_{L^p(w)}^p, \end{aligned}$$

from which (3.8) is obtained.

We now show that $(\mathcal{B}h_1)' = f_D'$ on D in the sense of distributions. We have $\mathcal{H}h_1 = f_D'$ almost everywhere on D by (3.5) and therefore $\mathcal{H}h_1 = f_D'$ on D in the sense of distributions; moreover, $\mathcal{H}h_1 = (\mathcal{B}h_1)'$ on $\mathbb{R}$ in the sense of distributions by Part (a) of Lemma 3.3, which implies that $\mathcal{H}h_1 = (\mathcal{B}h_1)'$ on D in the sense of distributions. As a consequence, $(\mathcal{B}h_1)' = f_D'$ on D in the sense of distributions.

We next obtain a solution of the mixed problem (1.1) in the upper half-plane. Since $(\mathcal{B}h_1)' = f_D'$ on D in the sense of distributions, there exists a constant C such that $f_D = \mathcal{B}h_1 + C$ almost everywhere on D. By Theorem 3.1, $u_{h_1} + C$ is harmonic in $\mathbb{R}^2_+$ and $\nabla(u_{h_1}+C)\cdot(0,-1) = h_1$ in the sense of non-tangential convergence; in particular, the latter implies that we have $\nabla(u_{h_1}+C)\cdot(0,-1) = f_N$ on N in the sense of non-tangential convergence. Moreover,

$$\|\mathcal{M}_\alpha(\nabla(u_{h_1}+C))\|_{L^p(w)} \lesssim \|h_1\|_{L^p(w)};$$

this estimate and (3.8), lead to (3.4). Finally, Lemma 3.3 gives that $u_{h_1} = \mathcal{B}h_1$ almost everywhere on $\mathbb{R}$ in the sense of non-tangential convergence and therefore $u_{h_1} + C = f_D$ almost everywhere on D in the sense of non-tangential convergence.

Proof of Part (b): Let f_D satisfy $f_D' \in L^p(w)$ and $f_N \in L^p(w)$. We proceed as in Part (a) to find another function h_2 such that

$$h_2 = f_N + p_D$$

where now p_D is zero outside D and is to be determined so that $h_2 \in L^p(w)$ and $(\mathcal{B}h_2)' = f_D'$ on D in the sense of distributions. Repeating the computations in Part (a) up to the identity

$$f_D'(x^2) - \mathcal{H}(f_N)(x^2) = \frac{1}{\pi}\int_0^\infty p_D(t^2)\frac{2t}{x^2-t^2}\,dt,$$

we continue heuristically as follows:

$$\begin{aligned} f_D'(x^2) - \mathcal{H}(f_N)(x^2) &= \frac{1}{\pi x}\int_0^\infty p_D(t^2)\,t\left(\frac{1}{x-t}+\frac{1}{x+t}\right)dt \\ &= \frac{1}{\pi x}\left(\int_0^\infty \frac{p_D(t^2)\,t}{x-t}\,dt - \int_{-\infty}^0 \frac{p_D(t^2)\,t}{x-t}\,dt\right) = \frac{1}{\pi x}\mathcal{H}q_2(x), \end{aligned}$$

with q_2 defined as

$$q_2(x) = \begin{cases} p_D(x^2)\,x & \text{for } x > 0, \\ -p_D(x^2)\,x & \text{for } x < 0. \end{cases}$$

Using again that $\mathcal{H}^2 = -\mathrm{Id}$, it follows that $q_2(x) = -\mathcal{H}\left[\pi\, t\,(f_D'(t^2) - \mathcal{H}(f_N)(t^2))\right](x)$ for $x \in \mathbb{R}$ and therefore

$$p_D(x) = -\frac{1}{\sqrt{x}}\mathcal{H}\left[\pi\, t\left(f_D'(t^2) - \mathcal{H}(f_N)(t^2)\right)\right](\sqrt{x}) \quad \text{for } x > 0. \tag{3.10}$$

The heuristic argument given above is justified since $x\,(f_D'(x^2) - \mathcal{H}(f_N)(x^2)) \in L^p(|x|^{1-p}w(x^2))$ and $|x|^{1-p}w(x^2) \in A_p$ (see line (3.13) below), giving that q_2 and $\mathcal{H}q_2$ are well-defined. We then have

$$h_2(x) = \begin{cases} f_N(x) & \text{for } x < 0, \\ -\frac{1}{\sqrt{x}}\mathcal{H}\left[\pi\, t(f_D'(t^2) - \mathcal{H}(f_N)(t^2))\right](\sqrt{x}) & \text{for } x > 0. \end{cases} \tag{3.11}$$

We next show that $h_2 \in L^p(w)$ and

$$\|h_2\|_{L^p(w),} \lesssim \|f_D'\|_{L^p(w)} + \|f_N\|_{L^p(w)}\,. \tag{3.12}$$

We have

$$\|h_2\|_{L^p(w)}^p = \|f_N\|_{L^p(w)}^p + \int_0^\infty |h_2(x)|^p w(x)\,dx,$$

and since $|x|^{1-p}w(x^2) \in A_p$ and $w \in A_p$, then

$$
\begin{aligned}
\int_0^\infty |h_2(x)|^p w(x)\,dx &= \int_0^\infty \left|\frac{1}{\sqrt{x}}\mathcal{H}\left[\pi\, t\left(f_D'(t^2) - \mathcal{H}(f_N)(t^2)\right)\right](\sqrt{x})\right|^p w(x)\,dx \\
&= 2\int_0^\infty \left|\mathcal{H}\left[\pi\, t\left(f_D'(t^2) - \mathcal{H}(f_N)(t^2)\right)\right](x)\right|^p |x|^{1-p} w(x^2)\,dx \\
&\lesssim \int_{-\infty}^\infty \left|x\left(f_D'(x^2) - \mathcal{H}(f_N)(x^2)\right)\right|^p |x|^{1-p} w(x^2)\,dx \qquad (3.13)\\
&= 2\int_0^\infty |f_D'(x^2) - \mathcal{H}(f_N)(x^2)|^p |x| w(x^2)\,dx \\
&= \int_0^\infty |f_D'(x) - \mathcal{H}(f_N)(x)|^p w(x)\,dx \lesssim \|f_D'\|_{L^p(w)}^p + \|f_N\|_{L^p(w)}^p\,,
\end{aligned}
$$

from which (3.12) follows.

The same reasoning used in Part (a) shows that $u_{h_2} + C$ is a solution of the mixed problem (1.1) in the upper half-plane, where C is a constant such that $f_D = \mathcal{B}h_2 + C$ almost everywhere on D, and that the estimate

$$\|\mathcal{M}_\alpha(\nabla(u_{h_2} + C))\|_{L^p(w)} \lesssim \|f_D'\|_{L^p(w)} + \|f_N\|_{L^p(w)}$$

holds.

