A SOLUTION TO A CONJECTURE ON THE NORM OF THE COPSON OPERATOR MINUS THE IDENTITY FOR MONOTONE SEQUENCES

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Abstract. We investigate the operator distance between the Identity and the Copson operator on the cone of nonnegative, nonincreasing sequences within the Lebesgue sequence spaces $\ell^p(\mathbb{N})$. By proving the exact value of this distance, we confirm a conjecture originally formulated by A. Ben Said, S. Boza, and J. Soria in "The norm of Hardy-type oscillation operators in the discrete and continuous settings" (*Analysis and Applications*, 2025).

1. Introduction

For a given sequence $\{x(n)\}_{n\geq 1}$ of real numbers, the Cesàro operator C and its dual, also known as the Copson operator, C^*, are defined as

$$Cx(n)=\frac{1}{n}\sum_{k=1}^{n}x(k) \quad \text{and} \quad C^*x(n)=\sum_{k=n}^{\infty}\frac{x(k)}{k},$$

respectively. The continuous counterparts of these operators are the classical Hardy average operator H and the adjoint operator H^*. Both operators, H and H^*, are given by

$$Hf(t)=\frac{1}{t}\int_0^t f(s)\,\mathrm{d}s \quad \text{and} \quad H^*f(t)=\int_t^{\infty}\frac{f(s)}{s}\,\mathrm{d}s,$$

assuming the integral expressions above converge for any such function f on $(0,\infty)$. The boundedness of the discrete operators C and C^* on $\ell^p(\mathbb{N})$, as well as the continuous operators H and H^* on $L^p(\mathbb{R}^+)$, follows from the classical Hardy inequalities [10].

G. Bennett formulated in [5] the question of determining the exact norm of $C-I$ as a map acting in $\ell^p(\mathbb{N})$, while he was interested in finding out the best constant for some equivalent norms for the Cesàro sequence space $\mathrm{ces}_p(\mathbb{N})$. By means of Hilbert space methods, the operator norm was previously shown to be 1 for $\ell^2(\mathbb{N})$ (see [9]). G. J. O. Jameson [11] conjectured that $\|C-I\|_{\ell^p(\mathbb{N})}=\frac{1}{p-1}$, for $1<p\leq 2$ and he obtained the result for $p=4/3$. Very recently, G. Sinnamon [15] used Jameson's method to prove the conjecture and, in fact, answered Bennett's question in the full range $1<p\leq\infty$ (with a different value of the norm for $2<p\leq\infty$). Specifically,

2020 *Mathematics Subject Classification.* Primary 26D15; Secondary 47B37.

Key words and phrases. Cesàro operator, Copson operator, monotone sequence, nonincreasing sequence, sharp constants, Hardy-type inequality.

he proved that if $m_p = \min\limits_{t\in[0,1/2]} \big(pt^{p-1} + (1-t)^p - t^p\big)$, then

$$\|C - I\|_{\ell^p(\mathbb{N})} = \begin{cases} \frac{1}{p-1}, & \text{if } 1 < p \le 2, \\ m_p^{-1/p}, & \text{if } 2 \le p < \infty, \\ 2, & \text{if } p = \infty. \end{cases}$$

The operator norm of $C^* - I$ is obtained via a duality argument;

$$\|C^* - I\|_{\ell^p(\mathbb{N})} = \begin{cases} 2, & \text{if } p = 1, \\ m_{p'}^{-1/p'}, & \text{if } 1 < p \le 2, \\ p - 1, & \text{if } 2 \le p < \infty, \end{cases}$$

where $p' = p/(p-1)$ is the conjugate exponent of p. In [2], the authors removed the dependency on p' for the norm of $C^* - I$ when $1 < p \le 2$ by establishing that $m_{p'}^{-1/p'} = M_p^{1/p}$ in this range, where $M_p = \max_{t\in[0,1/2]} \big(pt^{p-1} + (1-t)^p - t^p\big)$.

Regarding the cone of nonnegative sequences in $\ell^p(\mathbb{N})$, the authors in [2] proved the following results:

Theorem 1.1. [2, Remark 2.1, Theorem 2.4, and Proposition 2.5] *Let $1 < p \le \infty$ and let $x \in \ell^p(\mathbb{N})$ be a nonnegative sequence. If $1 < p \le 2$, then*

$$\|(C - I)x\|_p \le (p-1)^{-1}\|x\|_p, \tag{1}$$

and if $2 \le p \le \infty$, then

$$\|(C - I)x\|_p \le \|x\|_p. \tag{2}$$

Moreover, both inequalities (1) and (2) are sharp.

Theorem 1.2. [2, Theorem 3.3 and Proposition 3.6] *Let $1 \le p < \infty$ and $x \in \ell^p(\mathbb{N})$ be a nonnegative sequence. If $1 \le p \le 2$, then*

$$\|(C^* - I)x\|_p \le M_p^{1/p}\|x\|_p, \tag{3}$$

and if $2 \le p < \infty$, then

$$\|(C^* - I)x\|_p \le (p-1)\|x\|_p. \tag{4}$$

Furthermore, the constants in (3) and (4) are the best possible.

On the cone of nonnegative, nonincreasing sequences in $\ell^p(\mathbb{N})$, we have the following result for the operator $C - I$:

Theorem 1.3. [8, Theorem 2.2 and Proposition 2.4] *Let $1 < p \le \infty$ and $x \in \ell^p(\mathbb{N})$ be a nonnegative, nonincreasing sequence. If $1 < p \le 2$, then*

$$\|(C - I)x\|_p \le (p-1)^{-1}\|x\|_p, \tag{5}$$

if $2 \le p < \infty$, then

$$\|(C - I)x\|_p \le (p-1)^{-1/p}\|x\|_p, \tag{6}$$

and if $p = \infty$, then

$$\|(C - I)x\|_\infty \le \|x\|_\infty, \tag{7}$$

Moreover, the inequalities (5), (6), and (7) are sharp.

