SHARP BOUNDS FOR EQUIVALENT NORMS IN THE CLASSICAL CESÀRO AND COPSON FUNCTION SPACES

ACHRAF BEN SAID*, SANTIAGO BOZA**, AND JAVIER SORIA†

ABSTRACT. Recently, several authors have investigated the problem of determining optimal constants in the L^p-norm for Hardy-type inequalities. In this work, we continue this line of research by studying new norms on the classical Cesàro and Copson function spaces, $\mathrm{Ces}_p(\mathbb{R}^+)$ and $\mathrm{Cop}_p(\mathbb{R}^+)$, respectively. Furthermore, we establish the sharp constants associated with the corresponding norm equivalences. In particular, we improve a result due to K. Lešnika and L. Maligranda (J. Math. Anal. Appl., 424 (2015), 932–951).

1. INTRODUCTION

Denote $\mathcal{M}^+(\mathbb{R}^+)$ the class of all nonnegative measurable functions on $\mathbb{R}^+ = (0, \infty)$. Let $f \in \mathcal{M}^+(\mathbb{R}^+)$. Set

$$Hf(x) = \frac{1}{x}\int_0^x f(t)dt$$

and

$$H^*f(x) = \int_x^\infty \frac{f(t)}{t}dt,$$

provided that the integrals make sense for a function f on $\mathbb{R}^+$. These equalities define the classical Hardy operator H and its dual operator H^*. The boundedness of H on $L^p(\mathbb{R}^+)$, for $1 < p \leq \infty$, follows from Hardy's inequality (cf. [9, p. 240]):

$$\|Hf\|_{L^p(\mathbb{R}^+)} \leq p' \, \|f\|_{L^p(\mathbb{R}^+)}, \tag{1}$$

where $p' = \frac{p}{p-1}$ if $1 < p < \infty$, and $p' = 1$ if $p = \infty$. Moreover, p' is the best possible constant in (1).

However, H is not invertible on $L^p(\mathbb{R}^+)$ (see [15]), so no constant $c(p) > 0$ depending only on p exists such that the reverse inequality

$$\|Hf\|_{L^p(\mathbb{R}^+)} \geq c(p) \, \|f\|_{L^p(\mathbb{R}^+)} \tag{2}$$

holds in general, even for nonnegative functions $f \in L^p(\mathbb{R}^+)$. Note that since H is a linear injective operator, the functional $N(f) = \|H|f|\|_p$ defines a norm on $L^p(\mathbb{R}^+)$. If there were a constant $c(p)$ in (2) to exist for all nonnegative $f \in L^p(\mathbb{R}^+)$, (1)

2020 *Mathematics Subject Classification.* 26D15, 47B37, 46E30.

Key words and phrases. Cesàro function spaces, Copson function spaces, Hardy, optimal constants, sharp constants.

* The author was partially supported by grants PID2020-113048GB-I00 and PID2024-155917NB-I00, funded by MCIN/AEI/ 10.13039/501100011033.

** The author was partially supported by grants PID2020-113048GB-I00 and PID2024-155917NB-I00, funded by MCIN/AEI/10.13039/501100011033, and by grant 2021SGR 00087.

† The author was partially supported by grants PID2020-113048GB-I00, PID2024-155917NB-I00, and CEX-2023-001347-S, funded by MCIN/AEI/ 10.13039/501100011033, and Grupo UCM-970966.

and (2) together would imply that N defines a norm on $L^p(\mathbb{R}^+)$ equivalent to the standard L^p-norm; however, this is not the case (see [15]). Interestingly, replacing H by $H - I$ yields an operator that induces a norm on $L^p(\mathbb{R}^+)$ equivalent to the standard L^p-norm (see [3]).

The classical Cesàro function spaces $\operatorname{Ces}_p(\mathbb{R}^+)$ and the classical Copson function spaces $\operatorname{Cop}_p(\mathbb{R}^+)$ are defined, respectively, by

$$\operatorname{Ces}_p(\mathbb{R}^+) = \big\{ f : \|f\|_{\operatorname{Ces}_p(\mathbb{R}^+)} = \|H|f|\|_p < \infty \big\}, \qquad 1 < p \le \infty,$$

and

$$\operatorname{Cop}_p(\mathbb{R}^+) = \big\{ f : \|f\|_{\operatorname{Cop}_p(\mathbb{R}^+)} = \|H^*|f|\|_p < \infty \big\}, \qquad 1 \le p < \infty.$$

These spaces are Banach spaces that have been extensively studied; see, for example, [1, 5, 10, 14, 18] and the references therein.

Note that (1) together with (2) implies the strict inclusion

$$L^p(\mathbb{R}^+) \subsetneq \operatorname{Ces}_p(\mathbb{R}^+),$$

for all $1 < p \le \infty$.

In [5, Theorem 21.1] G. Bennett showed that $\operatorname{Ces}_p(\mathbb{R}^+)$ and $\operatorname{Cop}_p(\mathbb{R}^+)$ coincide for $1 < p < \infty$. He also derived estimates for the norms of the corresponding inclusion operators. Moreover, he proved that if $1 < p \le 2$, then

$$\|f\|_{\operatorname{Cop}_p(\mathbb{R}^+)} \le (p-1)^{1/p} \|f\|_{\operatorname{Ces}_p(\mathbb{R}^+)},$$

and if $2 \le p < \infty$, then

$$(p-1)^{1/p} \|f\|_{\operatorname{Ces}_p(\mathbb{R}^+)} \le \|f\|_{\operatorname{Cop}_p(\mathbb{R}^+)}.$$

Later, V. Kolyada completed the work started by G. Bennett proving the following result.

Theorem 1.1. [12, Theorem 1.1] *Let $1 < p < \infty$ and let $f \in \mathcal{M}^+(\mathbb{R}^+)$. If $1 < p \le 2$, then*

$$(p-1)\|f\|_{\operatorname{Ces}_p(\mathbb{R}^+)} \le \|f\|_{\operatorname{Cop}_p(\mathbb{R}^+)} \le (p-1)^{1/p} \|f\|_{\operatorname{Ces}_p(\mathbb{R}^+)}, \tag{3}$$

and if $2 \le p < \infty$, then

$$(p-1)^{1/p}\|f\|_{\operatorname{Ces}_p(\mathbb{R}^+)} \le \|f\|_{\operatorname{Cop}_p(\mathbb{R}^+)} \le (p-1) \|f\|_{\operatorname{Ces}_p(\mathbb{R}^+)}. \tag{4}$$

Moreover, all constants in (3) and (4) are the best possible.