For the proofs of Parts (c) and (d), we will denote $\widetilde{w}(t) = 2|x|w(x^2)$ and use the following identities:

$$\left\|g(\sqrt{\cdot})\chi_D\right\|_{L^{p,1}(w)} = \|g\chi_D\|_{L^{p,1}(\widetilde{w})} \quad \text{and} \quad \left\|g(\sqrt{\cdot})\chi_D\right\|_{L^{p,\infty}(w)} = \|g\chi_D\|_{L^{p,\infty}(\widetilde{w})}\,.$$

Since $\widetilde{w}$ is even, for g even, it holds that

$$\|g\|_{L^{p,1}(\widetilde{w})} = 2\,\|g\chi_D\|_{L^{p,1}(\widetilde{w})} \quad \text{and} \quad \|g\|_{L^{p,\infty}(\widetilde{w})} = 2\,\|g\chi_D\|_{L^{p,\infty}(\widetilde{w})}\,.$$

Proof of Part (c): Let f_D be such that $f_D' \in L^{p,1}(w)$ and $f_N \in L^{p,1}(w)$. We consider h_1 as given in (3.7) and show that, under the assumptions on w, $h_1 \in L^{p,\infty}(w)$ and

$$\|h_1\|_{L^{p,\infty}(w)} \lesssim \|f_D'\|_{L^{p,1}(w)} + \|f_N\|_{L^{p,1}(w)}\,. \qquad (3.14)$$

We have

$$\|h_1\|_{L^{p,\infty}(w)} \le \|f_N\|_{L^{p,\infty}(w)} + \|h_1\chi_D\|_{L^{p,\infty}(w)} \le \|f_N\|_{L^{p,1}(w)} + \|h_1\chi_D\|_{L^{p,\infty}(w)}\,.$$

Since $w \in A_p$ and $\widetilde{w} \in A_p^{\mathcal{R}}$, we obtain

$$
\begin{aligned}
\|h_1\chi_D\|_{L^{p,\infty}(w)} &= \left\|\mathcal{H}\left(f_D'(x^2) - \mathcal{H}(f_N)(x^2)\right)\chi_D\right\|_{L^{p,\infty}(\widetilde{w})} \\
&\lesssim \left\|f_D'(x^2) - \mathcal{H}(f_N)(x^2)\right\|_{L^{p,1}(\widetilde{w})} = 2\left\|\left(f_D'(x^2) - \mathcal{H}(f_N)(x^2)\right)\chi_D\right\|_{L^{p,1}(\widetilde{w})} \\
&= 2\,\|f_D' - \mathcal{H}(f_N)\chi_D\|_{L^{p,1}(w)} \lesssim \|f_D'\|_{L^{p,1}(w)} + \|f_N\|_{L^{p,1}(w)}\,,
\end{aligned}
$$

from which (3.14) follows.

Using Remark 3.2 and the same reasoning as in Part (a) shows that $u_{h_1} + C$ is a solution of the mixed problem (1.1) in the upper half-plane, where C is a constant such that $f_D = \mathcal{B}h_1 + C$ almost everywhere on D, and that the estimate

$$\|\mathcal{M}_\alpha(\nabla u_{h_1} + C)\|_{L^{p,\infty}(w)} \lesssim \|f_D'\|_{L^{p,1}(w)} + \|f_N\|_{L^{p,1}(w)}$$

holds.

Proof of Part (d): Let f_D be such that $f_D' \in \mathcal{L}^{p,1}(w)$ and $f_N \in \mathcal{L}^{p,1}(w)$. We consider h_2 as given in (3.11) and show that, under the assumptions on w, $h_2 \in \mathcal{L}^{p,\infty}(w)$ and

$$\|h_2\|_{\mathcal{L}^{p,\infty}(w)} \lesssim \|f_D'\|_{\mathcal{L}^{p,1}(w)} + \|f_N\|_{\mathcal{L}^{p,1}(w)}\,. \qquad (3.15)$$

We have

$$\|h_2\|_{\mathcal{L}^{p,\infty}(w)} \lesssim \|f_N\|_{\mathcal{L}^{p,\infty}(w)} + \|h_2\chi_D\|_{\mathcal{L}^{p,\infty}(w)} \leq \|f_N\|_{\mathcal{L}^{p,1}(w)} + \|h_2\chi_D\|_{\mathcal{L}^{p,\infty}(w)}.$$

Consider the operators

$$Sf(x) = \frac{\mathcal{H}(w(t^2)|t|f(t))(\sqrt{x})\chi_D(x)}{\sqrt{x}\,w(x)} \quad \text{and} \quad \widetilde{S}f(x) = \frac{\mathcal{H}(w(t^2)|t|f(t))(x)\chi_D(x)}{x\,w(x^2)}$$

and note that, by Part (a) of Lemma 2.2, it follows that

$$\|Sf\|_{L^{p,\infty}(w)} = \left\|\widetilde{S}f\right\|_{L^{p,\infty}(\widetilde{w})} = \left\|\frac{\mathcal{H}(\widetilde{w}f)\chi_D}{\widetilde{w}}\right\|_{L^{p,\infty}(\widetilde{w})} \lesssim \|f\|_{L^{p,1}(\widetilde{w})}.$$

Taking $f(x) = \frac{-\pi \operatorname{sign}(x)\left(f_D'(x^2) - \mathcal{H}(f_N)(x^2)\right)}{w(x^2)}$, we have $Sf = \frac{h_2\chi_D}{w}$; therefore

$$\begin{aligned}
\|h_2\chi_D\|_{\mathcal{L}^{p,\infty}(w)} &= \left\|\frac{h_2\chi_D}{w}\right\|_{L^{p,\infty}(w)} \lesssim \left\|\frac{f_D'(x^2) - \mathcal{H}(f_N)(x^2)}{w(x^2)}\right\|_{L^{p,1}(\widetilde{w})} \\
&= 2\left\|\left(\frac{f_D'(x^2) - \mathcal{H}(f_N)(x^2)}{w(x^2)}\right)\chi_D\right\|_{L^{p,1}(\widetilde{w})} = 2\left\|\frac{f_D' - \mathcal{H}(f_N)}{w}\chi_D\right\|_{L^{p,1}(w)} \\
&\lesssim \left\|\frac{f_D'}{w}\right\|_{L^{p,1}(w)} + \left\|\frac{f_N}{w}\right\|_{L^{p,1}(w)} = \|f_D'\|_{\mathcal{L}^{p,1}(w)} + \|f_N\|_{\mathcal{L}^{p,1}(w)},
\end{aligned}$$

where in the last inequality we have used Part (b) of Lemma 2.2.

Using Remark 3.2 and the same reasoning as in Part (a) shows that $u_{h_2} + C$ is a solution of the mixed problem (1.1) in the upper half-plane, where C is a constant such that $f_D = \mathcal{B}h_2 + C$ almost everywhere on D, and that the estimate

$$\|\mathcal{M}_\alpha(\nabla u_{h_2})\|_{\mathcal{L}^{p,\infty}(w)} \lesssim \|f_D'\|_{\mathcal{L}^{p,1}(w)} + \|f_N\|_{\mathcal{L}^{p,1}(w)}$$

holds.

□

3.3. **Proof of Corollary 1.6.** Proof of Part (a): Recalling that if $1 < p < \infty$, $|x|^a \in A_p$ if and only if $-1 < a < p-1$, the conditions on $w \equiv 1$ become $p > 2$ in Part (a) of Theorem 1.5 and $p < 2$ in Part (b) of Theorem 1.5. This gives the solvability result for all $p \neq 2$. As mentioned in the introduction, known examples show that the Zaremba problem in the upper half-plane with $N = (-\infty, 0)$ and $D = (0, \infty)$ is not solvable in $X = L^2(\mathbb{R})$ with $Y = L^2(\mathbb{R})$ (see, for instance, [15, 20]).

