ON SHARP CONSTANTS FOR INFINITELY MANY EQUIVALENT NORMS ON CESÀRO FUNCTION SPACES

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Abstract. Recent years have seen significant interest in finding equivalent norms on Cesàro function spaces together with their optimal equivalence constants. In this work, we continue this line of research by deriving sharp two-sided inequalities relating the L^p-norms of $\mathcal{C}_m f$ and $\mathcal{C}_n f$, where $\mathcal{C}_k$ is the generalized Cesàro operator of order $k \in \mathbb{N}$ acting on non-negative measurable functions f on $(0,\infty)$. A parallel study is also conducted for the corresponding adjoint operators.

1. Introduction

Let $\mathcal{M}(\mathbb{R}^+)$ denote the space of measurable functions on $\mathbb{R}^+ = (0,\infty)$, and let $\mathcal{M}^+(\mathbb{R}^+)$ be the subclass of non-negative functions in $\mathcal{M}(\mathbb{R}^+)$. For $m > 0$, the *generalized Cesàro operator of order* m, denoted by $\mathcal{C}_m$, and its dual operator $\mathcal{C}_m^*$ are defined for $f \in \mathcal{M}(\mathbb{R}^+)$ by

$$\mathcal{C}_m f(x) = \frac{m}{x^m} \int_0^x f(t)(x-t)^{m-1}\,dt, \quad x > 0,$$

and

$$\mathcal{C}_m^* f(x) = m \int_x^\infty \frac{f(t)}{t^m}(t-x)^{m-1}\,dt, \quad x > 0,$$

respectively, provided these integrals exist. By convention, for $m = 0$, both $\mathcal{C}_0$ and $\mathcal{C}_0^*$ are defined as the identity operator I, so that

$$\mathcal{C}_0 f = \mathcal{C}_0^* f = f.$$

Recall that for $m > 0$, the classical *Riemann–Liouville fractional integral operator* J_a^m with origin at $a \in \mathbb{R}$ is defined as

$$(J_a^m f)(x) = \frac{1}{\Gamma(m)} \int_a^x f(t)(x-t)^{m-1}\,dt, \quad x > a$$

(see, e.g., [10, Definition 2.1, p. 13]), where $\Gamma\colon (0,\infty) \to \mathbb{R}$ denotes the classical *Euler Gamma function*. Setting $a = 0$ yields a direct connection to $\mathcal{C}_m$, namely

$$\mathcal{C}_m f(x) = \frac{\Gamma(m+1)}{x^m}(J_0^m f)(x).$$

Analogously, the adjoint $\mathcal{C}_m^*$ connects to the *Weyl fractional integral operator* W^m *of order* $m > 0$ [18, p. 239], defined on $x > 0$ by

$$(W^m f)(x) = \frac{1}{\Gamma(m)} \int_x^\infty f(t)(t-x)^{m-1}\,dt.$$

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Indeed, the adjoint formulation takes the form

$$\mathcal{C}_m^* f = \Gamma(m+1) W^m \left(\frac{f(\cdot)}{(\cdot)^m} \right).$$

The generalized Cesàro operators $\mathcal{C}_m$ and their adjoints play an important role in operator theory and harmonic analysis and have been widely studied. For instance, a detailed study of the boundedness and a spectral analysis of these operators on Sobolev–Lebesgue spaces was carried out in [17]. For further studies regarding these operators, the reader is referred to [1, 9, 8, 20] and the references therein.

The boundedness of the Cesàro operator $\mathcal{C}_1$ from $L^p(\mathbb{R}^+)$ into itself is a straightforward consequence of the well-known *Hardy inequality*, see a nice survey about this inequality in [15]. The following generalized Hardy inequality, see [11, Theorem 329, p. 245], allows to show the boundedness of Cesàro operators $\mathcal{C}_m$ from $L^p(\mathbb{R}^+)$ into itself:

Let $m > 0$. For every $f \in L^p(\mathbb{R}^+)$ with $1 < p \leq \infty$, the operator $\mathcal{C}_m$ satisfies the bound

$$\left(\int_0^\infty \left| \frac{m}{x^m} \int_0^x f(t)(x-t)^{m-1}\, dt \right|^p dx \right)^{1/p} \leq \frac{\Gamma(m+1)\Gamma(1/p')}{\Gamma(m+1/p')} \left(\int_0^\infty |f(x)|^p\, dx \right)^{1/p}, \tag{1}$$

where $p' = p/(p-1)$ is the conjugate exponent of p. Moreover, the constant in (1) is the best possible. A closer (and dual) inequality is the following,

$$\left(\int_0^\infty \left| m \int_x^\infty (t-x)^{m-1} f(t) \frac{dt}{t^m} \right|^p dx \right)^{1/p} \leq \frac{\Gamma(m+1)\Gamma(1/p)}{\Gamma(m+1/p)} \left(\int_0^\infty |f(x)|^p\, dx \right)^{1/p}, \tag{2}$$

which holds for all $f \in L^p(\mathbb{R}^+)$ with $1 \leq p < \infty$. Also the constant in inequality (2) is optimal; see [11, Theorem 329, p. 245].

The classical *Cesàro function spaces* $\Lambda_1^p(\mathbb{R}^+)$ and the classical *Copson function spaces* $\Lambda_1^{*p}(\mathbb{R}^+)$ are defined, respectively, by

$$\Lambda_1^p(\mathbb{R}^+) = \left\{ f \in \mathcal{M}(\mathbb{R}^+) : \|f\|_{\Lambda_1^p(\mathbb{R}^+)} = \|\mathcal{C}_1|f|\|_p < \infty \right\}, \qquad 1 < p \leq \infty,$$

and

$$\Lambda_1^{*p}(\mathbb{R}^+) = \left\{ f \in \mathcal{M}(\mathbb{R}^+) : \|f\|_{\Lambda_1^{*p}(\mathbb{R}^+)} = \|\mathcal{C}_1^*|f|\|_p < \infty \right\}, \qquad 1 \leq p < \infty.$$

These spaces are Banach spaces that have been extensively studied; see, for example, [2, 7, 13, 16, 21] and the references therein.

Note that, in particular, inequality (1) implies that

$$L^p(\mathbb{R}^+) \subset \Lambda_1^p(\mathbb{R}^+)$$

for all $1 < p \leq \infty$. However, the reverse inclusion does not hold. Indeed, because the classical Hardy operator $\mathcal{C}_1$ is not invertible on $L^p(\mathbb{R}^+)$ (see [19]), there exists no constant $c(p) > 0$ depending solely on p such that the reverse inequality

$$\|\mathcal{C}_1 f\|_p \geq c(p)\, \|f\|_p \tag{3}$$

holds in general, even for non-negative functions $f \in L^p(\mathbb{R}^+)$. Interestingly, replacing $\mathcal{C}_1$ with $\mathcal{C}_1 - I$ yields an operator that is both continuous and bounded from below on $L^p(\mathbb{R}^+)$ for every $1 < p < \infty$ (see [4]).

Nevertheless, if we replace the $L^p(\mathbb{R}^+)$ norm with the $\Lambda_1^p(\mathbb{R}^+)$ norm in (3), we have the equivalence

$$\frac{\Gamma(p+1)^{1/p}}{p-1} \|f\|_{\Lambda_1^p(\mathbb{R}^+)} \leq \|\mathcal{C}_1 f\|_{\Lambda_1^p(\mathbb{R}^+)} \leq p' \|f\|_{\Lambda_1^p(\mathbb{R}^+)} \tag{4}$$

for all $f \in \mathcal{M}^+(\mathbb{R}^+)$ and $1 < p \leq \infty$. In the limiting case $p = \infty$ in (4), the constants in the lower and upper bounds are given by

$$\lim_{p\to\infty} \frac{\Gamma(p+1)^{1/p}}{p-1} = \frac{1}{e} \quad \text{and} \quad \lim_{p\to\infty} p' = 1,$$

respectively. It is worth emphasizing that the constants appearing in (4) are optimal. This line of research was initiated by K. Leśnik and L. Maligranda in [16] and subsequently completed by A. Ben Said, S. Boza, and J. Soria in [5].

Observe that since $\mathcal{C}_1$ is a linear injective operator, the functional

$$N(f) = \|\mathcal{C}_1(|f|)\|_{\Lambda_1^p(\mathbb{R}^+)}$$

defines a norm on $\Lambda_1^p(\mathbb{R}^+)$ that is equivalent to its standard norm for all $1 < p \leq \infty$, in view of (4).

In recent years, considerable attention has been devoted to the problem of renorming Cesàro function spaces. For instance, G. Bennett [7, Theorem 21.1] established that the spaces $\Lambda_1^p(\mathbb{R}^+)$ and $\Lambda_1^{*p}(\mathbb{R}^+)$ coincide for $1 < p < \infty$. In terms of norms, this equivalence ensures the existence of positive constants $c(p)$ and $C(p)$ such that

$$c(p)\|f\|_{\Lambda_1^{*p}(\mathbb{R}^+)} \leq \|f\|_{\Lambda_1^p(\mathbb{R}^+)} \leq C(p)\|f\|_{\Lambda_1^{*p}(\mathbb{R}^+)}$$

for all $f \in \mathcal{M}^+(\mathbb{R}^+)$ and $1 < p < \infty$. Bennett also provided explicit estimates for some of these equivalence constants. In particular, he proved that

$$\|f\|_{\Lambda_1^{*p}(\mathbb{R}^+)} \leq (p-1)^{1/p}\|f\|_{\Lambda_1^p(\mathbb{R}^+)}$$

for $1 < p \leq 2$, whereas the reverse inequality holds for $2 \leq p < \infty$:

$$(p-1)^{1/p}\|f\|_{\Lambda_1^p(\mathbb{R}^+)} \leq \|f\|_{\Lambda_1^{*p}(\mathbb{R}^+)}.$$

Later, V. Kolyada completed the work started by G. Bennett established that:

Theorem 1.1. [14, Theorem 1.1] *Let $1 < p < \infty$ and let $f \in \mathcal{M}(\mathbb{R}^+)$. If $1 < p \leq 2$, then*

$$(p-1)\|f\|_{\Lambda_1^p(\mathbb{R}^+)} \leq \|f\|_{\Lambda_1^{*p}(\mathbb{R}^+)} \leq (p-1)^{1/p}\|f\|_{\Lambda_1^p(\mathbb{R}^+)}, \tag{5}$$

and if $2 \leq p < \infty$, then

$$(p-1)^{1/p}\|f\|_{\Lambda_1^p(\mathbb{R}^+)} \leq \|f\|_{\Lambda_1^{*p}(\mathbb{R}^+)} \leq (p-1)\|f\|_{\Lambda_1^p(\mathbb{R}^+)}. \tag{6}$$

Moreover, all constants in (5) and (6) are the best possible.

In [5], A. Ben Said, S. Boza, and J. Soria also proved the following theorem.

Theorem 1.2. [5, Theorem 2.2] *Let $1 \leq p < \infty$ and let $f \in \mathcal{M}(\mathbb{R}^+)$. Then the following inequalities are sharp:*

$$\Gamma(p+1)^{1/p}\|f\|_{\Lambda_1^{*p}(\mathbb{R}^+)} \leq \|\mathcal{C}_1^* f\|_{\Lambda_1^{*p}(\mathbb{R}^+)} \leq p\|f\|_{\Lambda_1^{*p}(\mathbb{R}^+)}. \tag{7}$$

Let $C_p = \int_0^1 |\ln(t)+1|^p dt$. As a consequence of a result initiated in [3], A. Ben Said and G. Sinnamon in [6] obtained the result below:

Theorem 1.3. [6, Theorem 1.2] *Let $1 < p < \infty$ and $f \in \mathcal{M}(\mathbb{R}^+)$ be such that $\mathcal{C}_1^*|f(x)| < \infty$ for all $x > 0$. If $1 < p \leq 2$, then*

$$(p-1)\|f\|_{\Lambda_1^{*p}(\mathbb{R}^+)} \leq \|(\mathcal{C}_1^* - I)\mathcal{C}_1^*|f|\|_p \leq C_p^{1/p}\|f\|_{\Lambda_1^{*p}(\mathbb{R}^+)}, \tag{8}$$

and if $2 \leq p \leq \infty$, *then*

$$C_p^{1/p}\|f\|_{\Lambda_1^{*p}(\mathbb{R}^+)} \leq \|(\mathcal{C}_1^* - I)\mathcal{C}_1^*|f|\|_p \leq (p-1)\|f\|_{\Lambda_1^{*p}(\mathbb{R}^+)}. \tag{9}$$

The constants $p-1$ *and* $C_p^{1/p}$ *are optimal in both* (8) *and* (9).

