ON SHARP REVERSE HARDY-TYPE INEQUALITIES FOR OSCILLATION OPERATORS IN THE DISCRETE AND CONTINUOUS SETTINGS

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Abstract. Recently, several authors have studied the problem of determining optimal constants in L^p-norm for Hardy-type inequalities. In this work, we continue this investigation by establishing sharp lower bounds for the norm of the difference between the Hardy operator and the identity when acting on various cones in $L^p(\mathbb{R}^+)$, with $1 < p < \infty$. This operator quantifies the oscillation of a function relative to its average. We also obtain optimal lower bounds for its adjoint. In the discrete setting, we conduct a similar analysis for the difference between the Cesàro and the identity, the difference between the Cesàro and the left-shift operators, as well as for its adjoint. As an application, we identify norms on $L^p(\mathbb{R}^+)$ and on $\ell^p(\mathbb{N})$ that are equivalent to their standard p-norm, and we provide the exact optimal constants for these equivalences.

1. Introduction

For a sequence $\{x(n)\}_{n\geq 1} \subset \mathbb{R}$, the Cesàro operator C, the left-shift operator S, and their transposes are defined by

$$Cx(n) = \frac{1}{n}\sum_{k=1}^{n} x(k), \qquad C^*x(n) = \sum_{k=n}^{\infty} \frac{x(k)}{k},$$

$$Sx(n) = x(n+1), \qquad \text{and} \qquad S^*x(n) = x(n-1),$$

where we set $S^*x(1) = 0$.

The continuous counterparts of the C and C^* operators are the classical Hardy averaging operator H and its adjoint H^*. Both operators are defined by

$$Hf(t) = \frac{1}{t}\int_0^t f(s)\,ds \quad \text{and} \quad H^*f(t) = \int_t^\infty \frac{f(s)}{s}\,ds,$$

provided that the integrals make sense for a function f on $(0,\infty)$.

It is well known that these operators play a fundamental role in Analysis. For instance, the operator H is of great importance in the study of weighted norm inequalities for classical maximal operators arising in Harmonic Analysis. The boundedness of H on $L^p(\mathbb{R}^+)$, where $1 < p \leq \infty$, follows from Hardy's inequality (cf. [11,

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p. 240]):

$$\|Hf\|_{L^p(\mathbb{R}^+)} \leq p' \, \|f\|_{L^p(\mathbb{R}^+)}, \tag{1}$$

where $p' = \frac{p}{p-1}$ if $1 < p < \infty$, and $p' = 1$ if $p = \infty$. Moreover, p' is the best possible constant in (1).

It is unfortunate that this operator is not invertible on $L^p(\mathbb{R}^+)$, and therefore it is not possible to find a constant $c(p)$, depending only on p, such that a reverse $L^p(\mathbb{R}^+)$-Hardy inequality

$$\|Hf\|_{L^p(\mathbb{R}^+)} \geq c(p) \, \|f\|_{L^p(\mathbb{R}^+)}, \tag{2}$$

holds in general, even for nonnegative functions. However, replacing the L^p-norm with the BMO-norm yields the equivalence between Hf and f in the general setting (see [25]). Nevertheless, if f is positive and nonincreasing on $\mathbb{R}^+$, then the inequality (2) is valid with $c(p) = (p')^{1/p}$, where p' is defined as before. Moreover, the constant $c(p)$ is sharp. (cf. [20] and [21]).

A similar situation arises with the Cesàro operator C: there is no constant $k(p)$, depending only on p, such that

$$\|Cx\|_{\ell^p(\mathbb{N})} \geq k(p) \, \|x\|_{\ell^p(\mathbb{N})}$$

holds for all sequences x. However, when x is positive and nonincreasing, the inequality holds with $k(p) = \zeta(p)^{1/p}$, for $1 < p < \infty$, and $k(p) = 1$ when $p = \infty$, where $\zeta(p)$ denotes the Riemann zeta function evaluated at p. Furthermore, these constants $k(p)$ are optimal (cf. [5], [19], and [21]). It is worth noting that the constants differ between the discrete and continuous cases, whereas for the classical inequalities they coincide: $\|H\|_{L^p(\mathbb{R}^+)} = p'$ (cf. [11, p. 240]) and $\|C\|_{\ell^p(\mathbb{N})} = p'$ (see [18]).

The situation changes entirely, as we will show in this work, when we consider $H - I$ instead of H, since in this case there exists a constant $C(p)$ such that

$$\|(H - I)f\|_p \geq C(p) \, \|f\|_p, \tag{3}$$

where $f \in L^p(\mathbb{R}^+)$, with $1 < p < \infty$.

The operator $H - I$ arises naturally in other, widely separated contexts, for example, the complex Beurling-Ahlfors transform reduces to $H - I$ when restricted to radial functions, see [1] and [23]. When $p = 2$, $H - I$ is a Hilbert space isomorphism that is unitarily equivalent to the unilateral shift, see [10]. It has been linked to Laguerre polynomials in [17]. In the theory of interpolation of operators, it acts on rearrangements of functions (nonnegative, nonincreasing functions) to give equivalent norm in Lorentz spaces, see [4, p. 384].

Recent work on $H - I$, $H^* - I$, and their discrete analogues can be found, for instance, in [2, 3, 8, 9, 15, 22].

In particular, Kolyada [15] completed the work of [17] and [7] by determining the optimal constants in the L^p-norm equivalence between Hf and H^*f for nonnegative functions. A key idea in this work connects it with finding optimal upper and lower bounds for the operator $H - I$ restricted to nonnegative, nonincreasing functions.

Theorem 1.1. [15, Theorem 1.2]
Let $1 < p < \infty$ and let f be a nonnegative, nonincreasing function. If $1 < p \leq 2$, then

$$(p-1)\|(H - I)f\|_p \leq \|f\|_p \leq (p-1)^{1/p}\|(H - I)f\|_p, \tag{4}$$

and if $2 \leq p < \infty$*, then*

$$(p-1)^{1/p}\|(H-I)f\|_p \leq \|f\|_p \leq (p-1)\|(H-I)f\|_p. \tag{5}$$

The constants $p-1$ *and* $(p-1)^{1/p}$ *in* (4) *and* (5) *are the best possible.*

Regarding the operator $H^* - I$, the authors in [3] finalized the investigation initiated in [2] by proving the following theorem.

Theorem 1.2. [3, Theorem 1.2]
Let $1 \leq p < \infty$ *and let* f *be a nonnegative and nonincreasing. If* $1 < p \leq 2$ *then,*

$$C_p^{-1/p}\|(H^*-I)f\|_p \leq \|f\|_p \leq (p-1)^{-1}\|(H^*-I)f\|_p, \tag{6}$$

and if $2 \leq p < \infty$ *then,*

$$(p-1)^{-1}\|(H^*-I)f\|_p \leq \|f\|_p \leq C_p^{-1/p}\|(H^*-I)f\|_p. \tag{7}$$

The constants $(p-1)^{-1}$ *and* $C_p^{-1/p}$ *are optimal in both* (6) *and* (7)*, where the constant* C_p *is defined by* $C_p = \int_0^1 |1 + \ln x|^p dx$*.*

In the discrete setting the situation is completely different, as there is no constant $C(p) > 0$, with $1 < p < \infty$, such that

$$\|x\|_p \leq C(p)\,\|(C^* - I)x\|_p,$$

for all $x \in \ell^p$. In fact, no such constant exists even when x is restricted to the cone of nonnegative, nonincreasing sequences in ℓ^p. This failure stems from the non-injectivity of the operator $C^* - I$. Indeed, if a constant $C(p) > 0$ existed, then taking $x = (1, 0, 0, \ldots)$ would give

$$0 < C(p) = C(p)\|x\|_p \leq \|(C^* - I)x\|_p = 0,$$

a contradiction.

With regard to sharp reverse inequalities for the operator $C - I$, we have the following result due to G. J. O. Jameson:

Theorem 1.3. [13, Corollary 5]
Let $x \in \ell^2$*. Then the following inequality is sharp:*

$$\|x\|_2 \leq \sqrt{2}\|(C-I)x\|_2. \tag{8}$$

Sharp inequalities associated with the operator $C - S$ and its transpose have also been widely studied in the literature. In particular, G. Bennett [6], and S. Boza and J. Soria [9], obtained the following result.

Theorem 1.4 ([6, 9])**.** *Let* $1 < p < \infty$ *and let* $x \in \ell^p(\mathbb{N})$ *be a nonnegative, nonincreasing sequence. If* $1 < p \leq 2$*, then*

$$(p-1)\|(C-I)x\|_p \leq \|x\|_p \leq (p-1)^{1/p}\|(C-S)x\|_p. \tag{9}$$

If $2 \leq p < \infty$*, then*

$$(p-1)^{1/p}\|(C-I)x\|_p \leq \|x\|_p. \tag{10}$$

Furthermore, the constants obtained in (9) *and in* (10) *are sharp.*

V. Kolyada also contributed to this line of research by establishing the next theorem.

Theorem 1.5. [16, Theorem 3.1]
Let $2 \leq p < \infty$ and let $x = \{x(n)\}_{n\geq 1}$ be a nonnegative sequence. Then

$$\|C^* x\|_p \leq (p-1)\|Cx\|_p.$$

The constant is optimal.

By using the following identity (see [6, 9])

$$C \circ C^* = C + SC^*,$$

and taking into account that, for a positive sequence x, the sequence C^*x is positive and nonincreasing, it follows that Theorem 1.5 is equivalent to the result below.

Theorem 1.6. *Let $2 \leq p < \infty$ and let $x = \{x(n)\}_{n\geq 1} \in \ell^p(\mathbb{N})$ be a nonnegative, nonincreasing sequence. Then*

$$\|x\|_p \leq (p-1)\|(C-S)x\|_p. \tag{11}$$

The constant is the best possible.

The case $p = 2$ in (11) was obtained independently by G. Jameson in [12].