Let $\Gamma : (0, \infty) \to \mathbb{R}$ denotes the Euler Gamma function given by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

The following theorem, also due to V. Kolyada [13], extends a result by S. Boza and J. Soria in [7].

Theorem 1.2. [13, Theorem 1.3] *Let $1 < p < \infty$ and let $f \in \mathcal{M}^+(\mathbb{R}^+)$ be nonincreaisng function. If $1 < p \le 2$, then*

$$(p-1)\|Hf\|_p \le \|H^* f\|_p \le \left(\frac{\Gamma(p+1)}{p'}\right)^{1/p} \|Hf\|_p, \tag{5}$$

and if $2 \le p < \infty$, *then*

$$\left(\frac{\Gamma(p+1)}{p'}\right)^{1/p} \|Hf\|_p \le \|H^*f\|_p \le (p-1)\|Hf\|_p. \tag{6}$$

Moreover, all constants in (5) *and* (6) *are the best possible.*

Taking into account that $HH^* = H^*H$ (see [8, 11]) and that H^*f is nonnegative and nonincreasing for a nonnegative function f, we obtain the following result as a consequence of Theorem 1.2.

Theorem 1.3. *Let* $1 < p < \infty$ *and let* $f \in \mathcal{M}^+(\mathbb{R}^+)$ *such that* $H^*f(x) < \infty$ *for all* $x > 0$. *If* $1 < p \le 2$, *then*

$$(p-1)\|Hf\|_{\mathrm{Cop}_p(\mathbb{R}^+)} \le \|H^*f\|_{\mathrm{Cop}_p(\mathbb{R}^+)} \le \left(\frac{\Gamma(p+1)}{p'}\right)^{1/p} \|Hf\|_{\mathrm{Cop}_p(\mathbb{R}^+)},$$

and if $2 \le p < \infty$, *then*

$$\left(\frac{\Gamma(p+1)}{p'}\right)^{1/p} \|Hf\|_{\mathrm{Cop}_p(\mathbb{R}^+)} \le \|H^*f\|_{\mathrm{Cop}_p(\mathbb{R}^+)} \le (p-1)\|Hf\|_{\mathrm{Cop}_p(\mathbb{R}^+)}.$$

Let $C_p = \int_0^1 |\ln(t)+1|^p dt$. As a consequence of a result initiated in [2], the authors in [4] obtained the following theorem.

Theorem 1.4. [4, Theorem 1.2]
Let $1 < p < \infty$ *and let* $f \in \mathcal{M}^+(\mathbb{R}^+)$ *be such that* $H^*f(x) < \infty$ *for all* $x > 0$. *If* $1 < p \le 2$, *then*

$$(p-1)\|f\|_{\mathrm{Cop}_p(\mathbb{R}^+)} \le \|(H^*-I)H^*f\|_p \le C_p^{1/p}\|f\|_{\mathrm{Cop}_p(\mathbb{R}^+)}, \tag{7}$$

and if $2 \le p \le \infty$, *then*

$$C_p^{1/p}\|f\|_{\mathrm{Cop}_p(\mathbb{R}^+)} \le \|(H^*-I)H^*f\|_p \le (p-1)\|f\|_{\mathrm{Cop}_p(\mathbb{R}^+)}. \tag{8}$$

The constants $p-1$ *and* $C_p^{1/p}$ *are optimal in both* (7) *and* (8).

K. Leśnika and L. Maligranda [14] also contributed to this line of research by proving the following inequalities

$$\frac{p'}{e}\|f\|_{\mathrm{Ces}_p(\mathbb{R}^+)} \le \|Hf\|_{\mathrm{Ces}_p(\mathbb{R}^+)} \le p'\|f\|_{\mathrm{Ces}_p(\mathbb{R}^+)}, \tag{9}$$

for any nonnegative measurable function f and $1 < p \le \infty$. However, the constant on the left-hand side of (9) is not optimal for $1 < p < \infty$. The sharp constant in that inequality for this range of p is given in Theorem 2.1.

In the following section, we present Theorem 2.1, which improves (9), together with an analogous statement for $\mathrm{Cop}_p(\mathbb{R}^+)$ given in Theorem 2.2. In Section 3, we give the proofs of these theorems. Finally, we present an application of the main results of this work in Section 4.

2. MAIN RESULTS

The following theorems represent the main results of this paper.

Theorem 2.1. *Let* $1 < p < \infty$ *and let* $f \in \mathcal{M}^+(\mathbb{R}^+)$*. Then the following inequalities are sharp:*

$$\frac{\Gamma(p+1)^{1/p}}{(p-1)}\|f\|_{\mathrm{Ces}_p(\mathbb{R}^+)} \le \|Hf\|_{\mathrm{Ces}_p(\mathbb{R}^+)} \le p'\|f\|_{\mathrm{Ces}_p(\mathbb{R}^+)}, \tag{10}$$

and if $p = \infty$ *the constants in* (10) *are, respectively,* $\lim_{p\to\infty} \frac{\Gamma(p+1)^{1/p}}{(p-1)} = \frac{1}{e}$ *and* 1.