For each $m > 0$, natural generalizations of the Cesàro function spaces $\Lambda_1^p(\mathbb{R}^+)$ and the Copson function spaces $\Lambda_1^{*p}(\mathbb{R}^+)$ are the function spaces $\Lambda_m^p(\mathbb{R}^+)$ and $\Lambda_m^{*p}(\mathbb{R}^+)$, defined respectively by

$$\Lambda_m^p(\mathbb{R}^+) = \big\{f \in \mathcal{M}(\mathbb{R}^+) : \|f\|_{\Lambda_m^p(\mathbb{R}^+)} = \|\mathcal{C}_m|f|\|_p < \infty\big\}, \qquad 1 < p \leq \infty,$$

and

$$\Lambda_m^{*p}(\mathbb{R}^+) = \big\{f \in \mathcal{M}(\mathbb{R}^+) : \|f\|_{\Lambda_m^{*p}(\mathbb{R}^+)} = \|\mathcal{C}_m^*|f|\|_p < \infty\big\}, \qquad 1 \leq p < \infty.$$

The main goal of this paper is to establish the existence of positive constants c, C, k, K (depending only on $m, n \in \mathbb{N}$ and p) such that

$$c\,\|f\|_{\Lambda_n^p(\mathbb{R}^+)} \leq \|f\|_{\Lambda_m^p(\mathbb{R}^+)} \leq C\,\|f\|_{\Lambda_n^p(\mathbb{R}^+)} \tag{10}$$

and

$$k\,\|f\|_{\Lambda_n^{*p}(\mathbb{R}^+)} \leq \|f\|_{\Lambda_m^{*p}(\mathbb{R}^+)} \leq K\,\|f\|_{\Lambda_n^{*p}(\mathbb{R}^+)} \tag{11}$$

for all $f \in \mathcal{M}(\mathbb{R}^+)$. Furthermore, we aim to determine the optimal values for c, C, k, and K in (10) and (11).

Note that (10) and (11) imply, respectively, the space equalities

$$\Lambda_m^p(\mathbb{R}^+) = \Lambda_n^p(\mathbb{R}^+) \quad \text{for } 1 < p \leq \infty \tag{12}$$

and

$$\Lambda_m^{*p}(\mathbb{R}^+) = \Lambda_n^{*p}(\mathbb{R}^+) \quad \text{for } 1 \leq p < \infty \tag{13}$$

for any $m, n \in \mathbb{N}$. Combining (12) and (13) with Theorem 1.1, we deduce that

$$\Lambda_m^p(\mathbb{R}^+) = \Lambda_n^p(\mathbb{R}^+) = \Lambda_r^{*p}(\mathbb{R}^+) = \Lambda_s^{*p}(\mathbb{R}^+)$$

for all $1 < p < \infty$ and all $m, n, r, s \in \mathbb{N}$.

This paper is organized as follows. Section 2 presents our main results, whose formal proofs are detailed in Section 4. Section 3 collects the necessary preliminary concepts and technical tools. Finally, Section 5 summarizes all constants derived in this study in tabular form and poses a conjecture regarding those that remain open.

2. Main Results

Let B denote the Beta function, defined for $\mathrm{Re}(z_1), \mathrm{Re}(z_2) > 0$ by

$$B(z_1, z_2) = \int_0^1 v^{z_1-1}(1-v)^{z_2-1}\,dv = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}.$$

The following theorem establishes the continuous embeddings

$$\Lambda_n^p(\mathbb{R}^+) \hookrightarrow \Lambda_{n+k}^p(\mathbb{R}^+) \quad \text{for all } 1 < p \leq \infty$$

and

$$\Lambda_n^{*p}(\mathbb{R}^+) \hookrightarrow \Lambda_{n+k}^{*p}(\mathbb{R}^+) \quad \text{for all } 1 \leq p < \infty,$$

for every $n, k \in \mathbb{N}$, while also determining the exact norms of these inclusion operators.

Theorem 2.1. *Let $1 \le p \le \infty$, $n, k \in \mathbb{N}$, and $f \in \mathcal{M}^+(\mathbb{R}^+)$. If $1 < p \le \infty$, then*

$$\|\mathcal{C}_{n+k} f\|_p \le \frac{B\big(n + \frac{1}{p'}, k\big)}{B(n+1,k)} \|\mathcal{C}_n f\|_p \tag{14}$$

and if $1 \le p < \infty$, then

$$\|\mathcal{C}^*_{n+k} f\|_p \le \frac{B\big(n + \frac{1}{p}, k\big)}{B(n+1,k)} \|\mathcal{C}^*_n f\|_p. \tag{15}$$

Furthermore, the constants in (14) and (15) are sharp.

For all $n, k \in \mathbb{N}$, the following theorem proves the continuous embedding

$$\Lambda^p_{n+k}(\mathbb{R}^+) \hookrightarrow \Lambda^p_n(\mathbb{R}^+)$$

when $p = 2$ or $p = \infty$, and explicitly determines the norms of the corresponding inclusion operators.

Theorem 2.2. *Let $n, k \in \mathbb{N}$ and let $f \in \mathcal{M}^+(\mathbb{R}^+)$. Then*

$$\|\mathcal{C}_n f\|_\infty \le \frac{(n-1)^{n-1}}{n^{n-1}} \frac{(n+k)^{n+k-1}}{(n+k-1)^{n+k-1}} \|\mathcal{C}_{n+k} f\|_\infty \tag{16}$$

and

$$\|\mathcal{C}_n f\|_2^2 \le \left(\frac{n}{n+k}\right)^2 \frac{B\big(2(n-1)+1, 1\big)}{B\big(2(n+k-1)+1, 1\big)} \|\mathcal{C}_{n+k} f\|_2^2. \tag{17}$$

Moreover, the constants in (16) and in (17) are the best possible.

For all $n, k \in \mathbb{N}$, the following result proves the continuous embedding

$$\Lambda^{*p}_{n+k}(\mathbb{R}^+) \hookrightarrow \Lambda^{*p}_n(\mathbb{R}^+)$$

when $p = 1$ or $p = 2$, and explicitly provides the norms of the corresponding inclusion operators.

Theorem 2.3. *Let $n, k \in \mathbb{N}$ and let $f \in \mathcal{M}^+(\mathbb{R}^+)$. If $p = 1$ or $p = 2$, then*

$$\|\mathcal{C}^*_n f\|_p^p \le \frac{n^p}{(n+k)^p} \frac{p(n+k-1)+1}{p(n-1)+1} \|\mathcal{C}^*_{n+k} f\|_p^p. \tag{18}$$

Moreover, the constant in (18) is optimal.

The following theorem demonstrates the continuous embedding

$$\Lambda^p_2(\mathbb{R}^+) \hookrightarrow \Lambda^p_1(\mathbb{R}^+)$$

for all $1 < p < \infty$ and explicitly determines the norm of the corresponding inclusion operator.

Theorem 2.4. *Let $1 < p < \infty$ and let $f \in \mathcal{M}^+(\mathbb{R}^+)$. Then*

$$\|\mathcal{C}_1 f\|_p^p \le \frac{1}{2^p} \frac{B(1, p-1)}{B(p+1, p-1)} \|\mathcal{C}_2 f\|_p^p. \tag{19}$$

Moreover, the constant in (19) is sharp.

The following theorem proves the continuous embedding

$$\Lambda^{*p}_2(\mathbb{R}^+) \hookrightarrow \Lambda^{*p}_1(\mathbb{R}^+)$$

for all $1 < p < \infty$ and explicitly determines the norm of the corresponding inclusion operator.

Theorem 2.5. *Let $1 < p < \infty$ and let $f \in \mathcal{M}^+(\mathbb{R}^+)$. Then*

$$\|\mathcal{C}^*_1 f\|_p^p \le \frac{p+1}{2^p} \|\mathcal{C}^*_2 f\|_p^p. \tag{20}$$

Moreover, the constant in (20) is the best possible.

3. Auxiliary results

We begin by introducing an elementary lemma which will be used throughout the proofs of the main results in this paper.

Lemma 3.1. *Let $\{T_n\}_{n\geq 1}$ be a sequence of bounded operators on $L^p(\mathbb{R}^+)$, and let $\{C(n)\}_{n\geq 1}$ be a sequence of positive real numbers. If*

$$\|T_n f\|_p \leq \frac{C(n)}{C(n+1)}\|T_{n+1}f\|_p \quad (\textit{resp.} \ \geq)$$

holds for each $n \in \mathbb{N}$ and all $f \in \mathcal{M}^+(\mathbb{R}^+)$, then

$$\|T_n f\|_p \leq \frac{C(n)}{C(m)}\|T_m f\|_p \quad (\textit{resp.} \ \geq)$$

holds for all $f \in \mathcal{M}^+(\mathbb{R}^+)$ and for all $m, n \in \mathbb{N}$ with $m \geq n$.

Proof. The result follows immediately by induction on $m \geq n$. □

The following result is a crucial lemma that relates $\mathcal{C}_m$ with $\mathcal{C}_{m+1}$ and $\mathcal{C}_m^*$ with $\mathcal{C}_{m+1}^*$.

Lemma 3.2. *Let $m \in \mathbb{N} \cup \{0\}$ and let $f \in \mathcal{M}^+(\mathbb{R}^+)$. Then*

$$\mathcal{C}_{m+1}f(x) = \frac{m+1}{x^{m+1}}\int_0^x t^m \mathcal{C}_m f(t)\,dt \tag{21}$$

and

$$\mathcal{C}_{m+1}^* f(x) = (m+1)x^m \int_x^\infty \frac{\mathcal{C}_m^* f(t)}{t^{m+1}}\,dt, \tag{22}$$

for all $x > 0$.

Proof. We first establish identity (21). The base case $m = 0$ is trivial. Now, let $m \geq 1$ be a natural number and let $f \in \mathcal{M}^+(\mathbb{R}^+)$. Applying Tonelli's theorem to exchange the order of integration, we obtain

$$\begin{aligned}
\frac{1}{x^{m+1}}\int_0^x t^m \mathcal{C}_m f(t)\,dt &= \frac{m}{x^{m+1}}\int_0^x \int_0^t (t-u)^{m-1} f(u)\,du\,dt \\
&= \frac{m}{x^{m+1}}\int_0^x f(u)\left(\int_u^x (t-u)^{m-1}\,dt\right)du \\
&= \frac{1}{x^{m+1}}\int_0^x (x-u)^m f(u)\,du \\
&= \frac{1}{m+1}\mathcal{C}_{m+1}f(x).
\end{aligned}$$

We now turn to identity (22), where the base case $m = 0$ is again immediate. For any natural number $m \geq 1$ and $f \in \mathcal{M}^+(\mathbb{R}^+)$, a similar argument using Tonelli's theorem and the change of variables $s = u/t$ can be followed to prove

$$(m+1)x^m \int_x^\infty \frac{\mathcal{C}_m^* f(t)}{t^{m+1}}\,dt = \mathcal{C}_{m+1}^* f(x).$$

This concludes the proof □

Throughout what follows, we adopt the standard convention $0^0 = 1$. We now introduce an important family of test functions.

Lemma 3.3. *Let $1 < p \leq \infty$ and let $m, n \in \mathbb{N}$. If $1 < p < \infty$, then*

$$\lim_{\varepsilon \to 0^+} \frac{\|\mathcal{C}_m \chi_{(1,1+\varepsilon)}\|_p^p}{\|\mathcal{C}_n \chi_{(1,1+\varepsilon)}\|_p^p} = \left(\frac{m}{n}\right)^p \frac{B\big((m-1)p+1, p-1\big)}{B\big((n-1)p+1, p-1\big)} \tag{23}$$

and

$$\lim_{\varepsilon \to 0^+} \frac{\|\mathcal{C}_m^* \chi_{(1,1+\varepsilon)}\|_p^p}{\|\mathcal{C}_n^* \chi_{(1,1+\varepsilon)}\|_p^p} = \frac{m^p}{p(m-1)+1} \frac{p(n-1)+1}{n^p}, \tag{24}$$

If $p = \infty$, then

$$\lim_{\varepsilon \to 0^+} \frac{\|\mathcal{C}_m \chi_{(1,1+\varepsilon)}\|_\infty}{\|\mathcal{C}_n \chi_{(1,1+\varepsilon)}\|_\infty} = \frac{(m-1)^{m-1}}{m^{m-1}} \frac{n^{n-1}}{(n-1)^{n-1}}. \tag{25}$$

Proof. Let $m \in \mathbb{N}$ and $\varepsilon > 0$. We proceed to compute $\mathcal{C}_m \chi_{(1,1+\varepsilon)}$. For $0 < x \leq 1$, we clearly have

$$\mathcal{C}_m \chi_{(1,1+\varepsilon)}(x) = \frac{m}{x^m} \int_0^x 0 \cdot (x-t)^{m-1}\, dt = 0.$$

If $1 < x \leq 1 + \varepsilon$, it follows that

$$\mathcal{C}_m \chi_{(1,1+\varepsilon)}(x) = \frac{m}{x^m} \int_1^x (x-t)^{m-1}\, dt = \frac{(x-1)^m}{x^m},$$

and for $x > 1 + \varepsilon$, we obtain

$$\mathcal{C}_m \chi_{(1,1+\varepsilon)}(x) = \frac{m}{x^m} \int_1^{1+\varepsilon} (x-t)^{m-1}\, dt = \frac{(x-1)^m - (x-1-\varepsilon)^m}{x^m}.$$