A key observation here is that Theorem 1.4 together with Theorem 1.6 serves as the analogue of Theorem 1.1 in the continuous setting, where we have the same constants. A closer examination of inequalities (4) and (5) and their discrete counterparts, reveal the presence of $C - S$ instead of $C - I$. This contrast arises because, in the continuous case, we have the identity

$$H \circ H^* = H^* \circ H = H + H^*, \tag{12}$$

(see [14]) whereas in the discrete case, the corresponding relation is

$$C \circ C^* = C + SC^* = CS^* + C^* \neq C + C^*. \tag{13}$$

The goal of this paper is to determine the remaining optimal constants in (3) for all $1 < p < \infty$, in the settings where f is an arbitrary, nonnegative, or nonnegative and nonincreasing function. Moreover, we establish these results not only for the operator $H - I$, but also for $C - S$ and their corresponding transposes. We also carry out a similar study for the operator $C - I$.

In Section 2, we introduce the notations that will be used throughout the paper and present several auxiliary results needed for establishing the main theorems of this work. Section 3 is devoted to studying sharp lower bounds for $H - I$ on different cones in $L^p(\mathbb{R}^+)$, with $1 < p < \infty$. In Section 4, we carry out a similar analysis for the adjoint operator $H^* - I$. Section 5 turns to the discrete setting, where we investigate the operator $C - S$ on $\ell^p(\mathbb{N})$, again for $1 < p < \infty$, following the approach developed in the continuous case. Section 6 addresses the best lower bounds for the transpose operator $(C - S)^*$. Finally, Section 7 provides sharp lower bounds for the operator $C - I$ on $\ell^p(\mathbb{N})$, for $1 < p \leq 2$.

2. Preliminaries

In this section we fix the notations used throughout the paper and state several auxiliary results that will be useful for its development.

In what follows, we let X denote either $L^p(\mathbb{R}^+)$ or $\ell^p(\mathbb{N})$, with $1 < p < \infty$. The subsets X_+ and X_{dec} of X are defined by

$$X_+ = \{x \in X : x \text{ is nonnegative}\},$$

and

$$X_{\text{dec}} = \{x \in X : x \text{ is nonnegative and decreasing}\}.$$

In several of the arguments below, it is important that the power functions involved be defined on all of $\mathbb{R}$. For this purpose, we let

$$\mathbb{F} = \left\{ \frac{2i}{2j+1} : i, j \in \mathbb{N} \right\} \cap (1, 2),$$

and

$$\mathbb{E} = \left\{ \frac{2i}{2j+1} : i, j \in \mathbb{N} \right\} \cap (2, \infty).$$

The sets $\mathbb{F}$ and $\mathbb{E}$ are a dense subsets of $(1, 2]$ and $[2, \infty)$ respectively. The set $\mathbb{F}$ was first introduced in [2], while $\mathbb{E}$ was defined in [22]. If $p \in \mathbb{F} \cup \mathbb{E}$, it is easy to see that the function $t \mapsto t^p$ is well defined and continuously differentiable on $\mathbb{R}$. Its derivative is strictly increasing and

$$pa^{p-1}(b-a) \le b^p - a^p \le pb^{p-1}(b-a) \tag{14}$$

holds for all $a, b \in \mathbb{R}$.

Given $p \in (1, \infty)$, we use p' to denote its conjugate exponent, defined by $p' = \frac{p}{p-1}$.

Let $1 < p < \infty$, and consider the function $f_p : [0, 1/2] \to \mathbb{R}$ defined by

$$f_p(t) = p\, t^{p-1} + (1-t)^p - t^p. \tag{15}$$

Observe that if $p \in \mathbb{F} \cup \mathbb{E}$, then f_p can be extended to the entire real line $\mathbb{R}$. This function plays an important role in the development of this work. The following two lemmas related to the function f_p are of importance.

Lemma 2.1. [2, Lemma 3.1] *Let $1 < p < 2$. Consider the function f_p defined as in (15). Then, f_p has a unique critical point t_p in $(0, \frac{1}{2})$, $f_p(t_p) = M_p$, and M_p is a continuous function of p. If, in addition, $p \in \mathbb{F}$, we have that t_p is the unique critical point of f_p on $\mathbb{R}$ and $f_p(t) \le M_p$, for all $t \in \mathbb{R}$.*

Lemma 2.2. [22, Lemma 7] *Let $2 < p < \infty$. Consider the function f_p defined as in (15). Then, f_p has a unique critical point t_p in $(0, \frac{1}{2})$, $f_p(t_p) = m_p$, and m_p is a continuous function of p. If, in addition, $p \in \mathbb{E}$, we have that t_p is the unique critical point of f_p on $\mathbb{R}$ and $f_p(t) \ge m_p$, for all $t \in \mathbb{R}$.*

An important property of the constants $m_p^{-1/p}$, which we will use later, is that

$$m_{p'}^{-1/p'} = \max_{t \in [0,1/2]} f_{p'}(t)^{-1/p'} = \max_{t \in [0,1/2]} f_p(t)^{1/p} = M_p^{1/p}, \tag{16}$$

where $1 < p \le 2$, p' denotes the conjugate exponent of p, and f_p is defined as in (15). The proof of the identity (16) can be found in [2].

Other important lemmas are the following:

Lemma 2.3. *If $p \in (1, 2]$ and $0 < t \le s$, then*

(i)

$$f_1(s,t) = s^p \left[\left(1 - \frac{t}{s}\right)^p - 1 + \frac{pt}{s} \right] - t^p \le 0,$$

(ii)

$$f_2(s,t) = p\, t^{p-1}(t-s) - t^p + s^p \left[1 - \left(1 - \frac{t}{s}\right)^p \right] \le 0.$$

Proof. The proof is similar to that in [8, Lemma 3.1]. □

Lemma 2.4. *Let $1 < p < 2$. Then,*

$$F_p(t) = (p-1)t^p - pt^{p-1} - (t-1)^p + 1 \leq 0, \quad t \in [1, \infty),$$

and if $p \in \mathbb{F}$, then

$$F_p(t) = (p-1)t^p - pt^{p-1} - (t-1)^p + 1 \leq 0, \quad t \in [0, 1].$$

Proof. We have that,

$$\begin{aligned} F_p'(t) &= p(p-1)t^{p-1} - p(p-1)t^{p-2} - p(t-1)^{p-1} \\ &= p\{(p-1)(t^{p-1} - t^{p-2}) - (t-1)^{p-1}\}, \end{aligned}$$

for all $t > 1$. Let $t > 1$. As $t > t-1 > 0$ and $p - 2 < 0$, we get

$$0 < t^{p-2} < (t-1)^{p-2},$$

as $p - 1 < 1$ we get,

$$0 < (p-1)t^{p-2} < t^{p-2} < (t-1)^{p-2},$$

which means,

$$(p-1)t^{p-2} < (t-1)^{p-2} = (t-1)^{p-1}\frac{1}{(t-1)}.$$

Therefore,

$$(p-1)(t^{p-1} - t^{p-2}) = (p-1)t^{p-2}(t-1) < (t-1)^{p-1},$$

so,

$$(p-1)(t^{p-1} - t^{p-2}) - (t-1)^{p-1} < 0.$$

Then,

$$F_p'(t) = p\{(p-1)(t^{p-1} - t^{p-2}) - (t-1)^{p-1}\} < 0,$$

for all $t > 1$, which means that F_p is strictly decreasing in the interval $(1, \infty)$. Therefore, $F_p(t) < F_p(1) = 0$, for all $t > 1$.

Now let us analyze the case where $0 < t < 1$. Since $p \in \mathbb{F}$, we have that $(t-1)^p = (1-t)^p$. Then

$$F_p(t) = (p-1)t^p - pt^{p-1} - (1-t)^p + 1.$$

We compute the derivative of F_p:

$$\begin{aligned} F_p'(t) &= p(p-1)t^{p-1} - p(p-1)t^{p-2} + p(1-t)^{p-1} \\ &= p\{(p-1)(t^{p-1} - t^{p-2}) + (1-t)^{p-1}\} \\ &= p\{(p-1)t^{p-2}(t-1) + (1-t)^{p-1}\} \\ &= p\{-(p-1)t^{p-2}(1-t) + (1-t)^{p-1}\} \\ &= p\{(1-t)((1-t)^{p-2} - (p-1)t^{p-2})\} \\ &= p(1-t)G_p(t), \end{aligned}$$

where $G_p(t) = (1-t)^{p-2} - (p-1)t^{p-2}$. We prove that F_p' has exactly one zero in the interval $(0, 1)$. For this purpose, it is enough to check that G_p has the same property. Indeed, we have that

$$\begin{aligned} G_p'(t) &= (p-2)(1-t)^{p-3}(-1) - (p-1)(p-2)t^{p-3} \\ &= -(p-2)(1-t)^{p-3} + (p-1)t^{p-3}) > 0, \end{aligned}$$

for all $t \in (0,1)$. Then G_p is strictly increasing in the interval $(0,1)$. Observe that

$$\lim_{t\to 0^+} G_p(t) = -\infty \quad ; \quad \lim_{t\to 1^-} G_p(t) = \infty.$$

So G_p has only one zero in the interval $(0,1)$. Then F_p has only one local extremal in $(0,1)$. Since $F_p(0) = F_p(1) = 0$ and $F_p(1/2) = (2^p - p - 2)/2^p < 0$, we can conclude that F_p has local minimum in $(0,1)$ and

$$F_p(t) \le 0, \quad \forall t \in [0,1].$$

□

The identity (12) can equivalently be written as

$$(H - I)(H^* - I) = (H^* - I)(H - I) = I. \tag{17}$$

While a direct analogue of (17) does not hold in the discrete setting due to (13), a closely related equalities remain valid, as stated in the following lemma.

Lemma 2.5. *The following identities holds:*

$$(C^* - I)S^*(C - S) = I, \tag{18}$$

and

$$(C - I)(C - S)^* = S^*. \tag{19}$$

Proof. The operator C^* can be expressed as (see [6])

$$C^* = (C^* - I)S^*C. \tag{20}$$

Let $e_1 = (1, 0, 0, \dots)$. Since $(C^* - I)e_1 = 0$, we have

$$(C^* - I)S^*S = (C^* - I). \tag{21}$$

Combining (20) and (21), we obtain

$$(C^* - I)S^*(C - S) = (C^* - I)S^*C - (C^* - I)S^*S = C^* - (C^* - I) = I,$$

which proves (18).