For the operator $C^* - I$ acting on the cone of nonnegative, nonincreasing sequences in $\ell^p(\mathbb{N})$, we have the following theorem:

Theorem 1.4. [2, Proposition 3.6 and Theorem 3.8] *Let $1 \leq p < \infty$ and $x \in \ell^p(\mathbb{N})$ be a nonnegative sequence. If $p = 1$, then*

$$\|(C^* - I)x\|_1 \leq 2e^{-1}\|x\|_1, \tag{8}$$

and if $2 \leq p < \infty$, then

$$\|(C^* - I)x\|_p \leq (p-1)\|x\|_p. \tag{9}$$

Furthermore, the constants in (8) and (9) are optimal.

Further results on optimal constants for $C-I$, its dual C^*-I, and their continuous analogues can be found in [1, 7, 13, 14, 16].

Note that for $1 < p < 2$, the exact norm of $C^* - I$ restricted to the cone of nonnegative, nonincreasing sequences in $\ell^p(\mathbb{N})$, denoted $\|C^*-I\|_{\ell^p_{\mathrm{dec}}(\mathbb{N})}$, has remained open until now. Regarding this constant, Ben Said, Boza, and Soria [2] conjectured that

$$\|C^* - I\|_{\ell^p_{\mathrm{dec}}(\mathbb{N})} = \left(\int_0^1 |\ln t + 1|^p \, \mathrm{d}t\right)^{1/p} \stackrel{\mathrm{def}}{=} C_p^{1/p} \quad \text{for } 1 < p < 2. \tag{10}$$

In the same work [2], the authors posed an analogous conjecture for the dual of the continuous Hardy operator minus the Identity. This conjecture was recently confirmed by Ben Said and Sinnamon [4], who completed the work initiated in [2] by proving.

Theorem 1.5. [4, Theorem 1.2], [2, Proposition 4.4 and Theorem 4.5]
Let $1 \leq p < \infty$. Let f be a nonnegative and nonincreasing function. If $1 \leq p \leq 2$, then

$$(p-1)\|f\|_p \leq \|(H^* - I)f\|_p \leq C_p^{1/p}\|f\|_p, \tag{11}$$

and if $2 \leq p < \infty$, then

$$C_p^{1/p}\|f\|_p \leq \|(H^* - I)f\|_p \leq (p-1)\|f\|_p. \tag{12}$$

Moreover, the constants $p-1$ and $C_p^{1/p}$ are optimal in both (11) and (12).

Our main result, Theorem 3.1, confirms the validity of the conjecture in (10). Together, Theorem 3.1 and Theorem 1.4 yield the same upper constants for $C^* - I$ as those obtained for $H^* - I$ in Theorem 1.5. However, we cannot obtain lower bounds for $C^* - I$ analogous to those for $H^* - I$ in Theorem 1.4, as no such finite constants exist in this case (see [3]).

This paper is organized into two sections. In Section 2, we introduce several preliminary lemmas that will be used to establish our core results. Section 3 presents the main results of this paper and several noteworthy corollaries.

In what follows, for $m, n \in \mathbb{N}$, we adopt the convention that $\sum_{j=m}^{n} a(j) = 0$ whenever $n < m$.

2. Preliminaries

In this section, we collect several preliminary results required for the proofs of our main theorems.

We start with the following lemma, which will help establish the optimality of our target constant $C_p^{1/p}$.

Lemma 2.1. *Let $1 \leq p < \infty$. For each $m \in \mathbb{N}$, let e_m denote the m-th canonical basis vector of $\ell^p(\mathbb{N})$. Then,*

$$\lim_{m\to\infty} \frac{\|(C^* - I)C^* e_m\|_p^p}{\|C^* e_m\|_p^p} = C_p. \tag{13}$$

Proof. Let $1 \leq p < \infty$ and let m be a natural number. We have that

$$C^* e_m(n) = \sum_{k=n}^{\infty} \frac{e_m(k)}{k} = \begin{cases} \frac{1}{m}, & \text{if } n \leq m, \\ 0, & \text{if } n > m. \end{cases}$$

It is clear that $y = C^* e_m$ is nonnegative, nonincreasing sequence and it has p-norm $\|C^* e_m\|_p^p = m^{1-p}$, for all $m \in \mathbb{N}$. By applying C^* to y we obtain

$$C^* y = C^*(C^* e_m(n)) = \sum_{k=n}^{\infty} \frac{C^* e_m(k)}{k} = \begin{cases} \frac{1}{m} \sum_{k=n}^{m} \frac{1}{k}, & \text{if } n \leq m, \\ 0, & \text{if } n > m. \end{cases}$$

So

$$|(C^* - I)C^* e_m(n)| = \begin{cases} \frac{1}{m}\big| - 1 + \sum_{k=n}^{m} \frac{1}{k}\big|, & \text{if } n \leq m, \\ 0, & \text{if } n > m. \end{cases} \tag{14}$$

Therefore, we have to show that

$$\lim_{m\to\infty} \frac{\|(C^* - I)C^* e_m\|_p^p}{\|C^* e_m\|_p^p} = \lim_{m\to\infty} \frac{1}{m} \sum_{n=1}^{m} \left| -1 + \sum_{k=n}^{m} \frac{1}{k} \right|^p = C_p^p. \tag{15}$$

Let $\{f_m\}_{m\geq 1}$ be the sequence of function defined on $(0, 1]$ by

$$f_m(x) = \sum_{n=1}^{m} \Bigg(-1 + \sum_{k=\lceil mx \rceil}^{m} \frac{1}{k} \Bigg) \chi_{(\frac{n-1}{m}, \frac{n}{m}]}(x),$$

where $t \mapsto \lceil t \rceil$ is the ceiling function. Note that

$$\int_0^1 |f_m(x)|^p \, \mathrm{d}x = \sum_{n=1}^{m} \int_{\frac{n-1}{m}}^{\frac{n}{m}} \left| -1 + \sum_{k=\lceil mx \rceil}^{m} \frac{1}{k} \right|^p \mathrm{d}x = \frac{1}{m} \sum_{n=1}^{m} \left| -1 + \sum_{k=n}^{m} \frac{1}{k} \right|^p.$$