Theorem 2.2. *Let* $1 \le p < \infty$ *and let* $f \in \mathcal{M}^+(\mathbb{R}^+)$*. Then the following inequalities are sharp:*

$$\Gamma(p+1)^{1/p}\|f\|_{\mathrm{Cop}_p(\mathbb{R}^+)} \le \|H^*f\|_{\mathrm{Cop}_p(\mathbb{R}^+)} \le p\|f\|_{\mathrm{Cop}_p(\mathbb{R}^+)}. \tag{11}$$

Remark 2.3. Let $1 \le p \le \infty$ and let f be an arbitrary measurable function defined on $\mathbb{R}^+$. If $1 < p \le \infty$, observe that Theorem 2.1 shows, in particular, that the classical Cesàro function spaces enjoy the following property:

$$f \in \mathrm{Ces}_p(\mathbb{R}^+) \iff H|f| \in \mathrm{Ces}_p(\mathbb{R}^+).$$

Theorem 2.2 confirms that an analogous property holds for the classical Copson function spaces $\mathrm{Cop}_p(\mathbb{R}^+)$, with $1 \le p < \infty$; that is,

$$f \in \mathrm{Cop}_p(\mathbb{R}^+) \iff H^*|f| \in \mathrm{Cop}_p(\mathbb{R}^+).$$

3. PROOFS OF THE MAIN RESULTS

Proof of Theorem 2.1. Let let f be a nonnegative function. First, observe that an application of Tonelli's Theorem yields

$$\begin{aligned} H^2f(x) = H(Hf(x)) &= \frac{1}{x}\int_0^x \frac{1}{u}\int_0^u f(t)\,dt\,du \\ &= \frac{1}{x}\int_0^x f(t)\int_t^x \frac{1}{u}\,du\,dt = \frac{1}{x}\int_0^x f(t)\ln\left(\frac{x}{t}\right)dt, \end{aligned}$$

for all $x > 0$. Let $1 < p \le 2$. By using integration by parts we obtain that

$$\|Hf\|^p_{\mathrm{Ces}_p(\mathbb{R}^+)} = \|H^2f\|_p^p = p'\int_0^\infty \left(\frac{1}{x}\int_0^x f(u)du\right)\left(\frac{1}{x}\int_0^x f(t)\ln\left(\frac{x}{t}\right)dt\right)^{p-1}dx.$$

By applying Jensen's inequality, we have that

$$\left(\frac{1}{x}\int_0^x f(t)\ln\left(\frac{x}{t}\right)dt\right)^{p-1} \ge \left(\frac{1}{x}\int_0^x f(t)dt\right)^{p-2}\frac{1}{x}\int_0^x f(t)\ln^{p-1}\left(\frac{x}{t}\right)dt,$$

for all $x > 0$. Now, by using Tonelli's Theorem we get that

$$\int_0^x f(t)\ln\left(\frac{x}{t}\right)^{p-1}dt = (p-1)\int_0^x \left(\frac{1}{t}\int_0^t f(u)du\right)\ln^{p-2}\left(\frac{x}{t}\right)dt,$$

for all $x > 0$. Therefore,

$$\begin{aligned}
\|Hf\|^p_{\mathrm{Ces}_p(\mathbb{R}^+)} &\geq p\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^{p-1}\frac{1}{x}\int_0^x \left(\frac{1}{t}\int_0^t f(u)du\right)\ln^{p-2}\left(\frac{x}{t}\right)dtdx\\
&= p\int_0^\infty \left(\frac{1}{t}\int_0^t f(u)du\right)\int_t^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^{p-1}\ln^{p-2}\left(\frac{x}{t}\right)\frac{dx}{x}dt\\
&\geq p\int_0^\infty \left(\frac{1}{t}\int_0^t f(u)du\right)\left(\int_0^t f(x)dx\right)^{p-1}\int_t^\infty \frac{1}{x^p}\ln^{p-2}\left(\frac{x}{t}\right)dxdt\\
&= p\int_0^\infty \left(\frac{1}{t}\int_0^t f(x)dx\right)^p\int_t^\infty \left(\frac{t}{x}\right)^p\ln^{p-2}\left(\frac{x}{t}\right)\frac{dx}{t}dt\\
&= p\frac{\Gamma(p-1)}{(p-1)^{p-1}}\int_0^\infty \left(\frac{1}{t}\int_0^t f(x)dx\right)^p dt = \frac{\Gamma(p+1)}{(p-1)^p}\|f\|^p_{\mathrm{Ces}_p(\mathbb{R}^+)}.
\end{aligned}$$

Now we turn to the case $2 \leq p < \infty$. Let $x > 0$ and let $\varphi_x : (0, x] \to \mathbb{R}^+$ be the map defined by

$$\varphi_x(v) = \left(\int_0^v f(t)\ln\left(\frac{x}{t}\right)dt\right)^{p-1}.$$

By applying the Fundamental Theorem of Calculus we obtain that

$$\begin{aligned}
\varphi_x(x) = \varphi_x(x) - \varphi_x(0) &= \int_0^x \varphi'(v)dv\\
&= (p-1)\int_0^x \left(\int_0^v f(t)\ln\left(\frac{x}{t}\right)dt\right)^{p-2} f(v)\ln\left(\frac{x}{v}\right)dv\\
&\geq (p-1)\int_0^x \left(\int_0^v f(t)dt\right)^{p-2} f(v)\ln^{p-1}\left(\frac{x}{v}\right)dv.
\end{aligned}$$

So, we have that

$$\left(\int_0^x f(t)\ln\left(\frac{x}{t}\right)dt\right)^{p-1} \geq (p-1)\int_0^x \left(\int_0^v f(t)dt\right)^{p-2} f(v)\ln^{p-1}\left(\frac{x}{v}\right)dv, \tag{12}$$

for all $x > 0$. Now, by using integration by parts and (12) we get that

$$\begin{aligned}
\|H^2 f\|^p_p &= p'\int_0^\infty \left(\frac{1}{x}\int_0^x f(u)du\right)\left(\frac{1}{x}\int_0^x f(t)\ln^{p-1}\left(\frac{x}{t}\right)dt\right)dx\\
&\geq p\int_0^\infty \frac{1}{x^p}\left(\int_0^x f(u)du\right)\int_0^x \left(\int_0^v f(t)dt\right)^{p-2} f(v)\ln^{p-1}\left(\frac{x}{v}\right)dvdx\\
&= p\int_0^\infty \left(\int_0^v f(t)dt\right)^{p-2} f(v)\int_v^\infty \frac{1}{x^p}\left(\int_0^x f(u)du\right)\ln^{p-1}\left(\frac{x}{v}\right)dxdv.
\end{aligned}$$