As for (19), it follows from (13) that

$$(C - I)(C - S)^* = (C - I)(C^* - S^*) = CC^* - CS^* - C^* + S^* = S^*.$$

This completes the proof. □

3. Optimal lower bounds related to $H - I$

The aim of this section is to determine the optimal constants $B(p)$ in the inequality

$$\|f\|_p \le B(p)\,\|(H - I)f\|_p, \tag{22}$$

where f belongs to $L^p(\mathbb{R}^+)$ or $L^p_+(\mathbb{R}^+)$, and for the range $1 < p < \infty$. The best lower bounds for $H - I$ on nonnegative, nonincreasing functions in $L^p(\mathbb{R}^+)$, with $1 < p < \infty$, are given in [15].

We begin with the general case. The corresponding sharp upper bound was obtained by M. Strzelecki in [23]. In [24] the same author investigated the inequality in various cones of Lebesgue spaces, including weighted cases associated with power weights. Later, and independently, G. Sinnamon derived the same constant in [22], where the discrete setting was also considered. In these works, it was shown that the optimal constant $A(p)$ in the inequality

$$\|(H - I)f\|_p \le A(p)\,\|f\|_p$$

is given by

$$A(p) = \begin{cases} (p-1)^{-1}, & \text{if } 1 < p \leq 2, \\ m_p^{-1/p}, & \text{if } 2 \leq p < \infty, \end{cases} \tag{23}$$

where $m_p = \min_{t \in [0,1/2]} f_p(t)$ and f_p is defined as in (15).

The result below gives the exact value of $B(p)$ in (22) for a general function f.

Theorem 3.1. *Let $1 < p < \infty$ and let f be an arbitrary function. If $1 < p \leq 2$, then*

$$\|f\|_p \leq M_p^{1/p} \, \|(H-I)f\|_p, \tag{24}$$

whereas if $2 \leq p < \infty$, then

$$\|f\|_p \leq (p-1) \, \|(H-I)f\|_p. \tag{25}$$

Moreover, all the constants in (24) and (25) are the best possible.

Proof. Let $1 < p < \infty$. The map $A(p)$ in (23) implies, in particular, that the operator norm of $H-I$ is given by

$$\|H-I\|_p = A(p) = \begin{cases} (p-1)^{-1}, & \text{if } 1 < p \leq 2, \\ m_p^{-1/p}, & \text{if } 2 \leq p < \infty. \end{cases}$$

Taking duality into account and using (16), we obtain that

$$\|H^* - I\|_p = \|H-I\|_{p'} = \begin{cases} M_p^{1/p}, & \text{if } 1 < p \leq 2, \\ p-1, & \text{if } 2 \leq p < \infty. \end{cases} \tag{26}$$

We focus on proving only (24), since the proof of (25) is analogous. Assume that $1 < p \leq 2$, and let f be an arbitrary function. If $(H-I)f \notin L^p(\mathbb{R}^+)$, then (24) holds trivially. Otherwise, assuming $(H-I)f \in L^p(\mathbb{R}^+)$, by taking into account (17) and applying (26) yields

$$\|f\|_p = \|(H^*-I)(H-I)f\|_p \leq M_p^{1/p} \, \|(H-I)f\|_p.$$

Let us now prove the optimality of the constant $M_p^{1/p}$ in (24). By taking into account Lemma 2.1, let $t_p \in (0, 1/2)$ such that $f_p(t_p) = M_p$, where f_p is defined as in (15). Let $r = 1/t_p$ and let g be the function given by

$$g(x) = -\chi_{(0,1)}(x) + (r-1)x^{-r}\chi_{[1,\infty)}.$$

Then

$$Hg(x) = -\chi_{(0,1)}(x) - x^{-r}\chi_{[1,\infty)}(x).$$

Therefore,

$$|(H-I)g(x)| = rx^{-r}\chi_{[1,\infty)}(x).$$

Hence,

$$\frac{\|g\|_p^p}{\|(H-I)g\|_p^p} = \frac{rp - 1 + (r-1)^p}{r^p} = pt_p^{p-1} - t_p + (1-t_p^p) = M_p.$$

For the optimality of the constant $(p-1)$ in (25), note that the right-hand side of inequality (5) implies the existence of a sequence of nonnegative, nonincreasing functions $\{f_m\}_{m\in\mathbb{N}}$ such that

$$\lim_{m\to\infty} \frac{\|f_m\|_p}{\|(H-I)f_m\|_p} = (p-1).$$

This concludes the proof. □

Remark 3.2. Note that (23), together with (24) and (25), implies the existence of optimal constants $c(p), C(p) > 0$ such that

$$c(p)\,\|(H-I)f\|_p \le \|f\|_p \le C(p)\,\|(H-I)f\|_p,$$

for all $f \in L^p(\mathbb{R}^+)$, with $1<p<\infty$. It is now easy to see that $\|(H-I)f\|_p$ is an equivalent norm in $L^p(\mathbb{R}^+)$.

Let us now identify the best constant $B(p)$ in (22) in the case of nonnegative functions. The corresponding upper bounds can be found in [2, 8], where the authors proved that the best constant $A(p)$ in

$$\|(H-I)f\|_p \le A(p)\,\|f\|_p,$$

for nonnegative function $f \in L^p(\mathbb{R}^+)$, is given by

$$A(p) = \begin{cases} (p-1)^{-1}, & \text{if } 1<p\le 2, \\ 1, & \text{if } 2\le p<\infty. \end{cases}$$

Now, we are able to state the analogous result to Theorem 3.1, for nonnegative functions.

Theorem 3.3. *Let $1<p<\infty$ and let f be a nonnegative function. If $1<p\le 2$, then the following sharp estimate holds:*

$$\|f\|_p \le \|(H-I)f\|_p, \tag{27}$$

and if $2\le p<\infty$, then the following sharp estimate holds:

$$\|f\|_p \le (p-1)\|(H-I)f\|_p, \tag{28}$$

Proof. The proof of (27) is analogous to the proof of [8, Theorem 3.2], but using Lemma 2.3 instead of [8, Lemma 3.1], with the obvious modifications.

The function $f_r = \chi_{[1,1+r]}$, where $r>0$, shows the optimality of the constant 1 in (27) since $\lim_{r\to 0^+} \frac{\|(H-I)f_r\|_p}{\|f_r\|_p} = 1$.

The inequality (28) is a consequence of (25), and the optimality of the constant $(p-1)$ in (28) is established by considering the same functions used to prove its optimality in (25). □

4. Optimal lower bounds related to $H^* - I$

The aim of this section is to determine the optimal constants $B(p)$ in the inequality

$$\|f\|_p \le B(p)\,\|(H^*-I)f\|_p, \tag{29}$$

where f belongs to $L^p(\mathbb{R}^+)$ or $L^p_+(\mathbb{R}^+)$, and for the range $1<p<\infty$. The best lower bounds for H^*-I on nonnegative, nonincreasing functions in $L^p(\mathbb{R}^+)$, with $1<p<\infty$, are given in [3].

We begin with the general case. We have seen in (26) that the corresponding upper bound $A(p)$ in

$$\|(H^* - I)f\|_p \le A(p)\,\|f\|_p,$$

is given by

$$A(p) = \|H^* - I\|_p = \begin{cases} M_p^{1/p}, & \text{if } 1 < p \le 2, \\ p-1, & \text{if } 2 \le p < \infty. \end{cases} \tag{30}$$

The result below gives the exact value of $B(p)$ in (29) for a general function f.

Theorem 4.1. *Let $1 < p < \infty$ and let f be an arbitrary function. If $1 < p \le 2$, then*

$$(p-1)\|f\|_p \le \|(H^* - I)f\|_p, \tag{31}$$

whereas if $2 \le p < \infty$, then

$$m_p^{1/p}\|f\|_p \le \|(H^* - I)f\|_p. \tag{32}$$

Furthermore, all the constants in (31) and (32) are the best possible.

Proof. Let $1 < p < \infty$. We focus on proving only the inequality (31), since the proof of (32) is analogous. Assume that $1 < p \le 2$, and let f be an arbitrary function. If $(H^* - I)f \notin L^p(\mathbb{R}^+)$, then (31) holds trivially. Otherwise, assuming that $(H^* - I)f \in L^p(\mathbb{R}^+)$, by taking into account (17) and applying (23) yields

$$\|f\|_p = \|(H - I)(H^* - I)f\|_p \le (p-1)^{-1}\,\|(H - I)f\|_p.$$

We now establish the optimality of the constants in (31) and (32). We start with the case $1 < p \le 2$. Note that the right-hand side of inequality (6) implies the existence of a sequence of nonnegative, nonincreasing functions $\{f_m\}_{m\in\mathbb{N}}$ such that

$$\lim_{m\to\infty} \frac{\|f_m\|_p}{\|(H^* - I)f_m\|_p} = (p-1)^{-1}.$$

This shows that the constant $(p-1)$ in (31) is optimal.

We next establish that the constant $m_p^{1/p}$ in (32) is sharp. By taking into account Lemma 2.2, let $t_p \in (0, 1/2)$ such that $f_p(t_p) = m_p$, where f_p is defined as in (15). Let $r = 1/t_p$ and let g be the function given by

$$g(s) = r s^{-r} \chi_{(1,\infty)}(s).$$

We have that

$$\|g\|_p^p = r^p \int_1^\infty s^{-rp} ds = \frac{r^p}{rp-1},$$

and

$$H^* g(s) = \begin{cases} 1, & \text{if } \ 0 < s < 1, \\ s^{-r}, & \text{if } \quad s \ge 1. \end{cases}$$

So

$$|H^* g(s) - g(s)| = \begin{cases} 1, & \text{if } \ 0 < s < 1, \\ (r-1)s^{-r}, & \text{if } \quad s \ge 1. \end{cases}$$

Therefore

$$\frac{\|H^* g - g\|_p^p}{\|g\|_p^p} = \frac{1 + \frac{(r-1)^p}{rp-1}}{\frac{r^p}{rp-1}} = \frac{rp - 1 + (r-1)^p}{r^p} = m_p,$$

and this concludes the proof. $\square$

Remark 4.2. Note that (30), together with (31) and (32), implies the existence of optimal constants $c(p), C(p) > 0$ such that

$$c(p)\,\|(H^* - I)f\|_p \le \|f\|_p \le C(p)\,\|(H^* - I)f\|_p,$$

for all $f \in L^p(\mathbb{R}^+)$, with $1 < p < \infty$. It is now easy to see that $\|(H^* - I)f\|_p$ is an equivalent norm in $L^p(\mathbb{R}^+)$.