Proof of Part (b): We have that $|x|^a \in A_p^{\mathcal{R}}$ if and only if $-1 < a \leq p-1$. In particular, $|x| \in A_2^{\mathcal{R}}$ and the result follows from Part (c) of Theorem 1.5.

□

4. Solvability of the Zaremba problem in a Lipschitz graph domain

The goal of this section is to prove Theorem 1.4 on the solvability of the Zaremba problem in a general Lipschitz graph domain Ω as in (1.2) with N and D as in (1.3). The proof is based on Theorem 1.5 as well as on results regarding the solvability of the Neumann problem in a Lipschitz graph domain in the plane. In Section 4.1 we present some preliminaries along with such results. We give the proof of Theorem 1.4 in Section 4.2.

4.1. **Preliminaries and the Neumann problem in a Lipschitz graph domain in the plane.** Let Ω be a Lipschitz graph domain in the plane as in (1.2). Since Ω is simply connected, it is conformally equivalent to $\mathbb{R}^2_+$; let $\Phi : \mathbb{R}^2_+ \longrightarrow \Omega$ be a conformal map such that $\Phi(\infty) = \infty$. Then Φ extends as a homeomorphism from $\overline{\mathbb{R}^2_+}$ onto $\overline{\Omega}$ and $\Phi(x)$, $x \in \mathbb{R}$, is absolutely continuous when restricted to any finite interval; in particular, $\Phi'(x)$ exists for almost every $x \in \mathbb{R}$ and is locally integrable. Moreover, $\Phi'(x) \neq 0$ for almost every $x \in \mathbb{R}$, $\lim_{z\to x} \Phi'(z) = \Phi'(x)$ for almost every $x \in \mathbb{R}$ in the sense of non-tangential convergence and $|\Phi'| \in A_2$. If $\Phi'(x)$ exists and is not zero, then it is a vector tangent to $\partial\Omega$ at $\Phi(x)$. See Kenig [13, Theorems 1.1 and 1.10] for the proof of those properties and additional ones. Given N and D as in (1.3), we will assume that $\Phi((-\infty, 0)) = N$ and $\Phi((0,\infty)) = D$.

We next recall the definitions of r_Φ and p_Φ. Let $1 \le r_\Phi \le \infty$ be such that its conjugate exponent is defined by

$$r'_\Phi = \inf\{q \in (1,\infty) : |x\, \Phi'(x^2)| \in A_q \text{ and } |\Phi'(x)| \in A_q\}.$$

Let $1 \le p_\Phi \le \infty$ be such that its conjugate exponent is defined by

$$p'_\Phi = \inf\{q \in (1,\infty) : |\Phi'(x)| \in A_q\}.$$

Note that we have $r_\Phi \le p_\Phi$ and $r_\Phi < p_\Phi$ if $|\Phi'| \in A_{r'_\Phi}$ with $r_\Phi < \infty$.

Denote by $L^p(\partial\Omega)$ the space of measurable functions in $\partial\Omega$ that are p-integrable with respect to arc-length; similarly, $L^{p,1}(\partial\Omega)$ and $L^{p,\infty}(\partial\Omega)$ are the corresponding Lorentz spaces with respect to arc-length in $\partial\Omega$. Given a measurable function g defined in $\partial\Omega$, let Tg and $\widetilde{T}g$ be given by

$$\widetilde{T}g(x) = \Phi'(x) g(\Phi(x)) \quad \text{and} \quad Tg(x) = |\Phi'(x)| g(\Phi(x)), \qquad x \in \mathbb{R}.$$

We note that $(g \circ \Phi)' = \widetilde{T}g'$ since, for $x \in \mathbb{R}$,

$$(g\circ\Phi)(x) = g(\Phi_1(x), \Phi_2(x)) = g(\Phi_1(x), \gamma(\Phi_1(x))) = (g\circ\nu)(\Phi_1(x)),$$

and, by Definition 1.2,

$$\begin{aligned}(g\circ\Phi)'(x) &= (g\circ\nu)'(\Phi_1(x))\Phi_1'(x) = g'(\Phi(x))(1 + i\gamma'(\Phi_1(x))\Phi_1'(x) \\ &= g'(\Phi(x))\Phi'(x).\end{aligned}$$

Also, if $1 < p < \infty$, then T and $\widetilde{T}$ are bijections from $L^p(\partial\Omega)$ onto $L^p(|\Phi'|^{1-p})$, from $L^{p,1}(\partial\Omega)$ onto $\mathcal{L}^{p,1}(|\Phi'|)$ and from $L^{p,\infty}(\partial\Omega)$ onto $\mathcal{L}^{p,\infty}(|\Phi'|)$; in particular, it holds that

$$\|\widetilde{T}g\|_{L^p(|\Phi'|^{1-p})} = \|Tg\|_{L^p(|\Phi'|^{1-p})} = \|g\|_{L^p(\partial\Omega)}, \tag{4.1}$$

$$\|\widetilde{T}g\|_{\mathcal{L}^{p,1}(|\Phi'|)} = \|Tg\|_{\mathcal{L}^{p,1}(|\Phi'|)} = \|g\|_{L^{p,1}(\partial\Omega)}, \tag{4.2}$$

$$\|\widetilde{T}g\|_{\mathcal{L}^{p,\infty}(|\Phi'|)} = \|Tg\|_{\mathcal{L}^{p,\infty}(|\Phi'|)} = \|g\|_{L^{p,\infty}(\partial\Omega)}. \tag{4.3}$$

Given f_D and f_N and the corresponding mixed problem (1.1) in Ω, we will consider the following mixed problem in the upper half-plane (see Figure 1):

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2_+, \\ u = f_D \circ \Phi & \text{in } (0,\infty), \\ \nabla u \cdot \mathbf{n} = Tf_N & \text{in } (-\infty, 0). \end{cases} \tag{4.4}$$

We end this section by recalling results regarding the solvability of the Neumann problem in a Lipschitz graph domain. Consider the classical Neumann boundary value problem in Ω :

$$\Delta v = 0 \text{ on } \Omega \qquad \text{and} \qquad \nabla v \cdot \mathbf{n} = g \text{ on } \partial\Omega, \tag{4.5}$$

where $\mathbf{n}$ denotes the outward unit normal vector to $\partial\Omega$ and the equality $\nabla v \cdot \mathbf{n} = g$ is interpreted in the sense of non-tangential convergence.

Theorem 4.1 (Solvability of the Neumann problem in $L^p(\partial\Omega)$, Theorem 1.4 in [6])**.** *Let Ω be a Lipschitz graph domain in the plane as in (1.2) and $\Phi:\mathbb{R}^2_+\to\Omega$ a conformal map. If $1<p<p_\Phi$ and $g\in L^p(\partial\Omega)$ then $v_g := u_{Tg}\circ\Phi^{-1}$ is a solution of (4.5) and*

$$\|\mathcal{M}_\alpha(\nabla v_g)\|_{L^p(\partial\Omega)} \lesssim \|g\|_{L^p(\partial\Omega)}, \tag{4.6}$$

where $0<\alpha<\arctan(1/L)$ and the implicit constant is independent of g.