In the last equality, we applied Tonelli's Theorem. Then, using integration by parts once more, we obtain that

$$\begin{aligned} p\int_0^\infty \left(\int_0^v f(t)dt\right)^{p-2} f(v)\int_v^\infty \frac{1}{x^p}\left(\int_0^x f(u)du\right)\ln^{p-1}\left(\frac{x}{v}\right)dxdv \\ = p\int_0^\infty \frac{1}{v}\left(\int_0^v f(t)dt\right)^{p-1}\int_v^\infty \frac{1}{x^p}\left(\int_0^x f(u)du\right)\ln^{p-2}\left(\frac{x}{v}\right)dxdv \\ \geq p\int_0^\infty \frac{1}{v}\left(\int_0^v f(t)dt\right)^{p}\int_v^\infty \frac{1}{x^p}\ln^{p-2}\left(\frac{x}{v}\right)dxdv. \end{aligned}$$

Therefore

$$\begin{aligned} \|Hf\|^p_{\mathrm{Ces}_p(\mathbb{R}^+)} = \|H^2f\|_p^p &\geq p\int_0^\infty \frac{1}{v}\left(\int_0^v f(t)dt\right)^{p}\int_v^\infty \frac{1}{x^p}\ln^{p-2}\left(\frac{x}{v}\right)dxdv \\ &= p\int_0^\infty \left(\frac{1}{v}\int_0^v f(t)dt\right)^{p}\int_v^\infty \left(\frac{v}{x}\right)^p\ln^{p-2}\left(\frac{x}{v}\right)\frac{dx}{v}dv \\ &= p\frac{\Gamma(p-1)}{(p-1)^{p-1}}\int_0^\infty \left(\frac{1}{v}\int_0^v f(x)dx\right)^p dv = \frac{\Gamma(p+1)}{(p-1)^p}\|Hf\|_p^p \\ &= \frac{\Gamma(p+1)}{(p-1)^p}\|f\|^p_{\mathrm{Ces}_p(\mathbb{R}^+)}. \end{aligned}$$

We now consider the case $p=\infty$. We employ the following inequality, due to Robbins [17], which holds for all positive integers n:

$$\left(\frac{n}{e}\right)^n\sqrt{2\pi n}\,e^{\frac{1}{12n+1}} \leq n! \leq \left(\frac{n}{e}\right)^n\sqrt{2\pi n}\,e^{\frac{1}{12n}}.$$

Since the function $x\mapsto x^{-1/x}$ is strictly increasing on $[e,\infty)$, we have that

$$\frac{1}{\left(\frac{n}{e}\right)(2\pi n)^{\frac{1}{2n}}e^{\frac{1}{12n^2+n}}} \leq \frac{1}{(n!)^{1/n}} \leq \frac{1}{\left(\frac{n}{e}\right)(2\pi n)^{\frac{1}{2n}}e^{\frac{1}{12n}}},$$

for all $n\geq 3$. Hence,

$$\frac{n-1}{\left(\frac{n}{e}\right)(2\pi n)^{\frac{1}{2n}}e^{\frac{1}{12n^2+n}}} \leq \frac{n-1}{(n!)^{1/n}} \leq \frac{n-1}{\left(\frac{n}{e}\right)(2\pi n)^{\frac{1}{2n}}e^{\frac{1}{12n}}}, \tag{13}$$

for all $n\geq 3$. Finally, taking the limit as $n\to\infty$ in (13), we get

$$\lim_{n\to\infty}\frac{n-1}{\Gamma(n+1)^{1/n}} = e. \tag{14}$$

Now, if $f\in L^\infty(\mathbb{R}^+)$ is a nonnegative function, then, by (14) and (9), we have

$$\left(\lim_{n\to\infty}\frac{\Gamma(n+1)^{1/n}}{(n-1)}\right)\|f\|_{\mathrm{Ces}_\infty(\mathbb{R}^+)} = \frac{1}{e}\|f\|_{\mathrm{Ces}_\infty(\mathbb{R}^+)} \leq \|Hf\|_{\mathrm{Ces}_\infty(\mathbb{R}^+)}.$$

Regarding the right-hand side of (10), if $Hf\notin L^p(\mathbb{R}^+)$, the inequality holds trivially. Otherwise, it follows directly from (1).

Let us now examine the optimality of the constant on the left-hand side of (10). We begin with the case $1<p<\infty$. Consider the sequence

$$f_\epsilon = \chi_{(1,1+\epsilon)}, \quad \epsilon>0.$$

After some computations, we obtain that

$$Hf_\epsilon(x) = \frac{x-1}{x}\chi_{(1,1+\epsilon)}(x) + \frac{\epsilon}{x}\chi_{[1+\epsilon,\infty)}(x),$$

and

$$\begin{aligned} H^2 f_\epsilon(x) = \Big(\frac{x-\ln(x)-1}{x}\Big)&\chi_{(1,1+\epsilon)}(x) \\ &+ \frac{\epsilon(\ln(x)+1)-(\epsilon+1)\ln(1+\epsilon)}{x}\chi_{[1+\epsilon,\infty)}(x). \end{aligned}$$

By taking p-norm we have that

$$\begin{aligned} \frac{\|f_\epsilon\|^p_{\mathrm{Ces}_p(\mathbb{R}^+)}}{\|Hf_\epsilon\|^p_{\mathrm{Ces}_p(\mathbb{R}^+)}} &= \frac{\|Hf_\epsilon\|^p_p}{\|H^2 f_\epsilon\|^p_p} \\ &= \frac{\displaystyle\int_1^{1+\epsilon}\Big(1-\frac{1}{x}\Big)^p dx + \frac{\epsilon^p}{(p-1)}\frac{1}{(1+\epsilon)^{p-1}}}{\displaystyle\int_1^{1+\epsilon}\Big(1-\frac{1}{x}-\frac{\ln(x)}{x}\Big)^p dx + \int_{1+\epsilon}^{\infty}\big(\epsilon(\ln(x)+1)-(\epsilon+1)\ln(\epsilon+1)\big)^p\frac{dx}{x^p}} \\ &\geq \frac{\frac{\epsilon^p}{(p-1)}\frac{1}{(1+\epsilon)^{p-1}}}{\displaystyle\int_1^{1+\epsilon}\Big(1-\frac{1}{x}-\frac{\ln(x)}{x}\Big)^p dx + \int_{1+\epsilon}^{\infty}\big(\epsilon(\ln(x)+1)-(\epsilon+1)\ln(\epsilon+1)\big)^p\frac{dx}{x^p}} \\ &\geq \frac{1/(p-1)}{\frac{(1+\epsilon)^{p-1}}{\epsilon^p}\displaystyle\int_1^{1+\epsilon}\Big(1-\frac{1}{x}-\frac{\ln(x)}{x}\Big)^p dx + \int_1^{\infty}\Big(\ln(x)+1-\frac{\ln(\epsilon+1)}{\epsilon}\Big)^p\frac{dx}{x^p}}. \end{aligned}$$