Let us now identify the best constant $B(p)$ in (29) in the case of nonnegative functions. The corresponding upper bounds can be found in [2, 8], where the authors proved that the best constant $A(p)$ in

$$\|(H^* - I)f\|_p \le A(p)\,\|f\|_p,$$

for nonnegative function $f \in L^p(\mathbb{R}^+)$, is given by

$$A(p) = \begin{cases} M^{1/p}, & \text{if } 1 < p \le 2, \\ p-1, & \text{if } 2 \le p < \infty. \end{cases}$$

The result below provides us the exact value of $B(p)$ in (29) for function $f \in L^p_+(\mathbb{R}^+)$, with $1 < p < \infty$.

Theorem 4.3. *Let $1 < p < \infty$ and let f be an nonnegative function. If $1 < p \le 2$, then*

$$\|f\|_p \le (p-1)^{-1}\,\|(H^* - I)f\|_p, \tag{33}$$

and if $2 \le p < \infty$, then

$$\|f\|_p \le m_p^{-1/p}\,\|(H^* - I)f\|_p. \tag{34}$$

Furthermore, all the constants in (33) and (34) are the best possible.

Proof. This follows from Theorem 4.1, since the functions used to establish the optimality of the constants in (31) and (32) are nonnegative functions. $\square$

5. Optimal lower bounds related to $C - S$

The aim of this section is to determine the optimal constants $B(p)$ in the inequality

$$\|x\|_p \le B(p)\,\|(C - S)x\|_p, \tag{35}$$

where x belongs to $\ell^p(\mathbb{N})$ or $\ell^p_+(\mathbb{N})$, and for the range $1 < p < \infty$. The best lower bounds for $C-S$ on nonnegative, nonincreasing sequences in $\ell^p(\mathbb{N})$, with $1 < p < \infty$, are given in (9) and (11).

We begin with the general case. On the one hand, the authors in [13, 22] proved that the best constant $A(p)$ in

$$\|(C - I)x\|_p \le A(p)\,\|x\|_p, \tag{36}$$

valid for all $x \in \ell^p(\mathbb{N})$ and $1 < p < \infty$, coincides with that in the continuous setting; thus, $A(p)$ is given by (23). On the other hand, the author in [22] showed that the optimal constant $C(p)$ in

$$\|(C^* - I)x\|_p \le C(p)\,\|x\|_p, \tag{37}$$

which holds for all $x \in \ell^p(\mathbb{N})$ and $1 < p < \infty$, is the same as the corresponding constant in the continuous setting as well. Consequently, $C(p) = \|H^* - I\|_p$, where $\|H^* - I\|_p$ is given by (26).

The result below gives the exact value of $B(p)$ in (35) for a general sequence x.

Theorem 5.1. *Let $1 < p < \infty$ and let x be an arbitrary sequence. If $1 < p \le 2$, then*

$$\|x\|_p \le M_p^{1/p} \, \|(C-S)x\|_p, \tag{38}$$

and if $2 \le p < \infty$, then

$$\|x\|_p \le (p-1) \, \|(C-S)x\|_p. \tag{39}$$

Moreover, all the constants in (38) and (39) are the best possible.

Proof. Let $1 < p < \infty$. We are going to prove only (38), since the proof of (39) is analogous. Assume that $1 < p \le 2$, and let x be an arbitrary sequence. If $(C-S)x \notin \ell^p(\mathbb{N})$, then (38) holds trivially. Otherwise, assuming that the sequence $(C-S)x \in \ell^p(\mathbb{N})$, it follows that $S^*(C-S)x \in \ell^p(\mathbb{N})$. Using (18), together with (37) and the fact that $\|S^*\|_p = 1$, we obtain

$$\|x\|_p = \|(C^*-I)S^*(C-S)x\|_p \le \|C^*-I\|_p \, \|S^*\|_p \, \|(C-S)x\|_p = M_p^{1/p} \, \|(C-S)x\|_p.$$

We proceed in an analogous way to prove inequality (39).

What remains is to verify the optimality of the constants in (38) and (39). We start with the optimality of $M_p^{1/p}$ in (38). To show that (38) is sharp, we can simply use the exactness of the value from [22]. Let $0 < \epsilon < 1$ and let y such that

$$\|(C^*-I)y\|_p > (1-\varepsilon)M_p^{1/p}\|y\|_p.$$

This y can be chosen to be finitely non-zero, since such sequences are dense in $\ell^p(\mathbb{N})$, where $m \ge 1$ is a natural number. Let $e_m = (0, \cdots, 0, \overset{m)}{1}, 0, \cdots)$. Note that we can also assume that $y(1) = 0$, since $(C^*-I)e_1 = 0$. If we can find x such that $S^*(C-S)x = y$, then $x = (C^*-I)y$ and $\|y\|_p = \|(C-S)x\|_p$, so

$$\|x\|_p \ge (1-\varepsilon)M_p^{1/p}\|(C-S)x\|_p.$$

This will follow if for each $m \ge 1$, there exists x_m such that $(C-S)x_m = e_m$, hence $S^*(C-S)x_m = e_{m+1}$. This is achieved by

$$x_m(n) = \begin{cases} \frac{1}{m+1}, & \text{if} \quad 1 \le n \le m, \\[2ex] -\frac{m}{m+1}, & \text{if} \quad n = m+1, \\[2ex] 0, & \text{if} \quad n > m+1. \end{cases}$$

Regarding the sharpness of the constant $(p-1)$ in (39), observe that since the same constant $(p-1)$ is optimal in (11), there exists a sequence $\{x_m\}_{m\ge1} \subset \ell^p(\mathbb{N})$ of nonnegative, nonincreasing sequences such that

$$\lim_{m\to\infty} \frac{\|x_m\|_p}{\|(C-S)x_m\|_p} = p-1,$$

for all $2 \le p < \infty$. □

Let us now identify the best constant $B(p)$ in (35) in the case of nonnegative sequences. It was proved in [2] that the best constant $A(p)$ in

$$\|(C-I)x\|_p \le A(p) \, \|x\|_p,$$

for all nonnegative sequences $x \in \ell^p(\mathbb{N})$, is given by

$$A(p) = \begin{cases} (p-1)^{-1}, & \text{if } 1 < p \leq 2, \\ 1, & \text{if } 2 \leq p < \infty. \end{cases}$$

Now, we proceed to find out the best constant $B(p)$ in (35) for nonnegative sequences and $p \in (1, \infty)$, but before we are going to state two useful lemmas. The first one is the following.

Lemma 5.2. *Let* $1 < p < \infty$ *and* $x = \{x(n)\}_{n\geq 1} \in \ell^p(\mathbb{N})$ *be a real sequence. Then,*

$$\lim_{n\to\infty} n|Cx(n)|^p = 0.$$

Proof. Firstly, we are going to take a sequence u with finite support. Let $u \in \ell^p(\mathbb{N})$ with $\mathrm{Card}(\mathrm{supp}(u)) < \infty$. There is $M \geq 1$ such that

$$u(n) = 0, \quad \forall n > M.$$

Then,

$$nCu(n) = u(1) + u(2) + \cdots u(M), \quad \forall n > M.$$

Therefore,

$$n|Cu(n)| \leq |u(1)| + |u(2)| + \cdots + |u(M)| = K, \quad \forall n > M.$$

Thus,

$$n|Cu(n)|^p \leq \frac{K^p}{n^{p-1}} \to 0, \tag{40}$$

as $n \to \infty$.

Now let $x = \{x(n)\}_{n\geq 1} \in \ell^p(\mathbb{N})$ be a general real sequence and $\epsilon > 0$. We must show that there is $N \geq 1$ such that

$$n|Cx(n)|^p < \epsilon, \quad \forall n \geq N.$$

On the one hand, there is $u \in \ell^p(\mathbb{N})$ with finite support such that

$$\|x - u\|_p < \frac{\epsilon^{1/p}}{2^{1+1/p}}. \tag{41}$$

By applying Hölder's inequality we have that

$$\begin{aligned} |C(x-u)(n)| &= \frac{1}{n}\left|\sum_{k=1}^{n}(x-u)(k)\right| \leq \frac{1}{n}\sum_{k=1}^{n}|(x-u)(k)| \\ &\leq \frac{1}{n}\left(\sum_{k=1}^{n}|(x-u)(k)|^p\right)^{1/p}\left(\sum_{k=1}^{n}1\right)^{1/q} \\ &= \frac{1}{n}\left(\sum_{k=1}^{n}|(x-u)(k)|^p\right)^{1/p} n^{1/q} \\ &= \left(\sum_{k=1}^{n}|(x-u)(k)|^p\right)^{1/p} n^{-1/p}, \quad \forall n \geq 1. \end{aligned}$$

Equivalently,

$$n^{1/p}|C(x-u)(n)| \leq \left(\sum_{k=1}^{n}|(x-u)(k)|^p\right)^{1/p} \leq \|(x-u)\|_p, \quad \forall n \geq 1.$$

Therefore,

$$n|C(x-u)(n)|^p \le \|x-u\|_p^p < \frac{\epsilon}{2^p 2}, \quad \forall n \ge 1, \tag{42}$$

where we have used (41) in the last inequality. On the other hand, since u is a sequence with finite support, by using (40), there is $N_0 \ge 1$ such that

$$n|Cu(n)|^p < \frac{\epsilon}{2^p 2}, \quad \forall n \ge N_0. \tag{43}$$

Now, let $n \ge 1$. We have

$$\begin{aligned} |Cx(n)| &= |Cu(n) + C(x-u)(n)| \le |Cu(n)| + |C(x-u)(n)| \\ &\le 2\max\{|Cu(n)|, |C(x-u)(n)|\}, \end{aligned}$$

and then,

$$|Cx(n)|^p \le 2^p \max\{|Cu(n)|, |C(x-u)(n)|\}^p \le 2^p(|Cu(n)|^p + |C(x-u)(n)|^p).$$

Thus,

$$n|Cx(n)|^p \le 2^p n|Cu(n)|^p + 2^p n|C(x-u)(n)|^p, \quad \forall n \ge 1.$$

Let us take $N = N_0$, and let $n \ge N_0$. We have by (42) and (43) that

$$n|Cx(n)|^p \le 2^p n|Cu(n)|^p + 2^p n|C(x-u)(n)|^p \le 2^p \frac{\epsilon}{2^p 2} + 2^p \frac{\epsilon}{2^p 2} = \epsilon,$$

which concludes our proof. □

The second lemma of importance is the following.