Therefore, for all $x > 0$, we have

$$\mathcal{C}_m \chi_{(1,1+\varepsilon)}(x) = \frac{(x-1)^m}{x^m} \chi_{(1,1+\varepsilon]}(x) + \frac{(x-1)^m - (x-1-\varepsilon)^m}{x^m} \chi_{(1+\varepsilon,\infty)}(x). \tag{26}$$

Taking the L^p norm in (26) we get that

$$\begin{aligned} \|\mathcal{C}_m \chi_{(1,1+\varepsilon)}\|_p^p &= \int_1^{1+\varepsilon} \frac{(x-1)^{mp}}{x^{mp}}\, dx + \int_{1+\varepsilon}^{\infty} \frac{\big((x-1)^m - (x-1-\varepsilon)^m\big)^p}{x^{mp}} dx \\ &= I_1(\varepsilon) + I_2(\varepsilon). \end{aligned}$$

On the one hand, for $x \in (1, 1+\varepsilon)$, we have $1 \leq x^{mp} \leq (1+\varepsilon)^{mp}$, so:

$$\frac{1}{(1+\varepsilon)^{mp}} \int_1^{1+\varepsilon} (x-1)^{mp}\, dx \leq I_1(\varepsilon) \leq \int_1^{1+\varepsilon} (x-1)^{mp}\, dx.$$

Evaluating $\int_1^{1+\varepsilon} (x-1)^{mp}\, dx = \frac{\varepsilon^{mp+1}}{mp+1}$:

$$\frac{\varepsilon^{mp+1}}{(1+\varepsilon)^{mp}(mp+1)} \leq I_1(\varepsilon) \leq \frac{\varepsilon^{mp+1}}{mp+1}.$$

Divide the entire inequality chain by ε^p:

$$\frac{\varepsilon^{(m-1)p+1}}{(1+\varepsilon)^{mp}(mp+1)} \leq \frac{I_1(\varepsilon)}{\varepsilon^p} \leq \frac{\varepsilon^{(m-1)p+1}}{mp+1}.$$

Since $m \geq 1$ and $p > 1$, the exponent $(m-1)p+1 > 0$. Taking $\varepsilon \to 0^+$ sends both the left and right bounds to 0. By the Squeeze Theorem:

$$\lim_{\varepsilon \to 0^+} \frac{I_1(\varepsilon)}{\varepsilon^p} = 0.$$

On the other hand, by applying the change of variables $v = 1 - \frac{1}{x}$, we get

$$\begin{aligned} I_2(\varepsilon) &= \int_{1+\varepsilon}^{\infty} \frac{\big((x-1)^m - (x-1-\varepsilon)^m\big)^p}{x^{mp}}\,dx \\ &= \int_{1+\varepsilon}^{\infty} \left(\left(1-\frac{1}{x}\right)^m - \left(1-\frac{1}{x}-\frac{\varepsilon}{x}\right)^m\right)^p dx \\ &= \int_{\frac{\varepsilon}{1+\varepsilon}}^{1} \big(v^m - (v-\varepsilon(1-v))^m\big)^p \frac{dv}{(1-v)^2}. \end{aligned}$$

Dividing by ε^p we obtain that

$$\frac{I_2(\varepsilon)}{\varepsilon^p} = \int_{\frac{\varepsilon}{1+\varepsilon}}^{1} \left(\frac{v^m - (v-\varepsilon(1-v))^m}{\varepsilon}\right)^p \frac{dv}{(1-v)^2}.$$

Now define the function inside the integral as $K(\varepsilon, v)$:

$$K(\varepsilon, v) = \chi_{(\frac{\varepsilon}{1+\varepsilon},1)}(v) \cdot \left(\frac{v^m - (v-\varepsilon(1-v))^m}{\varepsilon}\right)^p \frac{1}{(1-v)^2}.$$

Fix $v \in (0,1)$ and define $g(y) = y^m$. By setting $h = \varepsilon(1-v)$, we observe that $h \to 0^+$ as $\varepsilon \to 0^+$. Expressing the difference quotient in terms of g, we have:

$$\lim_{\varepsilon\to 0^+} \frac{v^m - (v-\varepsilon(1-v))^m}{\varepsilon(1-v)} = \lim_{h\to 0^+} \frac{g(v) - g(v-h)}{h} = g'(v) = mv^{m-1}.$$

This means

$$\lim_{\varepsilon\to 0^+} \frac{v^m - (v-\varepsilon(1-v))^m}{\varepsilon} = mv^{m-1}(1-v).$$

Thus

$$\lim_{\varepsilon\to 0^+} K(\varepsilon, v) = \big(mv^{m-1}(1-v)\big)^p \frac{1}{(1-v)^2} = m^p v^{(m-1)p}(1-v)^{p-2}.$$

For a fixed $v \in (0,1)$ and $\varepsilon > 0$, by the Mean Value Theorem, there exists a number $\xi \in (v-\varepsilon(1-v), v)$ such that

$$\begin{aligned} K(\varepsilon, v) &\le \left(\frac{v^m - (v-\varepsilon(1-v))^m}{\varepsilon}\right)^p \frac{1}{(1-v)^2} \\ &= \big(m\xi^{m-1}(1-v)\big)^p \frac{1}{(1-v)^2} \\ &\le \big(mv^{m-1}(1-v)\big)^p \frac{1}{(1-v)^2} \\ &= m^p v^{(m-1)p}(1-v)^{p-2} =: h(v). \end{aligned}$$

Since $h \in L^1(0,1)$, we may apply the Dominated Convergence Theorem to conclude that

$$\begin{aligned} \lim_{\varepsilon\to 0^+} \frac{I_2(\varepsilon)}{\varepsilon^p} &= \lim_{\varepsilon\to 0^+} \int_0^1 K(\varepsilon, v)\,dv = \int_0^1 \lim_{\varepsilon\to 0^+} K(\varepsilon, v)\,dv \\ &= \int_0^1 m^p v^{(m-1)p}(1-v)^{p-2}\,dv = m^p B\big((m-1)p+1, p-1\big). \end{aligned}$$

Hence

$$\lim_{\varepsilon\to 0^+} \frac{\|\mathcal{C}_m \chi_{(1,1+\varepsilon)}\|_p^p}{\varepsilon^p} = \lim_{\varepsilon\to 0^+} \left(\frac{I_1(\varepsilon)}{\varepsilon^p} + \frac{I_2(\varepsilon)}{\varepsilon^p}\right) = m^p B\big((m-1)p+1, p-1\big),$$

for all $m \in \mathbb{N}$. Therefore

$$\lim_{\varepsilon\to 0^+} \frac{\|\mathcal{C}_m\chi_{(1,1+\varepsilon)}\|_p^p}{\|\mathcal{C}_n\chi_{(1,1+\varepsilon)}\|_p^p} = \lim_{\varepsilon\to 0^+} \left(\frac{\|\mathcal{C}_m\chi_{(1,1+\varepsilon)}\|_p^p}{\varepsilon^p} \frac{\varepsilon^p}{\|\mathcal{C}_n\chi_{(1,1+\varepsilon)}\|_p^p} \right)$$
$$= \left(\frac{m}{n}\right)^p \frac{B\big((m-1)p+1, p-1\big)}{B\big((n-1)p+1, p-1\big)},$$

for all $m, n \in \mathbb{N}$.

We now consider the case $p = \infty$ in order to prove (25). Suppose that $m \geq 2$. Recall from (26) that

$$\mathcal{C}_m\chi_{(1,1+\varepsilon)}(x) = \frac{(x-1)^m}{x^m}\chi_{(1,1+\varepsilon]}(x) + \frac{(x-1)^m - (x-1-\varepsilon)^m}{x^m}\chi_{(1+\varepsilon,\infty)}(x).$$

Observe that $x \mapsto \mathcal{C}_m\chi_{(1,1+\varepsilon)}(x)$ is continuous on $\mathbb{R}^+$ and increasing on $(0, 1+\varepsilon)$. Then, let $f_\varepsilon : (1+\varepsilon, \infty) \to \mathbb{R}$ be defined by

$$f_\varepsilon(x) = \frac{(x-1)^m - (x-1-\varepsilon)^m}{x^m}.$$

Setting $h_\varepsilon(x) = f_\varepsilon(1/x) = (1-x)^m - \big(1 - x(1+\varepsilon)\big)^m$ for $x \in \big(0, \frac{1}{1+\varepsilon}\big)$, we differentiate h_ε with respect to x:

$$h_\varepsilon'(x) = -m(1-x)^{m-1} + m(1+\varepsilon)\big(1 - x(1+\varepsilon)\big)^{m-1}.$$

Thus,

$$h_\varepsilon'(x) = 0 \iff (1+\varepsilon)^{\frac{1}{m-1}}\big(1 - x(1+\varepsilon)\big) = 1 - x,$$

which yields

$$x = \frac{(1+\varepsilon)^{\frac{1}{m-1}} - 1}{(1+\varepsilon)^{\frac{m}{m-1}} - 1}.$$

Consequently, f_ε has a unique critical point at

$$x_0 = \frac{(1+\varepsilon)^{\frac{m}{m-1}} - 1}{(1+\varepsilon)^{\frac{1}{m-1}} - 1} \in (1+\varepsilon, \infty).$$

A straightforward calculation gives

$$f_\varepsilon(x_0) = \frac{\varepsilon^m}{\Big((1+\varepsilon)^{\frac{m}{m-1}} - 1\Big)^{m-1}}.$$

Moreover, since the function $x \mapsto (1-1/x)^m$ is strictly increasing on $(1, 1+\varepsilon)$, we have

$$\left(1 - \frac{1}{x}\right)^m \leq \left(1 - \frac{1}{1+\varepsilon}\right)^m = \frac{\varepsilon^m}{(1+\varepsilon)^m} < \frac{\varepsilon^m}{\Big((1+\varepsilon)^{\frac{m}{m-1}} - 1\Big)^{m-1}}$$

for all $x \in (1, 1+\varepsilon)$. Therefore, the maximum of $\mathcal{C}_m\chi_{(1,1+\varepsilon)}$ is achieved at x_0, so

$$\|\mathcal{C}_m\chi_{(1,1+\varepsilon)}\|_\infty = \frac{\varepsilon^m}{\Big((1+\varepsilon)^{\frac{m}{m-1}} - 1\Big)^{m-1}}$$

for all $\varepsilon > 0$. Taking the limit as $\varepsilon \to 0^+$, we obtain

$$\lim_{\varepsilon\to 0^+} \frac{\varepsilon}{\|\mathcal{C}_m\chi_{(1,1+\varepsilon)}\|_\infty} = \lim_{\varepsilon\to 0^+} \left(\frac{(1+\varepsilon)^{\frac{m}{m-1}} - 1}{\varepsilon} \right)^{m-1} = \left(\frac{m}{m-1}\right)^{m-1}.$$

This implies that

$$\lim_{\varepsilon\to 0^+}\frac{\|\mathcal{C}_m\chi_{(1,1+\varepsilon)}\|_\infty}{\varepsilon}=\frac{(m-1)^{m-1}}{m^{m-1}} \tag{27}$$

for all $m\in\mathbb{N}$ with $m\geq 2$.

For the case $m=1$, by using (26), we have

$$\mathcal{C}_1\chi_{(1,1+\varepsilon)}(x)=\left(1-\frac{1}{x}\right)\chi_{(1,1+\varepsilon]}(x)+\frac{\varepsilon}{x}\chi_{(1+\varepsilon,\infty)}(x)\quad\text{for all } x>0.$$

The supremum is clearly attained at $x=1+\varepsilon$, yielding

$$\|\mathcal{C}_1\chi_{(1,1+\varepsilon)}\|_\infty=\mathcal{C}_1\chi_{(1,1+\varepsilon)}(1+\varepsilon)=\frac{\varepsilon}{1+\varepsilon}.$$

Consequently,

$$\lim_{\varepsilon\to 0^+}\frac{\|\mathcal{C}_1\chi_{(1,1+\varepsilon)}\|_\infty}{\varepsilon}=\lim_{\varepsilon\to 0^+}\frac{1}{1+\varepsilon}=1. \tag{28}$$

Combining (27) and (28) under the convention $0^0=1$, we conclude that

$$\lim_{\varepsilon\to 0^+}\frac{\|\mathcal{C}_m\chi_{(1,1+\varepsilon)}\|_\infty}{\varepsilon}=\frac{(m-1)^{m-1}}{m^{m-1}} \tag{29}$$

holds for all $m\in\mathbb{N}$ with $m\geq 1$. Applying (29), we then obtain

$$\lim_{\varepsilon\to 0^+}\frac{\|\mathcal{C}_m\chi_{(1,1+\varepsilon)}\|_\infty}{\|\mathcal{C}_n\chi_{(1,1+\varepsilon)}\|_\infty}=\lim_{\varepsilon\to 0^+}\left(\frac{\|\mathcal{C}_m\chi_{(1,1+\varepsilon)}\|_\infty}{\varepsilon}\frac{\varepsilon}{\|\mathcal{C}_n\chi_{(1,1+\varepsilon)}\|_\infty}\right)=\frac{(m-1)^{m-1}}{m^{m-1}}\frac{n^{n-1}}{(n-1)^{n-1}}.$$

for all $m,n\in\mathbb{N}$.

