SHARP EMBEDDINGS BETWEEN WEIGHTED PALEY-WIENER SPACES

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ABSTRACT. In this paper we address the problem of estimating the operator norm of the embeddings between multidimensional weighted Paley-Wiener spaces. These can be equivalently thought as Fourier uncertainty principles for bandlimited functions. By means of radial symmetrization mechanisms, we show that such problems can all be shifted to dimension one. We provide precise asymptotics in the general case and, in some particular situations, we are able to identify the sharp constants and characterize the extremizers. The sharp constant study is actually a consequence of a more general result we prove in the setup of de Branges spaces of entire functions, addressing the operator given by multiplication by z^k, $k \in \mathbb{N}$. Applications to sharp higher order Poincaré inequalities and other related extremal problems are discussed.

CONTENTS

1. INTRODUCTION

This paper touches on a few themes within analysis and approximation theory. We are inspired by the classical work of Holt and Vaaler [21], that approaches certain extremal problems in harmonic and complex analysis via the powerful theory of de Branges spaces of entire functions [5]. Our aim here is to study certain sharp inequalities for bandlimited functions that fall under the paradigm of *Fourier uncertainty*, i.e. that one cannot have an unrestricted control of a function and its Fourier transform

simultaneously. Different manifestations of Fourier uncertainty are ubiquitous in harmonic analysis; for a brief historical account and many important examples we refer the reader to the survey articles [3, 19] and the references therein. Other recent works that apply de Branges spaces techniques in connection to extremal problems in analysis include [6, 7, 10, 11, 12, 13, 25, 26, 27, 37].

Our normalization for the Fourier transform of a function $F \in L^1(\mathbb{R}^d)$ is

$$\widehat{F}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} e^{-2\pi i \boldsymbol{x} \cdot \boldsymbol{\xi}} \, F(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}.$$

As we shall see, some of the inequalities discussed in this paper can be equivalently formulated, via the Fourier transform, in terms of classical derivatives (i.e. *Poincaré inequalities*). When such a reformulation is available, our approach via de Branges spaces provides an alternative framework to look into the problem of finding sharp constants, in contrast to the classical Sturm-Liouville approach on the Fourier side; more details on §1.3 below.

The main results in this paper are the ones described in this introduction together with the general Theorems 24 and 25, presented in Section 8.

1.1. **Setup.** We start with our general setup, in the language of entire functions of several complex variables. Vectors in $\mathbb{R}^d$ or $\mathbb{C}^d$ are denoted here with bold font (e.g. $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$) and numbers in $\mathbb{R}$ or $\mathbb{C}$ with regular font (e.g. x, y, z). For $\boldsymbol{z} = (z_1, z_2, \ldots, z_d) \in \mathbb{C}^d$ we let $|\cdot|$ be the usual norm $|\boldsymbol{z}| := (|z_1|^2 + \ldots + |z_d|^2)^{1/2}$, and define a second norm $\|\cdot\|$ by

$$\|\boldsymbol{z}\| := \sup \left\{ \left| \sum_{n=1}^{d} z_n \, t_n \right| ; \ \boldsymbol{t} \in \mathbb{R}^d \text{ and } |\boldsymbol{t}| \leq 1 \right\}.$$

If $F : \mathbb{C}^d \to \mathbb{C}$ is an entire function of d complex variables, which is not identically zero, we say that F has *exponential type* if

$$\tau(F) := \limsup_{\|\boldsymbol{z}\| \to \infty} \|\boldsymbol{z}\|^{-1} \log |F(\boldsymbol{z})| < \infty.$$

In this case, the number $\tau(F)$ is called the exponential type of F. When $d = 1$ this is the classical definition of exponential type and, when $d \geq 2$, our definition is a particular case of a more general concept of exponential type with respect to a compact, convex and symmetric set $K \subset \mathbb{R}^d$ (cf. [35, pp. 111-112]). In our case, this convex set K is simply the unit Euclidean ball.

For each $\alpha > -1$ and $\delta > 0$, we let $\mathcal{H}_\alpha(d\,;\delta)$ be the Hilbert space of entire functions $F : \mathbb{C}^d \to \mathbb{C}$ of exponential type at most δ such that

$$\|F\|_{\mathcal{H}_\alpha(d\,;\delta)} := \left(\int_{\mathbb{R}^d} |F(\boldsymbol{x})|^2 \, |\boldsymbol{x}|^{2\alpha+2-d} \, \mathrm{d}\boldsymbol{x} \right)^{1/2} < \infty. \tag{1.1}$$

For generic $\alpha \geq \beta > -1$ observe that one has the inclusion $\mathcal{H}_\alpha(d\,;\delta) \subset \mathcal{H}_\beta(d\,;\delta)$, and from the closed graph theorem it follows that the inclusion map $I : \mathcal{H}_\alpha(d\,;\delta) \to \mathcal{H}_\beta(d\,;\delta)$ is a bounded operator. The purpose of this paper is to investigate the operator norm $\|I\|_{\mathcal{H}_\alpha(d\,;\delta) \to \mathcal{H}_\beta(d\,;\delta)}$ of this embedding, which can be equivalently thought of as the quest to find the sharp form of a Fourier uncertainty principle. In our study, it will be convenient to consider the following formulation:

Extremal Problem 1 (EP1): For $\alpha \geq \beta > -1$ and $\delta > 0$ real parameters, and $d \in \mathbb{N}$, find the value of

$$(\mathbb{EP}1)(\alpha, \beta\,;d\,;\delta) := \inf_{0 \neq F \in \mathcal{H}_\alpha(d;\delta)} \frac{\int_{\mathbb{R}^d} |F(\boldsymbol{x})|^2\, |\boldsymbol{x}|^{2\alpha+2-d}\,\mathrm{d}\boldsymbol{x}}{\int_{\mathbb{R}^d} |F(\boldsymbol{x})|^2\, |\boldsymbol{x}|^{2\beta+2-d}\,\mathrm{d}\boldsymbol{x}}. \tag{1.2}$$

This is the first of four extremal problems that will appear in this paper, and our choice of notation for the sharp constant aims to facilitate such references. In particular, note that $(\mathbb{EP}1)(\alpha, \beta\,;d\,;\delta) = \left(\|I\|_{\mathcal{H}_\alpha(d\,;\delta) \to \mathcal{H}_\beta(d\,;\delta)}\right)^{-2}$ and hence

$$(\mathbb{EP}1)(\alpha, \beta\,;d\,;\delta) > 0. \tag{1.3}$$

Inequality (1.3) can be viewed as a Fourier uncertainty principle. In fact, the Paley-Wiener theorem (see [35, Chapter III, Section 4] and [22, Theorem 1.7.5 and Theorem 1.7.7]) tells us that $F \in \mathcal{H}_\alpha(d\,;\delta)$ has (distributional) Fourier transform supported in the closed ball of radius $\delta/2\pi$ centered at the origin and, therefore, the Fourier uncertainty paradigm implies that the mass of F cannot be too concentrated around the origin.

We now move to describing our main results. We split them into three distinct parts.

1.2. **Main results.**

1.2.1. *Part I: Qualitative properties.* Observe that $F(\cdot) \in \mathcal{H}_\alpha(d\,;\delta)$ if and only if $F(\cdot/\delta) \in \mathcal{H}_\alpha(d\,;1)$. This change of variables plainly yields the relation

$$(\mathbb{EP}1)(\alpha, \beta\,;d\,;\delta) = \delta^{2\beta-2\alpha}\,(\mathbb{EP}1)(\alpha, \beta\,;d\,;1). \tag{1.4}$$

Our first non-trivial observation about the extremal problem (EP1) is as follows.

Theorem 1 (Dimension shifts)**.** *We have*

$$(\mathbb{EP}1)(\alpha, \beta\,;d\,;\delta) = (\mathbb{EP}1)(\alpha, \beta\,;1\,;\delta). \tag{1.5}$$

The proof of Theorem 1 is carried out in Section 3 and it relies on suitable radial symmetrization mechanisms and an auxiliary extremal problem that may be of independent interest. In view of (1.4) and (1.5), when it comes to the extremal problem (EP1), we can therefore restrict our attention to dimension $d = 1$ with any particular exponential type δ of our choice.

A function $0 \neq F \in \mathcal{H}_\alpha(d\,;\delta)$ is said to be an *extremizer* for $(\mathbb{EP}1)(\alpha, \beta\,;d\,;\delta)$ if it realizes the infimum in (1.2), i.e. if

$$\int_{\mathbb{R}^d} |F(\boldsymbol{x})|^2\, |\boldsymbol{x}|^{2\alpha+2-d}\,\mathrm{d}\boldsymbol{x} = (\mathbb{EP}1)(\alpha, \beta\,;d\,;\delta) \int_{\mathbb{R}^d} |F(\boldsymbol{x})|^2\, |\boldsymbol{x}|^{2\beta+2-d}\,\mathrm{d}\boldsymbol{x}.$$

We say that an entire function $F : \mathbb{C}^d \to \mathbb{C}$ is radial, if its restriction to $\mathbb{R}^d$ is radial. Our next result addresses the existence of extremizers.

Theorem 2 (Radial extremizers)**.** *There exists a radial extremizer for* $(\mathbb{EP}1)(\alpha, \beta\,;d\,;\delta)$.

We discuss the proof of this result in Section 4. In dimension $d = 1$ we go further and establish that any extremizer of $(\mathbb{EP}1)(\alpha, \beta\,;1\,;\delta)$ must be an even function; see Proposition 15.

Our next result is concerned with the continuity of $(\mathbb{EP}1)(\alpha, \beta\,;d\,;\delta)$ in all real-valued parameters.

Theorem 3 (Continuity). *The function $(\alpha, \beta, \delta) \mapsto (\mathbb{EP}1)(\alpha, \beta\,; d\,; \delta)$ is continuous in the range $\alpha \geq \beta > -1$ and $\delta > 0$.*

The proof of this result is carried out in Section 5.

1.2.2. *Part II: Asymptotics.* Finding the exact value of the constant $(\mathbb{EP}1)(\alpha, \beta\,; d\,; \delta)$ is not a simple task in general, and one is naturally led to study this object from an asymptotic point of view. As observed in §1.2.1, from now on let us restrict ourselves to the scenario $d = 1$ and $\delta = 1$, without loss of generality. That is, we are looking at [1]

$$(\mathbb{EP}1)(\alpha, \beta\,; 1\,; 1) := \inf_{0 \neq f \in \mathcal{H}_\alpha(1;1)} \frac{\int_\mathbb{R} |f(x)|^2\, |x|^{2\alpha+1}\, \mathrm{d}x}{\int_\mathbb{R} |f(x)|^2\, |x|^{2\beta+1}\, \mathrm{d}x}.$$

It is convenient to introduce a logarithmic scale, and look at the function $(\alpha, \beta) \mapsto \log\big((\mathbb{EP}1)(\alpha, \beta\,; 1\,; 1)\big)$. We aim to find a description of the form

$$\log\big((\mathbb{EP}1)(\alpha, \beta\,; 1\,; 1)\big) = M(\alpha, \beta) + R(\alpha, \beta)\,,$$

where M would be a main term as α, β move towards the limits -1 and/or ∞, and R would be a remainder term. Also, when $\alpha = \beta$ we know that $\log\big((\mathbb{EP}1)(\alpha, \beta\,; 1\,; 1)\big) = 0$, and we would like our M and R to reflect that, possibly under some reasonable control. This is the spirit of our next result.