Lemma 5.3. *Let $p \in \mathbb{F}$ and let $x \in \ell^p(\mathbb{N})$ be a nonnegative sequence. Then,*

$$\|x\|_p \le \|(C-S)x\|_p.$$

Proof. Let $x = \{x(n)\}_{n\ge1} \in \ell^p(\mathbb{N})$ and let $\{y(n)\}_{n\ge1}$ be defined by $y(n) = Cx(n)$, for all $n \ge 1$. Clearly,

$$Sx(n) = x(n+1) = (n+1)y(n+1) - ny(n), \quad \forall n \ge 1,$$

and,

$$z(n) = y(n) - Sx(n) = (n+1)(y(n) - y(n+1)), \quad \forall n \ge 1.$$

Let $n \in \mathbb{N}$, by applying (14) we get that,

$$(n+1)(y(n)^p - y(n+1)^p) \le (n+1)py(n)^{p-1}(y(n) - y(n+1)).$$

Thus,

$$\begin{aligned} ny(n)^p - (n+1)y(n+1)^p &\le (n+1)py(n)^{p-1}(y(n) - y(n+1)) - y(n)^p \\ &= py(n)^{p-1}z(n) - y(n)^p \\ &= py(n)^{p-1}(y(n) - Sx(n)) - y(n)^p \\ &= py(n)^p - py(n)^{p-1}Sx(n) - y(n)^p \\ &= (p-1)y(n)^p - py(n)^{p-1}Sx(n). \end{aligned}$$

If $Sx(n) \ne 0$ we may let $t = y(n)/Sx(n)$ and apply the Lemma 2.4 in the form,

$$(p-1)t^p - pt^{p-1} \le (t-1)^p - 1,$$

to get,

$$\begin{aligned}(p-1)y(n)^p - py(n)^{p-1}Sx(n) &= Sx(n)^p((p-1)t^p - pt^{p-1})\\ &\le Sx(n)^p((t-1)^p - 1)\\ &= (y(n)-Sx(n))^p - Sx(n)^p.\end{aligned}$$

If $Sx(n)=0$, the same inequality holds as $(p-1)y(n)^p \le y(n)^p$. Therefore,

$$(p-1)y(n)^p - py(n)^{p-1}Sx(n) \le (y(n)-Sx(n))^p - Sx(n)^p, \quad \forall n \ge 1.$$

Summing over n, we have,

$$\begin{aligned}y(1)^p - (N+1)y(N+1)^p &= \sum_{n=1}^{N} ny(n)^p - (n+1)y(n+1)^p\\ &\le \sum_{n=1}^{N}(y(n)-Sx(n))^p - Sx(n)^p.\end{aligned}$$

Letting $N\to\infty$ and using the Lemma 5.2, we have,

$$0 \le y(1)^p \le \sum_{n=1}^{\infty}(y(n)-Sx(n))^p - \sum_{n=1}^{\infty}Sx(n)^p.$$

Equivalently,

$$\sum_{n=1}^{\infty}x(n)^p \le \sum_{n=1}^{\infty}(Cx(n)-Sx(n))^p.$$

□

We finally prove the main estimate for nonnegative sequences:

Theorem 5.4. *Let $1<p<\infty$ and let $x\in\ell^p(\mathbb{N})$ be a nonnegative sequence. Then, the following inequalities are sharp:*

$$\|x\|_p \le \|(C-S)x\|_p, \tag{44}$$

if $1<p\le 2$, and

$$\|x\|_p \le (p-1)\|(C-S)x\|_p, \tag{45}$$

if $2\le p<\infty$.

Proof. The inequality (44) can be obtained as we did in (38), in this case by using Lemma 5.3.

It remains to show that the constant in (44) is optimal. By considering the sequence

$$y_m(n) = \begin{cases}1, & \text{if } n=m,\\ 0, & \text{if } n\ne m,\end{cases}$$

where $m\ge 1$. A straightforward calculation shows that:

$$\frac{\|y_m\|^p_{\ell^p(\mathbb{N})}}{\|(C-S)y_m\|^p_{\ell^p(\mathbb{N})}} = \frac{1}{1+\sum_{n=m}^{\infty}\frac{1}{n^p}} \to 1,$$

as $m\to\infty$, and this concludes our proof.

The inequality (45) is a direct consequence of the general estimate (39). To establish the sharpness of the constant, we showed in the proof of (39) the existence of a sequence $\{x_m\}_{m\geq 1}$, where each x_m is nonnegative and nonincreasing, such that

$$\lim_{m\to\infty} \frac{\|x_m\|_p}{\|(C-S)x_m\|_p} = p-1,$$

for all $2 \leq p < \infty$. This ends the proof. □

6. Optimal lower bounds related to $(C-S)^*$

The main goal of this section is to determine the optimal constants $B(p)$ in the inequality

$$\|x\|_p \leq B(p)\,\|(C-S)^*x\|_p, \tag{46}$$

where x belongs to $\ell^p(\mathbb{N})$, $\ell^p_+(\mathbb{N})$, or $\ell^p_{\text{dec}}(\mathbb{N})$, and for the range $1 < p < \infty$. We begin with the general case. It was proved in [22] that the best constant $A(p)$ in

$$\|(C^*-I)x\|_p \leq A(p)\,\|x\|_p,$$

valid for all $x \in \ell^p(\mathbb{N})$ and $1 < p < \infty$, coincides with that in the continuous setting; thus, $A(p) = \|H^* - I\|_p$ is given by (26).

We are now ready to present one of the main results of this section, which provides the exact value of $B(p)$ in (46) for a general sequence x and $p \in (1,\infty)$.

Theorem 6.1. *Let $1 < p < \infty$ and let x be an arbitrary sequence. If $1 < p \leq 2$, then*

$$(p-1)\|x\|_p \leq \|(C-S)^*x\|_p, \tag{47}$$

and if $2 \leq p < \infty$, then

$$m_p^{1/p}\|x\|_p \leq \|(C-S)^*x\|_p. \tag{48}$$

Furthermore, all the constants in (47) and (48) are the best possible.

Proof. Let $1 < p < \infty$. We are going to prove only (47), since the proof of (48) is analogous. Assume $1 < p \leq 2$, and let x be an arbitrary sequence. If the sequence $(C-S)^*x \notin \ell^p(\mathbb{N})$, then (47) holds trivially. Otherwise, assuming $(C-S)^*x \in \ell^p(\mathbb{N})$, by taking into account (19) and applying (36) yields

$$\|x\|_p = \|S^*x\|_p = \|(C-I)(C-S)^*x\|_p \leq (p-1)^{-1}\,\|(C-S)^*x\|_p.$$

It remains to show that the constants in (47) and in (48) are optimal.We start with the optimality of $(p-1)^{-1}$ in (47). Let $m \geq 1$ be a natural number and $s = s(\epsilon) = 1/p' - \epsilon$, where $p' = p/(p-1)$ and $0 < \epsilon < 1/p'$. Consider the sequence $x_{\epsilon,m} = \{x_{\epsilon,m}(n)\}_{n\in\mathbb{N}}$ given by

$$x_{\epsilon,m}(n) = x(n) = \begin{cases} (m+1)^{s-1}, & \text{if} \quad 1 \leq n \leq m, \\ n(n^{s-1} - (n+1)^{s-1}), & \text{if} \quad n > m. \end{cases}$$

Let us now estimate the ℓ^p-norm of x from below. Let $n > m$. We have that

$$n^{s-1} - (n+1)^{s-1} = (1-s)\int_n^{n+1} t^{s-2}dt \geq (1-s)(n+1)^{s-2}.$$

So,

$$n(n^{s-1} - (n+1)^{s-1}) \geq (1-s)\frac{n}{n+1}(n+1)^{s-1} \geq (1-s)\frac{m}{m+1}(n+1)^{s-1}.$$

Therefore,

$$\begin{aligned}\sum_{n=m+1}^{\infty} x(n)^p &\geq (1-s)^p\left(\frac{m}{m+1}\right)^p \sum_{n=m+1}^{\infty} \frac{1}{(n+1)^{p(1-s)}} \\ &= (1-s)^p\left(\frac{m}{m+1}\right)^p \sum_{n=m+2}^{\infty} \frac{1}{n^{p(1-s)}} \\ &\geq (1-s)^p\left(\frac{m}{m+1}\right)^p \sum_{n=m+2}^{\infty} \int_n^{n+1} \frac{dx}{x^{p(1-s)}} \\ &= (1-s)^p\left(\frac{m}{m+1}\right)^p \int_{m+2}^{\infty} \frac{dx}{x^{p(1-s)}} \\ &= (1-s)^p\left(\frac{m}{m+1}\right)^p \frac{1}{p(1-s)-1}\frac{1}{(m+2)^{p(1-s)-1}}.\end{aligned}$$