Remark 4.2. We note that Theorem 4.1 and interpolation give that the results of Theorem 4.1 also hold for $L^{p,\infty}(\partial\Omega)$ with $1<p<p_\Phi$.

4.2. **Proof of Theorem 1.4.** Let f_N be zero outside N and f_D be zero outside D. By the assumption $\Phi((-\infty,0))=N$ and $\Phi((0,\infty))=D$, Tf_N is zero in $(0,\infty)$ while $f_D\circ\Phi$ and $\widetilde{T}f_D'$ are zero in $(-\infty,0)$.

Proof of Part (a): We first note that the condition $1<p<r_\Phi$ is equivalent to

$$|x\,\Phi'(x^2)|\in A_{p'} \text{ and } |\Phi'(x)|\in A_{p'}. \tag{4.7}$$

In view of (4.1), for $f_N\in L^p(\partial\Omega)$ and f_D such that $f_D'\in L^p(\partial\Omega)$, we have $Tf_N\in L^p(|\Phi'|^{1-p})$ and $(f_D\circ\Phi)'=\widetilde{T}f_D'\in L^p(|\Phi'|^{1-p})$. Also note that, by (4.7), $w=|\Phi'|^{1-p}$ satisfies the conditions in Part (b) of Theorem 1.5. Let then u be the solution of (4.4) given by Theorem 1.5 corresponding to $w=|\Phi'|^{1-p}$ and define $v=u\circ\Phi^{-1}$. We will show that v is a solution of the mixed problem (1.1) and

$$\|\mathcal{M}_\alpha(\nabla v)\|_{L^p(\partial\Omega)} \lesssim \|f_D'\|_{L^p(\partial\Omega)} + \|f_N\|_{L^p(\partial\Omega)}. \tag{4.8}$$

According to the proof of Part (b) of Theorem 1.5, $u=u_{h_2}$ where

$$h_2(x)=\begin{cases} Tf_N(x) & \text{for } x<0,\\ -\frac{1}{\sqrt{x}}\mathcal{H}\left[\pi\, t\left(\widetilde{T}f_D'(t^2)-\mathcal{H}(Tf_N)(t^2)\right)\right](\sqrt{x}) & \text{for } x>0,\end{cases} \tag{4.9}$$

and $h_2\in L^p(|\Phi'|^{1-p})$. By Theorem 4.1, v is a solution of the Neumann problem in Ω with datum $T^{-1}h_2$; that is, v is harmonic in Ω and $\nabla v\cdot\mathbf{n}=T^{-1}h_2$ almost everywhere in $\partial\Omega$ in the sense of non-tangential convergence. Moreover,

$$\|\mathcal{M}_\alpha(\nabla v)\|_{L^p(\partial\Omega)} \lesssim \|T^{-1}h_2\|_{L^p(\partial\Omega)}.$$

We then have $\nabla v\cdot\mathbf{n}=f_N$ on N and since

$$\begin{aligned}\|T^{-1}h_2\|_{L^p(\partial\Omega)} = \|h_2\|_{L^p(|\Phi'|^{1-p})} &\lesssim \|\widetilde{T}f_D'\|_{L^p(|\Phi'|^{1-p})} + \|Tf_N\|_{L^p(|\Phi'|^{1-p})}\\ &= \|f_D'\|_{L^p(\partial\Omega)} + \|f_N\|_{L^p(\partial\Omega)}\end{aligned}$$

by (4.1) and (3.12), we obtain (4.8). Finally, $v=f_D$ since $v\circ\Phi=u=f_D\circ\Phi$.

Proof of Part (b): Recalling (4.2), for $f_N\in L^{r_\Phi,1}(\partial\Omega)$ and f_D such that $f_D'\in L^{r_\Phi,1}(\partial\Omega)$, we have $Tf_N\in\mathcal{L}^{r_\Phi,1}(|\Phi'|)$ and $(f_D\circ\Phi)'=\widetilde{T}f_D'\in\mathcal{L}^{r_\Phi,1}(|\Phi'|)$. Also note that, by (1.4), $w=|\Phi'|$ satisfies the conditions in Part (d) of Theorem 1.5 with $p=r_\Phi$. Let then u be the solution of (4.4) given by Theorem 1.5 corresponding to $w=|\Phi'|$ and define $v=u\circ\Phi^{-1}$. We will show that v is a solution of the mixed problem (1.1) and

$$\|\mathcal{M}_\alpha(\nabla v)\|_{L^{r_\Phi,\infty}(\partial\Omega)} \lesssim \|f_D'\|_{L^{r_\Phi,1}(\partial\Omega)} + \|f_N\|_{L^{r_\Phi,1}(\partial\Omega)}. \tag{4.10}$$

In this case, according to Part (d) of Theorem 1.5, $u=u_{h_2}$ with $h_2\in\mathcal{L}^{r_\Phi,\infty}(|\Phi'|)$ given by (4.9). Since $r_\Phi<p_\Phi$, Remark 4.2 give that v is harmonic in Ω and $\nabla v\cdot\mathbf{n}=T^{-1}h_2$ almost everywhere in $\partial\Omega$ in the sense of non-tangential convergence. Moreover,

$$\|\mathcal{M}_\alpha(\nabla v)\|_{L^{r_\Phi,\infty}(\partial\Omega)} \lesssim \|T^{-1}h_2\|_{L^{r_\Phi,\infty}(\partial\Omega)}. \tag{4.11}$$

As in Part (a), the boundary conditions hold and the estimate (4.10) follows from (4.11), (4.3), (3.15) and (4.2).

□

Remark 4.3. Let $r_\Phi < \infty$ and assume $|\Phi'| \in A_{r'_\Phi}$. Recalling the definition of r_Φ, the weight $|x\,\Phi'(x^2)|$ fails to be in the class $A_{r'_\Phi}$; however, it may belong to the larger class $A^{\mathcal{R}}_{r'_\Phi}$. Then, (1.4) is a natural condition for the endpoint r_Φ when compared to condition (4.7) corresponding to the case $1 < p < r_\Phi$.

An analogous proof for that of Part (a) of Theorem 1.4 leads to the following result.

Theorem 4.4. *Let Ω be as in (1.2), D and N be as in (1.3) and $\Phi : \mathbb{R}^2_+ \to \Omega$ be a conformal map such that $\Phi((-\infty, 0)) = N$ and $\Phi((0,\infty)) = D$. Then the mixed problem (1.1) is solvable in the setting $X = Y = L^p(\partial\Omega)$ and $1 < p < \infty$ under the condition*

$$|x||\Phi'(x^2)|^{1-p} \in A_p \text{ and } |\Phi'(x)|^{1-p} \in A_p. \tag{4.12}$$

Proof. We reason exactly as in the proof of Part (a) of Theorem 1.4 by observing that $w = |\Phi'|^{1-p}$ satisfies the conditions in Part (a) of Theorem 1.5 by the assumption (4.12) and using $u = u_{h_1}$ where

$$h_1(x) = \begin{cases} Tf_N(x) & \text{for } x < 0,\\ -\mathcal{H}\left(\widetilde{T}f'_D(t^2) - \mathcal{H}(Tf_N)(t^2)\right)(\sqrt{x}) & \text{for } x > 0. \end{cases} \tag{4.13}$$

□

Remark 4.5. We make some observations related to the conditions (4.7) and (4.12). To that end, we will use the fact that if $w_0 \in A_{q_0}$ and $w_1 \in A_{q_1}$ for some $1 \le q_0, q_1 < \infty$, $0 \le \theta \le 1$, $1/q = (1-\theta)/q_0 + \theta/q_1$ and $w_0^{(1-\theta)q/q_0} w_1^{\theta q/q_1}$, then $w \in A_q$.

