A WEIGHTED GENERALISATION OF CARLEMAN'S INEQUALITY

SERGI ARIAS AND SALVADOR RODRÍGUEZ-LÓPEZ

ABSTRACT. In this paper we present a generalisation of the classical inequality of Carleman, which we obtain by an elementary argument based on log-convexity and Hölder's inequality. As a consequence, we recover some other classical estimates such as the Pólya-Knopp inequality.

1. INTRODUCTION

The classical inequality by T. Carleman [4], known as Carleman's inequality, asserts that given any non-negative sequence $\{a_n\}_{n=0}^{\infty}$, it holds that

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k\right)^{1/n} \leq e\sum_{n=1}^{\infty} a_n.$$

Equality holds if, and only if, the sequence is identically zero and the constant e is the best possible, in the sense that it cannot be replaced by any smaller constant.

The sumands on the left hand side of Carleman's inequality correspond to the geometric mean of the first n terms in the sequence, which can be rewritten as

$$\left(\prod_{k=1}^{n} a_k\right)^{1/n} = e^{\frac{1}{n}\sum_{k=1}^{n}\log a_k}.$$

In that sense, some authors have provided integral generalizations of Carleman's inequality, such as the one obtained by K. Knopp [13] (also attributed to G. Pólya), known as the Pólya-Knopp inequality, which states that

$$\int_0^\infty \exp\left\{\frac{1}{x}\int_0^x \log f(t)\mathrm{d}t\right\}\mathrm{d}x < e\int_0^\infty f(x)\mathrm{d}x \tag{1.1}$$

holds for all positive function f. Some years later, L. Carleson [5] obtained a different extension of the inequality of Carleman, showing that the estimate

$$\int_0^\infty x^p e^{\frac{-m(x)}{x}}\mathrm{d}x \leq e^{p+1}\int_0^\infty x^p e^{-m'(x)}\mathrm{d}x,$$

holds for all convex function m on $\mathbb{R}_+ = [0,\infty)$ such that $m(0) = 0$, with $-1 < p < \infty$.

Weighted versions of the inequalities of Carleman and Pólya-Knopp have also been studied by several authors. See, for instance, the works by R. P. Boas [3], H. P. Heinig [11], J. A. Cochran and C. S. Lee [10], E. R. Love [14,18], B. Opic and P. Gurka [19], L. Pick and B. Opic [17], A. Čižmešija and J. Pečarić [7], S. Kaijser, L-E.Persson and A. Öberg [12], A. Čižmešija, J. Pečarić and L-E. Persson [9], D-C. Luor [15,16] or A. Čižmešija, S. Hussain and J. Pečarič [8].

In this paper, we obtain a generalization of Carleman's inequality, acting on decreasing functions, by using an elementary approach. Indeed, the argument used to prove our main theorem is based on

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log-convexity and Hölder's inequality. In this way, we recover some of the classical estimates such as Carleman's or the inequalities of Pólya-Knopp and Cochran-Lee [10] (where we realise that, in fact, turn out to be equivalent).

As a consequence of our main theorem some applications are obtained. For instance, we get in Corollary 3.7 an estimate for the harmonic mean operator in the cone of positive decreasing functions on weighted Lebesgue spaces. In addition, we include in Corollary 3.9 an estimate for the Laplace transform of an average operator, evaluated on decreasing functions. Finally, we also obtain in Corollary 3.13 an equivalent norm expression for rearrangement invariant spaces, in terms of a log-convex average of the decreasing rearrangement function.

The paper is organised as follows. In Section 2 we state and prove the main theorem of this article while in Section 3 we describe its applications. Finally, in Section 4 we extend particular cases of our result to positive functions.

2. Main result

Let us start by introducing the following proposition, which states that a certain average operator keeps the monotonicity.

Proposition 2.1. *Let μ be a locally finite Borel measure on $\mathbb{R}_+$ such that for all t, $\mu[0,t) > 0$ and let $f : \mathbb{R}_+ \to \mathbb{R}$ be an increasing (resp. decreasing) function. Then the function*

$$t \mapsto T_\mu f(t) := \frac{1}{\mu[0,t)} \int_{[0,t)} f(x)\mathrm{d}\mu(x),$$

is increasing (resp. decreasing) in $\mathbb{R}_+$.

Proof. We shall give the proof for f being an increasing function. The other case is proved similarly with minor modifications on the argument.

Let $0 \le s < t$. Consider the integral of $f(y) - f(x)$ over the rectangle $R = [0,s) \times [s,t)$ with respect to the product measure $\mu \times \mu$ and observe that for $0 \le x < s \le y < t$ it holds that $f(y) - f(x) \ge 0$. Hence

$$\int_R (f(y) - f(x))\,\mathrm{d}\mu(x)\mathrm{d}\mu(y) \ge 0.$$

Then Tonelli's theorem yields

$$\begin{aligned}\mu[0,s) \int_{[s,t)} f(y)\mathrm{d}\mu(y) &= \int_R f(y)\mathrm{d}\mu(x)\mathrm{d}\mu(y) \ge \int_R f(x)\mathrm{d}\mu(x)\mathrm{d}\mu(y) \\ &= \mu[s,t) \int_{[0,s)} f(x)\mathrm{d}\mu(x).\end{aligned}$$

Therefore, adding $\mu[0,s) \int_{[0,s)} f(y)\mathrm{d}\mu(y)$ to both sides and operating, this yields

$$T_\mu f(t) = \frac{1}{\mu[0,t)} \int_{[0,t)} f(x)\mathrm{d}\mu(x) \ge \frac{1}{\mu[0,s)} \int_{[0,s)} f(x)\mathrm{d}\mu(x) = T_\mu f(s).$$

□

Remark 2.2. *Notice that if μ is the Lebesgue measure in $[0,\infty)$ then T_μ corresponds to the classical Hardy operator*

$$Hf(t) = \frac{1}{t} \int_0^t f(x)\mathrm{d}x.$$

Definition 2.3. *We say that a positive function Φ defined on an interval $I \subseteq \mathbb{R}$ is log-convex if $\ln\Phi$ is a convex function. That is, for all $s \in [0,1]$ and all $x, y \in I$ it holds that*

$$\Phi(sx + (1-s)y) \le \Phi(x)^s \Phi(y)^{1-s}.$$

Examples. Here we have some examples of log-convex functions:

(1) If $a \geq 1$ the function e^{x^a} is log-convex on $(0,\infty)$.
(2) The function e^{-x} is log-convex on $\mathbb{R}$.
(3) More generally, if Ψ is convex on $I \subseteq \mathbb{R}$ then the function $\Phi(x) = e^{\Psi(x)}$ is log-convex on I.
(4) If $a \geq 0$ the function x^{-a} is log-convex on $(0,\infty)$.
(5) If Ψ is concave on $I \subseteq \mathbb{R}$ then the function $\Phi(x) = e^{-\Psi(x)}$ is log-convex on I. For example $\Psi(x) = e^{-x^2}$ on $\mathbb{R}$ or $\Psi(x) = \int_{-\infty}^{x} e^{-s^2} \mathrm{d}s$ on $[0,\infty)$.