We now study the pointwise convergence of the sequence $\{f_m\}_{m\geq 1}$. Let $x \in (0, 1]$. Then there exists $n \in \{1, \ldots, m\}$ such that $x \in \left(\frac{n-1}{m}, \frac{n}{m}\right]$. In this case, we have

$$f_m(x) = -1 + \sum_{k=\lceil mx \rceil}^{m} \frac{1}{k} = -1 + H_m - H_{\lceil mx \rceil} + \frac{1}{\lceil mx \rceil},$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ denotes the n-th harmonic number. It is known (see, e.g., [12]) that

$$H_n = \ln n + \gamma + \frac{1}{2n} - \varepsilon_n, \tag{16}$$

where γ is the Euler–Mascheroni constant and $0 \le \varepsilon_n \le \frac{1}{8n^2}$, which tends to 0 as $n \to \infty$. Applying (16), we obtain

$$f_m(x) = -1 + \ln\left(\frac{m}{\lceil mx \rceil}\right) + \frac{1}{2\lceil mx \rceil} + \frac{1}{2m} - \varepsilon_m + \varepsilon_{\lceil mx \rceil}.$$

Taking the limit as $m \to \infty$ in the expression above and using the fact that $\lim_{m\to\infty} \ln\left(\frac{m}{\lceil mx \rceil}\right) = -\ln(x)$, we conclude that

$$\lim_{m\to\infty} f_m(x) = -1 - \ln(x) \stackrel{\text{def}}{=} f(x),$$

for all $x \in (0, 1]$. We now seek a nonnegative function g such that $|f_m(x)|^p \le g(x)$ for all $x \in (0, 1]$ and $m \in \mathbb{N}$. Fixing $x \in (0, 1]$, there exists $n \in \{1, \dots, m\}$ such that $x \in \left(\frac{n-1}{m}, \frac{n}{m}\right]$, which implies $\lceil mx \rceil = n$. We have

$$\sum_{k=\lceil mx \rceil}^{m} \frac{1}{k} = \sum_{k=n}^{m} \frac{1}{k} = \frac{1}{n} + \sum_{k=n+1}^{m} \frac{1}{k} \le \frac{1}{n} + \sum_{k=n+1}^{m} \int_{k-1}^{k} \frac{\mathrm{d}t}{t} = \frac{1}{n} + \int_{n}^{m} \frac{\mathrm{d}t}{t} = \frac{1}{n} + \ln\left(\frac{m}{n}\right).$$

Since $x \le \frac{n}{m}$, it follows that $\frac{m}{n} \le \frac{1}{x}$, and hence $\ln\left(\frac{m}{n}\right) \le \ln\left(\frac{1}{x}\right)$. Therefore,

$$\sum_{k=\lceil mx \rceil}^{m} \frac{1}{k} \le \frac{1}{n} + \ln\left(\frac{m}{n}\right) \le 1 - \ln x.$$

Therefore,

$$f_m(x) = -1 + \sum_{k=\lceil mx \rceil}^{m} \frac{1}{k} \le -\ln x \le 1 - \ln x.$$

On the other hand, since $n \le m$, we have

$$f_m(x) = -1 + \sum_{k=\lceil mx \rceil}^{m} \frac{1}{k} = -1 + \sum_{k=n}^{m} \frac{1}{k} \ge -1 \ge -1 + \ln x.$$

Therefore, $|f_m(x)| \le 1 - \ln x$ for all $x \in (0, 1]$ and $m \in \mathbb{N}$. Hence,

$$|f_m(x)|^p \le (1 - \ln x)^p = g(x)$$

for all $x \in (0, 1]$ and $m \in \mathbb{N}$. Furthermore,

$$\begin{aligned}\int_0^1 g(x)\,\mathrm{d}x = \int_0^1 (1 - \ln x)^p\,\mathrm{d}x = \int_0^1 \left(\ln \frac{\mathrm{e}}{x}\right)^p \mathrm{d}x = \mathrm{e}\int_0^{1/\mathrm{e}} \left(\ln \frac{1}{u}\right)^p \mathrm{d}u \\ < \mathrm{e}\int_0^1 \left(\ln \frac{1}{u}\right)^p \mathrm{d}u = \mathrm{e}\,\Gamma(p+1) < \infty,\end{aligned}$$

where Γ denotes Euler's Gamma function. Thus, by the Lebesgue Dominated Convergence Theorem, we obtain

$$\begin{aligned}\lim_{m\to\infty} \frac{1}{m} \sum_{n=1}^{m} \left| -1 + \sum_{k=n}^{m} \frac{1}{k} \right|^p &= \lim_{m\to\infty} \int_0^1 |f_m(x)|^p\,\mathrm{d}x = \int_0^1 \lim_{m\to\infty} |f_m(x)|^p\,\mathrm{d}x \\ &= \int_0^1 |f(x)|^p\,\mathrm{d}x = \int_0^1 |1 + \ln x|^p\,\mathrm{d}x = C_p.\end{aligned}$$

This completes the proof of (15). □

We define the auxiliary vectors $\rho, d \in \mathbb{R}^m$ by

$$\rho(k) = \begin{cases} 0, & \text{if } k = 1, \\ \left(k \ln\left(\dfrac{k}{k-1}\right)\right)^{-1}, & \text{if } 1 < k \leq m, \end{cases}$$

and

$$d(k) = \begin{cases} \displaystyle\sum_{i=k}^{m-1} \frac{\rho(i+1) - \rho(i)}{i}, & \text{if } 1 \leq k \leq m, \\ 0, & \text{if } k = m. \end{cases}$$

The next result establishes several essential properties of the vectors d and ρ.

Lemma 2.2. *Let $m \in \mathbb{N}$. The vectors ρ and d defined above satisfy the following properties:*

(i) The vector ρ is nonnegative, strictly increasing, and bounded; explicitly,

$$0 \leq \rho(j-1) \leq \rho(j) < 1, \quad \text{for all } 1 < j \leq m. \tag{17}$$

(ii) The vector d is nonnegative and strictly decreasing, and satisfies the identity

$$\sum_{k=1}^{j} d(k) = j d(j) + \rho(j), \quad \text{for all } 1 \leq j \leq m. \tag{18}$$

Proof. We begin with the proof of (i). Define $f\colon (1,\infty) \to \mathbb{R}$ by

$$f(x) = x \ln\left(1 + \frac{1}{x-1}\right).$$

Differentiating f yields

$$f'(x) = \ln\left(1 + \frac{1}{x-1}\right) - \frac{1}{x-1} < 0 \quad \text{for all } x > 1,$$

where the inequality follows from the standard estimate $\ln(1+t) < t$ for $t > 0$. Thus, f is strictly decreasing on $(1,\infty)$, which implies that $1/f$ is strictly increasing. Consequently, the vector ρ is strictly increasing.