- The conditions (4.7) and (4.12) are incompatible for the same value of p: Suppose that $w_0(x) = |x\,\Phi'(x^2)| \in A_{p'}$ and $w_1(x) = |x||\Phi'(x^2)|^{1-p} \in A_p$; taking $q_0 = p'$, $q_1 = p$ and $\theta = 1/2$, we get that $|x| \in A_2$, which is false.
- Assume $r_\Phi < p_\Phi$; the set of values of p such that (4.12) holds is an open interval contained in (r_Φ, p_Φ) :

  In view of the definition of p_Φ, the condition $|\Phi'(x)|^{1-p} \in A_p$ in (4.12) implies that $p < p_\Phi$; the incompatibility of (4.7) and (4.12) gives that if (4.12) holds for p then $p \ge r_\Phi$.

  Assume that (4.12) is satisfied for p_0 and p_1 with $p_0 < p_1$ and let $p_0 < p < p_1$. If $0 \le \theta \le 1$ is such that $1/p' = (1-\theta)/p'_0 + \theta/p'_1$, we have

$$\left(|x|^{1-p'_0}|\Phi'(x^2)|\right)^{(1-\theta)p'/p'_0}\left(|x|^{1-p'_1}|\Phi'(x^2)|\right)^{\theta p'/p'_1} = |x|^{1-p'}|\Phi'(x^2)| \in A_{p'};$$

  moreover, $|\Phi'(x^2)| \in A_{p'}$ since $A_{p'_1} \subset A_{p'}$. Thus, (4.12) holds with p.

  Finally, we show that if (4.12) holds for p then (4.12) holds for q in a neighborhood of p. If (4.12) holds for p, then $|x|^{1-p'}\Phi'(x^2) \in A_{p'}$; hence, there exist A_1 weights u_0 and u_1 such that

$$|x|^{1-p'}\Phi'(x^2) = u_0 u_1^{1-p'}.$$

  Since $M\delta_0(x) \approx 1/|x|$, it follows that

$$|x|^{1-q'}\Phi'(x^2) \approx u_0 u_1^{1-p'}(M\delta_0)^{q'-p'}.$$

  Assume $q < p$ and let $r > 1$ be such that $u_0^r \in A_1$; then

$$u_0 u_1^{1-p'}(M\delta_0)^{q'-p'} = u_1^{1-p'}\left((u_0^r)^{\frac{1}{r}}(M\delta_0)^{q'-p'}\right)$$

is a weight in $A_{p'} \subset A_{q'}$ for q sufficiently near p such that $\frac{1}{r} + q' - p' < 1$. Since the condition $|\Phi'(x)|^{1-p} \in A_p$ is equivalent to $|\Phi'(x)| \in A_{p'}$, we also have $|\Phi'(x)| \in A_{q'}$. We have then shown that (4.12) holds for $q < p$ with q sufficiently closed to p. Reasoning similarly with u_1 instead of u_0, it follows that (4.12) holds for $q > p$ with q sufficiently closed to p.

5. Applications

In this section we present applications of Theorem 1.4 when the domain is a cone with vertex at the origin and, more generally, for Schwarz–Christoffel domains.

5.1. **The Zaremba problem for cones.** We will consider Ω to be a cone with vertex at the origin and aperture $\alpha\pi$ with $0 < \alpha < 2$, as shown in Figures 2.

Corollary 5.1. *If Ω is a cone with vertex at the origin and aperture $\alpha\pi$ for some $0 < \alpha < 2$, then the mixed problem (1.1) in Ω with N given by the left ray and D given by the right ray is solvable in the following settings:*

(a) $X = Y = L^p(\partial\Omega)$ for the following values of α and p :
- *(i) $0 < \alpha \leq \frac{1}{2}$ and $1 < p < \infty$,*
- *(ii) $\frac{1}{2} < \alpha \leq 1$ and $1 < p < \infty$ such that $p \neq \frac{2\alpha}{2\alpha-1}$,*
- *(iii) $1 < \alpha < 2$ and $1 < p < \frac{\alpha}{\alpha-1}$ such that $p \neq \frac{2\alpha}{2\alpha-1}$.*

(b) $X = L^{p,1}(\partial\Omega)$, $Y = L^{p,\infty}(\partial\Omega)$ for $\frac{1}{2} < \alpha < 2$ and $p = \frac{2\alpha}{2\alpha-1}$.

A different proof for the solvability of the Zaremba problem in a cone, based on sharp invertibility properties for a singular integral operator, is given in [1, Theorem 5] for values of $1 < p < \infty$ such that

$$p \neq \begin{cases} \frac{2-\alpha}{1-\alpha} & \text{if } 0 < \alpha \leq \frac{1}{2}, \\ \frac{2-\alpha}{1-\alpha}, \frac{2\alpha}{2\alpha-1} & \text{if } \frac{1}{2} < \alpha < 1, \\ 2 & \text{if } \alpha = 1, \\ \frac{2\alpha}{2\alpha-1}, \frac{\alpha}{\alpha-1} & \text{if } 1 < \alpha \leq \frac{3}{2}, \\ \frac{2\alpha}{2\alpha-1}, \frac{2\alpha}{2\alpha-3}, \frac{\alpha}{\alpha-1} & \text{if } \frac{3}{2} < \alpha < 2. \end{cases}$$

Corollary 5.1 improves [1, Theorem 5] by adding the solvability of the mixed problem in a cone in $L^p(\partial\Omega)$ for $p = \frac{2-\alpha}{1-\alpha}$ with $0 < \alpha < 1$ and in $L^{p,1}(\partial\Omega)$ for $p = \frac{2\alpha}{2\alpha-1}$ with $\frac{1}{2} < \alpha < 2$. On the other hand, for $1 < \alpha < 2$, the solvability of the mixed problem in a cone in $L^p(\partial\Omega)$ given in Corollary 5.1 is restricted to $1 < p < \frac{\alpha}{\alpha-1}$ with $\alpha \neq \frac{2\alpha}{2\alpha-1}$ while the range in [1, Theorem 5] includes all $p > \frac{\alpha}{\alpha-1}$ for $1 < \alpha \leq \frac{3}{2}$ and all $p > \frac{\alpha}{\alpha-1}$ with $p \neq \frac{2\alpha}{2\alpha-3}$ for $\frac{3}{2} < \alpha < 2$.

Proof of Corollary 5.1. Let Ω be a cone with vertex at the origin and aperture $\alpha\pi$ with $0 < \alpha < 2$. Consider $\Phi : \mathbb{R}^2_+ \to \mathbb{C}$ given by $\Phi(z) = e^{i\theta} z^\alpha$ for some $\theta \in \mathbb{R}$, where we have chosen the branch cut $\{iy : y \leq 0\}$ so that Φ is analytic in $\mathbb{R}^2_+$. Note that $\Phi((-\infty, 0)) = N$, $\Phi((0,\infty)) = D$ and $|\Phi'(x)| = \alpha\,|x|^{\alpha-1}$ for $x \in \mathbb{R} \setminus \{0\}$.