Now, on the one hand, by applying L'Hôpital's rule we get that

$$\lim_{\epsilon\to 0^+}\frac{1}{\epsilon^p}\int_1^{1+\epsilon}\Big(1-\frac{1}{x}-\frac{\ln(x)}{x}\Big)^p dx = 0,$$

and on the other hand, by using Dominated Convergence Theorem we obtain that

$$\begin{aligned} \lim_{\epsilon\to 0^+}\int_1^{\infty}\Big(\ln(x)+1-\frac{\ln(\epsilon+1)}{\epsilon}\Big)^p\frac{dx}{x^p} &= \int_1^{\infty}\lim_{\epsilon\to 0^+}\Big(\ln(x)+1-\frac{\ln(\epsilon+1)}{\epsilon}\Big)^p\frac{dx}{x^p} \\ &= \int_1^{\infty}\frac{\ln^p(x)}{x^p}dx = \frac{\Gamma(p+1)}{(p-1)^{p+1}}. \end{aligned}$$

Therefore, using (10), it follows that

$$\frac{(p-1)^p}{\Gamma(p+1)} \geq \sup_{f\geq 0, f\neq 0}\frac{\|f\|^p_{\mathrm{Ces}_p(\mathbb{R}^+)}}{\|Hf\|^p_{\mathrm{Ces}_p(\mathbb{R}^+)}} = \sup_{f\geq 0, f\neq 0}\frac{\|Hf\|^p_p}{\|H^2 f\|^p_p} \geq \sup_{\epsilon>0}\frac{\|Hf_\epsilon\|^p_p}{\|H^2 f_\epsilon\|^p_p} \geq \frac{(p-1)^p}{\Gamma(p+1)}.$$

In the case of $p=\infty$ we get that

$$\|Hf_\epsilon\|_\infty = Hf_\epsilon(1+\epsilon) = \frac{\epsilon}{\epsilon+1},$$

and

$$\|H^2 f_\epsilon\|_\infty = H^2 f_\epsilon\big((1+\epsilon)^{\frac{(1+\epsilon)}{\epsilon}}\big) = \frac{\epsilon}{(1+\epsilon)^{\frac{(1+\epsilon)}{\epsilon}}},$$

therefore,

$$\sup_{\epsilon>0} \frac{\|f_\epsilon\|_{\mathrm{Ces}_\infty(\mathbb{R}^+)}}{\|Hf_\epsilon\|_{\mathrm{Ces}_\infty(\mathbb{R}^+)}} = \sup_{\epsilon>0} = \frac{\|Hf_\epsilon\|_\infty}{\|H^2 f_\epsilon\|_\infty} = \sup_{\epsilon>0}(1+\epsilon)^{\frac{1}{\epsilon}} = \lim_{\epsilon\to 0^+}(1+\epsilon)^{\frac{1}{\epsilon}} = e.$$

This completes the proof that the constant on the left-hand side of (10) is the best possible for $1 < p \leq \infty$.

Now, we are going to see the sharpness of the constant p' on the right-hand side of (10). In order to do that, we consider the functions $f_\epsilon(x) = x^{\epsilon/p-1/p}\chi_{(0,1)}(x)$ and $g_\epsilon(x) = x^{\epsilon/p'-1/p'}\chi_{(0,1)}(x)$, where $1 < p < \infty$ and $0 < \epsilon < 1$. A straightforward computation shows that

$$\begin{aligned} H^2 f_\epsilon(x) &= \frac{1}{x}\int_0^x f_\epsilon(t)\ln\left(\frac{x}{t}\right)dt = \frac{1}{x}\int_0^x t^{\epsilon/p-1/p}\ln\left(\frac{x}{t}\right)dt \\ &= x^{\epsilon/p-1/p}\int_0^x \left(\frac{t}{x}\right)^{\epsilon/p-1/p}\ln\left(\frac{x}{t}\right)\frac{dt}{x} = x^{\epsilon/p-1/p}\int_0^1 t^{\epsilon/p-1/p}\ln\left(\frac{1}{t}\right)dt \\ &= x^{\epsilon/p-1/p}\frac{1}{(\epsilon/p+1/p')^2}, \end{aligned}$$

for all $0 < x < 1$. So, by using Hölder's inequality and (1), we get

$$\begin{aligned} \frac{\|Hf_\epsilon\|_{\mathrm{Ces}_p(\mathbb{R}^+)}}{\|f_\epsilon\|_{\mathrm{Ces}_p(\mathbb{R}^+)}} = \frac{\|H^2 f_\epsilon\|_p}{\|Hf_\epsilon\|_p} &\geq \frac{\displaystyle\int_0^1 H^2 f_\epsilon(x)\cdot g_\epsilon(x)dx}{\|Hf_\epsilon\|_p\|g_\epsilon\|_{p'}} \geq \frac{1}{p'}\frac{\displaystyle\int_0^1 H^2 f_\epsilon(x)\cdot g_\epsilon(x)dx}{\|f_\epsilon\|_p\|g_\epsilon\|_{p'}} \\ &= \frac{1}{p'}\frac{1}{(\epsilon/p+1/p')^{2p}}\frac{\displaystyle\int_0^1 x^{\epsilon/p-1/p}x^{\epsilon/p'-1/p'}dx}{\frac{1}{\epsilon^{1/p}}\frac{1}{\epsilon^{1/p'}}} \\ &= \frac{1}{p'}\frac{1}{(\epsilon/p+1/p')^{2p}} \to p', \end{aligned}$$

as $\epsilon \to 0^+$. In the case when $p = \infty$, we consider the function $f = \chi_{(0,1)}$ and an easy calculations show that

$$\|H^2 f\|_\infty = \|Hf\|_\infty = 1,$$

which means

$$\|Hf_\epsilon\|_{\mathrm{Ces}_\infty(\mathbb{R}^+)} = \|f_\epsilon\|_{\mathrm{Ces}_\infty(\mathbb{R}^+)} = 1.$$

This ends the proof. ☐

Before proving Theorem 2.2, we require the following inequality, which was independently established by G. Bennett [6] and P. F. Renaud [16]:

$$(p')^{1/p}\,\|f\|_p \leq \|Hf\|_p, \tag{15}$$

valid for any nonnegative, nonincreasing function $f \in L^p(\mathbb{R}^+)$, with $1 < p < \infty$. The constant $(p')^{1/p}$ is the best possible.