Theorem 4 (Asymptotics). *For $\alpha \geq \beta > -1$ we have*

$$
\begin{aligned}
\log\big((\mathbb{EP}1)(\alpha, \beta\,; 1\,; 1)\big) = {}& 2(\alpha - \beta) \log(\alpha + 2) + \log\left(\frac{\beta + 1}{\alpha + 1}\right) \\
&+ O\left(\left(\frac{(\alpha - \beta)(\alpha + 2)}{(\alpha + 1)}\right) \log\left(\frac{2(\alpha + 1)(\alpha - \beta + 1)}{(\alpha - \beta)(\alpha + 2)}\right)\right),
\end{aligned}
\tag{1.6}
$$

where the implied constant is universal.

We described our error term above in a unified expression for brevity, but a careful look at it reveals its behaviour in the different regimes. Let us briefly reflect on the possibilities. Writing $x = \alpha - \beta$ and $y = \alpha + 1$ for simplicity, note that $0 \leq x < y$ and $\frac{y(x+1)}{x(y+1)} \geq 1$. If α is away from -1, say $\alpha \geq -\frac{1}{2}$, then our error term is $O\big(x \log(\frac{2(x+1)}{x})\big)$, and this can be further understood as $O\big(x \log(\frac{1}{x})\big)$ for x small and $O(x)$ for x large. If $\alpha \leq -\frac{1}{2}$, our error term is $O\big(\frac{x}{y} \log(\frac{2y}{x})\big)$ which, in particular, is bounded. If $\alpha \geq \beta \geq -\frac{1}{2}$, since $\log\left(\frac{\beta+1}{\alpha+1}\right) = O(\alpha - \beta)$ (just observe that the derivative of $s \mapsto \log(s + 1)$ is bounded in this range and apply the mean value theorem), our expression becomes

$$\log\big((\mathbb{EP}1)(\alpha, \beta\,; 1\,; 1)\big) = 2(\alpha - \beta) \log(\alpha + 2) + O\left((\alpha - \beta) \log\left(\frac{2(\alpha - \beta + 1)}{\alpha - \beta}\right)\right).$$

Similarly, if $-1 < \beta \leq \alpha \leq -\frac{1}{2}$, we have

$$\log\big((\mathbb{EP}1)(\alpha, \beta\,; 1\,; 1)\big) = \log\left(\frac{\beta + 1}{\alpha + 1}\right) + O\left(\left(\frac{\alpha - \beta}{\alpha + 1}\right) \log\left(\frac{2(\alpha + 1)}{\alpha - \beta}\right)\right).$$

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[1]The reader may wonder why we choose the normalization $|x|^{2\alpha+1}$ instead of $|x|^{2\alpha}$ (which is perhaps more common in PDEs given that it would be related to the derivative or order α). The reason is that, as we shall see, these problems are related to the classical Bessel functions of the first kind J_α, and we opted for the simplicity of notation at this end.

If $\alpha \to \infty$ and $\beta \to -1$ simultaneously, there might be a legitimate match between the two components of the main term in (1.6).

The proof of Theorem 4 is carried out in Section 7. For this proof we establish upper and lower bounds that match the proposed asymptotic (1.6). The upper bound is obtained via a suitable example involving the reproducing kernel of the space. The lower bound is subtler, and we need two different mechanisms to treat the two different ranges: α and β near the upper endpoint ∞, and α and β near the lower endpoint -1. For the former we rely on quantitative versions of certain classical Fourier uncertainty principles, and run an optimization procedure to arrive at the desired bound. For the latter, where the classical Fourier transform is not available, we use a different strategy, relying on certain estimates involving the reproducing kernel of the space, and again running an optimization procedure on the parameters. It is interesting to notice that, at least as far as our setup goes, neither of these strategies would be sufficient to tackle the whole range on its own. The remaining cases, where α is near ∞ and β is near -1, are addressed via a comparison with a suitable test point between β and α.

1.2.3. *Part III: Sharp constants.* We continue to restrict our attention to the scenario $d = 1$ and $\delta = 1$, without loss of generality. In some special occasions we are able to identify the precise value of the constant $(\mathbb{EP}1)(\alpha, \beta \,; 1 \,; 1)$. Before proceeding to this description, let us fix some notation and review some facts from the theory of Bessel functions that shall be relevant for our purposes.