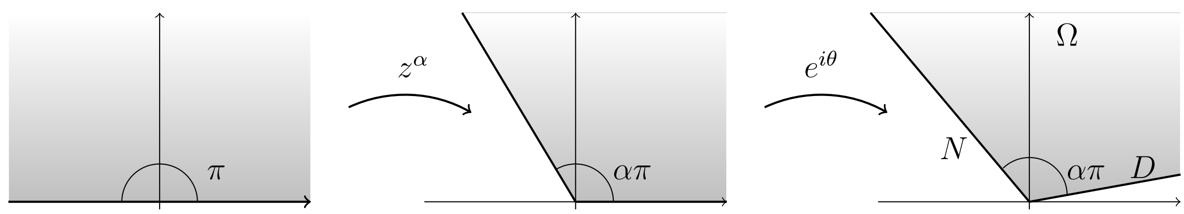

FIGURE 2. Cone with aperture $\alpha\pi$

Proof of Part (a): We will see under what conditions on α and p, Φ satisfies (4.7) or (4.12).

Condition (4.7): Since

$$|x\,\Phi'(x^2)|^{1-p} = \alpha^{1-p}|x|^{1-p+2(\alpha-1)(1-p)} \quad \text{and} \quad |\Phi'(x)|^{1-p} = \alpha^{1-p}|x|^{(\alpha-1)(1-p)},$$

we must have

$$-1 < 1-p+2(\alpha-1)(1-p) < p-1 \text{ and } -1 < (\alpha-1)(1-p) < p-1. \tag{5.1}$$

We then obtain

- $0 < \alpha \leq \frac{1}{2}$: (5.1) holds for all $1 < p < \infty$,
- $\frac{1}{2} < \alpha < 2$: (5.1) holds for $1 < p < \frac{2\alpha}{2\alpha-1}$.

Condition (4.12) : Since

$$|x||\Phi'(x^2)|^{1-p} = \alpha^{1-p}|x|^{1+2(\alpha-1)(1-p)} \quad \text{and} \quad |\Phi'(x)|^{1-p} = \alpha^{1-p}|x|^{(\alpha-1)(1-p)},$$

we must have

$$-1 < 1+2(\alpha-1)(1-p) < p-1 \text{ and } -1 < (\alpha-1)(1-p) < p-1. \tag{5.2}$$

This leads to

- $0 < \alpha \leq \frac{1}{2}$: there is no p such that (5.2) holds,
- $\frac{1}{2} < \alpha \leq 1$: (5.2) holds for $\frac{2\alpha}{2\alpha-1} < p < \infty$,
- $1 < \alpha < 2$: (5.2) holds for $\frac{2\alpha}{2\alpha-1} < p < \frac{\alpha}{\alpha-1}$.

From the analysis above we conclude the desired result.

Proof of Part (b): This will follow from Part (b) of Theorem 1.4 if we prove that $r_\Phi = \frac{2\alpha}{2\alpha-1}$ and (1.4) holds.

For $\frac{1}{2} < \alpha < 2$, we have

$$\begin{aligned} r'_\Phi &= \inf\{q \in [1,\infty) : |x\,\Phi'(x^2)| \in A_q \text{ and } |\Phi'(x)| \in A_q\} \\ &= \inf\{q \in [1,\infty) : |x\,\Phi'(x^2)|^{1-q'} \in A_{q'} \text{ and } |\Phi'(x)|^{1-q'} \in A_{q'}\} \\ &= \inf\{q \in [1,\infty) : -1 < 1-q'+2(\alpha-1)(1-q') < q'-1 \\ &\qquad \text{and } -1 < (\alpha-1)(1-q') < q'-1\} \\ &= \inf\{q \in [1,\infty) : 1 < q' < \tfrac{2\alpha}{2\alpha-1}\}, \end{aligned}$$

and therefore $r_\Phi = \frac{2\alpha}{2\alpha-1}$.

In this case, condition (1.4) holds if and only if

$$-1 < \alpha - 1 < r'_\Phi - 1 \text{ and } -1 < 1 + 2(\alpha-1) \leq r'_\Phi - 1,$$

which is true since $r'_\Phi = 2\alpha$. □

5.2. **The Zaremba problem for Schwarz–Christoffel domains.** In this section, we give examples of the Zaremba problem in domains Ω where the boundary $\partial\Omega$ is a polygonal curve with a finite number of line segments. More precisely, Ω is defined by a collection of vertices $w_1, w_2, \ldots w_{n-1} \in \mathbb{C}$, $w_n = \infty$ and interior angles $\alpha_1\pi, \alpha_2\pi, \ldots, \alpha_{n-1}\pi$, where $0 < \alpha_j < 2$ for $j = 1, \cdots, n-1$. The vertices are given in counterclockwise order with respect to the interior of Ω. In this case, Ω is called a *polygon* and we always assume that the polygon makes a total turn of 2π; in particular, this implies

$$\sum_{j=1}^{n-1} \alpha_j > n-2. \tag{5.3}$$

Figure 3 illustrates this type of domain.

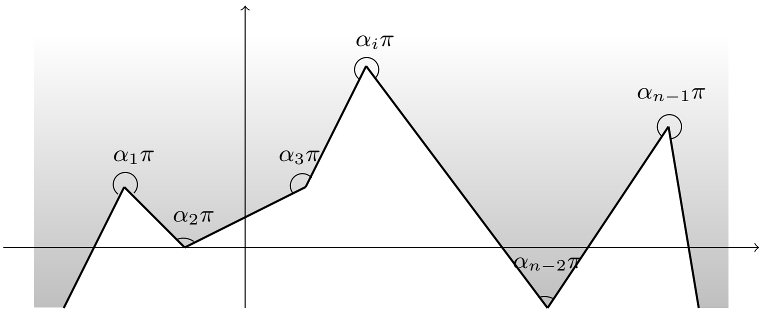

FIGURE 3. Unbounded polygon with $w_1, \dots, w_{n-1} \in \mathbb{C}$ and $w_n = \infty$

Dricoll–Trefethen [10, Theorem 2.1] proved that given a *polygon* Ω as described above, if $\Phi : \mathbb{R}^2_+ \to \Omega$ is a conformal mapping with $\Phi(\infty) = w_n$, then

$$\Phi(z) = A + B \int_{[z_0,z]} (\xi - x_1)^{\alpha_1 - 1} \cdots (\xi - x_{n-1})^{\alpha_{n-1}-1} d\xi, \tag{5.4}$$

where $A, B \in \mathbb{C}$, z_0 is a suitably chosen point in $\mathbb{R}^2_+$ or its boundary, $[z_0, z]$ is the straight line segment from z_0 to z, $x_1, \dots, x_{n-1} \in \mathbb{R}$ with $x_1 < \cdots < x_{n-1}$ and $\Phi(x_j) = w_j$ for $j = 1, \dots, n-1$. A mapping of this form is called a *Schwarz–Christoffel transformation* and we say that Ω is a *Schwarz–Christoffel* domain. We have

$$|\Phi'(x)| = |B||x - x_1|^{\alpha_1 - 1}|x - x_2|^{\alpha_2 - 1} \cdots |x - x_{n-1}|^{\alpha_{n-1}-1}. \tag{5.5}$$

Before stating the main result in this section (Corollary 5.4), on the solvability of the Zaremba problem in a Schwarz–Christoffel domain, we present two lemmas to determine the classes of weights to which $|x\, \Phi'(x^2)|$ and $|\Phi'(x)|$ belong.