Proof of Theorem 2.2. Let $1 \leq p < \infty$ and let f be a nonnegative function. First, observe that an application of Tonelli's Theorem yields

$$\begin{aligned} H^{*2}f(x) = H^*(H^*f(x)) &= \int_x^\infty \frac{1}{u}\int_u^\infty \frac{f(t)}{t} dt du dx \\ &= \int_x^\infty \frac{f(t)}{t}\int_x^t \frac{1}{u} du dt dx = \int_x^\infty \frac{f(t)}{t}\ln\left(\frac{t}{x}\right) dt dx, \end{aligned}$$

for all $x > 0$.

If $p = 1$, Tonelli's Theorem again yields

$$\begin{aligned} \|H^*f\|_{\mathrm{Cop}_1(\mathbb{R}^+)} = \|H^{*2}f\|_1 &= \int_0^\infty \int_x^\infty \frac{f(t)}{t}\ln\left(\frac{t}{x}\right) dt dx = \int_0^\infty \frac{f(t)}{t}\int_0^t \ln\left(\frac{t}{x}\right) dx dt \\ &= \int_0^1 \ln\left(\frac{1}{x}\right) dx \cdot \int_0^\infty f(t) dt = \int_0^\infty f(t) dt = \int_0^\infty \frac{f(t)}{t}\int_0^t dx dt \\ &= \int_0^\infty \int_x^\infty \frac{f(t)}{t} dt dx = \|H^*f\|_1 = \|f\|_{\mathrm{Cop}_1(\mathbb{R}^+)}. \end{aligned}$$

Now, let us assume that $1 < p \leq 2$. Then by using integration by parts we get

$$\|H^*f\|^p_{\mathrm{Cop}_p(\mathbb{R}^+)} = \|H^{*2}f\|^p_p = p\int_0^\infty \left(\int_x^\infty \frac{f(u)}{u} du\right)\left(\int_x^\infty \frac{f(t)}{t}\ln\left(\frac{t}{x}\right) dt\right)^{p-1} dx.$$

By taking into account Jensen's inequality we get that

$$\left(\int_x^\infty \frac{f(t)}{t}\ln\left(\frac{t}{x}\right) dt\right)^{p-1} \geq \left(\int_x^\infty \frac{f(t)}{t} dt\right)^{p-2}\int_x^\infty \frac{f(t)}{t}\ln^{p-1}\left(\frac{t}{x}\right) dt,$$

for all $x > 0$. Therefore,

$$\begin{aligned} \|H^{*2}f\|^p_p &\geq p\int_0^\infty \left(\int_x^\infty \frac{f(u)}{u} du\right)^{p-1}\int_x^\infty \frac{f(t)}{t}\ln^{p-1}\left(\frac{t}{x}\right) dt dx \\ &= p\int_0^\infty \frac{f(t)}{t}\int_0^t \left(\int_x^\infty \frac{f(u)}{u} du\right)^{p-1}\ln^{p-1}\left(\frac{t}{x}\right) dx dt, \end{aligned}$$

where we have used Tonelli's Theorem in the last equality. Now, by applying integration by parts we get

$$\begin{aligned} &\int_0^\infty \frac{f(t)}{t}\int_0^t \left(\int_x^\infty \frac{f(u)}{u} du\right)^{p-1}\ln^{p-1}\left(\frac{t}{x}\right) dx dt \\ &= (p-1)\int_0^\infty \left(\int_t^\infty \frac{f(u)}{u} du\right)\frac{1}{t}\int_0^t \left(\int_x^\infty \frac{f(u)}{u} du\right)^{p-1}\ln^{p-2}\left(\frac{t}{x}\right) dx dt \\ &\geq (p-1)\int_0^\infty \left(\int_t^\infty \frac{f(u)}{u} du\right)^p \frac{1}{t}\int_0^t \ln^{p-2}\left(\frac{t}{x}\right) dx dt \\ &= (p-1)\Gamma(p-1)\|H^*f\|^p_p = \Gamma(p+1)\|f\|^p_{\mathrm{Cop}_p(\mathbb{R}^+)}. \end{aligned}$$

Thus

$$\|H^*f\|^p_{\mathrm{Cop}_p(\mathbb{R}^+)} = \|H^{*2}f\|^p_p \geq \Gamma(p+1)\|H^*f\|^p_p = \Gamma(p+1)\|f\|^p_{\mathrm{Cop}_p(\mathbb{R}^+)}.$$

Now we turn to the case $2 \leq p < \infty$. Since f is a nonnegative function we get that the function H^*f is nonnegative and nonincreasing. Then, combining (15) with the left-hand side of (6), we obtain

$$\begin{aligned}(p')^{1/p}\|f\|_{\mathrm{Cop}_p(\mathbb{R}^+)} = (p')^{1/p}\|H^*f\|_p \leq \|HH^*f\|_p &\leq \left(\frac{p'}{\Gamma(p+1)}\right)^{1/p}\|H^{*2}f\|_p \\ &= \left(\frac{p'}{\Gamma(p+1)}\right)^{1/p}\|H^*f\|_{\mathrm{Cop}_p(\mathbb{R}^+)},\end{aligned}$$

from which it is follow that

$$\Gamma(p+1)^{1/p}\|f\|_{\mathrm{Cop}_p(\mathbb{R}^+)} \leq \|H^*f\|_{\mathrm{Cop}_p(\mathbb{R}^+)}.$$

This completes the proof of the left-hand side of (11).

Let $1 < p < \infty$. Concerning the right-hand side of (11), if $H^*f \notin L^p(\mathbb{R}^+)$, the inequality holds trivially. Otherwise, it follows directly from the fact that

$$\|H^*\|_p = p, \tag{16}$$

for $1 < p < \infty$ (cf. [9, p. 244]).