Hence,

$$\|x\|_p^p \geq m(m+1)^{p(s-1)} + (1-s)^p\left(\frac{m}{m+1}\right)^p \frac{1}{p(1-s)-1}\frac{1}{(m+2)^{p(1-s)-1}}.$$

Now, let $n > m$. We have that,

$$C^*x(n) = \sum_{k=n}^{\infty} \frac{x(k)}{k} = \sum_{k=n}^{\infty}(k^{s-1} - (k+1)^{s-1}) = n^{s-1}.$$

So, if $n > m+1$, we get

$$(C-S)^*x(n) = n^{s-1} - (n-1)^s + (n-1)n^{s-1} = n^s - (n-1)^s.$$

So,

$$\begin{aligned}(C-S)^*x(1) = C^*x(1) &= \sum_{k=1}^{m} \frac{x(k)}{k} + \sum_{k=m+1}^{\infty} \frac{x(k)}{k} \\ &= (m+1)^{s-1}\sum_{k=1}^{m}\frac{1}{k} + (m+1)^{s-1} = (m+1)^{s-1}\left(1 + \sum_{k=1}^{m}\frac{1}{k}\right),\end{aligned}$$

and

$$\begin{aligned}(C-S)^*x(n) &= \sum_{k=n}^{m} \frac{x(k)}{k} + \sum_{k=m+1}^{\infty} \frac{x(k)}{k} \\ &= (m+1)^{s-1}\sum_{k=n}^{m}\frac{1}{k} + (m+1)^{s-1} - (m+1)^{s-1} \\ &= (m+1)^{s-1}\left(\sum_{k=n}^{m}\frac{1}{k}\right),\end{aligned}$$

for all $2 \leq n \leq m$. Finally,

$$C^*x(m+1) - S^*x(m+1) = (m+1)^{s-1} - x(m) = (m+1)^{s-1} - (m+1)^{s-1} = 0.$$

Thus,

$$|(C-S)^*x(n)| = \begin{cases} (m+1)^{s-1}\big(1+\sum_{k=1}^m \frac{1}{k}\big), & \text{if} \quad n=1, \\ (m+1)^{s-1}\big(\sum_{k=n}^m \frac{1}{k}\big), & \text{if} \quad 2 \le n \le m, \\ 0, & \text{if} \quad n=m+1, \\ n^s-(n-1)^s, & \text{if} \quad n>m+1. \end{cases}$$

Let $n > m+1$. Then,

$$n^s-(n-1)^s = s\int_{n-1}^n t^{s-1}dt \le s(n-1)^{s-1}.$$

So,

$$\begin{aligned} \sum_{n=m+2}^\infty |(C-S)^*x(n)|^p &\le s^p \sum_{n=m+2}^\infty \frac{1}{(n-1)^{p(1-s)}} = s^p \sum_{n=m+1}^\infty \frac{1}{n^{p(1-s)}} \\ &\le s^p \sum_{n=m+1}^\infty \int_{n-1}^n \frac{dx}{x^{p(1-s)}} = s^p \int_m^\infty \frac{dx}{x^{p(1-s)}} \\ &= s^p \frac{1}{p(1-s)-1}\frac{1}{m^{p(1-s)-1}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|(C-S)^*x\|_p^p \le (m+1)^{p(s-1)}\left(1+\sum_{k=1}^m \frac{1}{k}\right)^p + (m+1)^{p(s-1)}\sum_{n=2}^m\left(\sum_{k=n}^m \frac{1}{k}\right)^p \\ + s^p\frac{1}{p(1-s)-1}\frac{1}{m^{p(1-s)-1}}. \end{aligned}$$

Hence, if we define

$$\begin{aligned} I(m,s) &= (m+1)^{p(s-1)}\Big(1+\sum_{k=1}^m \frac{1}{k}\Big)^p, \\ J(m,s) &= (m+1)^{p(s-1)}\sum_{n=2}^m\Big(\sum_{k=n}^m \frac{1}{k}\Big)^p, \\ K(m,s) &= s^p\frac{1}{p(1-s)-1}\frac{1}{m^{p(1-s)-1}}, \end{aligned}$$

we have

$$\begin{aligned} (p-1)^p &\le \frac{\|(C-S)^*x\|_p^p}{\|x\|_p^p} \\ &\le \frac{I(m,s)+J(m,s)+K(m,s)}{m(m+1)^{p(s-1)}+(1-s)^p\big(\frac{m}{m+1}\big)^p\frac{1}{p(1-s)-1}\frac{1}{(m+2)^{p(1-s)-1}}} \\ &= \frac{\big(\frac{m+2}{m+1}\big)^{p(1-s)}\frac{p(1-s)-1}{m+2}\big(1+\sum_{k=1}^m\frac{1}{k}\big)^p}{\frac{m}{m+2}\big(\frac{m+2}{m+1}\big)^{p(1-s)}(p(1-s)-1)+(1-s)^p\big(\frac{m}{m+1}\big)^p} \\ &\quad + \frac{\big(\frac{m+2}{m+1}\big)^{p(1-s)}\frac{p(1-s)-1}{m+2}\sum_{n=2}^m\big(\sum_{k=n}^m\frac{1}{k}\big)^p + s^p\big(\frac{m+2}{m}\big)^{p(1-s)-1}}{\frac{m}{m+2}\big(\frac{m+2}{m+1}\big)^{p(1-s)}(p(1-s)-1)+(1-s)^p\big(\frac{m}{m+1}\big)^p} \\ &= A(m,s)+B(m,s), \end{aligned} \tag{49}$$

for all $m \geq 1$ and $0 < \epsilon < 1/p'$. Note that

$$0 \leq \frac{1}{m+2}\left(1+\sum_{k=1}^{m}\frac{1}{k}\right)^p = \frac{1}{m+2}\left(2+\sum_{k=2}^{m}\frac{1}{k}\right)^p \leq \frac{1}{m+2}\left(2+\sum_{k=2}^{m}\int_{k-1}^{k}\frac{dx}{x}\right)^p$$
$$= \frac{1}{m+2}\left(2+\sum_{k=2}^{m}\int_{1}^{m}\frac{dx}{x}\right)^p = \frac{1}{m+2}\big(2+\ln(m)\big)^p = \frac{\ln^p(e^2 m)}{m+2}, \tag{50}$$

for all $m \geq 1$. Now, since

$$\lim_{m\to\infty}\frac{\ln^p(e^2 m)}{m+2} = 0,$$

we have by an application of the squeeze theorem in (50) that

$$\lim_{m\to\infty}\frac{1}{m+2}\left(1+\sum_{k=1}^{m}\frac{1}{k}\right)^p = 0.$$

Therefore,

$$A(\infty,s) = \lim_{m\to\infty}A(m,s) = \lim_{m\to\infty}\frac{\left(\frac{m+2}{m+1}\right)^{p(1-s)}\frac{p(1-s)-1}{m+2}\left(1+\sum_{k=1}^{m}\frac{1}{k}\right)^p}{\frac{m}{m+2}\left(\frac{m+2}{m+1}\right)^{p(1-s)}(p(1-s)-1)+(1-s)^p\left(\frac{m}{m+1}\right)^p}$$
$$= \frac{(p(1-s)-1)\cdot 0}{p(1-s)-1+(1-s)^p} = 0,$$

for all $s = 1/p' - \epsilon$, where $0 < \epsilon < 1/p'$. On the other hand, if $2 \leq n \leq m$ are natural numbers, then

$$\sum_{k=n}^{m}\frac{1}{k} \leq \sum_{k=n}^{m}\int_{k-1}^{k}\frac{dx}{x} = \int_{n-1}^{m}\frac{dx}{x} = \ln\left(\frac{m}{n-1}\right)$$

and

$$\sum_{k=n}^{m}\frac{1}{k} \geq \sum_{k=n}^{m}\int_{k}^{k+1}\frac{dx}{x} = \int_{n}^{m+1}\frac{dx}{x} = \ln\left(\frac{m+1}{n}\right).$$

So,

$$\ln\left(\frac{m+1}{n}\right) \leq \sum_{k=n}^{m}\frac{1}{k} \leq \ln\left(\frac{m}{n-1}\right),$$

for all natural numbers $2 \leq n \leq m$. Thus,

$$\ln^p\left(\frac{m+1}{n}\right) \leq \left(\sum_{k=n}^{m}\frac{1}{k}\right)^p \leq \ln^p\left(\frac{m}{n-1}\right),$$

for all natural numbers $2 \leq n \leq m$. Therefore,

$$\frac{1}{m}\sum_{n=2}^{m}\ln^p\left(\frac{m+1}{n}\right) \leq \frac{1}{m}\sum_{n=2}^{m}\left(\sum_{k=n}^{m}\frac{1}{k}\right)^p \leq \frac{1}{m}\sum_{n=2}^{m}\ln^p\left(\frac{m}{n-1}\right),$$

for all $m \geq 2$. Equivalently,

$$\frac{1}{m}\sum_{n=2}^{m+1}\ln^p\left(\frac{m+1}{n}\right) \leq \frac{1}{m}\sum_{n=2}^{m}\left(\sum_{k=n}^{m}\frac{1}{k}\right)^p \leq \frac{1}{m}\sum_{n=1}^{m-1}\ln^p\left(\frac{m}{n}\right) = \frac{1}{m}\sum_{n=1}^{m}\ln^p\left(\frac{m}{n}\right),$$

for all $m \geq 2$. Hence,

$$-\frac{\ln^p(m+1)}{m} + \frac{m+1}{m}\frac{1}{m+1}\sum_{n=1}^{m+1}\ln^p\left(\frac{m+1}{n}\right) \leq \frac{1}{m}\sum_{n=2}^{m}\left(\sum_{k=n}^{m}\frac{1}{k}\right)^p \leq \frac{1}{m}\sum_{n=1}^{m}\ln^p\left(\frac{m}{n}\right), \tag{51}$$

for all $m \geq 2$. Identifying these terms as suitable Riemann sums,

$$\lim_{m\to\infty}\frac{1}{m}\sum_{n=1}^{m}\ln^p\left(\frac{m}{n}\right) = \lim_{m\to\infty}\frac{1}{m+1}\sum_{n=1}^{m+1}\ln^p\left(\frac{m+1}{n}\right) = \int_0^1 \ln^p\left(\frac{1}{t}\right)dt = \int_0^\infty u^p e^{-u}du = \Gamma(p+1), \tag{52}$$

where we have used the change of variable $u = -\ln(t)$ to arrive at the Γ function. Then, applying the squeeze theorem to (51) together with (52), we conclude that

$$\lim_{m\to\infty}\frac{1}{m}\sum_{n=2}^{m}\left(\sum_{k=n}^{m}\frac{1}{k}\right)^p = \Gamma(p+1).$$