On the other hand, we have

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x \ln\left(\frac{x}{x-1}\right) = \ln\left(\lim_{x\to\infty}\left(1 + \frac{1}{x-1}\right)^x\right) = \ln(e) = 1.$$

Since f is strictly decreasing on $(1,\infty)$ with limit 1 at infinity, we get $1 < f(x) < \infty$ for all $x \in (1,\infty)$. Consequently, $0 < 1/f(x) < 1$ on $(1,\infty)$, which yields

$$\rho(j) < 1 \quad \text{for all } j \in \mathbb{N}.$$

We now turn to the proof of (ii). Let $k \in \{1, \ldots, m-1\}$. It is clear from the definition of d that

$$d(k) - d(k+1) = \frac{\rho(k+1) - \rho(k)}{k}. \tag{19}$$

Since ρ is strictly increasing, it follows immediately that d is strictly decreasing, which gives

$$0 = d(m) < d(m-1) < \cdots < d(2) < d(1).$$

To complete the proof of (ii), it remains to establish the Identity

$$\sum_{k=1}^{j} d(k) = jd(j) + \rho(j) \quad \text{for all } 1 \le j \le m.$$

We proceed by induction on j. The base case $j = 1$ holds trivially. Assume the Identity holds for some $1 \le j < m$. For $j+1$, we have

$$\begin{aligned}(j+1)d(j+1) + \rho(j+1) &= d(j+1) + jd(j+1) + \rho(j+1)\\ &= d(j+1) + jd(j) + \rho(j)\\ &= d(j+1) + \sum_{k=1}^{j} d(k) = \sum_{k=1}^{j+1} d(k),\end{aligned}$$

where the second equality follows from rewriting $d(j) - d(j+1) = \frac{\rho(j+1)-\rho(j)}{j}$ as

$$jd(j+1) + \rho(j+1) = jd(j) + \rho(j), \tag{20}$$

and the third equality applies the induction hypothesis. This completes the proof. □

In addition to the vectors ρ and d introduced previously, the following matrix will play a central role in our analysis. For $m \in \mathbb{N}$, consider the square matrix $T = (T_{i,j})_{i,j=1}^{m} \in \mathbb{R}^{m\times m}$ defined by

$$T_{i,j} = \begin{cases} d(\max\{i,j\}), & \text{if } i \ne j,\\ d(i) + \rho(i), & \text{if } i = j. \end{cases} \tag{21}$$

Explicitly, T has the symmetric form

$$T = \begin{pmatrix} d(1)+\rho(1) & d(2) & \cdots & d(m-1) & d(m)\\ d(2) & d(2)+\rho(2) & \cdots & d(m-1) & d(m)\\ d(3) & d(3) & \cdots & d(m-1) & d(m)\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ d(m-1) & d(m-1) & \cdots & d(m-1)+\rho(m-1) & d(m)\\ d(m) & d(m) & \cdots & d(m) & d(m)+\rho(m) \end{pmatrix}.$$

The following lemma establishes crucial properties of the matrix T.

Lemma 2.3. *For $m \in \mathbb{N}$, the matrix T defined in (21) is symmetric and nonnegative. Moreover, its operator p-norm on $\mathbb{R}^m$ satisfies*

$$\|T\|_p \le 1. \tag{22}$$

Proof. By definition, T is symmetric. Furthermore, since d and ρ are nonnegative by Lemma 2.2, all entries of T are nonnegative. Now, let $1 \le j \le m$. Then

$$\begin{aligned}\sum_{i=1}^{m} T_{i,j} &= \sum_{i=1}^{j-1} T_{i,j} + T_{j,j} + \sum_{i=j+1}^{m} T_{i,j} = \sum_{i=1}^{j-1} d(j) + \big(d(j)+\rho(j)\big) + \sum_{i=j+1}^{m} d(i)\\ &= jd(j) + \rho(j) + \sum_{i=j+1}^{m} d(i) = \sum_{i=1}^{m} d(i) = md(m) + \rho(m) = \rho(m),\end{aligned}$$

where we used (18) in the third and fifth equalities. Therefore,

$$\|T\|_1 = \rho(m) = \|T\|_\infty,$$

where the second equality follows from the symmetry of T. By the Riesz–Thorin interpolation theorem (see, e.g., [6, Corollary IV.2.3]),

$$\|T\|_p \le \rho(m) \le 1$$

for all $m \ge 1$, where the final inequality follows from (17). This ends the proof. □

The following proposition is of importance.

Proposition 2.4. *Let $1 \le p \le 2$, $m \in \mathbb{N}$ and $b \in \mathbb{R}^m$ a nonnegative vector. Then, the following inequality holds:*

$$\sum_{k=1}^{m} \left| \sum_{i=k}^{m} b(i) \left(-1 + \sum_{n=k}^{i} \frac{1}{n} \right) \right|^p \le \sum_{k=1}^{m} \left| \sum_{i=k}^{m} b(i) \ln \left(i \frac{(k-1)^{k-1}}{k^k} \right) \right|^p. \tag{23}$$

Proof. Fix $1 \le p \le 2$ and $m \in \mathbb{N}$, and let $b \in \mathbb{R}^m$ be a nonnegative vector. We define the vectors $A, B, \in \mathbb{R}^m$ for each $k \in \{1, \dots, m\}$ by

$$A(k) = \sum_{i=k}^{m} b(i) \ln \left(i \frac{(k-1)^{k-1}}{k^k} \right),$$
$$B(k) = \sum_{i=k}^{m} b(i) \left(-1 + \sum_{n=k}^{i} \frac{1}{n} \right).$$