Definition 2.4. *For every non-negative function v on $\mathbb{R}_+$ and all $s > 1$, we define*

$$p(v,s,x) := \frac{v(sx)}{v(x)}, \quad p_-(v,s) := \inf_{x>0} p(v,s,x), \quad p_+(v,s) := \sup_{x>0} p(v,s,x).$$

For all $t > 0$ we also set

$$p_-(v,s,t) = \inf_{0<x<t} p(v,s,x).$$

Remark 2.5. *Observe that in the case where v is increasing and $x < t$, it follows that*

$$1 \leq p_-(v,s) \leq p_-(v,s,t) \leq p(v,s,x) \leq p_+(v,s) \leq \infty.$$

We present the main theorem of this paper.

Theorem 2.6. *Let μ be a locally finite Borel measure on $\mathbb{R}_+$ and assume that for all t, $\mu[0,t) > 0$. Let $M(t) = \mu[0,t)$ and suppose that there exists $s_0(M) \geq 1$ such that for all $s > s_0(M)$ and for all $t > 0$,*

$$1 < p_-(M,s,t). \tag{2.1}$$

Let $w : (0,\infty) \to \mathbb{R}_+$ be such that for all $s > s_0(M)$,

$$p_+(w,s) < +\infty. \tag{2.2}$$

Let I be an interval of $\mathbb{R}$ and let $\Phi : I \to \mathbb{R}_+$ be a log-convex and decreasing (resp. increasing) function. Then, for all increasing (resp. decreasing) functions $f : \mathbb{R}_+ \to I$ and for all $t > 0$, it holds that

$$\int_0^t \Phi\left(T_\mu f(x)\right) w(x) \mathrm{d}x \leq \inf_{s>s_0(M)} \left(s p_+(w,s)\right)^{\frac{p_-(M,s)}{p_-(M,s)-1}} \int_0^t \Phi\left(f(x)\right) w(x) \mathrm{d}x, \tag{2.3}$$

provided the left hand side is finite.

Remark 2.7. *By using Proposition 2.1 we notice that the term $\Phi\left(T_\mu f(x)\right)$ on the left hand side in* (2.3) *is well defined since $T_\mu f(x) \in I$ for all $x > 0$ when f is a monotone function.*

Proof of Theorem 2.6. We shall only prove the case for which Φ is decreasing. The other case can be shown with minor modifications, and thus we leave the details to the reader.

Let $s > s_0(M)$. Since f is increasing we have that

$$\int_{[0,sx)} f \mathrm{d}\mu = \int_{[0,x)} f \mathrm{d}\mu + \int_{[x,sx)} f \mathrm{d}\mu \geq \int_{[0,x)} f \mathrm{d}\mu + \left(M(sx) - M(x)\right) f(x)$$

for all $x > 0$, from where

$$\begin{aligned} T_\mu f(sx) &\geq \frac{M(x)}{M(sx)} T_\mu(f)(x) + \frac{M(sx) - M(x)}{M(sx)} f(x) \\ &= \frac{1}{p(M,s,x)} T_\mu(f)(x) + \left(1 - \frac{1}{p(M,s,x)}\right) f(x). \end{aligned}$$

Next, since Φ is decreasing and log-convex, it follows that

$$\Phi\left(T_\mu(f)(sx)\right) \leq \Phi\left(T_\mu(f)(x)\right)^{\frac{1}{p(M,s,x)}} \Phi\left(f(x)\right)^{1-\frac{1}{p(M,s,x)}}. \tag{2.4}$$

A change of variables, (2.2) and (2.4) yield that, for all $t>0$ and $s>s_0(M)\geq 1$,

$$\begin{aligned} I(t) &:= \int_0^t \Phi\left(T_\mu(f)(x)\right) w(x)\mathrm{d}x = s\int_0^{t/s} \Phi\left(T_\mu(f)(sx)\right) w(sx)\mathrm{d}x \\ &\leq sp_+(w,s)\int_0^{t/s} \Phi\left(T_\mu(f)(sx)\right) w(x)\mathrm{d}x \\ &\leq sp_+(w,s)\int_0^{t} \Phi\left(T_\mu(f)(x)\right)^{\frac{1}{p(M,s,x)}} \Phi\left(f(x)\right)^{1-\frac{1}{p(M,s,x)}} w(x)\mathrm{d}x. \end{aligned} \tag{2.5}$$

Since f is increasing then $T_\mu f(x)\leq f(x)$ for all $x>0$. Therefore, as Φ is decreasing, we have that

$$\Phi(f(x)) \leq \Phi\left(T_\mu(f)(x)\right), \quad x>0. \tag{2.6}$$

Let us observe that plugging (2.6) directly in (2.5) one obtains that, for all $s>s_0(M)$,

$$I(t)\leq sp_+(w,s)I(t).$$

In particular, if $I(t)$ is finite and non zero, this implies that $sp_+(w,s)\geq 1$ for all $s>s_0(M)$.

Using (2.6) and the fact that $0<p_-(M,s,t)\leq p(M,s,x)$ for all $0<x<t$, we have that

$$\left(\frac{\Phi\left(T_\mu(f)(x)\right)}{\Phi(f(x))}\right)^{\frac{1}{p(M,s,x)}} \leq \left(\frac{\Phi\left(T_\mu(f)(x)\right)}{\Phi(f(x))}\right)^{\frac{1}{p_-(M,s,t)}}, \quad 0<x<t. \tag{2.7}$$

Therefore,

$$\begin{aligned} &\int_0^t \Phi\left(T_\mu(f)(x)\right)^{\frac{1}{p(M,s,x)}} \Phi\left(f(x)\right)^{1-\frac{1}{p(M,s,x)}} w(x)\mathrm{d}x \\ &= \int_0^t \left(\frac{\Phi\left(T_\mu(f)(x)\right)}{\Phi\left(f(x)\right)}\right)^{\frac{1}{p(M,s,x)}} \Phi\left(f(x)\right) w(x)\mathrm{d}x \\ &\leq \int_0^t \left(\frac{\Phi\left(T_\mu(f)(x)\right)}{\Phi\left(f(x)\right)}\right)^{\frac{1}{p_-(M,s,t)}} \Phi\left(f(x)\right) w(x)\mathrm{d}x \\ &\leq I(t)^{\frac{1}{p_-(M,s,t)}} \left(\int_0^t \Phi\left(f(x)\right) w(x)\mathrm{d}x\right)^{1-\frac{1}{p_-(M,s,t)}}, \end{aligned}$$

where the last step follows by Hölder's inequality (notice that $p_-(M,s,t)>1$ by (2.1)). This last chain of inequalities and (2.5) yield

$$I(t)\leq \left(sp_+(w,s)\right)^{\frac{p_-(M,s,t)}{p_-(M,s,t)-1}} \int_0^t \Phi\left(f(x)\right) w(x)\mathrm{d}x, \quad t>0,$$

as long as $I(t)$ is finite.