To see the optimality of the constant on the left-hand side of (11) we consider the sequence of nonnegative functions f_ϵ given by $f_\epsilon = \chi_{[1,1+\epsilon)}$, where $0 < \epsilon < 1$. Then

$$H^*f_\epsilon(x) = \begin{cases} \ln(1+\epsilon), & \text{if} \quad 0 < x \leq 1, \\ \ln\left(\frac{1+\epsilon}{x}\right), & \text{if} \quad 1 < x < 1+\epsilon, \end{cases}$$

and

$$H^{*2}f_\epsilon(x) = \begin{cases} \ln(1+\epsilon)\ln\left(\frac{1}{x}\right) + \frac{1}{2}\ln^2(1+\epsilon), & \text{if} \quad 0 < x \leq 1, \\ \frac{1}{2}\ln^2\left(\frac{1+\epsilon}{x}\right), & \text{if} \quad 1 < x < 1+\epsilon. \end{cases}$$

Therefore

$$\begin{aligned}\frac{\|H^*f_\epsilon\|_p^p}{\|H^{*2}f_\epsilon\|_p^p} &= \frac{\ln^p(1+\epsilon) + \displaystyle\int_1^{1+\epsilon}\ln^p\left(\frac{1+\epsilon}{x}\right)dx}{\ln^p(1+\epsilon)\displaystyle\int_0^1\left(\ln\left(\frac{1}{x}\right)+\frac{1}{2}\ln(1+\epsilon)\right)^p dx + \frac{1}{2^p}\int_1^{1+\epsilon}\ln^{2p}\left(\frac{1+\epsilon}{x}\right)dx} \\ &\geq \frac{1}{\displaystyle\int_0^1\left(\ln\left(\frac{1}{x}\right)+\frac{1}{2}\ln(1+\epsilon)\right)^p dx + 2^{-p}\frac{1+\epsilon}{\ln^p(1+\epsilon)}\int_{1/(1+\epsilon)}^1\ln^{2p}\left(\frac{1}{x}\right)dx},\end{aligned}$$

for all $0 < \epsilon < 1$. Now by applying L'Hôpital's rule we get that

$$\lim_{\epsilon\to 0^+}\frac{\displaystyle\int_{1/(1+\epsilon)}^1\ln^{2p}\left(\frac{1}{x}\right)dx}{\ln^p(1+\epsilon)} = \lim_{\epsilon\to 0^+}\frac{\ln^{2p}(1+\epsilon)(1+\epsilon)^{-2}}{p\ln^{p-1}(1+\epsilon)(1+\epsilon)^{-1}} = \lim_{\epsilon\to 0^+}\frac{\ln^{p+1}(1+\epsilon)}{p(1+\epsilon)} = 0.$$

On the other hand, by using Dominated Convergence Theorem we obtain that

$$\begin{aligned}\lim_{\epsilon\to 0^+}\int_0^1\left(\ln\left(\frac{1}{x}\right)+\frac{1}{2}\ln(1+\epsilon)\right)^p dx &= \int_0^1\lim_{\epsilon\to 0^+}\left(\ln\left(\frac{1}{x}\right)+\frac{1}{2}\ln(1+\epsilon)\right)^p dx \\ &= \Gamma(p+1).\end{aligned}$$

Hence

$$\frac{1}{\Gamma(p+1)} \geq \lim_{\epsilon\to 0^+}\frac{\|f_\epsilon\|_{\mathrm{Cop}_p(\mathbb{R}^+)}^p}{\|H^*f_\epsilon\|_{\mathrm{Cop}_p(\mathbb{R}^+)}^p} = \lim_{\epsilon\to 0^+}\frac{\|H^*f_\epsilon\|_p^p}{\|H^{*2}f_\epsilon\|_p^p} \geq \frac{1}{\Gamma(p+1)}.$$

Thus

$$\lim_{\epsilon\to 0^+} \frac{\|f_\epsilon\|^p_{\mathrm{Cop}_p(\mathbb{R}^+)}}{\|H^* f_\epsilon\|^p_{\mathrm{Cop}_p(\mathbb{R}^+)}} = \frac{1}{\Gamma(p+1)},$$

for all $1 < p < \infty$. This concludes the proof that the estimate on the left-hand side of (11) is the best possible for $1 < p < \infty$.

Now, let us see the optimality of the constant on the right-hand side of (11) for $1 < p < \infty$. In order to do that, we are going to consider the sequence of functions $f_\epsilon(x) = x^{-\epsilon/p-1/p}\chi_{[1,\infty)}(x)$ and $g_\epsilon(x) = x^{-\epsilon/p'-1/p'}\chi_{[1,\infty)}(x)$, where $0 < \epsilon < 1$. Let $1 \le x < \infty$, then

$$\begin{aligned} H^{*2} f_\epsilon(x) &= \int_x^\infty t^{-\epsilon/p-1/p-1} \ln\left(\frac{t}{x}\right) dt = x^{-\epsilon/p-1/p} \int_x^\infty \left(\frac{t}{x}\right)^{-\epsilon/p-1/p-1} \ln\left(\frac{t}{x}\right) \frac{dt}{x} \\ &= x^{-\epsilon/p-1/p} \int_1^\infty t^{-\epsilon/p-1/p-1} \ln(t) dt = \frac{x^{-\epsilon/p-1/p}}{(\epsilon/p+1/p)^2} = \frac{f_\epsilon(x)}{(\epsilon/p+1/p)^2}. \end{aligned}$$