We now proceed to establish the limit asserted in (24). A direct evaluation yields

$$\mathcal{C}_m^*\chi_{(1,1+\epsilon)}(x)=m\int_1^{1+\varepsilon}\frac{(t-x)^{m-1}}{t^m}\,dt\cdot\chi_{(0,1]}(x)+m\int_x^{1+\varepsilon}\frac{(t-x)^{m-1}}{t^m}\,dt\cdot\chi_{(1,1+\varepsilon]}(x).$$

Taking the L^p-norm, we can decompose the expression into two integrals, $I_1(\varepsilon)$ and $I_2(\varepsilon)$:

$$\begin{aligned}\|\mathcal{C}_m^*\chi_{(1,1+\varepsilon)}\|_p^p&=m^p\int_0^1\left(\int_1^{1+\varepsilon}\frac{(t-x)^{m-1}}{t^m}\,dt\right)^p dx+m^p\int_1^{1+\varepsilon}\left(\int_x^{1+\varepsilon}\frac{(t-x)^{m-1}}{t^m}\,dt\right)^p dx\\&=:I_1(\varepsilon)+I_2(\varepsilon).\end{aligned}$$

To analyze $I_1(\varepsilon)$, define the auxiliary function $g_\varepsilon(x)$ on $(0,1)$ by

$$g_\varepsilon(x):=\frac{m}{\varepsilon}\int_1^{1+\varepsilon}\frac{(t-x)^{m-1}}{t^m}\,dt.$$

By the Lebesgue Differentiation Theorem, as $\varepsilon\to 0^+$, we have the pointwise convergence

$$\lim_{\varepsilon\to 0^+}g_\varepsilon(x)=m(1-x)^{m-1}\quad\text{for all } x\in(0,1),$$

which implies that

$$\lim_{\varepsilon\to 0^+}(g_\varepsilon(x))^p=m^p(1-x)^{p(m-1)}.$$

For $0<\varepsilon<1$, observe that $t^{-m}\leq 1$ for $t\in(1,1+\varepsilon)$. Since $t\leq 1+\varepsilon<2$, we have

$$t-x\leq 2-x<2\quad\text{for all } x\in(0,1)\text{ and } t\in(1,1+\varepsilon).$$

This provides a uniform upper bound for the integrand:

$$m\frac{(t-x)^{m-1}}{t^m}\leq m(t-x)^{m-1}\leq m(2-x)^{m-1}.$$

Integrating over $(1, 1+\varepsilon)$ gives

$$g_\varepsilon(x) \le m(2-x)^{m-1},$$

and consequently,

$$(g_\varepsilon(x))^p \le m^p(2-x)^{p(m-1)} =: w(x) \quad \text{for all } x \in (0,1).$$

Since $w(x)$ is continuous on the compact interval $[0,1]$, then $w \in L^1(0,1)$. Applying the Dominated Convergence Theorem, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \frac{I_1(\varepsilon)}{\varepsilon^p} &= \lim_{\varepsilon\to 0^+} \int_0^1 (g_\varepsilon(x))^p\,dx = \int_0^1 \lim_{\varepsilon\to 0^+} (g_\varepsilon(x))^p\,dx \\ &= m^p \int_0^1 (1-x)^{p(m-1)}\,dx = \frac{m^p}{p(m-1)+1}.\end{aligned}$$

Next, to estimate $I_2(\varepsilon)$, note that for $x \in (1, 1+\varepsilon)$ and $t \in (x, 1+\varepsilon)$,

$$\frac{m(t-x)^{m-1}}{t^m} \le m(t-x)^{m-1} \le m\varepsilon^{m-1}.$$

Integrating this bound over $(x, 1+\varepsilon)$ yields

$$\int_x^{1+\varepsilon} \frac{m(t-x)^{m-1}}{t^m}\,dt \le \int_x^{1+\varepsilon} m\varepsilon^{m-1}\,dt \le \int_1^{1+\varepsilon} m\varepsilon^{m-1}\,dt = m\varepsilon^m.$$

Thus, $I_2(\varepsilon)/\varepsilon^p$ can be bounded as

$$0 \le \frac{I_2(\varepsilon)}{\varepsilon^p} \le \frac{1}{\varepsilon^p} \int_1^{1+\varepsilon} m^p\varepsilon^{mp}\,dx = m^p\varepsilon^{(m-1)p+1}.$$

Since $(m-1)p+1 > 0$, taking $\varepsilon \to 0^+$ and applying the Squeeze Theorem gives

$$\lim_{\varepsilon\to 0^+} \frac{I_2(\varepsilon)}{\varepsilon^p} = 0.$$

Combining our results for $I_1(\varepsilon)$ and $I_2(\varepsilon)$, we find that

$$\lim_{\varepsilon\to 0^+} \frac{\|\mathcal{C}_m^*\chi_{(1,1+\varepsilon)}\|_p^p}{\varepsilon^p} = \lim_{\varepsilon\to 0^+} \frac{I_1(\varepsilon)}{\varepsilon^p} + \lim_{\varepsilon\to 0^+} \frac{I_2(\varepsilon)}{\varepsilon^p} = \frac{m^p}{p(m-1)+1}.$$

Finally, taking the ratio for m and $m+1$, we conclude that

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \frac{\|\mathcal{C}_m^*\chi_{(1,1+\varepsilon)}\|_p^p}{\|\mathcal{C}_n^*\chi_{(1,1+\varepsilon)}\|_p^p} &= \lim_{\varepsilon\to 0^+} \left(\frac{\|\mathcal{C}_m^*\chi_{(1,1+\varepsilon)}\|_p^p}{\varepsilon^p} \frac{\varepsilon^p}{\|\mathcal{C}_n^*\chi_{(1,1+\varepsilon)}\|_p^p} \right) \\ &= \frac{m^p}{p(m-1)+1} \frac{p(n-1)+1}{n^p},\end{aligned}$$

for all $m, n \in \mathbb{N}$. This completes the proof. □

The following lemma is of importance.

Lemma 3.4. *For any natural numbers $1 \le k \le m$, let $\Theta_{m,k}\colon (0,1] \to \mathbb{R}$ be defined by*

$$\Theta_{m,k}(t) = \frac{1}{2} \cdot \frac{t^{m-k}\int_0^t (1-s)^{m-1}(t-s)^{k-1}\,ds + \int_0^t (t-s)^{m-1}(1-s)^{k-1}\,ds}{\int_0^t (1-s)^{m-1}(t-s)^{m-1}\,ds}.$$

Then $\Theta_{m,k}$ is non-increasing on $(0,1]$.

Proof. Applying the change of variables $s = rt$ (so that $ds = t\,dr$), we obtain

$$\Theta_{m,k}(t) = \frac{1}{2}\frac{\int_0^1 (1-rt)^{m-1}(1-r)^{k-1}\,dr + \int_0^1 (1-r)^{m-1}(1-rt)^{k-1}\,dr}{\int_0^1 (1-rt)^{m-1}(1-r)^{m-1}\,dr}.$$

In order to see that $\Theta_{m,k}$ is non-increasing on $(0,1]$ we will show that the function $\Phi_{m,k}$ defined by

$$\Phi_{m,k}(t) := 2\Theta_{m,k}(-t)$$

is non-decreasing on $[-1,0)$. We have that

$$\Phi_{m,k}(t) = \frac{\int_0^1 (1+rt)^{m-1}(1-r)^{k-1}\,dr + \int_0^1 (1-r)^{m-1}(1+rt)^{k-1}\,dr}{\int_0^1 (1+rt)^{m-1}(1-r)^{m-1}\,dr}.$$

An application of the binomial theorem followed by straightforward integration yields

$$\int_0^1 (1+rt)^{m-1}(1-r)^{k-1}\,dr = \sum_{j=0}^{m-1} \frac{(m-1)!(k-1)!}{(m-1-j)!(k+j)!}t^j,$$

$$\int_0^1 (1+rt)^{k-1}(1-r)^{m-1}\,dr = \sum_{j=0}^{k-1} \frac{(m-1)!(k-1)!}{(k-1-j)!(m+j)!}t^j,$$

and

$$\int_0^1 (1+rt)^{m-1}(1-r)^{m-1}\,dr = \sum_{j=0}^{m-1} \frac{(m-1)!(m-1)!}{(m-1-j)!(m+j)!}t^j.$$

Let $P_{m,k}$ and $Q_{m,k}$ be the polynomials of degree $m-1$ defined by

$$\begin{aligned} P_{m,k}(t) &= \int_0^1 (1+rt)^{m-1}(1-r)^{k-1}\,dr + \int_0^1 (1+rt)^{k-1}(1-r)^{m-1}\,dr \\ &= \sum_{j=0}^{k-1}\left(\frac{(m-1)!(k-1)!}{(m-1-j)!(k+j)!} + \frac{(m-1)!(k-1)!}{(k-1-j)!(m+j)!}\right)t^j \\ &\quad + \sum_{j=k}^{m-1}\frac{(m-1)!(k-1)!}{(m-1-j)!(k+j)!}t^j = \sum_{j=0}^{m-1}\alpha_j t^j. \end{aligned}$$

$$Q_{m,k}(t) = \int_0^1 (1+rt)^{m-1}(1-r)^{m-1}\,dr = \sum_{j=0}^{m-1}\frac{(m-1)!(m-1)!}{(m-1-j)!(m+j)!}t^j = \sum_{j=0}^{m-1}\beta_j t^j.$$

We have that

$$\Phi_{m,k}(t) = \frac{P_{m,k}(t)}{Q_{m,k}(t)} = \frac{\sum_{j=0}^{m-1}\alpha_j t^j}{\sum_{j=0}^{m-1}\beta_j t^j}.$$

If $k = m$, we trivially obtain that $\Phi_{m,m}(t) = 2$. In the case when $k = m-1$, we have that

$$\begin{aligned}
P_{m,m-1}(t) &= (m-1)!(m-2)!\sum_{j=0}^{m-2}\left(\frac{1}{(m-1-j)!(m-1+j)!}+\frac{1}{(m-2-j)!(m+j)!}\right)t^j \\
&\quad + (m-1)!(m-2)!\frac{t^{m-1}}{(2m-2)!} \\
&= (m-1)!(m-2)!\sum_{j=0}^{m-2}\frac{2m-1}{(m-1-j)!(m+j)!}t^j + (m-1)!(m-2)!\frac{t^{m-1}}{(2m-2)!} \\
&= (m-1)!(m-2)!\sum_{j=0}^{m-1}\frac{2m-1}{(m-1-j)!(m+j)!}t^j \\
&= \frac{2m-1}{m-1}\sum_{j=0}^{m-1}\frac{(m-1)!(m-1)!}{(m-1-j)!(m+j)!}t^j = \frac{2m-1}{m-1}Q_{m,m-1}(t).
\end{aligned}$$

Consequently, $\Phi_{m,m-1}(t) = \frac{P_{m,m-1}(t)}{Q_{m,m-1}(t)} = \frac{2m-1}{m-1}$ is a constant function, which is trivially non-decreasing on $[-1, 0)$.

Now assume that $1 \le k < m-1$. Let $A = \{A(j)\}_{j=0}^{m-1}$ be the vector defined by

$$A(j) = \frac{\alpha_j}{\beta_j} = \begin{cases} \frac{(k-1)!}{(m-1)!}\left(\frac{(m+j)!}{(k+j)!}+\frac{(m-1-j)!}{(k-1-j)!}\right), & \text{if} \quad 0 \le j \le k-1, \\ \\ \frac{(k-1)!}{(m-1)!}\frac{(m+j)!}{(k+j)!}, & \text{if} \quad k \le j \le m-1. \end{cases}$$

We are going to show that A is an increasing vector. Let $0 \le j \le k-2$.

$$\begin{aligned}
A(j+1) - A(j) &= (m-k)\frac{(k-1)!}{(m-1)!}\left(\frac{(m+j)!}{(k+j+1)!}-\frac{(m-2-j)!}{(k-1-j)!}\right) \\
&= (m-k)\frac{(k-1)!}{(m-1)!}\left(T_1 - T_2\right),
\end{aligned}$$

where

$$\begin{aligned}
T_1 &= \frac{(m+j)!}{(k+j+1)!} = (m+j)(m+j-1)\cdots(k+j+2) \\
&= \prod_{i=1}^{m-k-1}(k+j+1+i) = \prod_{i=1}^{m-k-1} T_1(i)
\end{aligned}$$

and

$$\begin{aligned}
T_2 &= \frac{(m-2-j)!}{(k-1-j)!} = (m-2-j)(m-3-j)\cdots(k-j) \\
&= \prod_{i=1}^{m-k-1}(k-j-1+i) = \prod_{i=1}^{m-k-1} T_2(i).
\end{aligned}$$

Since $T_1(i) \ge T_2(i)$ for all $1 \le i \le m-k-1$, we get $T_1 - T_2 \ge 0$. Therefore

$$A(j+1) - A(j) = (m-k)\frac{(k-1)!}{(m-1)!}(T_1 - T_2) \ge 0, \quad \text{for all } 0 \le j \le k-2.$$

For $j = k-1$, similar reasoning as above yields:

$$A(k) - A(k-1) = \frac{(k-1)!(m-k)}{(m-1)!}\left(\frac{(m+k-1)!}{(2k)!} - (m-k-1)!\right) \geq 0.$$

Finally, for $k \leq j \leq m-2$, we have that

$$A(j+1) - A(j) = \frac{(k-1)!(m-k)}{(m-1)!}\frac{(m+j)!}{(k+j+1)!} \geq 0.$$

Therefore

$$A(j+1) \geq A(j), \quad \text{for all } 0 \leq j \leq m-2,$$

which means that A is an increasing vector. Direct application of [12, Theorem 4.4] implies that $\Phi_{m,k}$ is increasing on $[-1,0)$, which in turn yields that $\Theta_{m,k}$ is decreasing on $(0,1]$. This completes the proof. □

The following lemma will be needed.