Observe now that $p_-(M,s,t)$ is decreasing as a function of t and bounded below by $p_-(M,s)$. Therefore

$$\inf_{t>0} p_-(M,s,t) = \lim_{t\to+\infty} p_-(M,s,t) = p_-(M,s),$$

from where we get that

$$\sup_{t>0}\frac{p_-(M,s,t)}{p_-(M,s,t)-1} = \lim_{t\to+\infty}\frac{p_-(M,s,t)}{p_-(M,s,t)-1} = \frac{p_-(M,s)}{p_-(M,s)-1}.$$

Hence, using the fact $sp_+(w,s)\geq 1$ shown above for all $s>s_0(M)$, we see that

$$I(t)\leq \left(sp_+(w,s)\right)^{\frac{p_-(M,s)}{p_-(M,s)-1}} \int_0^t \Phi\left(f(x)\right) w(x)\mathrm{d}x$$

for all $t > 0$ and $s > s_0(M)$, which yields

$$I(t) \leq \inf_{s>s_0(M)} (sp_+(w,s))^{\frac{p_-(M,s)}{p_-(M,s)-1}} \int_0^t \Phi(f(x))\, w(x)\mathrm{d}x, \quad t > 0.$$

$\square$

Remark 2.8. *Notice that the weight w in Theorem 2.6 is not required to be locally integrable in $[0,\infty)$ so that, for instance, we are allowed to set $w(x) = 1/x$.*

If w is supposed to be locally integrable in $[0,\infty)$ then one can replace the constant $sp_+(w,s)$ by

$$q_+(w,s,t) := \sup_{0<r<t/s} \frac{W(sr)}{W(r)}$$

in (2.3)*, where $W(x) = \int_0^x w(u)\mathrm{d}u$. This comes from the fact that (see* [6, Corollary 2.7]*) for all non negative locally integrable functions F,G it holds that*

$$\sup_{h\downarrow,\ h\geq 0} \frac{\int_0^\infty h(x)F(x)\mathrm{d}x}{\int_0^\infty h(x)G(x)\mathrm{d}x} = \sup_{r>0} \frac{\int_0^r F(x)\mathrm{d}x}{\int_0^r G(x)\mathrm{d}x}. \tag{2.8}$$

To show this observe that since F is increasing, $\Phi(T_\mu F)$ is decreasing and positive. Thus taking $F(x) = w(sx)\chi_{[0,t/s)}(x)$ and $G(x) = w(x)\chi_{[0,t)}(x)$, the identity in (2.8) *yields that we can substitute $sp_+(w,s)$ in* (2.5) *by $q_+(w,s,t)$.*

Next let us give some examples of the measures and weights satisfying the hypotheses of the main theorem.

Examples 2.9. *We present some examples of functions M satisfying condition* (2.1)*.*

(1) *If $M(x) = x^\alpha(1+|\log x|)^\gamma$ with $\alpha > 0$ and $\gamma \geq 0$ then*

$$p_-(M,s,t) \geq p_-(M,s) = s^\alpha(1+\log s)^{-\gamma}$$

for all $t > 0$. Since $x > 1+\log(x)$ when $x > 1$ we see that $p_-(M,s) > 1$ for all $s > 1$ in the case $\gamma \leq \alpha$, so we can pick $s_0(M) = 1$. In the case where $\alpha < \gamma$, $s_0(M)$ is the unique $s > 1$ that solves the equation $s^{\alpha/\gamma} = 1+\log s$.

In particular, notice that the function $M(x) = x$, related to the Lebesgue measure on $(0,\infty)$, is within that group.

(2) *If $M(x) = 1-e^{-x}$, and hence $\mathrm{d}\mu(x) = e^{-x}\mathrm{d}x$, then by monotonicity*

$$p_-(M,s,t) = \inf_{0<x<t} \frac{1-e^{-sx}}{1-e^{-x}} = \frac{1-e^{-st}}{1-e^{-t}} > 1$$

for all $s > 1$. So we can pick $s_0(M) = 1$.

Examples 2.10. *We present some examples of weights w satisfying condition* (2.2)*.*

(1) *If $w(x) = x^\alpha(1+|\log x|)^\gamma$ with $\alpha \geq 0$ and $\gamma \geq 0$ then for all $s > 1$*

$$p_+(w,s) = \sup_{x>0} \frac{w(sx)}{w(x)} = s^\alpha(1+\log s)^\gamma < \infty.$$

In addition, we notice that if we pick $M(x) = x$ then $p_-(M,s) = s$, as seen in the previous set of examples. Hence under those choices of w and M the constant in (2.3) *becomes*

$$\inf_{s>1}\left(s^{1+\alpha}(1+\log s)^\gamma\right)^{\frac{s}{s-1}} = \inf_{s>1} e^{\frac{s}{s-1}((1+\alpha)\log s+\gamma\log(1+\log s))} = e^{1+\alpha+\gamma}. \tag{2.9}$$

(2) *If $w(x) = e^{-\lambda x}$ for some $\lambda > 0$ then $p_+(w,s) = \sup_{x>0} e^{-\lambda sx+\lambda x} = \left(\sup_{x>0} e^{-\lambda x}\right)^{s-1} = 1$ for all $s > 1$.*

Under the extra hypothesis that the log-convex function Φ is strictly monotone (and hence invertible), we obtain the following corollary from Theorem 2.6, where the estimate is now in terms of the weighted integral of f.

Corollary 2.11. *Let w, μ, I and Φ be as in Theorem 2.6. Assume in adddition that $\Phi : I \to \mathbb{R}_+$ is strictly monotonic and take $p > 0$. Then for all $t > 0$ the inequality*

$$\int_0^t \Phi\left(T_\mu(\Phi^{-1} f)(x)\right)^p w(x)\mathrm{d}x \leq \inf_{s > s_0(M)} (sp_+(w,s))^{\frac{p_-(M,s)}{p_-(M,s)-1}} \int_0^t f(x)^p w(x)\mathrm{d}x \tag{2.10}$$

holds for all decreasing function $f : \mathbb{R}_+ \to \Phi(I)$, provided the left hand side is finite.

Proof. Applying Theorem 2.6 to the function $\Phi^{-1} \circ f$ we get (2.10) for $p = 1$.

To obtain (2.10) for the remaining values of p we apply the case $p = 1$ for the function $\Psi = \Phi^p$, which satisfies the same conditions as Φ, and apply the change of variables $g = f^{1/p}$. □

3. Applications

3.1. **Integral inequalities.** By taking μ to be the Lebesgue measure on $\mathbb{R}_+$, $\Phi(x) = e^x$ with $I = \mathbb{R}$ and $w(x) = 1$ in Corollary 2.11, we recover (1.1) for every positive and decreasing function f such that $f \in L^1(\mathbb{R}_+)$. This can be extended to hold for all positive integrable function f (see Section 4), recovering the Pólya-Knopp inequality.

Actually, it happens that the inquality of Pólya and Knopp is equivalent to an inequality of Cochran and Lee [10, Theorem 1], which is given by

$$\int_0^\infty \exp\left(\frac{p}{x^p}\int_0^x t^{p-1}\log f(t)\mathrm{d}t\right) x^\gamma \mathrm{d}x \leq e^{\frac{\gamma+1}{p}} \int_0^\infty f(x) x^\gamma \mathrm{d}x,$$

where $p > 0$, $\gamma \in \mathbb{R}$ and f is a positive function in $L^1(\mathbb{R}_+, x^\gamma \mathrm{d}x)$.

Indeed, (1.1) follows from the Cochran-Lee inequality as it corresponds to the special case $\gamma = 0$ and $p = 1$.