To establish inequality (23), we have to show that

$$\|B\|_p \le \|A\|_p.$$

To this end, we will demonstrate that $B = TA$, where T is the matrix defined in (21). We have to show that

$$B(j) = \sum_{k=1}^{m} T_{j,k} A(k), \tag{24}$$

for all $1 \le j \le m$. Let $j \in \{1, \cdots, m\}$. Then

$$\sum_{k=1}^{m} T_{j,k} A(k) = \sum_{k=1}^{m} T_{j,k} \sum_{i=k}^{m} b(i) \ln \left(i \frac{(k-1)^{k-1}}{k^k} \right)$$
$$= \sum_{i=1}^{m} b(i) \sum_{k=1}^{i} T_{j,k} \ln \left(i \frac{(k-1)^{k-1}}{k^k} \right).$$

to prove (24) it is enough to show that

$$\sum_{k=1}^{i} T_{j,k} \ln \left(i \frac{(k-1)^{k-1}}{k^k} \right) = \begin{cases} -1 + \sum_{n=j}^{i} \frac{1}{n}, & \text{if } j \le i \le m, \\ 0, & \text{if } 1 \le i < j. \end{cases} \tag{25}$$

Notice that for $1 \leq i < j$, we have

$$\begin{aligned}\sum_{k=1}^{i} T_{j,k} \ln\left(i\frac{(k-1)^{k-1}}{k^k}\right) &= d(j)\sum_{k=1}^{i} \ln\left(i\frac{(k-1)^{k-1}}{k^k}\right)\\ &= d(j)\left(i\ln(i) - \sum_{k=1}^{i}\big(k\ln(k) - (k-1)\ln(k-1)\big)\right)\\ &= d(j)\big(i\ln(i) - i\ln(i)\big) = 0,\end{aligned}$$

where we have used that $x\ln x = 0$ for $x = 0$. Then, For $j \leq i \leq m$, we obtain

$$\begin{aligned}\sum_{k=1}^{i} T_{j,k} \ln\left(i\frac{(k-1)^{k-1}}{k^k}\right) &= \sum_{k=1}^{j-1} T_{j,k} \ln\left(i\frac{(k-1)^{k-1}}{k^k}\right) + T_{j,j}\ln\left(i\frac{(j-1)^{j-1}}{j^j}\right)\\ &\quad + \sum_{k=j+1}^{i} T_{j,k}\ln\left(i\frac{(k-1)^{k-1}}{k^k}\right)\\ &= d(j)\sum_{k=1}^{j} \ln\left(i\frac{(k-1)^{k-1}}{k^k}\right) + \rho(j)\ln\left(i\frac{(j-1)^{j-1}}{j^j}\right)\\ &\quad + \sum_{k=j+1}^{i} d(k)\ln\left(i\frac{(k-1)^{k-1}}{k^k}\right)\\ &= d(j)j\ln\left(\frac{i}{j}\right) + \rho(j)\ln\left(i\frac{(j-1)^{j-1}}{j^j}\right)\\ &\quad + \sum_{k=j+1}^{i} d(k)\ln\left(i\frac{(k-1)^{k-1}}{k^k}\right).\end{aligned}$$

That is

$$\begin{aligned}\sum_{k=1}^{i} T_{j,k} \ln\left(i\frac{(k-1)^{k-1}}{k^k}\right) &= d(j)j\ln\left(\frac{i}{j}\right) + \rho(j)\ln\left(i\frac{(j-1)^{j-1}}{j^j}\right)\\ &\quad + \sum_{k=j+1}^{i} d(k)\ln\left(i\frac{(k-1)^{k-1}}{k^k}\right).\end{aligned} \tag{26}$$

We now focus on the third summand on the right-hand side of the expression above.

$$\begin{aligned}\sum_{k=j+1}^{i} d(k)\ln\left(i\frac{(k-1)^{k-1}}{k^k}\right) &= \ln(i)\sum_{k=j+1}^{i} d(k) + \sum_{k=j+1}^{i} d(k)(k-1)\ln(k-1)\\ &\quad - \sum_{k=j+1}^{i} d(k)k\ln(k)\\ &= \ln(i)\sum_{k=j+1}^{i} d(k) + \sum_{k=j}^{i-1} d(k+1)k\ln(k) - \sum_{k=j+1}^{i} d(k)k\ln(k)\end{aligned}$$

$$
\begin{aligned}
&= \ln(i) \sum_{k=j+1}^{i} d(k) + d(j+1)j\ln(j) - d(i)i\ln(i) + \sum_{k=j+1}^{i-1} \big(d(k+1) - d(k)\big)k\ln(k) \\
&= \ln(i) \sum_{k=j+1}^{i} d(k) + d(j+1)j\ln(j) - d(i)i\ln(i) + \sum_{k=j+1}^{i-1} \big(\rho(k) - \rho(k+1)\big)\ln(k) \\
&= \ln(i) \sum_{k=j+1}^{i} d(k) + d(j+1)j\ln(j) - d(i)i\ln(i) \\
&\qquad + \rho(j+1)\ln(j+1) - \rho(i)\ln(i-1) + \sum_{k=j+2}^{i-1} \rho(k)\ln\left(\frac{k}{k-1}\right) \\
&= \ln(i) \sum_{k=j+1}^{i} d(k) + \big(d(j+1)j + \rho(j+1)\big)\ln(j) - \big(d(i)i + \rho(i)\big)\ln(i) \\
&\qquad + \rho(j+1)\ln\left(\frac{j+1}{j}\right) + \rho(i)\ln\left(\frac{i}{i-1}\right) + \sum_{k=j+2}^{i-1} \rho(k)\ln\left(\frac{k}{k-1}\right) \\
&= \ln(i) \sum_{k=j+1}^{i} d(k) + \big(d(j)j + \rho(j)\big)\ln(j) - \big(d(i)i + \rho(i)\big)\ln(i) \\
&\qquad + \rho(j+1)\ln\left(\frac{j+1}{j}\right) + \rho(i)\ln\left(\frac{i}{i-1}\right) + \sum_{k=j+2}^{i-1} \rho(k)\ln\left(\frac{k}{k-1}\right) \\
&= \ln(i) \sum_{k=j+1}^{i} d(k) + \ln(j) \sum_{k=1}^{j} d(k) - \ln(i) \sum_{k=1}^{i} d(k) + \sum_{k=j+1}^{i} \rho(k)\ln\left(\frac{k}{k-1}\right) \\
&= -\ln\left(\frac{i}{j}\right) \sum_{k=1}^{j} d(k) + \sum_{k=j+1}^{i} \rho(k)\ln\left(\frac{k}{k-1}\right) \\
&= -\ln\left(\frac{i}{j}\right)\big(jd(j) + \rho(j)\big) + \sum_{k=j+1}^{i} \rho(k)\ln\left(\frac{k}{k-1}\right),
\end{aligned}
$$