For $\nu > -1$ let $A_\nu : \mathbb{C} \to \mathbb{C}$ and $B_\nu : \mathbb{C} \to \mathbb{C}$ be the real entire functions defined by

$$A_\nu(z) := \sum_{n=0}^\infty \frac{(-1)^n \left(\frac{1}{2}z\right)^{2n}}{n!(\nu+1)(\nu+2)\dots(\nu+n)} \tag{1.7}$$

and

$$B_\nu(z) := \sum_{n=0}^\infty \frac{(-1)^n \left(\frac{1}{2}z\right)^{2n+1}}{n!(\nu+1)(\nu+2)\dots(\nu+n+1)}. \tag{1.8}$$

These functions are related to the classical Bessel functions of the first kind by the identities

$$\begin{aligned} A_\nu(z) &= \Gamma(\nu+1) \left(\tfrac{1}{2}z\right)^{-\nu} J_\nu(z), \\ B_\nu(z) &= \Gamma(\nu+1) \left(\tfrac{1}{2}z\right)^{-\nu} J_{\nu+1}(z). \end{aligned} \tag{1.9}$$

Both A_ν and B_ν have only real, simple zeros and have no common zeros (note that, in the simplest case $\nu = -1/2$, we have $A_{-1/2}(z) = \cos z$ and $B_{-1/2}(z) = \sin z$). The function A_ν is even while B_ν is odd, and we have $\tau(A_\nu) = \tau(B_\nu) = 1$. Moreover, they satisfy the system of differential equations

$$A_\nu'(z) = -B_\nu(z) \quad \text{and} \quad B_\nu'(z) = A_\nu(z) - (2\nu+1)z^{-1}B_\nu(z). \tag{1.10}$$

Our main result in this section is a complete solution for the extremal problem (EP1) in the cases where $\alpha = \beta + k$ for $k \in \mathbb{N}$. The answer is given in terms of the smallest positive solution of a certain explicit determinant equation involving Bessel functions. Let

$$0 < j_{\nu,1} < j_{\nu,2} < j_{\nu,3} < \dots$$

denote the sequence of positive zeros of the Bessel function J_ν, and define the meromorphic function

$$C_\nu(z) := \frac{B_\nu(z)}{A_\nu(z)}.$$

When $\nu = -1/2$, we simply have $C_{-1/2}(z) = \tan z$.

Theorem 5 (Sharp constants). *Let $\beta > -1$, let $k \in \mathbb{N}$ and set $\lambda_0 := \big((\mathbb{EP}1)(\beta + k, \beta\,;1\,;1)\big)^{1/2k}$.*

(i) *If $k = 1$ we have $\lambda_0 = \mathsf{j}_{\beta,1}$.*

(ii) *If $k \geq 2$, set $\ell := \lfloor k/2 \rfloor$. Then λ_0 is the smallest positive solution of the equation*

$$A_\beta(\lambda) \det \mathcal{V}_\beta(\lambda) = 0\,,$$

where $\mathcal{V}_\beta(\lambda)$ is the $\ell \times \ell$ matrix with entries

$$\big(\mathcal{V}_\beta(\lambda)\big)_{mj} = \sum_{r=0}^{k-1} \omega^{r(4\ell - 2m - 2j + 3)} C_\beta\big(\omega^r \lambda\big) \qquad (1 \leq m, j \leq \ell)\,,$$

and $\omega := e^{\pi i/k}$.

REMARK: For $k \geq 2$, we shall verify in our proof that $0 < \lambda_0 < \mathsf{j}_{\beta,\ell+1}$. The function $\lambda \mapsto \det \mathcal{V}_\beta(\lambda)$ is, in principle, a meromorphic function of the variable λ that is real-valued on $\mathbb{R}$, but we show that $\lambda \mapsto A_\beta(\lambda) \det \mathcal{V}_\beta(\lambda)$ is in fact continuous on the interval $(0, \mathsf{j}_{\beta,\ell+1})$.

We also fully classify the extremizers for $(\mathbb{EP}1)(\beta + k, \beta\,;1\,;1)$ when $k \in \mathbb{N}$. When $k = 1$ they are the complex multiples of $f(z) = A_\beta(z)/(z^2 - \mathsf{j}_{\beta,1}^2)$ and, when $k \geq 2$, they have the form

$$f(z) = \sum_{n=1}^{\infty} a_n \frac{\mathsf{j}_{\beta,n} A_\beta(z)}{(z^2 - \mathsf{j}_{\beta,n}^2)}\,,$$

where $(a_1, a_2, \ldots, a_\ell)$ belongs to the kernel of a certain $\ell \times \ell$ matrix (here $\ell = \lfloor k/2 \rfloor$) and each a_n, for $n > \ell$, is given in terms of $a_1, \ldots, a_\ell$; see Theorem 25 for details.

The novelty in our sharp constant approach here is to connect the extremal problem (EP1) with the rich theory of de Branges spaces of entire functions. As a matter of fact, Theorem 5 ends up being a special case of the much more general Theorem 24, that we prove in Section 8. Theorems 24 and 25 address the extremal problem (EP4), related to the operator given by multiplication by z^k in a general de Branges space, and are certainly among the key results of this paper as well. We have opted to postpone this discussion to Section 8 in order to minimize the technical considerations in this introduction (we shall need a brief overview of the de Branges space theory, which we do in §2.2), but the reader that is familiar with the theory may take a direct look at this section, which is of independent interest.