To see the converse, we fix $\gamma \in \mathbb{R}$, $p > 0$ and a positive function f in $L^1(\mathbb{R}_+, x^\gamma \mathrm{d}x)$. Applying (1.1) to the function $f(x^{1/p})x^{\gamma/p}p^{-1}x^{1/p-1}$ we get that

$$\begin{aligned}\int_0^\infty &\exp\left(\frac{1}{x}\int_0^x \log f(t^{1/p})\mathrm{d}t\right)\exp\left(\frac{1}{x}\int_0^x \log t^{\gamma/p}\mathrm{d}t\right)\exp\left(\frac{1}{x}\int_0^x \log\frac{t^{1/p-1}}{p}\mathrm{d}t\right)\mathrm{d}x \\ &\leq e\int_0^\infty f(x^{1/p})x^{\gamma/p}\frac{x^{1/p-1}}{p}\mathrm{d}x. \end{aligned} \tag{3.1}$$

By changing variables we notice that the right hand side in (3.1) satisfies

$$e\int_0^\infty f(x^{1/p})x^{\gamma/p}\frac{x^{1/p-1}}{p}\mathrm{d}x = e\int_0^\infty f(x)x^\gamma \mathrm{d}x. \tag{3.2}$$

Regarding the left hand side in (3.1), we study the three factors appearing in the integrand. On the one hand, a change of variables yields

$$\exp\left(\frac{1}{x}\int_0^x \log f(t^{1/p})\mathrm{d}t\right) = \exp\left(\frac{p}{x}\int_0^{x^{1/p}} t^{p-1}\log f(t)\mathrm{d}t\right), \tag{3.3}$$

while direct computation gives

$$\exp\left(\frac{1}{x}\int_0^x \log t^{\gamma/p}\mathrm{d}t\right) = x^{\gamma/p}e^{-\gamma/p} \tag{3.4}$$

and

$$\exp\left(\frac{1}{x}\int_0^x \log\frac{t^{1/p-1}}{p}\right) = \frac{x^{1/p-1}}{p}e^{-(1/p-1)}. \tag{3.5}$$

Plugging (3.2), (3.3), (3.4) and (3.5) into (3.1) we obtain that

$$\int_0^\infty \exp\left(\frac{p}{x}\int_0^{x^{1/p}} t^{p-1}\log f(t)\mathrm{d}t\right) x^{\gamma/p}\frac{x^{1/p-1}}{p}e^{-(1/p-1-\gamma/p)}\mathrm{d}x \le e\int_0^\infty f(x)x^\gamma \mathrm{d}x.$$

A last change of variables yields

$$\int_0^\infty \exp\left(\frac{p}{x^p}\int_0^x t^{p-1}\log f(t)\mathrm{d}t\right) x^\gamma \mathrm{d}x \le e^{\frac{\gamma+1}{p}}\int_0^\infty f(x)x^\gamma \mathrm{d}x.$$

Note that the constant in the inequality of Cochran and Lee is the same that we would get by taking $w(x) = x^\gamma$ from the main theorem when $\gamma \ge -1$, albeit only for decreasing and positive functions.

3.2. **The discrete case: Carleman's inequality.** Let us start by introducing some notation. For all non-negative function M on $\mathbb{R}_+$ and all $k \in \mathbb{N}$ we define

$$\Delta_k(M) := M(k) - M(k-1),$$

while for all locally integrable weight w in $[0,\infty)$ we will write $W(x) := \int_0^x w(u)\mathrm{d}u$.

We get the following discrete version of Theorem 2.6.

Corollary 3.1. *Let w, μ, M, I and Φ be as in Theorem 2.6 (the case where Φ is decreasing). Assume in addition that the weight w is locally integrable in $[0,\infty)$. Then, for all decreasing sequence $\{a_n\}_n$ of positive numbers such that $1/a_i \in I$ for all $i \ge 1$, and all integer $N \ge 1$, it holds that*

$$\sum_{k=1}^N \Phi\left(\frac{\sum_{n=1}^k \frac{\Delta_n(M)}{a_n}}{M(k)}\right)\Delta_k(W) \le \inf_{s>s_0(M)} (sp_+(w,s))^{\frac{p_-(M,s)}{p_-(M,s)-1}} \sum_{k=1}^N \Phi(1/a_k)\Delta_k(W). \tag{3.6}$$

Furthermore, if Φ is strictly monotone then for all decreasing sequence $\{a_n\}_n$ of positive numbers such that $\{a_n\}_n \subseteq \Phi(I)$ and all integer $N \ge 1$, it holds that

$$\sum_{k=1}^N \Phi\left(\frac{\sum_{n=1}^k \Phi^{-1}(a_n)\Delta_n(M)}{M(k)}\right)\Delta_k(W) \le \inf_{s>s_0(M)} (sp_+(w,s))^{\frac{p_-(M,s)}{p_-(M,s)-1}} \sum_{k=1}^N a_k\Delta_k(W). \tag{3.7}$$

Proof. We notice that the function

$$f(x) = \sum_{n\ge 1} \chi_{[n-1,n)}(x)\frac{1}{a_n}$$

is increasing in $\mathbb{R}_+$ and for all integer $k \ge 1$ it holds that

$$T_\mu f(k) = \frac{\sum_{n=1}^k \frac{\Delta_n(M)}{a_n}}{M(k)}.$$

We know from Proposition 2.1 that if $x \in [k-1,k)$ then $T_\mu f(x) \le T_\mu f(k)$. Using this property jointly with the fact that Φ is decreasing we obtain that the inequality

$$\sum_{k=1}^N \Phi(T_\mu f(k))\Delta_k W \le \int_0^N \Phi(T_\mu f(x))w(x)\mathrm{d}x$$

holds for all integer $N \geq 1$. The last inequality and Theorem 2.6 yield that, for every integer $N \geq 1$, it holds that

$$\sum_{k=1}^{N} \Phi\left(\frac{\sum_{n=1}^{k} \frac{\Delta_n(M)}{a_n}}{M(k)}\right)\Delta_k(W) \leq \inf_{s>s_0(M)} (sp_+(w,s))^{\frac{p_-(M,s)}{p_-(M,s)-1}} \sum_{k=1}^{N} \Phi(1/a_k)\Delta_k(W).$$

The inequality in (3.7) is obtained from (3.6) by the change of variables $b_n = \Phi(1/a_n)$. □

By taking particular choices of Φ we obtain more concrete estimates.

Corollary 3.2. *Let w, μ and M be as in Theorem 2.6 and assume in addition that w is locally integrable in $[0,\infty)$. Then the following estimates are valid for all decreasing sequence $\{a_n\}_n$ of positive numbers.*

(i) For all integer $N \geq 1$,

$$\sum_{k=1}^{N} \left(\prod_{n=1}^{k} a_n^{\frac{\Delta_n(M)}{M(k)}}\right)\Delta_k(W) \leq \inf_{s>s_0(M)} (sp_+(w,s))^{\frac{p_-(M,s)}{p_-(M,s)-1}} \sum_{k=1}^{N} a_k\Delta_k(W). \tag{3.8}$$

(ii) For all integer $N \geq 1$ and $\alpha \geq 0$,

$$\sum_{k=1}^{N} \left(\frac{M(k)}{\sum_{n=1}^{k} \frac{\Delta_n(M)}{a_n}}\right)^{\alpha} \Delta_k(W) \leq \inf_{s>s_0(M)} (sp_+(w,s))^{\frac{p_-(M,s)}{p_-(M,s)-1}} \sum_{k=1}^{N} a_k^{\alpha}\Delta_k(W). \tag{3.9}$$