where we used (19) in the fourth equality, (20) in the seventh equality, and (18) in the eighth equality. To summarize, we have proved the following equality:

$$
\sum_{k=j+1}^{i} d(k)\ln\left(i\frac{(k-1)^{k-1}}{k^k}\right) = -\ln\left(\frac{i}{j}\right)\big(jd(j) + \rho(j)\big) + \sum_{k=j+1}^{i} \rho(k)\ln\left(\frac{k}{k-1}\right).
$$

Substituting this expression into (26), we obtain

$$
\begin{aligned}
\sum_{k=1}^{i} T_{j,k}\ln\left(i\frac{(k-1)^{k-1}}{k^k}\right) &= d(j)j\ln\left(\frac{i}{j}\right) + \rho(j)\ln\left(i\frac{(j-1)^{j-1}}{j^j}\right) \\
&\quad + \sum_{k=j+1}^{i} d(k)\ln\left(i\frac{(k-1)^{k-1}}{k^k}\right)
\end{aligned}
$$

$$= d(j)j \ln\left(\frac{i}{j}\right) + \rho(j) \ln\left(i\frac{(j-1)^{j-1}}{j^j}\right)$$
$$- \ln\left(\frac{i}{j}\right)(jd(j) + \rho(j)) + \sum_{k=j+1}^{i} \rho(k) \ln\left(\frac{k}{k-1}\right)$$
$$= -\rho(j)(j-1) \ln\left(\frac{j}{j-1}\right) + \sum_{k=j+1}^{i} \rho(k) \ln\left(\frac{k}{k-1}\right)$$
$$= -\rho(j)j \ln\left(\frac{j}{j-1}\right) + \sum_{k=j}^{i} \rho(k) \ln\left(\frac{k}{k-1}\right) = -1 + \sum_{k=j}^{i} \frac{1}{k}.$$

This ends the proof of (25). Consequently,

$$\|B\|_p = \|TA\|_p \leq \|T\|_p \|A\|_p \leq \|A\|_p,$$

where we have used $\|T\|_p \leq 1$ from (22). The proof is complete. □

Let S denote the left-shift operator defined by

$$(Sx)(n) = x(n+1)$$

for any sequence of real numbers $x = \{x(n)\}_{n\geq 1}$. The next two results due to G. Bennett [5] are needed to deduce a consequence of our main theorem.

Proposition 2.5. [5, Corollary 10.13] *Let $1 < p \leq 2$ and suppose that x is a nonnegative, nonincreasing sequence. Then*

$$\|x\|_p \leq (p-1)^{1/p} \|(C-S)x\|_p. \tag{27}$$

Moreover, $(p-1)^{1/p}$ is the best constant possible in (27).

Lemma 2.6. [5, Lemma 4.7] *Let $p > 1$. Then,*

$$\lim_{n\to\infty} n^{p-1} \sum_{k=n}^{\infty} \frac{1}{k^p} = \frac{1}{p-1}. \tag{28}$$

3. Main Results

In this section, we present our principal results and their corollaries.

Our main theorem is stated below. Although the cases $p = 1$ and $p = 2$, including the optimality of the constants $C_1 = 2/e$ and $C_2 = 1$, were established in [2], we include them here because our approach provides a new, unified proof.

Theorem 3.1. *Let $1 \leq p \leq 2$ and let $x \in \ell^p(\mathbb{N})$ be a nonnegative, nonincreasing sequence. Then*

$$\|(C^* - I)x\|_p \leq C_p^{1/p} \|x\|_p, \tag{29}$$

where $C_p^{1/p}$ is defined in (10). Moreover, the constant $C_p^{1/p}$ is optimal in (29).

Proof. Let $1 \leq p \leq 2$ and let $x \in \ell^p(\mathbb{N})$ be a nonnegative, nonincreasing sequence. Since finitely supported sequences are dense in $\ell^p(\mathbb{N})$, it suffices to prove the result for x with finite support. Then, there is a natural number m such that

$$x(1) \geq x(2) \geq \cdots \geq x(m) > 0.$$

and

$$x(n) = 0,$$

for all $n > m$. Let $f_x : (0, \infty) \to \mathbb{R}$ be the simple function given by

$$f_x(s) = \sum_{k=1}^{m} x(k)\chi_{(k-1,k]}(s).$$

It is clear that f_x is a nonnegative, nonincreasing and $\|f_x\|_{L^p(\mathbb{R}^+)} = \|x\|_{\ell^p}$. Applying inequality (11), we obtain

$$\|(H^* - I)f_x\|_{L^p(\mathbb{R}^+)} \leq C(p)^{1/p}\|f_x\|_{L^p(\mathbb{R}^+)} = C(p)^{1/p}\|x\|_{\ell^p}.$$

Therefore, in order to prove (29) it suffices to prove that

$$\|(C^* - I)x\|_{\ell^p} \leq \|(H^* - I)f_x\|_{L^p(\mathbb{R}^+)}. \tag{30}$$

Now, if $k - 1 < s \leq k$ we have that

$$\begin{aligned} H^* f_x(s) &= \int_s^\infty f_x(t)\frac{\mathrm{d}t}{t} = x(k)\int_s^k \frac{\mathrm{d}t}{t} + \sum_{n=k+1}^{m} x(n)\int_{n-1}^{n} \frac{\mathrm{d}t}{t} \\ &= x(k)\ln\left(\frac{k}{s}\right) + \sum_{n=k+1}^{m} x(n)\ln\left(\frac{n}{n-1}\right). \end{aligned}$$