An interesting feature of Theorem 5 is that we exploit the fact that extremizers are even functions (see Proposition 15) to arrive at a determinant of order $\ell = \lfloor k/2 \rfloor$; this suits well the treatment for small k. As an illustration, for $k = 2$, $\big((\mathbb{EP}1)(\beta + 2, \beta\,;1\,;1)\big)^{1/4}$ is the first positive root of

$\lambda \mapsto A_\beta(\lambda)\big(C_\beta(\lambda) - iC_\beta(i\lambda)\big)$, while, for $k = 3$, $\big((\mathbb{EP}1)(\beta + 3, \beta\,;1\,;1)\big)^{1/6}$ is the first positive root of

$$\lambda \mapsto A_\beta(\lambda)\big(C_\beta(\lambda) - C_\beta(\omega\lambda) + C_\beta(\omega^2\lambda)\big), \tag{1.11}$$

where $\omega = e^{\pi i/3}$. For example, when $\beta = -1/2$, the first few values are:

	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$
$\big((\mathbb{EP}1)(-\frac{1}{2} + k, -\frac{1}{2}\,;1\,;1)\big)^{1/2k}$	1	$\pi/2$	$2.36\ldots$	π	$3.90\ldots$	$4.67\ldots$	$5.43\ldots$	$6.18\ldots$

With a quick look at the table above, the fact that $\big((\mathbb{EP}1)(\frac{5}{2}, -\frac{1}{2}\,;1\,;1)\big)^{1/6} = \pi$ stands out as rather curious. This is due to the fact that, when $\beta = -1/2$ and $k = 3$, the function in (1.11), reduces to

$$\lambda \mapsto -\frac{\sin\lambda\big(\cos\lambda - \cosh(\sqrt{3}\lambda)\big)}{\cos\lambda + \cosh(\sqrt{3}\lambda)}.$$

With the dilation relation (1.4) in mind, one has $(\mathbb{EP}1)(\frac{5}{2}, -\frac{1}{2}\,;1\,;\pi) = 1$, which is equivalent to the particularly neat sharp inequality: if $f \in L^2(\mathbb{R})$ and $\mathrm{supp}(\widehat{f}) \subset [-\frac{1}{2}, \frac{1}{2}]$, then

$$\int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x \leq \int_{-\infty}^{\infty} |f(x)|^2\,x^6\,\mathrm{d}x. \tag{1.12}$$

As an illustration of Theorem 25, the extremizer for (1.12) is unique (up to multiplication by a complex constant) and is given by

$$f(x) = \sum_{n \geq 1} \frac{\left(n - \frac{1}{2}\right)^2}{\left(\left(n - \frac{1}{2}\right)^6 - 1\right)\left(x^2 - \left(n - \frac{1}{2}\right)^2\right)}\frac{\cos\pi x}{}.$$

1.3. **Application 1: sharp higher order Poincaré inequalities.** Some particular cases of Theorem 5 can be equivalently formulated in the framework of sharp Poincaré inequalities via the Fourier transform. For $m \in \mathbb{Z}_{\geq 0}$, let $W_0^{m,2}(B_r)$ be the usual L^2-Sobolev space of order m and zero boundary data, i.e. the closure of $C_c^\infty(B_r)$ in $W^{m,2}(B_r)$, where $B_r \subset \mathbb{R}^d$ is the ball of radius r centered at $\mathbf{0}$.

Corollary 6. *Let $m, n \in \mathbb{Z}_{\geq 0}$, with $m \geq n$. For any $g \in W_0^{m,2}(-r, r)$ we have*

$$\int_{-r}^{r} |g^{(n)}(x)|^2\,\mathrm{d}x \leq \frac{r^{2(m-n)}}{(\mathbb{EP}1)(m - \frac{1}{2}, n - \frac{1}{2}\,;1\,;1)}\int_{-r}^{r} |g^{(m)}(x)|^2\,\mathrm{d}x. \tag{1.13}$$

This inequality is sharp. The value of $(\mathbb{EP}1)(m - \frac{1}{2}, n - \frac{1}{2}\,;1\,;1)$ is given by Theorem 5.

REMARK: When $r = \frac{1}{2\pi}$, the extremizers of (1.13) are the Fourier transforms of the extremizers of Theorem 25 , applied to the situation of Theorem 5 (with $\alpha = m - \frac{1}{2}$ and $\beta = n - \frac{1}{2}$); the extremizers for general r are obtained via a suitable dilation.

Poincaré inequalities have been extensively studied in many different contexts within PDEs and calculus of variations. In fact, Corollary 6 has been previously obtained via different methods. The cases $(m, n) = (1, 0), (2, 1)$ date back to Steklov [34]. The cases $(m + 1, m)$, with $m \geq 2$, date back to Janet [24]; see also [30]. The sharp constant in the generic integer case (m, n) has been recently characterized in the work of Yu. P. Petrova [32] (see also [31, Theorem 7]), also as the smallest positive

solution of a certain explicit determinant equation. Petrova's determinant is different than ours (in particular, it has larger order $k = m - n$ and relies on some specific properties of Bessel functions for certain simplifications) but, of course, it must lead to the same answer in these particular cases. The formulation in Theorem 5 has the advantage of being amenable to a generalization to the broader setup of de Branges spaces in Theorem 24. For a detailed account on the history of the inequalites (1.13) (also known as Steklov-type inequalities) and some related variants, we refer the reader to the survey article by A. I. Nazarov and A. P. Shcheglova [31] and the references therein.

It is also possible to recast our multidimensional extremal problem (EP1), via the Fourier transform, in a problem involving the classical gradient and Laplacian.