Proof. The first inequality is obtained from (3.7) by taking $\Phi(x) = e^{-x}$ and $I = \mathbb{R}$. The second statement follows by applying (3.6) with $\Phi(x) = x^{-\alpha}$ and $I = (0,\infty)$. □

Remark 3.3. *We notice that Carleman's inequality can be derived from* (3.8)*. Indeed, if we pick μ to be the Lebesgue measure in $[0,\infty)$ and w to be identically* 1 *then* (3.8) *and* (2.9) *yield the inequality*

$$\sum_{k=1}^{N} \left(\prod_{n=1}^{k} a_n\right)^{1/k} \leq e\sum_{k=1}^{N} a_k$$

for all integer $N \geq 1$ and all decreasing sequence $\{a_n\}_n$ of positive numbers. Letting $N \to \infty$ we obtain the desired inequality, that is,

$$\sum_{k=1}^{\infty} \left(\prod_{n=1}^{k} a_n\right)^{1/k} \leq e\sum_{k=1}^{\infty} a_k$$

for all positive sequence $\{a_n\}_n$. Here we should notice that the left hand side in Carleman's inequality attains its maximum when the terms in the sequence are arranged in decreasing order (see Section 4).

Remark 3.4. *Some variations of Carleman's inequality can be also obtained from Corollary 3.2. For instance, let us consider the measure $\mathrm{d}\mu(t) := \alpha t^{\alpha-1}\mathrm{d}t$, with $\alpha \geq 1$, so that $M(t) = t^{\alpha}$ and $p_-(M,s) = s^{\alpha}$ (see the first example in Examples 2.9). Then, given $\beta > 0$, we can apply* (3.8) *for the weight function*

$$w(x) := \sum_{k=1}^{\infty} k^{\beta}\chi_{(k-1,k]}(x)$$

to obtain that the inequality

$$\sum_{k=1}^{\infty} \left(\prod_{n=1}^{k} a_n^{\frac{n^{\alpha}-(n-1)^{\alpha}}{k^{\alpha}}}\right) k^{\beta} \leq \inf_{s>1} (sp_+(w,s))^{\frac{s^{\alpha}}{s^{\alpha}-1}} \sum_{k=1}^{\infty} a_k k^{\beta} \tag{3.10}$$

holds for all positive decreasing sequence $\{a_k\}_k$*. Here we notice that* (2.2) *is satisfied, since*

$$p_+(w,s) = \sup_{k\geq 1}\left(\sup_{k-1<x\leq k}\frac{w(sx)}{w(x)}\right) = \sup_{k\geq 1}\frac{1}{k^\beta}\left(\sup_{k-1<x\leq k} w(sx)\right) \leq \sup_{k\geq 1}\frac{(sk+1)^\beta}{k^\beta} = (s+1)^\beta$$

for all $s>1$*. This recovers an inequality by E.R. Love for decreasing sequences* [18, Theorem 1] *(see also* [1, p. 40]*), albeit with a larger constant.*

Remark 3.5. *If we pick* μ *to be the Lebesgue measure in* $[0,\infty)$*,* w *to be identically* 1 *and* $\alpha>0$*, then* (3.9) *yields*

$$\sum_{k=1}^N\left(\frac{k}{\sum_{n=1}^k\frac{1}{a_n}}\right)^\alpha \leq e\sum_{k=1}^N a_n^\alpha,$$

for all integer $N\geq 1$ *and all positive decreasing sequence* $(a_n)_n$*.*

For instance, by choosing $a_n = 1/(n+q)^\nu$ *for fixed real numbers* $q,\nu>0$ *and letting* $N\to\infty$ *we obtain the inequality*

$$\sum_{k=1}^\infty\left(\frac{k}{\sum_{n=1}^k(n+q)^\nu}\right)^\alpha \leq e\sum_{k=1}^\infty\frac{1}{(k+q)^{\nu\alpha}},$$

where the series appearing in the right hand side is the so-called Hurwitz zeta function $\zeta(\nu\alpha,q)$*.*

3.3. **The harmonic mean operator.** For a collection of positive numbers $a_1,\dots,a_k$ we define its harmonic mean by the quantity

$$\frac{k}{\sum_{n=1}^k\frac{1}{a_n}}.$$

Let us now generalize the concept of the harmonic mean for a collection of positive numbers to functions defined on $(0,\infty)$.

Definition 3.6. *Let* μ *be as in Theorem 2.6. For all function* $f:(0,\infty)\to\mathbb{R}$ *such that* $1/f\in L^1_{\mathrm{loc}}(\mathrm{d}\mu)$*, we define the harmonic mean operator as*

$$\mathscr{H}_\mu f(x) := \frac{\mu[0,x)}{\int_0^x\frac{1}{f(s)}\mathrm{d}\mu(s)}, \quad x>0.$$

As a consequence of our study we get the following boundedness property on the cone of decreasing functions in $L^p(w)$, the weighted Lebesgue space.

Corollary 3.7. *Let* w *and* μ *be as in Theorem 2.6 and set* $p>0$*. Then for all* $t>0$ *the inequality*

$$\int_0^t(\mathscr{H}_\mu f(x))^p w(x)\mathrm{d}x \leq \inf_{s>1}(sp_+(w,s))^{\frac{s}{s-1}}\int_0^t f(x)^p w(x)\mathrm{d}x$$

holds for all positive decreasing functions f *such that* $1/f\in L^1_{\mathrm{loc}}(\mathrm{d}\mu)$*, provided the left hand side is finite.*

Proof. The statement follows by applying Corollary 2.11 with $\Phi(x)=1/x$. ☐

Remark 3.8. *We observe that* $\mathscr{H}_\mu f\geq f$ *for all positive decreasing function in* $(0,\infty)$ *such that* $1/f\in L^1_{\mathrm{loc}}(\mathrm{d}\mu)$*. Hence we deduce from the previous Corollary that, given* w *and* μ *as in Theorem 2.6, it holds that*

$$\left\|\mathscr{H}_\mu f\right\|_{L^p(w)} \approx \|f\|_{L^p(w)}$$

for all $p>0$ *and all positive decreasing function* f *in* $(0,\infty)$*.*

3.4. **The Laplace transform.** Let s be a complex number and let f be a function defined on $\mathbb{R}_+$. We define the Laplace transform of f by

$$\mathscr{L}[f](s) := \int_0^\infty f(t)e^{-st}\mathrm{d}t.$$

Corollary 3.9. *Let μ, I and Φ be as in Theorem 2.6 and set $p>0$. Then it holds that*

$$\mathscr{L}[\Phi(T_\mu(\Phi^{-1}\circ f))^p](\lambda) \leq \left(\inf_{s>s_0(M)} s^{\frac{p_-(M,s)}{p_-(M,s)-1}}\right)\mathscr{L}[f^p](\lambda),$$

for all decreasing function $f:\mathbb{R}_+\to\Phi(I)$ and all $\lambda>0$, provided the left hand side is finite.

Proof. The statement follows by applying Corollary 2.11 for the weight $w(x)=e^{-\lambda x}$ (look at the second example in Examples 2.10) and letting $t\to\infty$ in (2.10). □

3.5. **Rearrangement invariant spaces.** In this section we will denote by (R,ν) a totally σ-finite measure space which is nonatomic and $\nu(R)=\infty$.

Let us start by recalling some basic definitions related to rearrangement invariant function spaces. We follow mainly the exposition in [2].