Lemma 5.2 ([6, Section 5.1], [7, Lemma 2.12]). *Let $\beta_1, \beta_2, \dots, \beta_M \in (-1, \infty)$, $x_1, x_2, \dots, x_M \in \mathbb{R}$ be such that $x_1 < x_2 < \cdots < x_M$ and*

$$w(x) = |x - x_1|^{\beta_1}|x - x_2|^{\beta_2} \cdots |x - x_M|^{\beta_M}.$$

Then $w \in A_q^{\mathcal{R}}$ with

$$q = \max\{1, \beta_1 + 1, \beta_2 + 1 \dots, \beta_M + 1, 1 + \sum_{j=1}^{M} \beta_j\}$$

and $w \notin A_p^{\mathcal{R}}$ whenever $p < q$.

Lemma 5.3. *Let Φ be a Schwarz–Christoffel transformation as given in (5.4) with $x_1 < x_2 < \cdots < x_{n-1}$ and $0 < \alpha_j < 2$ for $j = 1, \cdots, n-1$.*

(a) $|\Phi'(x)| \in A_{q_0}^{\mathcal{R}}$ with

$$q_0 = \max\{1, \alpha_1, \alpha_2, \dots, \alpha_{n-1}, 2 - n + \sum_{j=1}^{n-1} \alpha_j\} \tag{5.6}$$

and $w \notin A_p^{\mathcal{R}}$ whenever $p < q_0$.

(b) If $x_1 > 0$, then $|x\, \Phi'(x^2)| \in A_{q_1}^{\mathcal{R}}$ with

$$q_1 = \max\{2, 4 - 2n + 2\sum_{j=1}^{n-1} \alpha_j\} \tag{5.7}$$

and $w \notin A_p^{\mathcal{R}}$ whenever $p < q_1$.

(c) If $x_1 = 0$, *then* $|x\,\Phi'(x^2)| \in A^{\mathcal{R}}_{q_2}$ *with*

$$q_2 = \max\{1, 2\alpha_1, \alpha_2, \cdots, \alpha_{n-1}, 4 - 2n + 2\sum_{j=1}^{n-1}\alpha_j\} \tag{5.8}$$

and $w \notin A^{\mathcal{R}}_p$ *whenever* $p < q_2$.

Proof. Part (a) follows directly from (5.5) and Lemma 5.2. For parts (b) and (c), we have

$$|x\,\Phi'(x^2)| = |B||x|^{2-1}\prod_{j=1}^{n-1}|x - \sqrt{x_j}|^{\alpha_j - 1}|x + \sqrt{x_j}|^{\alpha_j - 1},$$

and

$$|x\,\Phi'(x^2)| = |B||x|^{2\alpha_1-1}\prod_{j=2}^{n-1}|x - \sqrt{x_j}|^{\alpha_j - 1}|x + \sqrt{x_j}|^{\alpha_j - 1},$$

respectively. The desired results then follow by applying Lemma 5.2 and recalling that $\alpha_j < 2$ for $j = 1, \cdots, n-1$. $\square$

We next state and prove the main result of this section:

Corollary 5.4. *Let* Ω *be a Schwarz–Christoffel domain as in Fig 3 with* $\Phi : \mathbb{R}^2_+ \to \Omega$ *given by* (5.4)*, where* $0 \le x_1 < x_2 < \cdots < x_{n-1}$. *Consider the mixed problem* (1.1) *with* $N = \Phi((-\infty, 0))$ *and* $D = \Phi((0, \infty))$.

(a) If $x_1 > 0$ *then*

$$r_\Phi = \frac{\max\{2, 4 - 2n + 2\sum_{j=1}^{n-1}\alpha_j\}}{\max\{2, 4 - 2n + 2\sum_{j=1}^{n-1}\alpha_j\} - 1}.$$

The mixed problem (1.1) *is solvable in* $X = Y = L^p(\partial\Omega)$ *with* $1 < p < r_\Phi$ *and in* $X = L^{r_\Phi,1}(\partial\Omega)$ *with* $Y = L^{r_\Phi,\infty}(\partial\Omega)$.

(b) If $x_1 = 0$ *then*

$$r_\Phi = \frac{\max\{1, 2\alpha_1, \alpha_2, \cdots, \alpha_{n-1}, 4 - 2n + 2\sum_{j=1}^{n-1}\alpha_j\}}{\max\{1, 2\alpha_1, \alpha_2, \cdots, \alpha_{n-1}, 4 - 2n + 2\sum_{j=1}^{n-1}\alpha_j\} - 1}.$$

The mixed problem (1.1) *is solvable in* $X = Y = L^p(\partial\Omega)$ *with* $1 < p < r_\Phi$ *and, if* $r_\Phi < p_\Phi$, *in* $X = L^{r_\Phi,1}(\partial\Omega)$ *with* $Y = L^{r_\Phi,\infty}(\partial\Omega)$.

Proof. We first note that, by (2.3), we have

$$r'_\Phi = \inf\{q \in (1,\infty) : |x\,\Phi'(x^2)| \in A^{\mathcal{R}}_q \text{ and } |\Phi'(x)| \in A_q\}. \tag{5.9}$$

and

$$r'_\Phi = \inf\{q \in (1,\infty) : |x\,\Phi'(x^2)| \in A^{\mathcal{R}}_q \text{ and } |\Phi'(x)| \in A^{\mathcal{R}}_q\}. \tag{5.10}$$

Using (5.3), we obtain

$$2 - n + \sum_{j=1}^{n-1}\alpha_j < 4 - 2n + 2\sum_{j=1}^{n-1}\alpha_j. \tag{5.11}$$

Let q_0, q_1 and q_2 be as in the statement of Lemma 5.3.

Proof of Part (a): We have to prove that $r'_\Phi = q_1$, $|x\,\Phi'(x^2)| \in A^{\mathcal{R}}_{r'_\Phi}$ and $|\Phi'(x)| \in A_{r'_\Phi}$; then the desired result follows from Theorem 1.4.

By (5.11) and since $\alpha_j < 2$ for $j = 1, \cdots, n-1$, we have $q_0 < q_1$. This inequality and the inclusions (2.3) along with Part (a) of Lemma 5.3, Part (b) of Lemma 5.3 and (5.9) give

$$|\Phi'(x)| \in A_{q_1}, \quad |x\,\Phi'(x)| \in A^{\mathcal{R}}_{q_1}, \quad r'_\Phi = q_1. \tag{5.12}$$

Proof of Part (b): We have to prove that $r'_\Phi = q_2$, $|x\,\Phi'(x^2)| \in A^{\mathcal{R}}_{r'_\Phi}$ and $|\Phi'(x)| \in A_{r'_\Phi}$; then the desired result follows from Theorem 1.4.