Lemma 3.5. *Let $m \in \mathbb{N}$ and let $(u,v) \in \mathbb{R}^2$ be a non-negative vector. Then*

$$\frac{1}{2}\left(\frac{1}{u^m}\frac{1}{v^k}\int_0^{\min\{u,v\}}(u-x)^{m-1}(v-x)^{k-1}\,dx + \frac{1}{v^m}\frac{1}{u^k}\int_0^{\min\{u,v\}}(v-x)^{m-1}(u-x)^{k-1}\,dx\right)$$
$$\geq \frac{2m-1}{m+k-1}\frac{1}{u^m}\frac{1}{v^m}\int_0^{\min\{u,v\}}(u-x)^{m-1}(v-x)^{m-1}\,dx, \tag{30}$$

for all $1 \leq k \leq m$.

Proof. Let $m \in \mathbb{N}$ and $1 \leq k \leq m$. For all $1 \leq j \leq m$, define

$$K_{m,j}(u,v) = \frac{1}{2}\left(\frac{1}{u^m v^j}\int_0^{\min\{u,v\}}(u-x)^{m-1}(v-x)^{j-1}dx + \frac{1}{v^m u^j}\int_0^{\min\{u,v\}}(v-x)^{m-1}(u-x)^{j-1}dx\right).$$

Then, (30) is equivalent to

$$K_{m,k}(u,v) \geq \frac{2m-1}{m+k-1}K_{m,m}(u,v), \tag{31}$$

for all $u,v > 0$ and all $1 \leq k \leq m$. Since $K_{m,k}$ is symmetric in u and v, without loss of generality we may assume that $v \leq u$. Let $t = \frac{v}{u} \in (0,1]$. Then (31) is equivalent to

$$\frac{1}{2}\left(\frac{1}{u^{m+k}}\frac{1}{t^k}\int_0^{tu}(u-x)^{m-1}(tu-x)^{k-1}\,dx + \frac{1}{t^m}\frac{1}{u^{m+k}}\int_0^{tu}(tu-x)^{m-1}(u-x)^{k-1}\,dx\right)$$
$$\geq \frac{2m-1}{m+k-1}\frac{1}{u^{2m}}\frac{1}{t^m}\int_0^{tu}(u-x)^{m-1}(tu-x)^{m-1}\,dx,$$

for all $0 < u < \infty$ and $t \in (0,1]$. Now, by doing the change of variable $x = us$, we get that (31) is equivalent to prove that

$$\frac{1}{2}\frac{t^{m-k}\int_0^t(1-s)^{m-1}(t-s)^{k-1}\,ds + \int_0^t(t-s)^{m-1}(1-s)^{k-1}\,ds}{\int_0^t(1-s)^{m-1}(t-s)^{m-1}\,ds} \geq \frac{2m-1}{m+k-1}, \tag{32}$$

for all $t \in (0,1]$. Let $\Theta_{m,k} : (0,1] \to \mathbb{R}$ be the function given by

$$\Theta_{m,k}(t) = \frac{1}{2}\frac{t^{m-k}\int_0^t(1-s)^{m-1}(t-s)^{k-1}\,ds + \int_0^t(t-s)^{m-1}(1-s)^{k-1}\,ds}{\int_0^t(1-s)^{m-1}(t-s)^{m-1}\,ds}.$$

Note that

$$\Theta_{m,k}(1) = \frac{\int_0^1 (1-s)^{m+k-2}\,ds}{\int_0^1 (1-s)^{2m-2}\,ds} = \frac{2m-1}{m+k-1}.$$

Thus, to prove (32), it suffices to show that $\Theta_{m,k}$ is non-increasing on $(0,1]$, which is precisely the content of Lemma 3.4. □

4. Proofs of main results

Let $m, n \in \mathbb{N}$, and let T_n and T_m be two positive, bounded linear operators on $L^p(\mathbb{R}^+)$. Before presenting the main proofs, we note that any inequality of the form

$$\|T_n f\|_p \le C(n,m,p)\|T_m f\|_p \tag{33}$$

needs only to be established for functions $f \in C_c^+(\mathbb{R}^+)$, the space of continuous, non-negative functions with compact support in $\mathbb{R}^+$. Indeed, for an arbitrary non-negative measurable function f, there exists a sequence $\{f_i\}_{i\in\mathbb{N}} \subset C_c^+(\mathbb{R}^+)$ such that

$$0 \le f_1(x) \le f_2(x) \le \cdots \le f_i(x) \uparrow f(x) \quad \text{pointwise almost everywhere as } i \to \infty.$$

Since T_n and T_m are positive operators, positivity implies monotonicity:

$$0 \le T_n f_1(x) \le T_n f_2(x) \le \cdots \le T_n f_i(x) \uparrow T_n f(x) \quad \text{pointwise almost everywhere as } i \to \infty.$$

By the Monotone Convergence Theorem, for almost every $x > 0$:

$$\lim_{i\to\infty} T_n f_i(x) = T_n f(x) \quad \text{and} \quad \lim_{i\to\infty} T_m f_i(x) = T_m f(x).$$

Applying the Monotone Convergence Theorem once more to the increasing sequences $|T_n f_i(x)|^p \uparrow |T_n f(x)|^p$ and $|T_m f_i(x)|^p \uparrow |T_m f(x)|^p$, we obtain:

$$\lim_{i\to\infty} \|T_n f_i\|_p = \|T_n f\|_p \quad \text{and} \quad \lim_{i\to\infty} \|T_m f_i\|_p = \|T_m f\|_p.$$

Assuming that (33) holds for each $f_i \in C_c^+(\mathbb{R}^+)$, taking $i \to \infty$ yields:

$$\|T_n f\|_p = \lim_{i\to\infty} \|T_n f_i\|_p \le C(n,m,p) \lim_{i\to\infty} \|T_m f_i\|_p = C(n,m,p)\|T_m f\|_p.$$

Therefore, the inequality

$$\|\mathcal{C}_n f\|_p \le C(n,m,p)\|\mathcal{C}_m f\|_p$$

holds for every non-negative measurable function f. If $\|\mathcal{C}_m f\|_p = \infty$, the right-hand side is infinite, making the inequality trivially true. If $\|\mathcal{C}_m f\|_p < \infty$, the inequality guarantees that $\|\mathcal{C}_n f\|_p$ is also finite and bounded appropriately.

Proof of Theorem 2.1. We begin with inequality (14) in the case $n = m \in \mathbb{N}$ and $k = 1$. Let $f \in C_c^+(\mathbb{R}^+)$. We have seen in (21) that

$$\mathcal{C}_{m+1} f(x) = \frac{m+1}{x^{m+1}} \int_0^x t^m \mathcal{C}_m f(t)\,dt,$$

for all $x > 0$. By applying the change of variable $u = t/x$ in the identity above gives

$$\mathcal{C}_{m+1} f(x) = (m+1) \int_0^1 u^m \mathcal{C}_m f(ux)\,du, \tag{34}$$

for all $x > 0$. We start with the case $p = \infty$. Taking (34) into account, we get the following.

$$\mathcal{C}_{m+1} f(x) = (m+1) \int_0^1 u^m \mathcal{C}_m f(ux)\,du \le \|\mathcal{C}_m f\|_\infty (m+1) \int_0^1 u^m\,du = \|\mathcal{C}_m f\|_\infty,$$

for all $x > 0$. This implies

$$\|\mathcal{C}_{m+1} f\|_\infty \le \|\mathcal{C}_m f\|_\infty,$$

for all $f \in C_c^+(\mathbb{R}^+)$.

Now, we turn to the case when $1 < p < \infty$. Taking the L^p-norm on both sides of the identity (34) and applying Minkowski's integral inequality, we find that

$$\begin{aligned}\|\mathcal{C}_{m+1}f\|_p &= (m+1)\left(\int_0^\infty \left(\int_0^1 u^m \mathcal{C}_m f(ux)\,du\right)^p dx\right)^{1/p}\\ &\le (m+1)\int_0^1 \left(\int_0^\infty \left(u^m \mathcal{C}_m f(ux)\right)^p dx\right)^{1/p} du\\ &= (m+1)\int_0^1 u^m \left(\int_0^\infty (\mathcal{C}_m f(ux))^p\,dx\right)^{1/p} du.\end{aligned}$$

Making the substitution $y = ux$ inside the inner integral (yielding $dx = u^{-1}dy$) decouples the variables, leading to

$$\begin{aligned}\|\mathcal{C}_{m+1}f\|_p &\le (m+1)\int_0^1 u^{m-1/p}\,du \left(\int_0^\infty (\mathcal{C}_m f(y))^p\,dy\right)^{1/p}\\ &= \frac{m+1}{m+1-1/p}\|\mathcal{C}_m f\|_p = \frac{m+1}{m+1/p'}\|\mathcal{C}_m f\|_p.\end{aligned}$$

Thus, we establish the norm recurrence relation

$$\|\mathcal{C}_{m+1}f\|_p \le \frac{\Gamma(m+2)\Gamma\left(m+\frac{1}{p'}\right)}{\Gamma(m+1)\Gamma\left(m+1+\frac{1}{p'}\right)}\|\mathcal{C}_m f\|_p = \frac{\frac{\Gamma\left(m+\frac{1}{p'}\right)}{\Gamma(m+1)}}{\frac{\Gamma\left(m+1+\frac{1}{p'}\right)}{\Gamma(m+2)}}\|\mathcal{C}_m f\|_p \tag{35}$$

for all $1 < p \le \infty$ and for all $f \in C_c^+(\mathbb{R}^+)$. Combining (35) and Lemma 3.1, we obtain

$$\|\mathcal{C}_{n+k}f\|_p \le \frac{\Gamma(n+k+1)\Gamma\left(n+\frac{1}{p'}\right)}{\Gamma(n+1)\Gamma\left(n+k+\frac{1}{p'}\right)}\|\mathcal{C}_n f\|_p = \frac{B\left(n+\frac{1}{p'},\,k\right)}{B(n+1,\,k)}\|\mathcal{C}_n f\|_p,$$

for all $f \in C_c^+(\mathbb{R}^+)$.

To establish the sharpness of the constant in (14), we proceed by contradiction. Suppose that there exists a constant $0 < C < B\left(n+\frac{1}{p'},k\right)/B(n+1,\,k)$ such that

$$\|\mathcal{C}_{n+k}f\|_p \le C\|\mathcal{C}_n f\|_p,$$

for all $f \in \mathcal{M}^+(\mathbb{R}^+)$. Iterating inequality (35) yields

$$\begin{aligned}\|\mathcal{C}_{n+k}|g|\|_p \le C\|\mathcal{C}_n|g|\|_p &\le C\frac{\Gamma(n+1)\Gamma\left(n-1+\frac{1}{p'}\right)}{\Gamma(n)\Gamma\left(n+\frac{1}{p'}\right)}\|\mathcal{C}_{n-1}|g|\|_p\\ &\le \cdots \le C\cdot\Gamma\left(\frac{1}{p'}\right)\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{p'}\right)}\|\mathcal{C}_0|g|\|_p\\ &= C\cdot\Gamma\left(\frac{1}{p'}\right)\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{p'}\right)}\|g\|_p,\end{aligned}$$

for all $g \in L^p(\mathbb{R}^+)$ with $1 < p \leq \infty$. Consequently,

$$\|\mathcal{C}_{n+k}g\|_p \leq \|\mathcal{C}_{n+k}|g|\|_p \leq C \cdot \Gamma\left(\frac{1}{p'}\right)\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{p'}\right)}\|g\|_p,$$

for all $g \in L^p(\mathbb{R}^+)$ with $1 < p \leq \infty$. This implies that the operator norm of $\mathcal{C}_{n+k}$ satisfies

$$\begin{aligned}\|\mathcal{C}_{n+k}\|_{L^p(\mathbb{R}^+)} &\leq C \cdot \Gamma\left(\frac{1}{p'}\right)\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{p'}\right)} < \frac{\Gamma(n+k+1)\Gamma\left(n+\frac{1}{p'}\right)}{\Gamma(n+1)\Gamma\left(n+k+\frac{1}{p'}\right)} \cdot \Gamma\left(\frac{1}{p'}\right)\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{p'}\right)} \\ &= \frac{\Gamma(n+k+1)\Gamma(1/p')}{\Gamma(n+k+1/p')},\end{aligned}$$

which contradicts the sharpness of inequality (1).