Corollary 7. *Let $d \geq 2$. Let $m, n \in \mathbb{Z}_{\geq 0}$, $m_1, n_1 \in \{0, 1\}$ be such that $2m + m_1 \geq 2n + n_1$. For any $g \in W_0^{2m+m_1,2}(B_r)$ we have*

$$\int_{B_r} \left|\nabla^{n_1}\left(\Delta^n g\right)(\boldsymbol{x})\right|^2 \mathrm{d}\boldsymbol{x} \leq \frac{r^{2(2m+m_1-2n-n_1)}}{(\mathbb{EP}1)(2m + m_1 - 1 + \frac{d}{2}, 2n + n_1 - 1 + \frac{d}{2}; 1; 1)} \int_{B_r} \left|\nabla^{m_1}\left(\Delta^m g\right)(\boldsymbol{x})\right|^2 \mathrm{d}\boldsymbol{x}.$$

(1.14)

This inequality is sharp. The value of $(\mathbb{EP}1)(2m+m_1-1+\frac{d}{2}, 2n+n_1-1+\frac{d}{2}; 1; 1)$ is given Theorem 5.

REMARK: In (1.14) note that we have used Theorem 1 and (1.4). When $r = \frac{1}{2\pi}$, the radial extremizers of (1.14) are given by the Fourier transforms of the lifts (see §2.3 and §4.4) of the extremizers of Theorem 25, applied to the situation of Theorem 5 (with $\alpha = 2m+m_1-1+\frac{d}{2}$ and $\beta = 2n+n_1-1+\frac{d}{2}$); the radial extremizers for general r are obtained via a suitable dilation.

In the particular situations of Corollaries 6 and 7, it turns out that Poincaré inequalities and Fourier uncertainty are simply the two sides of the same coin. Our takeaway point here is that the de Branges space approach on the entire function side provides an alternative framework to look into the classical problem of finding the sharp constants in these Poincaré inequalities and, in fact, allows for a robust generalization of it.

1.4. **Application 2: a related extremal problem.** The classical Beurling-Selberg extremal problem in approximation theory can be described as follows: given a function $f : \mathbb{R} \to \mathbb{R}$, one wants to find a real entire function M of prescribed exponential type such that $M(x) \geq f(x)$ for all $x \in \mathbb{R}$, with the norm $\|M - f\|_{L^1(\mathbb{R})}$ minimal. The original situation considered by Beurling in the late 1930s was with $f(x) = \operatorname{sgn}(x)$. This type of problem has been extensively studied in the literature with several interesting applications in analytic number theory, e.g. [8, 9, 13, 15, 16, 21, 36]. Recently, the first author and Littmann [14] studied a variant of the Beurling-Selberg extremal problem for $f(x) = \operatorname{sgn}(x)$, with the additional constraint that the majorant M is non-decreasing in $(-\infty, 0)$ and non-increasing in $(0, \infty)$, and used such an extremal function to give a simple Fourier analysis proof of the (non-sharp) weighted Hilbert's inequality, introduced by Montgomery and Vaughan [28].

The following related question arose in discussions between the first author and J. D. Vaaler in 2015. This is the analogue, with an additional monotonicity constraint, of a question solved by Holt and Vaaler in [21].

Extremal Problem 2 (EP2) (Radial non-increasing delta majorant): Let $\mathcal{R}^+(d\,;2\delta)$ be the class of real entire functions $M : \mathbb{C}^d \to \mathbb{C}$ that verify the following properties: (i) M has exponential type at most 2δ; (ii) M is non-negative and radial non-increasing on $\mathbb{R}^d$; (iii) $M(\mathbf{0}) \geq 1$. Find the value of

$$(\mathbb{EP}2)(d\,;\delta) := \inf_{M \in \mathcal{R}^+(d;2\delta)} \int_{\mathbb{R}^d} M(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

We show that the problem (EP2) is related to the problem (EP1) in the following way.

Theorem 8. *There exist extremizers for* $(\mathbb{EP}2)(d\,;\delta)$ *and*

$$(\mathbb{EP}2)(d\,;\delta) = \frac{\omega_{d-1}}{d}\,(\mathbb{EP}1)(\tfrac{d}{2},0\,;1\,;\delta)\,,$$

where $\omega_{d-1} = 2\,\pi^{d/2}\,\Gamma(d/2)^{-1}$ *is the surface area of the unit sphere* $\mathbb{S}^{d-1} \subset \mathbb{R}^d$.

From Theorem 5 we have the exact answer of the extremal problem (EP2) when the dimension d is even. The characterization of the extremizers is obtained via Theorem 25 and the radial symmetrization considerations of Theorem 1. Here is a table with the first values when $\delta = 1$:

	$d=2$	$d=4$	$d=6$	$d=8$	$d=10$	$d=12$	$d=14$	$d=16$
$\left((\mathbb{EP}2)(d\,;1)\right)^{1/d}$	$4.26\ldots$	$4.76\ldots$	$5.23\ldots$	$5.66\ldots$	$6.07\ldots$	$6.45\ldots$	$6.81\ldots$	$7.15\ldots$

Finding the exact answer in the cases when d is odd seems to be a subtler task. For a conjecture related to the case $d = 1$, see the recent work of Chirre, Dimitrov and Quesada-Herrera [17].

1.5. **Notation.** A function $F : \mathbb{C}^d \to \mathbb{C}$ is said to be *real entire* if it is entire and its restriction to $\mathbb{R}^d$ is real-valued. If $F : \mathbb{C}^d \to \mathbb{C}$ is entire, we define the entire function $F^* : \mathbb{C}^d \to \mathbb{C}$ by $F^*(\boldsymbol{z}) := \overline{F(\overline{\boldsymbol{z}})}$. We denote by $\mathbb{C}^+ = \{z \in \mathbb{C}\,;\, \mathrm{Im}(z) > 0\}$ the open upper half-plane. Throughout the text we work with the constants $\omega_{d-1} := 2\,\pi^{d/2}\,\Gamma(d/2)^{-1}$ (surface area of the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$) and $c_\nu := \pi\,2^{-2\nu-1}\,\Gamma(\nu+1)^{-2}$.