Definition 3.10. *Let $f:R\to\mathbb{C}$ be a measurable function and $(X,\|\cdot\|_X)$ be a Banach function space over (R,ν).*

- *We define the distribution function of f by*
$$d_f(s) := \nu\{x\in R : |f(x)|>s\},\quad s\geq 0.$$
- *The decreasing rearrangement of f, denoted by f^*, is the function defined on $[0,\infty)$ as*
$$f^*(t) := \inf\{s\geq 0 : d_f(s)\leq t\}.$$
- *We say that X is a rearrangement-invariant function space if whenever f belongs to X and g is a measurable function in (R,ν) such that $d_f=d_g$, then g belongs to X and $\|f\|_X=\|g\|_X$.*
- *Consider on X the norm given by*
$$\|f\|_{X'} := \sup_{\|g\|_X\leq 1}\int_R |f(x)g(x)|\,\mathrm{d}\nu(x).$$
We call $X'=(X,\|\cdot\|_{X'})$ the associate space of X.

Let us first recall some properties satisfied by the decreasing rearrangement function f^*.

Proposition 3.11. *(*[2, Propositions 1.7 and 1.8 in Chapter 2]*) Let μ be a measure in $\mathbb{R}_+$ such that $(\mathbb{R}_+,\mu)$ is a totally σ-finite measure space and set f for a measurable function.*

P.1 *f^* is a non-negative, decreasing and right continuous function on $[0,\infty)$.*

P.2 *The functions f and its decreasing rearrangement f^* are equimeasurable, in the sense that their distribution functions coincide.*

P.3 *We have that*
$$\int_{\mathbb{R}_+}|f(x)|\,\mathrm{d}\mu(x) = \int_0^\infty f^*(x)\mathrm{d}x.$$

We shall also state without proof, some important properties of rearrangement invariant spaces that will be helpful for our study. The proof can be found in [2, Chapter 2, Section 4].

Proposition 3.12. *Assume that X is a rearrangement-invariant space over (R,ν).*

(1) It holds that
$$\|f\|_X = \sup_{\|g\|_{X'}\leq 1}\int_0^\infty f^*(x)g^*(x)\mathrm{d}x. \tag{3.11}$$

(2) (Hölder's inequality) If $f \in X$ and $g \in X'$ then

$$\int_0^\infty f^*(x)g^*(x)\mathrm{d}x \leq \|f\|_X \|g\|_{X'}. \tag{3.12}$$

(3) (Luxemburg representation theorem) There exists a rearrangement-invariant space $(\overline{X}, \|\cdot\|_{\overline{X}})$ over $(\mathbb{R}_+, |\cdot|)$, where $|\cdot|$ denotes the Lebesgue measure on R_+, such that for all $f \in X$

$$\|f\|_X = \|f^*\|_{\overline{X}}. \tag{3.13}$$

(4) For every function f in X, its decreasing rearrangement f^* is the unique non-negative, decreasing and right continuous function on $[0,\infty)$ which satisfies (3.13).

The following corollary is a consequence of our main theorem.

Corollary 3.13. *Let μ, I and Φ be as in Theorem 2.6. Assume in addition that $\Phi : I \to \mathbb{R}_+$ is strictly monotonic and let $f : \mathbb{R}_+ \to \Phi(I)$ be a decreasing function. Then, for all non negative decreasing function g it holds that*

$$\int_0^\infty \Phi\left(T_\mu\left(\Phi^{-1}f\right)(x)\right) g(x)\mathrm{d}x \leq \inf_{s>s_0(M)} s^{\frac{p_-(M,s)}{p_-(M,s)-1}} \int_0^\infty f(x)g(x)\mathrm{d}x. \tag{3.14}$$

In particular, for all rearrangement-invariant Banach function space X it holds that

$$\|f\|_X \leq \left\|\Phi\left(T_\mu\left(\Phi^{-1}f^*\right)\right)\right\|_{\overline{X}} \leq \inf_{s>s_0(M)} s^{\frac{p_-(M,s)}{p_-(M,s)-1}} \|f\|_X \tag{3.15}$$

for all $f \in X$ such that $f^* : \mathbb{R}_+ \to \Phi(I)$, provided the middle term is finite.

Proof. We apply (2.8) with $F = \Phi(T_\mu(\Phi^{-1} \circ f))$, $G = f$ and $h = g$ to get that

$$\int_0^\infty \Phi\left(T_\mu\left(\Phi^{-1}f\right)(x)\right) g(x)\mathrm{d}x \leq \left(\sup_{r>0} \frac{\int_0^r \Phi\left(T_\mu\left(\Phi^{-1}f\right)(x)\right)\mathrm{d}x}{\int_0^r f(x)\mathrm{d}x}\right) \int_0^\infty f(x)g(x)\mathrm{d}x.$$

Next we know by Corollary 2.11 that

$$\sup_{r>0} \frac{\int_0^r \Phi\left(T_\mu\left(\Phi^{-1}f\right)(x)\right)\mathrm{d}x}{\int_0^r f(x)\mathrm{d}x} \leq \inf_{s>s_0(M)} s^{\frac{p_-(M,s)}{p_-(M,s)-1}},$$

from where (3.14) is deduced.

To see (3.15) let us start by proving that the right hand side of the inequality holds. Indeed, we use (3.11), (3.14), (3.12) and (3.13) to get that

$$\begin{aligned}
\left\|\Phi\left(T_\mu\left(\Phi^{-1}f^*\right)(x)\right)\right\|_{\overline{X}} &= \sup_{\|g\|_{\overline{X}'}\leq 1} \int_0^\infty \Phi\left(T_\mu\left(\Phi^{-1}f^*\right)(x)\right) g^*(x)\mathrm{d}x \\
&\leq \left(\inf_{s>s_0(M)} s^{\frac{p_-(M,s)}{p_-(M,s)-1}}\right) \sup_{\|g\|_{\overline{X}'}\leq 1} \int_0^\infty f^*(x)g^*(x)\mathrm{d}x \\
&= \left(\inf_{s>s_0(M)} s^{\frac{p_-(M,s)}{p_-(M,s)-1}}\right) \|f\|_X.
\end{aligned}$$

To show that the left hand side of (3.15) holds we notice that

$$\Phi[T_\mu(\Phi^{-1}f^*)] \geq f^*. \tag{3.16}$$

Hence we use (3.11), (3.16), (3.12) and (3.13) to get that

$$\begin{aligned}\|f\|_X &= \sup_{\|g\|_{\overline{X}'}\le 1}\int_0^\infty f^*(x)g^*(x)\mathrm{d}x\\ &\le \sup_{\|g\|_{\overline{X}'}\le 1}\int_0^\infty \Phi(T_\mu(\Phi^{-1}f^*)(x))g^*(x)\mathrm{d}x\\ &\le \left\|\Phi(T_\mu(\Phi^{-1}f^*))\right\|_{\overline{X}}\sup_{\|g\|_{\overline{X}'}\le 1}\|g^*\|_{\overline{X}'}\\ &\le \left\|\Phi(T_\mu(\Phi^{-1}f^*))\right\|_{\overline{X}}.\end{aligned}$$