By (2.3), we have

$$p'_\Phi = \inf\{q \in (1,\infty) : |\Phi'(x)| \in A^{\mathcal{R}}_q\}.$$

This and Part (a) of Lemma 5.3 imply that $p'_\Phi = q_0$. Then, the hypothesis $r_\Phi < p_\Phi$ give $|\Phi'(x)| \in A_{r'_\Phi}$

The inequality (5.11) leads to $q_0 \leq q_2$. This inequality and the inclusions (2.3) along with Part (a) of Lemma 5.3, Part (c) of Lemma 5.3 and (5.10) give

$$|\Phi'(x)| \in A^{\mathcal{R}}_{q_2}, \quad |x\,\Phi'(x)| \in A^{\mathcal{R}}_{q_2}, \quad r'_\Phi = q_2. \tag{5.13}$$

$\square$

Remark 5.5. Assume $r_\Phi < \infty$; then the condition $|\Phi'(x)| \in A_{r'_\Phi}$ is not possible in Part (b) of Corollary 5.4 if $r_\Phi = p_\Phi$. Indeed, this implies that $|\Phi'(x)| \in A_{p'_\Phi}$, which may only happen if $p'_\Phi = 1$.

References

[1] H. Awala, I. Mitrea, and K. Ott. On the solvability of the Zaremba problem in infinite sectors and the invertibility of associated singular integral operators. In *Recent applications of harmonic analysis to function spaces, differential equations, and data science*, Appl. Numer. Harmon. Anal., pages 719–751. Birkhäuser/Springer, Cham, 2017.

[2] A. Azzam and E. Kreyszig. On solutions of elliptic equations satisfying mixed boundary conditions. *SIAM J. Math. Anal.*, 13(2):254–262, 1982.

[3] R. Brown. The mixed problem for Laplace's equation in a class of Lipschitz domains. *Comm. Partial Differential Equations*, 19(7-8):1217–1233, 1994.

[4] R. Brown and L. Croyle. Estimates for the L^q-mixed problem in $C^{1,1}$-domains. *Complex Var. Elliptic Equ.*, 66(2):181–193, 2021.

[5] R. Brown and I. Mitrea. The mixed problem for the Lamé system in a class of Lipschitz domains. *J. Differential Equations*, 246(7):2577–2589, 2009.

[6] M. J. Carro, V. Naibo, and C. Ortiz-Caraballo. The Neumann problem in graph Lipschitz domains in the plane. *Math. Ann.*, 385(1-2):17–57, 2023.

[7] M. J. Carro and C. Ortiz-Caraballo. On the Dirichlet problem on Lorentz and Orlicz spaces with applications to Schwarz-Christoffel domains. *J. Differential Equations*, 265(5):2013–2033, 2018.

[8] H. Chung, R. Hunt, and D. Kurtz. The Hardy-Littlewood maximal function on $L(p, q)$ spaces with weights. *Indiana Univ. Math. J.*, 31(1):109–120, 1982.

[9] B. Dahlberg and C. Kenig. Hardy spaces and the Neumann problem in L^p for Laplace's equation in Lipschitz domains. *Ann. of Math. (2)*, 125(3):437–465, 1987.

[10] T. Driscoll and L. Trefethen. *Schwarz-Christoffel mapping*, volume 8 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge University Press, Cambridge, 2002.

[11] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[12] R. Hunt, B. Muckenhoupt, and R. Wheeden. Weighted norm inequalities for the conjugate function and Hilbert transform. *Trans. Amer. Math. Soc.*, 176:227–251, 1973.

[13] C. Kenig. Weighted H^p spaces on Lipschitz domains. *Amer. J. Math.*, 102(1):129–163, 1980.

[14] C. Kenig. *Harmonic analysis techniques for second order elliptic boundary value problems*, volume 83 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1994.

[15] L. Lanzani, L. Capogna, and R. M. Brown. The mixed problem in L^p for some two-dimensional Lipschitz domains. *Math. Ann.*, 342(1):91–124, 2008.

[16] G. Lieberman. Mixed boundary value problems for elliptic and parabolic differential equations of second order. *J. Math. Anal. Appl.*, 113(2):422–440, 1986.

[17] G. Lieberman. Optimal Hölder regularity for mixed boundary value problems. *J. Math. Anal. Appl.*, 143(2):572–586, 1989.

[18] I. Mitrea and M. Mitrea. The Poisson problem with mixed boundary conditions in Sobolev and Besov spaces in non-smooth domains. *Trans. Amer. Math. Soc.*, 359(9):4143–4182, 2007.

[19] M. Mitrea. Mixed boundary-value problems for Maxwell's equations. *Trans. Amer. Math. Soc.*, 362(1):117–143, 2010.

[20] K. Ott and R. Brown. The mixed problem for the Laplacian in Lipschitz domains. *Potential Anal.*, 38(4):1333–1364, 2013.

[21] K. Ott and R. Brown. Correction to: "The mixed problem for the Laplacian in Lipschitz domains". *Potential Anal.*, 54(1):213–217, 2021.

[22] G. Savaré. Regularity and perturbation results for mixed second order elliptic problems. *Comm. Partial Differential Equations*, 22(5-6):869–899, 1997.

[23] J. Sykes. *L^p regularity of solutions of the mixed boundary problem for Laplace's equation on a Lipschitz graph domain*. ProQuest LLC, Ann Arbor, MI, 1999. Thesis (Ph.D.)–University of Kentucky.

[24] J. Sykes and R. Brown. The mixed boundary problem in L^p and Hardy spaces for Laplace's equation on a Lipschitz domain. In *Harmonic analysis and boundary value problems (Fayetteville, AR, 2000)*, volume 277 of *Contemp. Math.*, pages 1–18. Amer. Math. Soc., Providence, RI, 2001.

[25] J. L. Taylor, K. A. Ott, and R. M. Brown. The mixed problem in Lipschitz domains with general decompositions of the boundary. *Trans. Amer. Math. Soc.*, 365(6):2895–2930, 2013.

[26] S. Zaremba. Sur un probleme mixte relatif a l'equation de Laplace. *Bull. Int. Acad. Cracovie, Classe Math. Nat., Ser. A*, pages 313–344, 1910.

[27] S. Zaremba. On a mixed problem for Laplace's equation. *Uspehi Matem. Nauk (N.S.)*, 1(3-4(13-14)):125–146, 1946.

MARÍA JESÚS CARRO. DEPARTMENT OF ANALYSIS AND APPLIED MATHEMATICS, UNIVERSIDAD COMPLUTENSE DE MADRID, PLAZA DE LAS CIENCIAS 3, 28040 MADRID, SPAIN. INSTITUTO DE CIENCIAS MATEMÁTICAS ICMAT, MADRID, SPAIN

Email address: `mjcarro@ucm.es`

TERESA LUQUE. DEPARTMENT OF ANALYSIS AND APPLIED MATHEMATICS, UNIVERSIDAD COMPLUTENSE DE MADRID, PLAZA DE LAS CIENCIAS 3, 28040 MADRID, SPAIN. INSTITUTO DE CIENCIAS MATEMÁTICAS ICMAT, MADRID, SPAIN

Email address: `t.luque@ucm.es`

VIRGINIA NAIBO. DEPARTMENT OF MATHEMATICS, KANSAS STATE UNIVERSITY. 138 CARDWELL HALL, 1228 N. MARTIN LUTHER KING JR. DR., KS 66506, USA.

Email address: `vnaibo@ksu.edu`