2. Preliminaries

In this section we collect a few auxiliary results that shall be relevant for our purposes later on. In doing so, we also establish some of the notation and terminology that will be used. Some of the following results are direct quotes from the current literature that we state as lemmas for the convenience of the reader.

2.1. **Fourier uncertainty.** We first highlight the following particular version of Fourier uncertainty that follows from classical results.

Lemma 9. *If* $f \in L^2(\mathbb{R})$ *and* $\mathrm{supp}(\widehat{f}) \subset [-\frac{1}{2\pi}, \frac{1}{2\pi}]$ *we have*

$$\int_{[-s,s]} |f(x)|^2\,\mathrm{d}x \leq e^{Cr}\left(1 - e^{-Cs}\right)\int_{[-r,r]^c} |f(x)|^2\,\mathrm{d}x \tag{2.1}$$

for any $r, s > 0$, *where* C *is a universal constant.*

Proof. We first claim that it suffices to prove that, for any $r > 0$,

$$\int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x \leq e^{Cr} \int_{[-r,r]^c} |f(x)|^2 \, \mathrm{d}x, \tag{2.2}$$

which in turn is equivalent to

$$\int_{[-r,r]} |f(x)|^2 \, \mathrm{d}x \leq (e^{Cr} - 1) \int_{[-r,r]^c} |f(x)|^2 \, \mathrm{d}x.$$

Hence, for numbers $r, s \geq 0$ we have

$$e^{Cr} \int_{[-r,r]^c} |f(x)|^2 \, \mathrm{d}x \geq \int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x = \int_{[-s,s]} |f(x)|^2 \, \mathrm{d}x + \int_{[-s,s]^c} |f(x)|^2 \, \mathrm{d}x$$

$$\geq \left(1 + \frac{1}{e^{Cs} - 1} \right) \int_{[-s,s]} |f(x)|^2 \, \mathrm{d}x \, ,$$

which plainly implies (2.1).

From now on we slightly abuse the notation and the constant C can possibly change from line to line. In order to arrive at (2.2), recall first the Fourier uncertainty principle of Amrein and Berthier [2] and Nazarov [29] (see also the work of Jaming [23]): for $E, F \subset \mathbb{R}$ sets of finite Lebesgue measure and $f \in L^2(\mathbb{R})$ we have

$$\int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x \leq C \, e^{C|E| \, |F|} \left(\int_{E^c} |f(x)|^2 \, \mathrm{d}x + \int_{F^c} |\widehat{f}(x)|^2 \, \mathrm{d}x \right).$$

Specializing to our case, with $E = [-r, r]$ and $F = [-\frac{1}{2\pi}, \frac{1}{2\pi}]$, we get

$$\int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x \leq C \, e^{Cr} \int_{[-r,r]^c} |f(x)|^2 \, \mathrm{d}x. \tag{2.3}$$

For a better bound when r is small, we recall the Fourier uncertainty principle of Donoho and Stark [18, Theorem 2]: for $E, F \subset \mathbb{R}$ measurable sets and $f \in L^2(\mathbb{R})$ we have

$$\|f\|_2 - \|f \cdot \chi_{E^c}\|_2 - \|\widehat{f} \cdot \chi_{F^c}\|_2 \leq |E|^{1/2} \, |F|^{1/2} \, \|f\|_2.$$

Choosing $E = [-r, r]$ and $F = [-\frac{1}{2\pi}, \frac{1}{2\pi}]$ we get, for r small,

$$\int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x \leq \frac{1}{(1 - C\sqrt{r})^2} \int_{[-r,r]^c} |f(x)|^2 \, \mathrm{d}x. \tag{2.4}$$

Note that $(1 - C\sqrt{r})^{-2} = 1 + O(\sqrt{r})$ for r small. Hence, (2.3) and (2.4) lead to (2.2). $\qquad \square$

2.2. **De Branges spaces.** Our extremal problem (EP1) is related to the the beautiful theory of de Branges spaces of entire functions [5], that we now briefly review.

2.2.1. *Overview.* Given a *Hermite-Biehler* function $E : \mathbb{C} \to \mathbb{C}$, i.e. an entire function that verifies

$$|E^*(z)| < |E(z)| \tag{2.5}$$

for all $z \in \mathbb{C}^+$, the de Branges space $\mathcal{H}(E)$ associated to E is the space of entire functions $f : \mathbb{C} \to \mathbb{C}$ such that

$$\|f\|_{\mathcal{H}(E)}^2 := \int_{\mathbb{R}} |f(x)|^2 \, |E(x)|^{-2} \, \mathrm{d}x < \infty \,,$$

and such that f/E and f^*/E have bounded type and non-positive mean type[2] in $\mathbb{C}^+$. This turns out to be a reproducing kernel Hilbert space with inner product given by

$$\langle f, g \rangle_{\mathcal{H}(E)} := \int_{\mathbb{R}} f(x) \, \overline{g(x)} \, |E(x)|^{-2} \, \mathrm{d}x.$$