□

Remark 3.14. *In the proof of the corollary we have implicitly used the identity*

$$[\Phi\left(T_\mu\left(\Phi^{-1}\circ f^*\right)\right)]^* = \Phi\left(T_\mu\left(\Phi^{-1}\circ f^*\right)\right),$$

although the function $\Phi\left(T_\mu\left(\Phi^{-1}\circ f^*\right)\right)$ might not be right continuous. However, the identity is still satisfied. Indeed, we notice that the function $\Phi\left(T_\mu\left(\Phi^{-1}\circ f^*\right)\right)$ is monotone and, therefore, it has at most a countable amount of discontinuities. More precisely, it is continuous in $\mathbb{R}_+\setminus E$, where E is a set of measure zero. Then we can construct a non-negative, decreasing and right continuous function Ψ which equals $\Phi\left(T_\mu\left(\Phi^{-1}f^*\right)\right)$ almost everywhere. For instance, we set

$$\Psi(x) := \begin{cases}\Phi\left(T_\mu\left(\Phi^{-1}f^*\right)(x)\right) & \text{if } x\notin E,\\ \lim_{y\to x^+}\Phi\left(T_\mu\left(\Phi^{-1}f^*\right)(y)\right) & \text{if } x\in E.\end{cases}$$

Then $\Psi^* = \Psi$ since Ψ is decreasing and right continuous, and we deduce that

$$\begin{aligned}\int_0^\infty [\Phi\left(T_\mu\left(\Phi^{-1}\circ f^*\right)\right)]^*(x)g^*(x)\mathrm{d}x &= \int_0^\infty \Psi^*(x)g^*(x)\mathrm{d}x\\ &= \int_0^\infty \Phi\left(T_\mu\left(\Phi^{-1}f^*\right)(x)\right)g^*(x)\mathrm{d}x.\end{aligned}$$

Remark 3.15. *Note that, in the previous result, no extra assumption on the space X is needed. This contrast, with the boundedness of the maximal function $f^{**}(t) = t^{-1}\int_0^t f^*(s)\mathrm{d}s$, which requires the upper Boyd index of X to be strictly smaller than* 1 *(See e.g.* [2, Theorem 5.15].

4. Some extensions to positive functions

In this section we want to illustrate how the main result of our paper can be extended to some families of non-negative functions, and in particular, how we can recover the well known inequality of Pólya and Knopp.

The following result gives some equivalent formulations of the main theorem of this paper when $\Phi(x) = e^x$, w is identically one and μ is the Lebesgue measure in $\mathbb{R}_+$. We write T instead of T_μ for the Lebesgue measure.

Proposition 4.1. *The following two statements are equivalent.*

I) The inequality

$$\int_0^\infty \exp\left(T[\log f](x)\right)\mathrm{d}x \le e\int_0^\infty f(x)\mathrm{d}x$$

holds for all decreasing function $f:\mathbb{R}_+\to(0,\infty)$.

II) The inequality

$$\int_0^\infty \exp\left(T[\log f](x)\right)\mathrm{d}x \leq e\int_0^\infty f(x)\mathrm{d}x$$

holds for all function $f:\mathbb{R}_+\to(0,\infty)$.

Proof. Let us prove the equivalence by showing that I) $\Rightarrow$ II).

First of all, we notice that it is enough to show II) for all positive bounded function. Indeed, if f is a positive function and II) is satisfied for all bounded functions from the increasing sequence $(f_n)_{n\geq 1} = (f\chi_{\{|f(x)|\leq n\}})_{n\geq 1}$, then the inequality would be deduced for f by using the Monotone Convergence Theorem.

Furthermore, we can show II) by reducing the study to positive bounded functions of the form $f:\mathbb{R}_+\to(0,1]$. Indeed, we notice that the inequality in II) is homogeneous, in the sense that for a given $\lambda>0$, the inequality holds for a positive function f if, and only if, it is satisfied for λf. Therefore, if f is a positive bounded function and II) is satisfied for $f/\|f\|_\infty$, then the homogeneity of the inequality yields the desired result for f.

So let us check that II) is satisfied for all positive bounded functions of the form $f:\mathbb{R}_+\to(0,1]$, assuming that I) is valid. To do so we show first that the inequality

$$\int_0^x \log f(y)\mathrm{d}y \leq \int_0^x \log f^*(y)\mathrm{d}y \tag{4.1}$$

holds for all $x>0$.

By applying Tonelli's Theorem we notice that

$$-\int_0^x \log f(y)\mathrm{d}y = \int_0^x\int_{f(y)}^1 \frac{\mathrm{d}s}{s}\mathrm{d}y = \int_0^1 |\{0<y<x,\ f(y)\leq s\}|\,\frac{\mathrm{d}s}{s} \tag{4.2}$$

is satisfied for all $x>0$. Next we see that for all $x>0$ and all $0<s<1$ we have that

$$|\{0<y<x,\ f(y)\leq s\}| = x - \left|\left\{y:\chi_{[0,x]}f(y)>s\right\}\right|$$

and applying Proposition 3.11 **P.2** we get that

$$\left|\left\{y>0:\chi_{[0,x]}f(y)>s\right\}\right| = \left|\left\{y>0:(\chi_{[0,x]}f)^*(y)>s\right\}\right| \leq |\{x>y>0:\ f^*(y)>s\}|. \tag{4.3}$$

Combining them we obtain that

$$|\{0<y<x,\ f(y)\leq s\}| \geq |\{0<y<x,\ f^*(y)\leq s\}|. \tag{4.4}$$

By plugging this last estimate in (4.2) we deduce that

$$-\int_0^x \log f(y)\mathrm{d}y \geq \int_0^1 |\{0<y<x,\ f^*(y)\leq s\}|\,\frac{\mathrm{d}s}{s} = -\int_0^x \log f^*(y)\mathrm{d}y,$$

where in the last identity one should notice that $f^*\leq 1$ when $\|f\|_\infty\leq 1$. Hence (4.1) follows.

Using (4.1) jointly with the monotonicity of the exponential function and the integral, applying I) to f^* and Proposition 3.11 **P.3** we deduce that

$$\begin{aligned}\int_0^\infty \exp\left(T[\log f](x)\right)\mathrm{d}x &\leq \int_0^\infty \exp\left(T[\log f^*](x)\right)\mathrm{d}x\\ &\leq e\int_0^\infty f^*(x)\mathrm{d}x = e\int_0^\infty f(x)\mathrm{d}x.\end{aligned}$$

□

Remark 4.2. *We notice that Theorem 4.1 I) follows from Theorem 2.6 by letting $t\to\infty$ (notice that the constant in* (2.3) *does not depend on t). Hence, using the formulation in Theorem 4.1 II) we obtain* (1.1) *for all positive functions f.*

Remark 4.3 (Changing the measure)**.** *By slightly modifying the argument above, changing the Lebesgue measure by* $\mathrm{d}\mu(t) = u(t)\mathrm{d}t$ *with* u *strictly positive, locally integrable on* $[0,\infty)$ *and continuous on* $(0,\infty)$*, one can show that for all* $x > 0$ *and for all* $f : \mathbb{R}_+ \to (0,\infty)$ *such that* $\|f\|_\infty \leq 1$ *it holds that*

$$\int_0^x (\log f(y))\, u(y)\mathrm{d}y \leq \int_0^{U(x)} \log(f)_u^*(y)\mathrm{d}y.$$