We now turn to inequality (15). We begin with the case $p = 1$. Let $f \in \mathcal{M}^+(\mathbb{R}^+)$. Applying identity (22) and Tonelli's theorem, we obtain

$$\begin{aligned}\|\mathcal{C}^*_{m+1}f\|_1 &= \int_0^\infty \mathcal{C}^*_{m+1}f(x)\,dx = (m+1)\int_0^\infty x^m \int_x^\infty \frac{\mathcal{C}^*_m f(t)}{t^{m+1}}\,dt\,dx \\ &= (m+1)\int_0^\infty \frac{\mathcal{C}^*_m f(t)}{t^{m+1}}\left(\int_0^t x^m\,dx\right)dt = \int_0^\infty \mathcal{C}^*_m f(t)\,dt = \|\mathcal{C}^*_m f\|_1\end{aligned}$$

for all $m \in \mathbb{N}$. Consequently, this yields

$$\|\mathcal{C}^*_{n+k}f\|_1 = \|\mathcal{C}^*_n f\|_1 \tag{36}$$

for all $n, k \in \mathbb{N}$. The proof of inequality (15) for $1 < p < \infty$, along with its sharpness, proceeds in a manner completely analogous to that of (14). One simply replaces the use of (21) with (22), and substitutes (2) for (1) when establishing optimality. This completes the proof. □

Proof of Theorem 2.2. We begin with inequality (16) in the case $n = m \in \mathbb{N}$ and $k = 1$. The operators $\mathcal{C}_m$ and $\mathcal{C}_{m+1}$ can be expressed as integral operators by

$$\mathcal{C}_m f(x) = \int_0^x f(t)K_m(x,t)\,dt \quad \text{for } x > 0,$$

and

$$\mathcal{C}_{m+1}f(u) = \int_0^u f(t)K_{m+1}(u,t)\,dt \quad \text{for } u > 0,$$

where the respective kernels are given by

$$K_m(x,t) = \frac{m(x-t)^{m-1}}{x^m} \quad \text{for } 0 < t \leq x,$$

and

$$K_{m+1}(u,t) = \frac{(m+1)(u-t)^m}{u^{m+1}} \quad \text{for } 0 < t \leq u.$$

For a fixed $x > 0$, we set $u_x = \frac{m+1}{m}x$. Since $u_x > x$, the inclusion $(0,x) \subset (0,u_x)$ follows immediately. Let R be the function defined on $(0,x)$ by

$$R(t) = \frac{K_m(x,t)}{K_{m+1}(u_x,t)} = \frac{m}{m+1}\frac{u_x^{m+1}}{x^m}\frac{(x-t)^{m-1}}{(u_x-t)^m}.$$

Substituting $u_x = \frac{m+1}{m}x$ we get

$$R(t) = \left(\frac{m+1}{m}\right)^m x\frac{(x-t)^{m-1}}{\left(\frac{m+1}{m}x-t\right)^m} = \left(\frac{m}{m+1}\right)^m xg(t),$$

where $g(t) = (x-t)^{m-1}\left(\frac{m+1}{m}x - t\right)^{-m}$. Now, taking derivative, and simple computations lead to

$$\left(\frac{m+1}{m}x - t\right)^{2m} \frac{d}{dt} g(t) = (x-t)^{m-2}\left(\frac{m+1}{m}x - t\right)^{m-1}\left(\frac{x}{m} - t\right).$$

Since $g'(t) > 0$ for $0 < t < \frac{x}{m}$ and $g'(t) < 0$ for $\frac{x}{m} < t < x$, we have that g attains its unique maximum on $(0,x)$ at $t_0 = \frac{x}{m}$. Evaluating $g(t_0)$ gives

$$g(t_0) = \frac{\left(\frac{m-1}{m}\right)^{m-1} x^{m-1}}{x^m} = \left(\frac{m-1}{m}\right)^{m-1} \frac{1}{x}.$$

Now, substituting $g(t_0)$ back into $R(t)$:

$$\max_{t\in(0,x)} R(t) = R(t_0) = \left(\frac{m+1}{m}\right)^m x \left(\frac{m-1}{m}\right)^{m-1} \frac{1}{x} = \frac{(m+1)^m (m-1)^{m-1}}{m^{2m-1}} := L_m.$$

This establishes that

$$K_m(x,t) \le L_m K_{m+1}\left(\frac{m+1}{m}x, t\right)$$

for all $t \in (0,x)$. Thus

$$\begin{aligned}
\mathcal{C}_m f(x) &= \int_0^x f(t) K_m(x,t)\, dt \le L_m \int_0^x f(t) K_{m+1}\left(\frac{m+1}{m}x, t\right) dt \\
&\le L_m \int_0^{\frac{m+1}{m}x} f(t) K_{m+1}\left(\frac{m+1}{m}x, t\right) dt = L_m \mathcal{C}_{m+1} f\left(\frac{m+1}{m}x\right) \\
&\le L_m \|\mathcal{C}_{m+1} f\|_\infty,
\end{aligned}$$

for all $x > 0$. Therefore

$$\|\mathcal{C}_m f\|_\infty \le \frac{(m-1)^{m-1}}{m^{m-1}} \frac{(m+1)^m}{m^m} \|\mathcal{C}_{m+1} f\|_\infty. \tag{37}$$

This establishes (16) for the case $n = m$ and $k = 1$. Combining (37) and Lemma 3.1, we obtain

$$\|\mathcal{C}_n f\|_\infty \le \frac{(n-1)^{n-1}}{n^{n-1}} \frac{(n+k)^{n+k-1}}{(n+k-1)^{n+k-1}} \|\mathcal{C}_{n+k} f\|_\infty.$$

This completes the proof of (16). The optimality of the constant in (16) follows from (25).

We now consider the case $p = 2$. Since $\|\mathcal{C}_m f\|_2 = \|\mathcal{C}_m^* f\|_2$ for every $m \in \mathbb{N}$, applying inequality (18) (the reader may check that its proof is independent) yields

$$\begin{aligned}
\|\mathcal{C}_n f\|_2^2 = \|\mathcal{C}_n^* f\|_2^2 &\le \frac{n^2}{2(n-1)+1} \frac{2(n+k-1)+1}{(n+k)^2} \|\mathcal{C}_{n+k}^* f\|_2^2 \\
&= \frac{n^2}{2(n-1)+1} \frac{2(n+k-1)+1}{(n+k)^2} \|\mathcal{C}_{n+k} f\|_2^2 \\
&= \frac{n^2}{(n+k)^2} \frac{B\big(2(n-1)+1, 1\big)}{B\big(2(n+k-1)+1, 1\big)} \|\mathcal{C}_{n+k} f\|_2^2.
\end{aligned}$$

This completes the proof of inequality (17). The optimality of the constant in (17) is an immediate consequence of (23). □

Proof of Theorem 2.3. The case $p = 1$ is a direct consequence of the identity (36). Let us now turn to the case $p = 2$. We start proving inequality (18) when $k = 1$. We have to show that

$$\|\mathcal{C}_m^* f\|_2^2 \leq \frac{m^2}{2(m-1)+1} \frac{2m+1}{(m+1)^2} \|\mathcal{C}_{m+1}^* f\|_2^2, \tag{38}$$

for all $f \in C_c^+(\mathbb{R}^+)$ and for all $m \in \mathbb{N}$. Let $m \in \mathbb{N}$ and let $f \in C_c^+(\mathbb{R}^+)$. Then, applying integration by parts and differentiation under the integral sign yields

$$\begin{aligned}
(m+1)^{-2}\|\mathcal{C}_{m+1}^* f\|_2^2 &= \int_0^\infty \left(\int_x^\infty \frac{f(t)}{t^{m+1}} (t-x)^m \, dt \right)^2 dx \\
&= \int_0^\infty x^{2m} \left(\int_x^\infty \frac{f(t)}{t^{m+1}} \left(\frac{t}{x} - 1 \right)^m dt \right)^2 dx \\
&= \frac{2m}{2m+1} \int_0^\infty x^{2m+1} \left(\int_x^\infty \frac{f(t)}{t^{m+1}} \left(\frac{t}{x} - 1 \right)^m dt \right) \left(\int_x^\infty \frac{f(u)}{u^{m+1}} \left(\frac{u}{x} - 1 \right)^{m-1} \frac{u}{x^2} du \right) dx \\
&= \frac{2m}{2m+1} \int_0^\infty \left(\int_x^\infty \frac{f(t)}{t^{m+1}} (t-x)^m dt \right) \left(\int_x^\infty \frac{f(u)}{u^m} (u-x)^{m-1} \, du \right) dx.
\end{aligned}$$

Using (22), we obtain

$$\begin{aligned}
\int_x^\infty \frac{f(t)}{t^{m+1}} (t-x)^m \, dt = \frac{\mathcal{C}_{m+1}^* f(x)}{m+1} &= x^m \int_x^\infty \frac{\mathcal{C}_m^* f(t)}{t^{m+1}} \, dt \\
&= m x^m \int_x^\infty \frac{1}{t^{m+1}} \left(\int_t^\infty \frac{f(s)}{s^m} (s-t)^{m-1} \, ds \right) dt,
\end{aligned}$$

for all $x > 0$. Hence,

$$\begin{aligned}
&(m+1)^{-2}\|\mathcal{C}_{m+1}^* f\|_2^2 \\
&\quad = \frac{2m^2}{2m+1} \int_0^\infty x^m \int_x^\infty \frac{1}{t^{m+1}} \int_t^\infty \frac{f(s)}{s^m} (s-t)^{m-1} ds \, dt \left(\int_x^\infty \frac{f(u)}{u^m} (u-x)^{m-1} du \right) dx.
\end{aligned}$$

It follows from Tonelli's theorem that

$$\begin{aligned}
&(m+1)^{-2}\|\mathcal{C}_{m+1}^* f\|_2^2 \\
&\quad = \frac{2m^2}{2m+1} \int_0^\infty \left(\int_t^\infty \frac{f(s)}{s^m} (s-t)^{m-1} \, ds \right) \frac{1}{t^{m+1}} \int_0^t x^m \int_x^\infty \frac{f(u)}{u^m} (u-x)^{m-1} \, du \, dx \, dt \\
&\quad \geq \frac{2m^2}{2m+1} \int_0^\infty \left(\int_t^\infty \frac{f(s)}{s^m} (s-t)^{m-1} \, ds \right) \frac{1}{t^{m+1}} \int_0^t x^m \int_t^\infty \frac{f(u)}{u^m} (u-x)^{m-1} \, du \, dx \, dt.
\end{aligned}$$

Now, for any $t > 0$, using the change of variables $v = x/t$ (so that $dv = dx/t$), we have

$$\begin{aligned}
\frac{1}{t^{m+1}} \int_0^t x^m \int_t^\infty \frac{f(u)}{u^m} (u-x)^{m-1} \, du \, dx &= \frac{1}{t^{m+1}} \int_t^\infty \frac{f(u)}{u^m} \int_0^t x^m (u-x)^{m-1} \, dx \, du \\
&= \int_t^\infty \frac{f(u)}{u^m} \int_0^1 v^m (u - tv)^{m-1} \, dv \, du \\
&= \int_t^\infty \frac{f(u)}{u} \int_0^1 v^m \left(1 - \frac{t}{u} v \right)^{m-1} dv \, du.
\end{aligned}$$

Evaluating the elementary integrals in terms of the Beta function yields

$$\begin{aligned}\int_0^1 v^m\left(1-\frac{t}{u}v\right)^{m-1}dv &= \int_0^1 v^m\left(1-v+\left(1-\frac{t}{u}\right)v\right)^{m-1}dv\\ &= \sum_{j=0}^{m-1}\binom{m-1}{j}\left(1-\frac{t}{u}\right)^j\int_0^1 v^{m+j}(1-v)^{m-1-j}dv\\ &= \sum_{j=0}^{m-1}\frac{(m-1)!(m+j)!}{j!(2m)!}\left(1-\frac{t}{u}\right)^j.\end{aligned}$$