So

$$H^* f_x(s) = \sum_{k=1}^{m}\left(x(k)\ln\left(\frac{k}{s}\right) + \sum_{n=k+1}^{m} x(n)\ln\left(\frac{n}{n-1}\right)\right)\chi_{(k-1,k]}(s).$$

Hence

$$(H^* - I)f_x(s) = \sum_{k=1}^{m}\left(x(k)\ln\left(\frac{k}{s}\right) - 1 + \sum_{n=k+1}^{m} x(n)\ln\left(\frac{n}{n-1}\right)\right)\chi_{(k-1,k]}(s).$$

Thus

$$\|(H^* - I)f_x\|^p_{L^p(\mathbb{R}^+)} = \sum_{k=1}^{m}\int_{k-1}^{k}\left|x(k)\left(\ln\left(\frac{k}{s}\right) - 1\right) + \sum_{n=k+1}^{m} x(n)\ln\left(\frac{n}{n-1}\right)\right|^p \mathrm{d}s.$$

On the other hand, we have that

$$\|(C^* - I)x\|^p_{\ell^p} = \sum_{k=1}^{m}\left| - x(k) + \sum_{n=k}^{m}\frac{x(n)}{n}\right|^p.$$

Then, proving the inequality (30) is equivalent to proving the following inequality

$$\begin{aligned} &\sum_{k=1}^{m}\int_{k-1}^{k}\left|x(k)\left(\ln\left(\frac{k}{s}\right) - 1\right) + \sum_{n=k+1}^{m} x(n)\ln\left(\frac{n}{n-1}\right)\right|^p \mathrm{d}s \\ &\qquad \geq \sum_{k=1}^{m}\left| - x(k) + \sum_{n=k}^{m}\frac{x(n)}{n}\right|^p. \end{aligned}$$

Now, by taking into account that $t \mapsto |t|^p$ is a convex function, we have that

$$\begin{aligned}
&\sum_{k=1}^{m} \int_{k-1}^{k} \left| x(k)\left(\ln\left(\frac{k}{s}\right) - 1\right) + \sum_{n=k+1}^{m} x(n) \ln\left(\frac{n}{n-1}\right)\right|^p \mathrm{d}s \\
&\quad \geq \sum_{k=1}^{m} \left| \int_{k-1}^{k} \left(x(k)\left(\ln\left(\frac{k}{s}\right) - 1\right) + \sum_{n=k+1}^{m} x(n) \ln\left(\frac{n}{n-1}\right)\right) \mathrm{d}s\right|^p \\
&\quad = \sum_{k=1}^{m} \left| x(k)(1-k) \ln\left(\frac{k}{k-1}\right) + \sum_{n=k+1}^{m} x(n) \ln\left(\frac{n}{n-1}\right)\right|^p.
\end{aligned}$$

So, to prove (30), it suffices to prove that the following inequality holds

$$\begin{aligned}
&\sum_{k=1}^{m} \left| x(k)(1-k) \ln\left(\frac{k}{k-1}\right) + \sum_{n=k+1}^{m} x(n) \ln\left(\frac{n}{n-1}\right)\right|^p \\
&\qquad\qquad \geq \sum_{k=1}^{m} \left| - x(k) + \sum_{n=k}^{m} \frac{x(n)}{n}\right|^p. \qquad (31)
\end{aligned}$$

Now, since x is a nonnegative, nonincreasing sequence, there is a nonnegative vector $b = \{b(i)\}_{i=1}^{m}$ such that $x(k) = \sum_{i=k}^{m} b(i)$, for all $k = 1, \cdots, m$. On the one hand, we have that

$$\begin{aligned}
&\sum_{k=1}^{m} \left| x(k)(1-k) \ln\left(\frac{k}{k-1}\right) + \sum_{n=k+1}^{m} x(n) \ln\left(\frac{n}{n-1}\right)\right|^p \\
&\quad = \sum_{k=1}^{m} \left| \sum_{i=k}^{m} b(i)(1-k) \ln\left(\frac{k}{k-1}\right) + \sum_{n=k+1}^{m} \sum_{i=n}^{m} b(i) \ln\left(\frac{n}{n-1}\right)\right|^p \\
&\quad = \sum_{k=1}^{m} \left| \sum_{i=k}^{m} b(i)(1-k) \ln\left(\frac{k}{k-1}\right) + \sum_{i=k+1}^{m} b(i) \sum_{n=k+1}^{i} \ln\left(\frac{n}{n-1}\right)\right|^p \\
&\quad = \sum_{k=1}^{m} \left| \sum_{i=k}^{m} b(i)(1-k) \ln\left(\frac{k}{k-1}\right) + \sum_{i=k+1}^{m} b(i) \ln\left(\frac{i}{k}\right)\right|^p \\
&\quad = \sum_{k=1}^{m} \left| \sum_{i=k}^{m} b(i)(1-k) \ln\left(\frac{k}{k-1}\right) + \sum_{i=k}^{m} b(i) \ln\left(\frac{i}{k}\right)\right|^p \\
&\quad = \sum_{k=1}^{m} \left| \sum_{i=k}^{m} b(i)\left(\ln\left(\frac{i}{k}\right) - (k-1) \ln\left(\frac{k}{k-1}\right)\right)\right|^p.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\sum_{k=1}^{m} \left| - x(k) + \sum_{n=k}^{m} \frac{x(n)}{n}\right|^p &= \sum_{k=1}^{m} \left| - \sum_{i=k}^{m} b(i) + \sum_{n=k}^{m} \frac{1}{n} \sum_{i=n}^{m} b(i)\right|^p \\
&= \sum_{k=1}^{m} \left| - \sum_{i=k}^{m} b(i) + \sum_{i=k}^{m} b(i) \sum_{n=k}^{i} \frac{1}{n}\right|^p \\
&= \sum_{k=1}^{m} \left| \sum_{i=k}^{m} b(i)\left(- 1 + \sum_{n=k}^{i} \frac{1}{n}\right)\right|^p.
\end{aligned}$$

Therefore, the inequality (31) is equivalent to

$$\sum_{k=1}^{m}\left|\sum_{i=k}^{m} b(i)\left(\ln\left(\frac{i}{k}\right)-(k-1)\ln\left(\frac{k}{k-1}\right)\right)\right|^p \geq \sum_{k=1}^{m}\left|\sum_{i=k}^{m} b(i)\left(-1+\sum_{n=k}^{i}\frac{1}{n}\right)\right|^p,$$

which is the statement of Proposition 2.4.