Consequently,

$$\lim_{m\to\infty}\frac{1}{m+2}\sum_{n=2}^{m}\left(\sum_{k=n}^{m}\frac{1}{k}\right)^p = \lim_{m\to\infty}\frac{m}{(m+2)}\frac{1}{m}\sum_{n=2}^{m}\left(\sum_{k=n}^{m}\frac{1}{k}\right)^p = \Gamma(p+1).$$

Then, using the above limit, we obtain

$$\begin{aligned} B(\infty,s) &= \lim_{m\to\infty} B(m,s) \\ &= \lim_{m\to\infty}\frac{\left(\frac{m+2}{m+1}\right)^{p(1-s)}\frac{p(1-s)-1}{m+2}\sum_{n=2}^{m}\left(\sum_{k=n}^{m}\frac{1}{k}\right)^p + s^p\left(\frac{m+2}{m}\right)^{p(1-s)-1}}{\frac{m}{m+2}\left(\frac{m+2}{m+1}\right)^{p(1-s)}(p(1-s)-1) + (1-s)^p\left(\frac{m}{m+1}\right)^p} \\ &= \frac{(p(1-s)-1)\Gamma(p+1)+s^p}{p(1-s)-1+(1-s)^p}, \end{aligned}$$

for all $0 < \epsilon < 1/p'$, where $s = 1/p' - \epsilon$. Therefore, it follows that

$$\begin{aligned} \lim_{\epsilon\to 0^+}(A(\infty,s)+B(\infty,s)) = \lim_{\epsilon\to 0^+} B(\infty,s) &= \lim_{\epsilon\to 0^+}\frac{(p(1-s)-1)\Gamma(p+1)+s^p}{p(1-s)-1+(1-s)^p} \\ &= \lim_{\epsilon\to 0^+}\frac{(p(1/p+\epsilon)-1)\Gamma(p+1)+(1/p'-\epsilon)^p}{p(1/p+\epsilon)-1+(1/p+\epsilon)^p} \\ &= (p/p')^p = (p-1)^p. \end{aligned} \tag{53}$$

By taking into account (53) and using the squeeze theorem in (49), we conclude that

$$(p-1)^p \leq \lim_{\epsilon\to 0^+}\lim_{m\to\infty}\frac{\|(C-S)^*x\|_p^p}{\|x\|_p^p} \leq \lim_{\epsilon\to 0^+}\lim_{m\to\infty}(A(m,s)+B(m,s)) = (p-1)^p,$$

which means that

$$\lim_{\epsilon\to 0^+}\lim_{m\to\infty}\frac{\|(C-S)^*x\|_p^p}{\|x\|_p^p} = (p-1)^p.$$

This proves that the constant $(p-1)$ is sharp in (47).

Now, let us see the optimality of the constant $m_p^{-1/p}$ in (48). By taking into account Lemma 2.2, we set $r = 1/t_p$, where $t_p \in (0, 1/2)$ is such that $f_p(t_p) = m_p$ and f_p is defined as in (15). Note that $r > 2$. Fix an integer $m > 1$. Let us define

$$y(n) = \begin{cases} 0, & \text{if} \quad 1 \le n \le m, \\ n(n^{-r} - (n+1)^{-r}), & \text{if} \qquad n > m. \end{cases}$$

Then, with $C^* y(n) = \sum_{k=n}^{\infty} \frac{y(k)}{k}$ and $z(n) = |(C-S)^* y(n)|$,

$$C^* y(n) = \begin{cases} (m+1)^{-r}, & \text{if} \quad 1 \le n \le m+1, \\ n^{-r}, & \text{if} \qquad n > m+1, \end{cases}$$

and

$$z(n) = \begin{cases} (m+1)^{-r}, & \text{if} \quad n \le m+1, \\ \\ (n-1)^{1-r} - n^{1-r}, & \text{if} \quad n > m+1. \end{cases}$$

Now, if $n \ge m+1$ we get

$$n^{-r} - (n+1)^{-r} = r \int_n^{n+1} t^{-r-1} dt \ge r(n+1)^{-r-1}.$$

Thus,

$$n(n^{-r} - (n+1)^{-r}) \ge rn(n+1)^{-r-1} = r\frac{n}{n+1}(n+1)^{-r} \ge r\frac{m}{m+1}(n+1)^{-r}.$$

As a consequence,

$$\begin{aligned} \|y\|_p^p &= \sum_{n=m+1}^{\infty} |y(n)|^p \ge r^p \frac{m^p}{(m+1)^p} \sum_{n=m+1}^{\infty} \frac{1}{(n+1)^{rp}} = r^p \frac{m^p}{(m+1)^p} \sum_{n=m+2}^{\infty} \frac{1}{n^{rp}} \\ &\ge r^p \frac{m^p}{(m+1)^p} \sum_{n=m+2}^{\infty} \int_n^{n+1} \frac{1}{x^{rp}} dx = r^p \frac{m^p}{(m+1)^p} \int_{m+2}^{\infty} \frac{1}{x^{rp}} dx \\ &= r^p \frac{m^p}{(m+1)^p} \frac{1}{rp-1} (m+2)^{1-rp}. \end{aligned} \tag{54}$$

On the other hand, we have that if $n > m+1$, then

$$z(n) = (n-1)^{1-r} - n^{1-r} = (r-1) \int_{n-1}^{n} t^{-r} dt \le (r-1)(n-1)^{-r}.$$

Hence,

$$\begin{aligned} \sum_{n=m+2}^{\infty} z(n)^p &\le (r-1)^p \sum_{n=m+2}^{\infty} \frac{1}{(n-1)^{rp}} \\ &= (r-1)^p \sum_{n=m+1}^{\infty} \frac{1}{n^{rp}} \\ &\le (r-1)^p \sum_{n=m+1}^{\infty} \int_{n-1}^{n} \frac{1}{x^{rp}} dx = (r-1)^p \int_m^{\infty} \frac{1}{x^{rp}} dx \\ &= \frac{(r-1)^p}{rp-1} m^{1-rp}. \end{aligned} \tag{55}$$

Therefore,

$$\|z\|_p^p \le (m+1)^{1-rp} + \frac{(r-1)^p}{rp-1} m^{1-rp}.$$

By using (54) and (55), we conclude that

$$\begin{aligned} m_p^{-1} &\ge \frac{\|y\|_p^p}{\|(C-S)^* y\|_p^p} \ge \frac{r^p \frac{m^p}{(m+1)^p} \frac{1}{rp-1} (m+2)^{1-rp}}{(m+1)^{1-rp} + \frac{(r-1)^p}{rp-1} m^{1-rp}} \\ &= \frac{r^p \left(\frac{m}{m+1}\right)^p}{(rp-1)\left(\frac{m+1}{m+2}\right)^{1-rp} + (r-1)^p \left(\frac{m}{m+2}\right)^{1-rp}} \\ &\to \frac{r^p}{rp-1+(r-1)^p} = m_p^{-1}, \end{aligned}$$

as $m \to \infty$. This completes the proof. □

Let us now identify the best constant $B(p)$ in (46) in the case of nonnegative sequences. The best constant $A(p)$ in the inequality

$$\|(C^* - I)x\|_p \le A(p)\, \|x\|_p,$$

for nonnegative sequences $x \in \ell^p(\mathbb{N})$, is given by

$$A(p) = \begin{cases} M_p^{1/p}, & \text{if } 1 < p \le 2, \\ p-1, & \text{if } 2 \le p < \infty. \end{cases}$$

This result is established in [2]. The following result provides us the exact value of $B(p)$ in (46) for nonnegative sequences and $p \in (1,2]$.

Theorem 6.2. *Let $1 < p < \infty$ and let $x \in \ell^p(\mathbb{N})$ be an nonnegative sequence. If $1 < p \le 2$, then*

$$\|x\|_p \le (p-1)^{-1}\, \|(C-S)^* x\|_p, \tag{56}$$

and if $2 \le p < \infty$, then

$$\|x\|_p \le m_p^{-1/p}\, \|(C-S)^* x\|_p. \tag{57}$$

Furthermore, all the constants in (56) and (57) are the best possible.

Proof. This follows from Theorem 6.1, since the sequences used to verify the optimality of both constants in its proof are nonnegative. □

We now study $B(p)$ in (46) under the assumption that x is nonnegative and nonincreasing. Concerning the best constant $A(p)$ in the inequality

$$\|(C^* - I)x\|_p \le A(p)\, \|x\|_p,$$

which holds for all nonnegative and nonincreasing sequences $x \in \ell^p(\mathbb{N})$, for the range $1 < p < \infty$, it is currently known only for the range $2 \le p < \infty$. In this case, as shown in [2], we have $A(p) = p - 1$.

We now present the exact value of $B(p)$ in the inequality valid for all nonnegative, nonincreasing sequences, with $1 < p \le 2$.

Theorem 6.3. *Let $1 < p \le 2$ and let $x \in \ell^p(\mathbb{N})$ be an nonnegative and nonincreasing sequence. Then,*

$$\|x\|_p \le (p-1)^{-1}\, \|(C-S)^* x\|_p.$$

The constant is the best possible.