Thus,

$$\frac{1}{t^{m+1}}\int_0^t x^m\int_t^\infty \frac{f(u)}{u^m}(u-x)^{m-1}\,du\,dx = \sum_{j=0}^{m-1}\frac{(m-1)!(m+j)!}{j!(2m)!}\int_t^\infty \frac{f(u)}{u^{j+1}}(u-t)^j\,du,$$

for all $t>0$. Therefore,

$$\begin{aligned}&(m+1)^{-2}\|\mathcal{C}^*_{m+1}f\|_2^2\\ &\quad\geq \frac{2m^2}{2m+1}\int_0^\infty\left(\int_t^\infty\frac{f(s)}{s^m}(s-t)^{m-1}\,ds\right)\frac{1}{t^{m+1}}\int_0^t x^m\int_t^\infty\frac{f(u)}{u^m}(u-x)^{m-1}\,du\,dx\,dt\\ &\quad= \frac{2m^2}{2m+1}\sum_{j=0}^{m-1}\frac{(m-1)!(m+j)!}{j!(2m)!}\int_0^\infty\int_t^\infty\frac{f(s)}{s^m}(s-t)^{m-1}\,ds\int_t^\infty\frac{f(u)}{u^{j+1}}(u-t)^j\,du\,dt\\ &\quad= \frac{1}{2m+1}\sum_{j=0}^{m-1}\frac{m!(m+j)!}{j!(2m-1)!}\int_0^\infty\int_t^\infty\frac{f(s)}{s^m}(s-t)^{m-1}\,ds\int_t^\infty\frac{f(u)}{u^{j+1}}(u-t)^j\,du\,dt.\\ &\quad= \frac{1}{2m+1}\sum_{j=0}^{m-1}\frac{m!(m+j)!}{j!(2m-1)!}\int_0^\infty\int_0^\infty\frac{f(s)}{s^m}\frac{f(u)}{u^{j+1}}\int_0^{\min\{u,s\}}(s-t)^{m-1}(u-t)^j dtduds\\ &\quad= \frac{1}{2(2m+1)}\sum_{j=0}^{m-1}\frac{m!(m+j)!}{j!(2m-1)!}\left(\int_0^\infty\int_0^\infty\frac{f(s)}{s^m}\frac{f(u)}{u^{j+1}}\int_0^{\min\{u,s\}}(s-t)^{m-1}(u-t)^j dtduds\right.\\ &\qquad\left.+\int_0^\infty\int_0^\infty\frac{f(u)}{u^m}\frac{f(s)}{s^{j+1}}\int_0^{\min\{u,s\}}(u-t)^{m-1}(s-t)^j dtduds\right)\\ &\quad\geq \frac{2m-1}{2m+1}\sum_{j=0}^{m-1}\frac{m!(m+j)!}{j!(2m-1)!}\frac{1}{m+j}\int_0^\infty\int_0^\infty\frac{f(u)}{u^m}\frac{f(s)}{s^m}\int_0^{\min\{u,s\}}(u-x)^{m-1}(s-x)^{m-1}dtduds\end{aligned}$$

where we have applied Lemma 3.5 in the last inequality. Note that

$$\begin{aligned}\sum_{j=0}^{m-1}\frac{m!(m+j)!}{j!(2m-1)!}\frac{1}{m+j} &= \frac{m!(m-1)!}{(2m-1)!}\sum_{j=0}^{m-1}\frac{(m+j-1)!}{j!(m-1)!} = \frac{1}{\binom{2m-1}{m}}\sum_{j=0}^{m-1}\binom{m+j-1}{j}\\ &= \frac{1}{\binom{2m-1}{m}}\binom{2m-1}{m-1} = \frac{1}{\binom{2m-1}{m}}\binom{2m-1}{m} = 1.\end{aligned}$$

Therefore,

$$(m+1)^{-2}\|\mathcal{C}^*_{m+1}f\|_2^2 \geq \frac{2m-1}{2m+1}\int_0^\infty\int_0^\infty \frac{f(u)}{u^m}\frac{f(s)}{s^m}\int_0^{\min\{u,s\}}(u-x)^{m-1}(s-x)^{m-1}dtduds$$
$$= \frac{2m-1}{2m+1}m^{-2}\|\mathcal{C}^*_m f\|_2^2.$$

That is

$$\|\mathcal{C}^*_m f\|_2^2 \leq \left(\frac{m}{m+1}\right)^2 \frac{2m+1}{2(m-1)+1}\|\mathcal{C}^*_{m+1}f\|_2^2. \tag{39}$$

Combining (39) and Lemma 3.1, we obtain

$$\|\mathcal{C}^*_n f\|_2^2 \leq \frac{n^2}{2(n-1)+1}\frac{2(n+k-1)+1}{(n+k)^2}\|\mathcal{C}^*_{n+k}f\|_2^2.$$

This proves inequality (18) when $p=2$. The proof that the constant in (18) is sharp follows directly from (24). □

Proof of Theorem 2.4. Let $1<p<\infty$ and let $f\in C_c^+(\mathbb{R}^+)$. We start by assuming that $1<p<2$. By using integration by parts we obtain that

$$2^{-p}\|\mathcal{C}_2 f\|_p^p = \int_0^\infty x^{-2p}\left(\int_0^x f(t)(x-t)\,dt\right)^p dx$$
$$= \frac{p}{2p-1}\int_0^\infty \frac{1}{x^{2p-1}}\left(\int_0^x f(t)(x-t)\,dt\right)^{p-1}\left(\int_0^x f(u)\,du\right)dx.$$

By applying Jensen's inequality, we get that

$$\left(\int_0^x f(t)(x-t)\,dt\right)^{p-1} \geq \left(\int_0^x f(u)\,du\right)^{p-2}\left(\int_0^x f(t)(x-t)^{p-1}\,dt\right),$$

for all $x>0$. Now, by using integration by parts we obtain that

$$\int_0^x f(t)(x-t)^{p-1}\,dt = (p-1)\int_0^x\left(\int_0^t f(w)\,dw\right)(x-t)^{p-2}\,dt,$$

for all $x>0$. So

$$\left(\int_0^x f(t)(x-t)\,dt\right)^{p-1} \geq (p-1)\left(\int_0^x f(u)\,du\right)^{p-2}\int_0^x\left(\int_0^t f(w)\,dw\right)(x-t)^{p-2}\,dt,$$

for all $x > 0$. Therefore

$$\begin{aligned}
2^{-p}\|\mathcal{C}_2 f\|_p^p &= \frac{p}{2p-1}\int_0^\infty \frac{1}{x^{2p-1}}\bigg(\int_0^x f(t)(x-t)\,dt\bigg)^{p-1}\bigg(\int_0^x f(u)\,du\bigg)\,dx \\
&\geq \frac{p(p-1)}{2p-1}\int_0^\infty \frac{1}{x^{2p-1}}\bigg(\int_0^x f(u)\,du\bigg)^{p-1}\int_0^x\bigg(\int_0^t f(w)\,dw\bigg)(x-t)^{p-2}\,dt\,dx \\
&= \frac{p(p-1)}{2p-1}\int_0^\infty \bigg(\int_0^t f(w)\,dw\bigg)\int_t^\infty\bigg(\int_0^x f(u)\,du\bigg)^{p-1}\frac{(x-t)^{p-2}}{x^{2p-1}}\,dx\,dt \\
&\geq \frac{p(p-1)}{2p-1}\int_0^\infty \bigg(\int_0^t f(w)\,dw\bigg)^{p}\int_t^\infty\frac{(x-t)^{p-2}}{x^{2p-1}}\,dx\,dt \\
&= \frac{p(p-1)}{2p-1}\int_0^\infty \bigg(\frac{1}{t}\int_0^t f(w)\,dw\bigg)^{p}\int_t^\infty\frac{(x/t-1)^{p-2}}{(x/t)^{2p-1}}\,\frac{dx}{t}\,dt \\
&= \frac{p(p-1)}{2p-1}\int_0^\infty \bigg(\frac{1}{t}\int_0^t f(w)\,dw\bigg)^{p}\int_1^\infty\frac{(u-1)^{p-2}}{u^{2p-1}}\,du\,dt \\
&= \frac{p(p-1)}{2p-1}B(p,p-1)\int_0^\infty \bigg(\frac{1}{t}\int_0^t f(w)\,dw\bigg)^{p}dt \\
&= \frac{p(p-1)}{2p-1}\frac{\Gamma(p)\Gamma(p-1)}{\Gamma(2p-1)}\|\mathcal{C}_1 f\|_p^p \\
&= \frac{B(p+1,p-1)}{B(1,p-1)}\|\mathcal{C}_1 f\|_p^p,
\end{aligned}$$

where we have applied of Tonelli's theorem again in the first equality and a change of variable $u = x/t$ in the third equality. Hence

$$\|\mathcal{C}_1 f\|_p^p \leq \frac{1}{2^p}\frac{B(1,p-1)}{B(p+1,p-1)}\|\mathcal{C}_2 f\|_p^p,$$

for all $1 < p < 2$ and for all $f \in C_c^+(\mathbb{R}^+)$.

Now we turn to the case $2 \leq p < \infty$. Let $x > 0$ and let $\varphi_x : (0,x] \to \mathbb{R}^+$ be the function defined by

$$\varphi_x(v) = \bigg(\int_0^v f(t)(x-t)\,dt\bigg)^{p-1}.$$

By applying the Fundamental Theorem of Calculus we obtain that

$$\begin{aligned}
\bigg(\int_0^x f(t)(x-t)\,dt\bigg)^{p-1} &= \varphi_x(x) = \varphi_x(x) - \varphi_x(0) = \int_0^x \varphi_x'(v)\,dv \\
&= (p-1)\int_0^x\bigg(\int_0^v f(t)(x-t)\,dt\bigg)^{p-2} f(v)(x-v)\,dv \\
&\geq (p-1)\int_0^x\bigg(\int_0^v f(t)\,dt\bigg)^{p-2} f(v)(x-v)^{p-1}\,dv,
\end{aligned}$$

for all $x > 0$. Therefore

$$\begin{aligned}
2^{-p}\|\mathcal{C}_2 f\|_p^p &= \frac{p}{2p-1}\int_0^\infty \frac{1}{x^{2p-1}}\bigg(\int_0^x f(u)\,du\bigg)\bigg(\int_0^x f(t)(x-t)\,dt\bigg)^{p-1}dx \\
&\geq \frac{p(p-1)}{2p-1}\int_0^\infty \frac{1}{x^{2p-1}}\bigg(\int_0^x f(u)\,du\bigg)\int_0^x\bigg(\int_0^v f(t)\,dt\bigg)^{p-2} f(v)(x-v)^{p-1}\,dv\,dx \\
&= \frac{p(p-1)}{2p-1}\int_0^\infty f(v)\bigg(\int_0^v f(t)\,dt\bigg)^{p-2}\int_v^\infty \frac{(x-v)^{p-1}}{x^{2p-1}}\bigg(\int_0^x f(u)\,du\bigg)\,dx\,dv \\
&= \frac{p(p-1)}{2p-1}\int_0^\infty \bigg(\int_0^v f(t)\,dt\bigg)^{p-1}\int_v^\infty \frac{(x-v)^{p-2}}{x^{2p-1}}\bigg(\int_0^x f(u)\,du\bigg)\,dx\,dv \\
&\geq \frac{p(p-1)}{2p-1}\int_0^\infty \bigg(\int_0^v f(t)\,dt\bigg)^{p}\int_v^\infty \frac{(x-v)^{p-2}}{x^{2p-1}}\,dx\,dv \\
&= \frac{B(p+1,p-1)}{B(1,p-1)}\|\mathcal{C}_1 f\|_p^p,
\end{aligned}$$

where we have used integration by parts (taking $ds = f(v)\big(\int_0^v f(t)dt\big)^{p-2}dv$) in the third equality. Thus

$$\|\mathcal{C}_1 f\|_p^p \leq \frac{1}{2^p}\frac{B(1,p-1)}{B(p+1,p-1)}\|\mathcal{C}_2 f\|_p^p,$$

for all $2 \leq p < \infty$ and for all $f \in C_c^+(\mathbb{R}^+)$. This completes the proof of the inequality in (19). The proof of the optimality of the constant in (19) follows from (23). □

Proof of Theorem 2.5. Let $1 < p < \infty$ and let $f \in C_c^+(\mathbb{R}^+)$. We start by assuming that $1 < p < 2$. Integration by parts leads to

$$\begin{aligned}
2^{-p}\|\mathcal{C}_2^* f\|_p^p &= \int_0^\infty\bigg(\int_x^\infty f(t)(t-x)\frac{dt}{t^2}\bigg)^p dx = \int_0^\infty x^p\bigg(\frac{1}{x}\int_x^\infty f(t)(t-x)\frac{dt}{t^2}\bigg)^p dx \\
&= \frac{p}{p+1}\int_0^\infty x^{p+1}\bigg(\frac{1}{x}\int_x^\infty f(t)(t-x)\frac{dt}{t^2}\bigg)^{p-1}\bigg(\frac{1}{x^2}\int_x^\infty \frac{f(v)}{v}\,dv\bigg)\,dx \\
&= \frac{p}{p+1}\int_0^\infty\bigg(\int_x^\infty \frac{f(t)}{t}\bigg(1-\frac{x}{t}\bigg)\,dt\bigg)^{p-1}\bigg(\int_x^\infty \frac{f(v)}{v}\,dv\bigg)\,dx.
\end{aligned}$$