The optimality of the constant $C_p^{1/p}$ is proven in Lemma 2.1 by considering the sequence $\{C^*e_m\}_{m\geq 1}$. □

Let $C^{*2} = C^* \circ C^*$. Theorem 3.1 has the following equivalent formulation.

Theorem 3.2. *Let $1 \leq p \leq 2$ and let x be a nonnegative sequence such that $C^*x \in \ell^p(\mathbb{N})$. Then the following inequality holds and is sharp:*

$$\|(C^{*2}-C^*)x\|_p \leq C_p^{1/p}\|C^*x\|_p. \tag{32}$$

Proof. Let x be a nonnegative sequence and set $y = C^*x$. Clearly, y is a nonnegative, nonincreasing sequence, and by hypothesis, $y \in \ell^p(\mathbb{N})$. Applying (29) directly to y yields (32).

Conversely, let $x = \{x(k)\}_{k\geq 1} \in \ell^p(\mathbb{N})$ be a nonnegative, nonincreasing sequence, and define $y = \{y(k)\}_{k\geq 1}$ by

$$y(k) = k\big(x(k)-x(k+1)\big), \quad k \geq 1. \tag{33}$$

Since x is nonnegative and nonincreasing, y is clearly nonnegative. By assumption, $x = C^*y \in \ell^p(\mathbb{N})$. Applying (32) directly to y completes the proof of (29).

To establish optimality, let $\{x_n\}_{n\geq 1}$ be a sequence in $\ell^p(\mathbb{N})$ of nonnegative, nonincreasing sequences such that

$$\lim_{n\to\infty}\frac{\|(C^*-I)x_n\|_p}{\|x_n\|_p} = C_p^{1/p}.$$

Considering $\{y_n\}_{n\geq 1}$ defined as in (33), we have $C^*y_n = x_n \in \ell^p(\mathbb{N})$ for each $n \geq 1$, which gives

$$\lim_{n\to\infty}\frac{\|(C^*-I)C^*y_n\|_p}{\|C^*y_n\|_p} = \lim_{n\to\infty}\frac{\|(C^*-I)x_n\|_p}{\|x_n\|_p} = C_p^{1/p}.$$

Conversely, to verify optimality from the dual perspective, let $\{y_n\}_{n\geq 1} \subset \ell^p(\mathbb{N})$ be a sequence of nonnegative sequences satisfying

$$\lim_{n\to\infty}\frac{\|(C^*-I)C^*y_n\|_p}{\|C^*y_n\|_p} = C_p^{1/p}.$$

Defining $x_n = C^*y_n \in \ell^p(\mathbb{N})$ for each $n \geq 1$, it follows that

$$\lim_{n\to\infty}\frac{\|(C^*-I)x_n\|_p}{\|x_n\|_p} = \lim_{n\to\infty}\frac{\|(C^*-I)C^*y_n\|_p}{\|C^*y_n\|_p} = C_p^{1/p},$$

which completes the proof. □

By combining Proposition 2.5 and Theorem 3.1, we obtain the optimal constant in an inequality relating $C-S$ and C^*-I for decreasing sequences.

Corollary 3.3. *Let $1 < p \leq 2$ and let $x \in \ell^p(\mathbb{N})$ be a nonnegative, nonincreasing sequence. Then*

$$\|(C^* - I)x\|_p \leq C_p^{1/p}(p-1)^{1/p}\|(C-S)x\|_p \tag{34}$$

Furthermore, the constant $C(p)^{1/p}(p-1)^{1/p}$ is the best possible.

Proof. Let $1 < p \leq 2$ and let $x \in \ell^p(\mathbb{N})$ be a nonnegative, nonincreasing sequence. Applying (29) and (27), we obtain

$$\|(C^* - I)x\|_p \leq C(p)^{1/p}\|x\|_p \leq C(p)^{1/p}(p-1)^{1/p}\|(C-S)x\|_p.$$

To see the optimality of the constant in (34) we consider the sequence $\{C^*e_m\}_{m\geq 1}$. We have seen in (14) that

$$|(C^* - I)C^*e_m| = \begin{cases} \left|\frac{1}{m} - 1 + \sum_{k=n}^{m} \frac{1}{k}\right|, & \text{if } n \leq m, \\ 0, & \text{if } n > m. \end{cases}$$

A straightforward calculation yields

$$|(C-S)C^*e_m(n)| = \begin{cases} 0, & \text{if } n < m, \\ \frac{1}{n}, & \text{if } n \geq m. \end{cases}$$

$$\lim_{m\to\infty} \frac{\|(C^* - I)C^*e_m\|_p^p}{\|(C-S)C^*e_m\|_p^p} = \lim_{m\to\infty} \frac{\frac{1}{m^p}\sum_{n=1}^{m}\left|-1+\sum_{k=n}^{m}\frac{1}{k}\right|^p}{\sum_{k=m}^{\infty}\frac{1}{n^p}} =$$

$$= \frac{\lim_{m\to\infty}\frac{1}{m}\sum_{n=1}^{m}\left|-1+\sum_{k=n}^{m}\frac{1}{k}\right|^p}{\lim_{m\to\infty} m^{p-1}\sum_{k=m}^{\infty}\frac{1}{n^p}} = C(p)(p-1).$$

□

Acknowledgements: The first author extends his sincere gratitude to his PhD supervisors, Santiago Boza and Javier Soria, for their continuous guidance and support throughout his doctoral studies. Additionally, he thanks the Institute of Mathematics of the Czech Academy of Sciences for hosting his research visit, during which part of this work was conducted.

Funding: The first author was partially supported by grants PID2020-113048GB-I00 and PID2024-155917NB-I00, funded by MCIN/AEI/10.13039/501100011033. The second author was partially supported by RVO: 67985840, Institute of Mathematics of the Czech Academy of Sciences.

Data availability: No data were used for the research described in the article.

Conflicts of Interest: The authors declare that they have no conflict of interest.

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