Now, by using (16) and Hölder's inequality, we obtain

$$\begin{aligned} \frac{\|H^* f_\epsilon\|_{\mathrm{Cop}_p(\mathbb{R}^+)}}{\|f_\epsilon\|_{\mathrm{Cop}_p(\mathbb{R}^+)}} &= \frac{\|H^{*2} f_\epsilon\|_p}{\|H^* f_\epsilon\|_p} \ge \frac{1}{p} \frac{\|H^{*2} f_\epsilon\|_p}{\|f_\epsilon\|_p} \ge \frac{1}{p} \frac{\displaystyle\int_0^\infty H^{*2} f_\epsilon(x) \cdot g_\epsilon(x) dx}{\|f_\epsilon\|_p \|g_\epsilon\|_{p'}} \\ &= \frac{1}{p(\epsilon/p+1/p)^2} \frac{\displaystyle\int_1^\infty x^{-\epsilon-1} dx}{\frac{1}{\epsilon^{1/p}} \frac{1}{\epsilon^{1/p'}}} = \frac{1}{p(\epsilon/p+1/p)^2} = \frac{p}{1+\epsilon} \to p, \end{aligned}$$

as $\epsilon \to 0^+$. This concludes the proof. □

4. Applications

Let $1 < p \le \infty$, and let

$$|||\cdot|||_{\mathrm{Ces}_p(\mathbb{R}^+)} : \mathrm{Ces}_p(\mathbb{R}^+) \to [0, \infty)$$

defined by

$$|||f|||_{\mathrm{Ces}_p(\mathbb{R}^+)} = \|Hf\|_{\mathrm{Ces}_p(\mathbb{R}^+)}.$$

Since $\|\cdot\|_{\mathrm{Ces}_p(\mathbb{R}^+)}$ is a norm on $\mathrm{Ces}_p(\mathbb{R}^+)$ and H is an injective linear operator, it follows that $|||\cdot|||_{\mathrm{Ces}_p(\mathbb{R}^+)}$ defines a norm on $\mathrm{Ces}_p(\mathbb{R}^+)$, which, by Theorem 2.1, is equivalent to the standard norm $\|\cdot\|_{\mathrm{Ces}_p(\mathbb{R}^+)}$ on the classical Cesàro function space $\mathrm{Ces}_p(\mathbb{R}^+)$.

An analogous construction holds for the Copson function spaces $\mathrm{Cop}_p(\mathbb{R}^+)$, with $1 \le p < \infty$. Let

$$|||\cdot|||_{\mathrm{Cop}_p(\mathbb{R}^+)} : \mathrm{Cop}_p(\mathbb{R}^+) \to [0, \infty)$$

given by

$$|||f|||_{\mathrm{Cop}_p(\mathbb{R}^+)} = \|H^* f\|_{\mathrm{Cop}_p(\mathbb{R}^+)}.$$

Since $\|\cdot\|_{\mathrm{Cop}_p(\mathbb{R}^+)}$ is a norm on $\mathrm{Cop}_p(\mathbb{R}^+)$ and H^* is an injective linear operator, the functional $N(\cdot) = |||\cdot|||_{\mathrm{Cop}_p(\mathbb{R}^+)}$ defines a norm on $\mathrm{Cop}_p(\mathbb{R}^+)$, which, by Theorem 2.2, is equivalent to the standard norm $\|\cdot\|_{\mathrm{Cop}_p(\mathbb{R}^+)}$.

References

[1] S. V. Astashkin and L. Maligranda, *Structure of Cesàro function spaces*, Indag. Math. (N.S.), 20 (2009), 329–379.

[2] A. Ben Said, S. Boza, and J. Soria, *The norm of Hardy-type oscillation operators in the discrete and continuous settings.* To appear in Analysis and Applications.

[3] A. Ben Said, S. Boza, and J. Soria, *On sharp reverse Hardy-type inequalities for oscillation operators in the discrete and continuous settings.* Preprint.

[4] A. Ben Said and G. Sinnamon, *Best bounds for the dual Hardy operator minus identity on decreasing functions*, Ann. Mat. Pura Appl. 205 (2026), 313–325.

[5] G. Bennett, *Factorizing the Classical Inequalities*, Mem. Amer. Math. Soc., 120 (1996). Providence, RI.

[6] G. Bennett, *Lower bounds for matrices*, Linear Algebra Appl., 82 (1986), 81–98.

[7] S. Boza and J. Soria, *Averaging operators on decreasing or positive functions: Equivalence and optimal bounds*, J. Approx. Theory, 237 (2019), 135–152.

[8] A. Brown, P. R. Halmos, and A. L. Shields, *Cesàro operators*, Acta Sci. Math. (Szeged), 26 (1965), 125–137.

[9] G. H. Hardy, J. Littlewood, and G. Pólya, *Inequalities*, 2nd Ed., Cambridge Univ. Press, 1967.

[10] A. A. Jagers, *A note on Cesàro sequence spaces*, Nieuw Arch. Wiskd., (3) 22 (1974), 113–124.

[11] N. Kaiblinger, L. Maligranda, and L.-E. Persson, *Norms in weighted L^2-spaces and Hardy operators*, Function spaces, the fifth conference, Poznań 1998, Lecture Notes in Pure Appl. Math. 213 (2000), 205–216. New York: Marcel Dekker.

[12] V. I. Kolyada, *Optimal relationships between L^p-norms for the Hardy operator and its dual*, Ann. Mat. Pura Appl., 193 (2014), 423–430.

[13] V. I. Kolyada, *On the optimal relationships between L^p-norms for the Hardy operator and its dual for decreasing functions*, J. Approx. Theory, 252 (2020), 105362.

[14] K. Leŝnika and L. Maligranda, *Abstract Cesàro spaces. Duality*, J. Math. Anal. Appl., 424 (2015), 932–951.

[15] M. Milman, *A note on reversed Hardy inequalities and Gehring's lemma*, Comm. Pure Appl. Math., L (1997), 0311–0315.

[16] P. F. Renaud, *A reversed Hardy inequality*, Bull. Austral. Math. Soc., 34 (1986), 225–232.

[17] H. Robbins, *A Remark on Stirling's Formula*, American Math. Monthly. 62 (1955), 26–29.

[18] G. Sinnamon, *The level functions in rearrangement invariant spaces*, Pub. Mat., 45 (2001), 175–198.

Department of Analysis and Applied Mathematics, Complutense University of Madrid, 28040 Madrid, Spain.
Email address: `achbensa@ucm.es`

Department of Mathematics, EETAC, Polytechnical University of Catalonia, 08860 Castelldefels, Spain.
Email address: `santiago.boza@upc.edu`

Department of Analysis and Applied Mathematics, Complutense University of Madrid, 28040 Madrid, Spain and ICMAT.
Email address: `javier.soria@ucm.es`