Now, applying Jensen's inequality, we have that

$$\bigg(\int_x^\infty \frac{f(t)}{t}\bigg(1-\frac{x}{t}\bigg)\,dt\bigg)^{p-1} \geq \bigg(\int_x^\infty \frac{f(v)}{v}\,dv\bigg)^{p-2}\int_x^\infty \frac{f(t)}{t}\bigg(1-\frac{x}{t}\bigg)^{p-1}dt,$$

for all $x > 0$. Therefore

$$\begin{aligned}
2^{-p}\|\mathcal{C}_2^* f\|_p^p &= \frac{p}{p+1}\int_0^\infty \left(\int_x^\infty \frac{f(v)}{v}\,dv\right)\left(\int_x^\infty \frac{f(t)}{t}\left(1-\frac{x}{t}\right)dt\right)^{p-1} dx \\
&\geq \frac{p}{p+1}\int_0^\infty \left(\int_x^\infty \frac{f(v)}{v}\,dv\right)^{p-1}\int_x^\infty \frac{f(t)}{t}\left(1-\frac{x}{t}\right)^{p-1} dt\,dx \\
&= \frac{p}{p+1}\int_0^\infty \frac{f(t)}{t}\int_0^t \left(\int_x^\infty \frac{f(v)}{v}\,dv\right)^{p-1}\left(1-\frac{x}{t}\right)^{p-1} dx\,dt \\
&= \frac{p(p-1)}{p+1}\int_0^\infty \left(\int_t^\infty \frac{f(u)}{u}\,du\right)\int_0^t \left(\int_x^\infty \frac{f(v)}{v}\,dv\right)^{p-1}\left(1-\frac{x}{t}\right)^{p-2}\frac{x}{t^2}\,dx\,dt \\
&\geq \frac{p(p-1)}{p+1}\int_0^\infty \left(\int_t^\infty \frac{f(u)}{u}\,du\right)^{p}\int_0^t \left(1-\frac{x}{t}\right)^{p-2}\frac{x}{t}\,\frac{dx}{t}\,dt \\
&= \frac{p(p-1)}{p+1}\int_0^\infty \left(\int_t^\infty \frac{f(u)}{u}\,du\right)^{p}\int_0^1 (1-u)^{p-2}u\,du\,dt \\
&= \frac{p(p-1)}{p+1}B(p-1,2)\int_0^\infty \left(\int_t^\infty \frac{f(u)}{u}\,du\right)^{p} dt \\
&= \frac{p(p-1)}{p+1}\frac{\Gamma(p-1)\Gamma(2)}{\Gamma(p+1)}\int_0^\infty \left(\int_t^\infty \frac{f(u)}{u}\,du\right)^{p} dt = \frac{1}{p+1}\|\mathcal{C}_1^* f\|_p^p,
\end{aligned}$$

where in the second equality we have applied Tonelli's theorem, in the third equality we used integration by parts (with $ds = f(t)/t\,dt$), and in the fourth equality we took into account the change of variables $u = x/t$. Hence

$$\|\mathcal{C}_1^* f\|_p^p \leq \frac{p+1}{2^p}\|\mathcal{C}_2^* f\|_p^p,$$

for all $1 < p < 2$ and for all $f \in \mathcal{M}^+(\mathbb{R}^+)$.

Now we turn to the case $2 \leq p < \infty$. Let $x > 0$ and let $\varphi_x : (0, x] \to \mathbb{R}^+$ be the function defined by

$$\varphi_x(v) = \left(\int_v^\infty f(t)(t-x)\frac{dt}{t^2}\right)^{p-1}.$$

By taking the Fundamental Theorem of Calculus into account, we have

$$\begin{aligned}
\left(\int_x^\infty f(t)(t-x)\frac{dt}{t^2}\right)^{p-1} &= -\big(\varphi_x(\infty) - \varphi_x(x)\big) = -\int_x^\infty \varphi_x'(v)\,dv \\
&= (p-1)\int_x^\infty \left(\int_v^\infty f(t)(t-x)\frac{dt}{t^2}\right)^{p-2} f(v)(v-x)\,\frac{dv}{v^2} \\
&= (p-1)\int_x^\infty \left(\int_v^\infty \frac{f(t)}{t}\left(1-\frac{x}{t}\right)dt\right)^{p-2}\frac{f(v)}{v}\left(1-\frac{x}{v}\right)dv \\
&\geq (p-1)\int_x^\infty \left(\int_v^\infty \frac{f(t)}{t}\,dt\right)^{p-2}\frac{f(v)}{v}\left(1-\frac{x}{v}\right)^{p-1} dv,
\end{aligned}$$

for all $x > 0$. Hence

$$\begin{aligned}
2^{-p}\|\mathcal{C}_2^* f\|_p^p &= \frac{p}{p+1}\int_0^\infty \left(\int_x^\infty \frac{f(u)}{u}\,du\right)\left(\int_x^\infty \frac{f(t)}{t}\left(1-\frac{x}{t}\right)dt\right)^{p-1} dx \\
&\geq \frac{p(p-1)}{p+1}\int_0^\infty \left(\int_x^\infty \frac{f(u)}{u}\,du\right)\int_x^\infty \left(\int_v^\infty \frac{f(t)}{t}\,dt\right)^{p-2}\frac{f(v)}{v}\left(1-\frac{x}{v}\right)^{p-1} dv\,dx \\
&= \frac{p(p-1)}{p+1}\int_0^\infty \left(\int_x^\infty \frac{f(u)}{u}\,du\right)\int_x^\infty \left(\int_v^\infty \frac{f(t)}{t}\,dt\right)^{p-1}\left(1-\frac{x}{v}\right)^{p-2}\frac{x}{v^2}\,dv\,dx \\
&= \frac{p(p-1)}{p+1}\int_0^\infty \left(\int_v^\infty \frac{f(t)}{t}\,dt\right)^{p-1}\int_0^v \left(\int_x^\infty \frac{f(u)}{u}\,du\right)\left(1-\frac{x}{v}\right)^{p-2}\frac{x}{v^2}\,dx\,dv \\
&\geq \frac{p(p-1)}{p+1}\int_0^\infty \left(\int_v^\infty \frac{f(t)}{t}\,dt\right)^{p}\int_0^v \left(1-\frac{x}{v}\right)^{p-2}\frac{x}{v^2}\,dx\,dv \\
&= \frac{p(p-1)}{p+1}\int_0^\infty \left(\int_v^\infty \frac{f(t)}{t}\,dt\right)^{p}\int_0^v \left(1-\frac{x}{v}\right)^{p-2}\frac{x}{v}\,\frac{dx}{v}\,dv \\
&= \frac{p(p-1)}{p+1}\int_0^\infty \left(\int_v^\infty \frac{f(t)}{t}\,dt\right)^{p}\int_0^1 (1-u)^{p-2}u\,du\,dv \\
&= \frac{p(p-1)}{p+1}B(p-1,2)\|\mathcal{C}_1^* f\|_p^p = \frac{1}{p+1}\|\mathcal{C}_1^* f\|_p^p,
\end{aligned}$$

where we have applied integration by parts in the second equality, Tonelli's theorem in the third equality, the change of variable $u = x/t$ in the fourth equality. Therefore

$$\|\mathcal{C}_1^* f\|_p^p \leq \frac{p+1}{2^p}\|\mathcal{C}_2^* f\|_p^p,$$

for all $2 \leq p < \infty$ and for all $f \in C_c^+(\mathbb{R}^+)$. This completes the proof of the inequality in (20). The proof that the constant in (19) is optimal follows directly from (24). □

5. Further Comments

The table below summarizes our results for the norms of the inclusion operators

$$\Lambda_n^p(\mathbb{R}^+) \hookrightarrow \Lambda_{n+k}^p(\mathbb{R}^+) \quad (1 < p \leq \infty),$$
$$\Lambda_n^{*p}(\mathbb{R}^+) \hookrightarrow \Lambda_{n+k}^{*p}(\mathbb{R}^+) \quad (1 \leq p < \infty).$$

	$\|i\|_{\Lambda_n^p(\mathbb{R}^+)\to\Lambda_{n+k}^p(\mathbb{R}^+)}$ $(1 < p \leq \infty)$	$\|i\|_{\Lambda_n^{*p}(\mathbb{R}^+)\to\Lambda_{n+k}^{*p}(\mathbb{R}^+)}$ $(1 \leq p < \infty)$
$(n,k) \in \mathbb{N}\times\mathbb{N}$	$\dfrac{B\left(n+\frac{1}{p'},k\right)}{B(n+1,k)}$	$\dfrac{B\left(n+\frac{1}{p},k\right)}{B(n+1,k)}$

Let $n, k \in \mathbb{N}$. We have seen in Theorem 2.2 that the norm of the inclusion operator

$$\|i\|_{\Lambda_{n+k}^\infty(\mathbb{R}^+)\to\Lambda_n^\infty(\mathbb{R}^+)} = \frac{(n-1)^{n-1}}{n^{n-1}}\frac{(n+k)^{n+k-1}}{(n+k-1)^{n+k-1}}.$$

The following table summarizes the norms of the inclusion operators:

$$\Lambda^p_{n+k}(\mathbb{R}^+) \hookrightarrow \Lambda^p_n(\mathbb{R}^+) \quad \text{for all } 1 < p < \infty,$$
$$\Lambda^{*p}_{n+k}(\mathbb{R}^+) \hookrightarrow \Lambda^{*p}_n(\mathbb{R}^+) \quad \text{for all } 1 \leq p < \infty,$$

for all $n, k \in \mathbb{N}$.

	$\|i\|_{\Lambda^p_{n+k}(\mathbb{R}^+)\to\Lambda^p_n(\mathbb{R}^+)}$ $(1 < p < \infty)$			$\|i\|_{\Lambda^{*p}_{n+k}(\mathbb{R}^+)\to\Lambda^{*p}_n(\mathbb{R}^+)}$ $(1 \leq p < \infty)$		
	$1 < p < 2$	$p = 2$	$2 < p < \infty$	$1 \leq p < 2$	$p = 2$	$2 < p < \infty$
$(n,k) = (1,1)$	$C(n,k,p)$			$C^*(n,k,p)$		
$(n,k) \in \mathbb{N}\times\mathbb{N}$ $(n,k) \neq (1,1)$	?	$C(n,k,p)$	?	?	$C^*(n,k,p)$	?

Here, the symbol "?" indicates that the exact value of the corresponding norm remains unknown, and $C(n,k,p)$ and $C^*(n,k,p)$ are defined for $n,k \in \mathbb{N}$ and $1 < p < \infty$ by

$$C(n,k,p) = \left(\frac{n}{n+k}\right)^p \frac{B\big((n-1)p+1, p-1\big)}{B\big((n+k-1)p+1, p-1\big)}$$

and

$$C^*(n,k,p) = \left(\frac{n}{n+k}\right)^p \frac{p(n+k-1)+1}{p(n-1)+1}.$$

By adapting the arguments from Theorems 2.4 and 2.5, together with the semigroup property of the Riemann–Liouville fractional integral operators (see [10, Theorem 2.2]), it can be shown that

$$0 < \|i\|_{\Lambda^p_{n+k}(\mathbb{R}^+)\to\Lambda^p_n(\mathbb{R}^+)} < \infty \tag{40}$$

and

$$0 < \|i\|_{\Lambda^{*p}_{n+k}(\mathbb{R}^+)\to\Lambda^{*p}_n(\mathbb{R}^+)} < \infty \tag{41}$$

for all $1 < p < \infty$ and $n,k \in \mathbb{N}$, with $p \neq 2$ and $(n,k) \neq (1,1)$. Unfortunately, this approach does not yield the exact values of these inclusion norms.

In any case, since the operator $\mathcal{C}_m$ and its adjoint $\mathcal{C}^*_m$ are defined for all real number $m > 0$ (and, for instance, the inequalities (1) and (2) both hold for any $m > 0$), and in view of the tables above, it is natural to state the following general conjecture, which generalizes the main results of this paper:

Conjecture. Let $1 < p < \infty$ and $n, k > 0$ be real numbers. Then:

$$\|i\|_{\Lambda^p_n(\mathbb{R}^+)\to\Lambda^p_{n+k}(\mathbb{R}^+)} = \frac{B\big(n+\frac{1}{p'},k\big)}{B(n+1,k)},$$

$$\|i\|_{\Lambda^{*p}_n(\mathbb{R}^+)\to\Lambda^{*p}_{n+k}(\mathbb{R}^+)} = \frac{B\big(n+\frac{1}{p},k\big)}{B(n+1,k)},$$

$$\|i\|^p_{\Lambda^p_{n+k}(\mathbb{R}^+)\to\Lambda^p_n(\mathbb{R}^+)} = \left(\frac{n}{n+k}\right)^p \frac{B\big(p(n-1)+1,1\big)}{B\big(p(n+k-1)+1,1\big)},$$

$$\|i\|^p_{\Lambda^{*p}_{n+k}(\mathbb{R}^+)\to\Lambda^{*p}_n(\mathbb{R}^+)} = \left(\frac{n}{n+k}\right)^p \frac{p(n+k-1)+1}{p(n-1)+1}.$$

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