Proof. We have seen in Theorem 6.1

$$\|x\|_p \leq (p-1)^{-1}\|(C^* - S^*)x\|_p,$$

for all $x \in \ell^p(\mathbb{N})$. In order to prove the optimality of the constant $(p-1)^{-1}$ in the previous inequality we chose the sequence

$$x(n) = \begin{cases} (m+1)^{s-1}, & \text{if} \quad 1 \leq n \leq m, \\ n(n^{s-1} - (n+1)^{s-1}), & \text{if} \quad n > m, \end{cases}$$

where $m \geq 1$ is a natural number and $s = s(\epsilon) = 1/p' - \epsilon$, where $0 < \epsilon < 1/p'$. It is obvious that x is nonnegative. Let us see that x is a nonincreasing sequence. We start with the case $n = m$. So, we have to prove that $x(m) \geq x(m+1)$.

$$\begin{aligned} x(m) \geq x(m+1) &\iff (m+1)^{s-1} \geq (m+1)((m+1)^{s-1} - (m+2)^{s-2}) \\ &\iff (m+1)(m+2)^{s-1} \geq m(m+1)^{s-1} \\ &\iff (m+1)^{2-s} \geq m(m+2)^{1-s} \\ &\iff \frac{(m+1)^2}{m+2} \geq m\left(\frac{m+1}{m+2}\right)^s, \end{aligned}$$

for all $m \geq 1$. Equivalently,

$$\frac{m^2}{m+1} \geq (m-1)\left(\frac{m}{m+1}\right)^s \iff \left(\frac{m}{m+1}\right)^{1-s} \geq \frac{m-1}{m},$$

for all $m \geq 2$. But since, for all $0 < a < b < 1$ and $0 < r < 1$, we have that $0 < a < a^r < b^r < 1$, then

$$\left(\frac{m}{m+1}\right)^{1-s} \geq \left(\frac{m-1}{m}\right)^{1-s} \geq \frac{m-1}{m},$$

for all $m \geq 2$, which means that $x(m) \geq x(m+1)$, for all $m \geq 1$. Now, consider the function $f : (1,\infty) \to \mathbb{R}^+$ defined by

$$f(t) = t(t^{s-1} - (t+1)^{s-1}) = t^s - t(t+1)^{s-1}.$$

Its derivative is given by

$$\begin{aligned} f'(t) &= st^{s-1} - (t+1)^{s-1} - (s-1)t(t+1)^{s-2} \\ &= (t+1)^{s-1}\left[s\left(\frac{t}{t+1}\right)^{s-1} - 1 - (s-1)\frac{t}{t+1}\right] \\ &= (1+t)^{s-1}h(t), \end{aligned}$$

where $h : (1,\infty) \to \mathbb{R}$ is defined by

$$h(t) = s\left(\frac{t}{t+1}\right)^{s-1} - 1 - (s-1)\frac{t}{t+1}.$$

To show that $f'(t) \leq 0$ for all $t > 1$, it suffices to prove that

$$g(z) = sz^{s-1} - 1 - (s-1)z \leq 0, \quad \text{for all } z \in [1/2, 1],$$

where $z = \frac{t}{t+1}$.

Now, observe that the second derivative of g is

$$g''(z) = \frac{d}{dz}\left(\frac{d}{dz}g(z)\right) = \frac{d}{dz}\big(s(s-1)z^{s-2} - (s-1)\big) = s(s-1)(s-2)z^{s-3} > 0,$$

for all $z \in (1/2, 1)$, since $s - 1 = 1/p' - \epsilon - 1 < 0$. Therefore, g is strictly convex on $(1/2, 1)$. Note that

$$g(1) = s - 1 - (s - 1) = 0.$$

If we prove that $g(1/2) \leq 0$, then by convexity we would have

$$g(t) \leq \max\{g(1), g(1/2)\} = 0,$$

for all $t \in [1/2, 1]$.

We now show that $g(1/2) < 0$. Consider the function $u : [0, 1/2] \to \mathbb{R}$ defined by

$$u(y) = y2^{2-y} - 1 - y.$$

We compute

$$\frac{d^2}{dy^2}u(y) = \frac{d}{dy}\left(\frac{d}{dy}u(y)\right) = \frac{d}{dy}\left(2^{2-y}(1 - y\ln 2) - 1\right) = -2^{2-y}\ln 2(2 - y\ln 2) < 0,$$

for all $y \in (0, 1/2)$. This means that u' is strictly decreasing. Since

$$u'(1/2) = 2^{3/2}(1 - \ln\sqrt{2}) - 1 \approx 0.8482 > 0,$$

we obtain

$$u'(y) > 0,$$

for all $y \in (0, 1/2)$. Hence, u is strictly increasing on $[0, 1/2]$. Now,

$$u(y) \leq u(1/2) = \sqrt{2} - 3/2 < 0,$$

for all $y \in [0, 1/2]$. So,

$$g(1/2) = \frac{1}{2}\left(s2^{2-s} - 1 - s\right) = \frac{1}{2}u(s) \leq 0,$$

since $0 < s = 1/p' - \epsilon < 1/p' \leq \frac{1}{2}$, for all $0 < \epsilon < 1/p'$. Therefore, f is nonnegative and nonincreasing on $(1, \infty)$, and hence x is also nonnegative and nonincreasing. □

7. Optimal lower bounds related to $C - I$

We begin by recalling the inequality (8), due to G. J. O. Jameson, which states that

$$\|x\|_2 \leq \sqrt{2}\,\|(C - I)x\|_2,$$

for all $x \in \ell^2$. The main goal of this section is to extend this result, at least, to the range $1 < p \leq 2$. More precisely, we determine the exact value of the best constant $A(p)$ in the inequality

$$\|x\|_p \leq A(p)\,\|(C - I)x\|_p,$$

valid for all $x \in \ell^p(\mathbb{N})$, with $1 < p \leq 2$.

The following result will be needed before we can state the main theorem of this section.

Proposition 7.1. *Let $1 \leq p \leq 2$, and let $x \in \ell^p(\mathbb{N})$. Then*

$$\|(C - S)^*x\|_p \leq 2^{1/p}\|x\|_p. \tag{58}$$

Moreover, the constant is the best possible.

Proof. Let $1 \leq p \leq 2$. First of all, observe that by considering the sequence

$$e_1 = (1, 0, 0, 0, \dots),$$

we have

$$(C - S)^* e_1 = (1, -1, 0, 0, \dots).$$

It follows that

$$\frac{\|(C - S)^* e_1\|_p}{\|e_1\|_p} = 2^{1/p}.$$

This establishes the optimality of the constant in (58).

Now, on the one hand, it is well known (see [11]) that $\|C^*\|_1 = 1$. On the other hand, we trivially have $\|S^*\|_1 = 1$. Therefore, for all $x \in \ell^1$,

$$\begin{aligned} \|(C - S)^* x\|_1 &= \|C^* x - S^* x\|_1 \leq \|C^* x\|_1 + \|S^* x\|_1 \\ &\leq (\|C^*\|_1 + \|S^*\|_1) \|x\|_1 = 2 \|x\|_1. \end{aligned} \tag{59}$$

For the case $p = 2$, it was shown in [13] that

$$\|(C - S)^* x\|_2 \leq \sqrt{2} \|x\|_2, \quad x \in \ell^2. \tag{60}$$

Now, let $1 < p < 2$. We have, by taking $\theta = 2(1 - 1/p)$, that

$$\frac{1 - \theta}{1} + \frac{\theta}{2} = 1 - 2\left(1 - \frac{1}{p}\right) + \left(1 - \frac{1}{p}\right) = 1 - \left(1 - \frac{1}{p}\right) = \frac{1}{p}.$$

By applying the Riesz-Thorin theorem [4, Chapter 4, Corollary 2.3] and taking into account (59) and (60), we obtain that

$$\|(C - S)^*\|_p \leq \|(C - S)^*\|_1^{1-\theta} \|(C - S)^*\|_2^{\theta} = 2^{1-\theta} 2^{\theta/2} = 2^{1/p}.$$

□

Remark 7.2. Combining (47) with inequality (58), we obtain

$$(p - 1) \|x\|_p \leq \|(C - S)^* x\|_p \leq 2^{1/p} \|x\|_p$$

for all $x \in \ell^p(\mathbb{N})$ with $1 < p \leq 2$. It is now easy to see that $\|(C - S)^* x\|_p$ is an equivalent norm in $\ell^p(\mathbb{N})$.

We are now in a position to state the main result of this section.

Theorem 7.3. *Let $1 < p \leq 2$, and let x be an arbitrary sequence. Then the following inequality is sharp:*

$$\|x\|_p \leq 2^{1/p} \|(C - I)x\|_p. \tag{61}$$

Proof. Recall the equality (18):

$$(C^* - I) S^* (C - S) = I.$$

Taking the transpose of this identity yields

$$(C - S)^* S (C - I) = I. \tag{62}$$

Now, let x be an arbitrary sequence. If $(C - I)x \notin \ell^p(\mathbb{N})$, then (61) holds trivially. Otherwise, assuming that $(C - I)x \in \ell^p(\mathbb{N})$, it follows that $S(C - I)x \in \ell^p(\mathbb{N})$. Using (62), together with (58) and the fact that $\|S\|_p = 1$, we obtain

$$\|x\|_p = \|(C - S)^* S (C - I) x\|_p \leq \|(C - S)^*\|_p \|S\|_p \|(C - I) x\|_p = 2^{1/p} \|(C - I) x\|_p.$$

It remains to show that the constant $2^{1/p}$ in (61) is optimal. To this end, consider the sequence $x = (1, -1, 0, 0, \dots)$. Then

$$(C - I)x = (1, 0, 0, \dots),$$

and hence

$$\frac{\|x\|_p}{\|(C - I)x\|_p} = 2^{1/p}.$$

This completes the proof. □

Remark 7.4. From (23) and (61), it follows that

$$(p - 1)\,\|(C - I)x\|_p \le \|x\|_p \le 2^{1/p}\,\|(C - I)x\|_p$$

for all $x \in \ell^p(\mathbb{N})$, $1 < p \le 2$. It is now easy to see that $\|(C - I)x\|_p$ is an equivalent norm in $\ell^p(\mathbb{N})$.

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