Here $(f)_u^*$ *denotes the non-increasing rearrangement of* f *with respect to the measure* μ *and* $U(x) := \int_0^x u(s)\mathrm{d}s$*. Thus, applying Corollary 2.11, the previous observation and* ***P.3*** *in Proposition 3.11 yield*

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^{U^{-1}(x)} (\log f(y))\, u(y)\mathrm{d}y\right)\mathrm{d}x \leq e\int_0^\infty f(x)u(x)\mathrm{d}x,$$

where $U^{-1}(x)$ *denotes the inverse of the strictly increasing function* U*. A change of variables, and the homogeneity of the expression yields*

$$\int_0^{U^{-1}(\infty)} \exp\left(\frac{1}{U(x)}\left(\int_0^x (\log f(y))\, u(y)\mathrm{d}y\right)\right)u(x)\mathrm{d}x \leq e\int_0^\infty f(x)u(x)\mathrm{d}x,$$

for all $f : \mathbb{R}_+ \to (0,\infty)$*.*

This, in particular, recovers Cochran-Lee's inequality with $u(x) = px^{p-1}$*.*

Remark 4.4 (Changing the function Φ)**.** *Assume that* $\Phi : \mathbb{R} \to \mathbb{R}_+$ *is a strictly increasing, continuous and log-convex function, such that* $\Phi(0) > 0$, $\lim_{t\to-\infty}\Phi(t) = 0$*. Then, for all positive* $f : \mathbb{R}_+ \to (0,\infty)$ *such that* $\|f\|_\infty \leq \Phi(0)$*, then*

$$\Phi\left(\frac{1}{x}\int_0^x \Phi^{-1}(f(y))\mathrm{d}y\right) \leq \Phi\left(\frac{1}{x}\int_0^x \Phi^{-1}(f^*(y))\mathrm{d}y\right). \tag{4.5}$$

Indeed, for all $x > 0$*, by* (4.4)*, it holds that*

$$\begin{aligned}-\int_0^x \Phi^{-1}(f(y))\mathrm{d}y &= \int_0^x\int_{f(y)}^{\Phi(0)} \mathrm{d}\Phi^{-1}(t)\mathrm{d}y = \int_0^{\Phi(0)} |\{0<y<x,\, f(y)\leq t\}|\,\mathrm{d}\Phi^{-1}(t)\\ &\geq -\int_0^x \Phi^{-1}(f^*(y))\mathrm{d}y\end{aligned}$$

So, multiplying by $-1/x$ *both sides, and using the monotonicity of* Φ*,* (4.5) *follows. Therefore we can deduce that*

$$S_\Phi f(x) = \Phi\left(\frac{1}{x}\int_0^x \Phi^{-1}(|f(y)|)\mathrm{d}y\right) \tag{4.6}$$

satisfies, for all $f : \mathbb{R}_+ \to \mathbb{C}$*, with* $\|f\|_\infty \leq \Phi(0)$

$$S_\Phi f(x) \leq S_\Phi f^*(x), \qquad x > 0.$$

Moreover, given any weight $w \in L^1_{\mathrm{loc}}([0,\infty))$ *satisfying the hypotheses in Theorem 2.6, we can apply Corollary 2.11 to get that for all* $f \in \Lambda^1(w)$ *with* $\|f\|_\infty \leq \Phi(0)$ *it holds that*

$$\|S_\Phi f\|_{L^1(w)} \leq \inf_{s>1}\left(s\sup_{x>0}\frac{w(sx)}{w(x)}\right)^{\frac{s}{s-1}}\|f\|_{\Lambda^1(w)},$$

where $\Lambda^1(w)$ *denotes the weighted Lorentz space given by*

$$\Lambda^1(w) := \left\{f : \mathbb{R}_+ \to \mathbb{C} : f \text{ is measurable and } \|f\|_{\Lambda^1(w)} := \int_0^\infty f^*(s)w(s)\mathrm{d}s < +\infty\right\}.$$

We can apply this argument to functions of the type

$$\Phi(t) = e^{t^{2k+1}}, \qquad k \in \mathbb{N}.$$

In the case of $\Phi(x) = e^x$, the homogeneity on the expression, allows one to remove the assumption $\|f\|_\infty \le 1$.

Remark 4.5. *One can obtain a similar result for power functions of the type $\Phi(t) = t^{-\alpha}$ with $\alpha > 0$. Lets start by considering the case $\alpha = 1$. Note that, in this case, for a given function $f > 0$ we have that*

$$\begin{aligned}
\int_0^x \frac{1}{f(s)}\mathrm{d}s &= \int_0^x \int_{f(s)/2}^{f(s)} \frac{\mathrm{d}t}{t^2}\mathrm{d}s = \int_0^\infty |\{0 < s < x : \frac{f(s)}{2} \le t < f(s)\}| \frac{\mathrm{d}t}{t^2} \\
&= \int_0^\infty (d_g(t) - d_g(2t)) \frac{\mathrm{d}t}{t^2} = \int_0^\infty (d_{g^*}(t) - d_{g^*}(2t)) \frac{\mathrm{d}t}{t^2} \\
&= \int_0^\infty |\{0 < s < x : \frac{g^*(s)}{2} \le t < g^*(s)\}| \frac{\mathrm{d}t}{t^2} \\
&= \int_0^x \frac{1}{g^*(s)}\mathrm{d}s,
\end{aligned}$$

where $g = \chi_{(0,x)} f$. Therefore, since $g^*(t) \le f^*(t)$ it follows that

$$\int_0^x \frac{1}{f(s)}\mathrm{d}s = \int_0^x \frac{1}{g^*(s)}\mathrm{d}s \ge \int_0^x \frac{1}{f^*(s)}\mathrm{d}s,$$

from where inequality (4.5) *holds for $\Phi(t) = t^{-1}$.*

For the general case $\alpha > 0$, note first that if we write $r_\alpha = (1+\alpha)^{1/\alpha}$ then one has that for all $s > 0$

$$s^{-\alpha} = \int_{s/r_\alpha}^s t^{-\alpha-1}\mathrm{d}t.$$

So arguing as the case for $\alpha = 1$ we deduce that inequality (4.5) *holds for $\Phi(t) = t^{-\alpha}$.*

Then Corollary 2.11 yields that the average operator

$$S_\alpha f(x) = \left(\mathscr{H}_1(|f|^{1/\alpha})\right)^\alpha \tag{4.7}$$

where $\mathscr{H}_1$ denotes the harmonic mean operator of Definition 3.6 associated to the Lebesgue measure of $(0,\infty)$, satisfies that, for all non-negative function f

$$S_\alpha f(x) \le S_\alpha f^*(x), \quad x > 0.$$

Therefore, for every weight w as in Theorem (2.6) *and for any $f \in \Lambda^1(w)$ it holds that*

$$\|S_\alpha f\|_{L^1(w)} \le \inf_{s>1} \left(s \sup_{x>0} \frac{w(sx)}{w(x)} \right)^{\frac{s}{s-1}} \|f\|_{\Lambda^1(w)}.$$

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DEPARTMENT OF MATHEMATICS, STOCKHOLM UNIVERSITY, SE-106 91 STOCKHOLM, SWEDEN
E-mail address: `arias@math.su.se`

DEPARTMENT OF MATHEMATICS, STOCKHOLM UNIVERSITY, SE-106 91 STOCKHOLM, SWEDEN
E-mail address: `s.rodriguez-lopez@math.su.se`