Associated to E, one can consider a pair of real entire functions A and B such that $E(z) = A(z) - iB(z)$. These companion functions are given by

$$A(z) := \frac{1}{2}\big(E(z) + E^*(z)\big) \quad \text{and} \quad B(z) := \frac{i}{2}\big(E(z) - E^*(z)\big),$$

and note that they can only have real roots, by the Hermite-Biehler condition. The reproducing kernel, that we denote by $K(w, \cdot)$, is given by (see [5, Theorem 19])

$$K(w, z) = \frac{B(z)A(\overline{w}) - A(z)B(\overline{w})}{\pi(z - \overline{w})} \,, \tag{2.6}$$

and, when $z = \overline{w}$, one has

$$K(\overline{z}, z) = \frac{B'(z)A(z) - A'(z)B(z)}{\pi}. \tag{2.7}$$

The set of functions $\Gamma_A := \{K(\xi, \cdot) \,; A(\xi) = 0\}$ is always an orthogonal set in $\mathcal{H}(E)$. If $A \notin \mathcal{H}(E)$, the set Γ_A is an orthogonal basis of $\mathcal{H}(E)$ and, if $A \in \mathcal{H}(E)$, the only elements of $\mathcal{H}(E)$ that are orthogonal to Γ_A are the constant multiples of A. The same statements hold for the set $\Gamma_B := \{K(\xi, \cdot) \,; B(\xi) = 0\}$ and these are specializations of a more general theorem in the theory; see [5, Theorem 22]. In particular, if $A \notin \mathcal{H}(E)$, for every $f \in \mathcal{H}(E)$ we have

$$f(z) = \sum_{A(\xi)=0} \frac{f(\xi)}{K(\xi, \xi)} \, K(\xi, z) \quad \text{and} \quad \|f\|_{\mathcal{H}(E)}^2 = \sum_{A(\xi)=0} \frac{|f(\xi)|^2}{K(\xi, \xi)}. \tag{2.8}$$

Analogous formulas would hold with A replaced by B. These formulas are fundamental for our purposes, as they provide a general analogue of the theory of Fourier series to this broad setup.

2.2.2. *A class of homogeneous de Branges spaces.* Let $\nu > -1$. A de Branges space $\mathcal{H}(E)$ is said to be homogeneous of order ν if, for all $0 < a < 1$ and all $f \in \mathcal{H}(E)$, the function $z \mapsto a^{\nu+1} f(az)$ belongs to $\mathcal{H}(E)$ and has the same norm as f. Such spaces were characterized by L. de Branges in [4] (see also [5, Section 50]).

For $\nu > 1$, let A_ν and B_ν be the real entire functions defined in (1.7) and (1.8). The function

$$E_\nu(z) = A_\nu(z) - iB_\nu(z)$$

---

[2]A function f, analytic in $\mathbb{C}^+$, has *bounded type* if it can be written as the quotient of two functions that are analytic and bounded in $\mathbb{C}^+$. If f has bounded type in $\mathbb{C}^+$, from its Nevanlinna factorization [5, Theorems 9 and 10] one has $v(f) := \limsup_{y \to \infty} y^{-1} \log |f(iy)| < \infty$. The number $v(f)$ is called the *mean type* of f.

turns out to be a Hermite-Biehler function with no real zeros and $\mathcal{H}(E_\nu)$ is homogeneous of order ν. Moreover we have $\tau(A_\nu) = \tau(B_\nu) = \tau(E_\nu) = 1$. Note also that $A_\nu, B_\nu \notin \mathcal{H}(E_\nu)$ (just observe the behaviour of the Bessel function J_ν at infinity) and hence the formulas in (2.8) hold. See [5, Section 50], [21, Section 5] or [12, Sections 3 and 4] for further details. We gather other relevant facts about the spaces $\mathcal{H}(E_\nu)$ in the next lemma, which is contained in [21, Eqs. (5.1), (5.2) and Lemma 16].

Lemma 10. *Let $\nu > -1$. The following properties hold:*

(i) *There exist positive constants a_ν, b_ν such that*

$$a_\nu |x|^{2\nu+1} \leq |E_\nu(x)|^{-2} \leq b_\nu |x|^{2\nu+1}$$

for all $x \in \mathbb{R}$ with $|x| \geq 1$.

(ii) *For $f \in \mathcal{H}(E_\nu)$ we have the identity*

$$\int_\mathbb{R} |f(x)|^2 \, |E_\nu(x)|^{-2} \, \mathrm{d}x = c_\nu \int_\mathbb{R} |f(x)|^2 \, |x|^{2\nu+1} \, \mathrm{d}x \,, \tag{2.9}$$

with $c_\nu = \pi \, 2^{-2\nu-1} \, \Gamma(\nu+1)^{-2}$.

(iii) *An entire function f belongs to $\mathcal{H}(E_\nu)$ if and only if f has exponential type at most 1 and*

$$\int_\mathbb{R} |f(x)|^2 \, |x|^{2\nu+1} \, \mathrm{d}x < \infty.$$

By Lemma 10 note that our space $\mathcal{H}_\nu(1\,;1)$ defined in (1.1) is exactly the same as $\mathcal{H}(E_\nu)$, with norms differing just by a multiplicative constant c_ν given in (2.9). This is the key identity that connects the theory of de Branges spaces to our weighted Paley-Wiener spaces.

2.3. **Lifts and radial symmetrization.** We now recall two basic constructions from the work of Holt and Vaaler [21, Section 6].

If $f : \mathbb{C} \to \mathbb{C}$ is an even entire function with power series representation

$$f(z) = \sum_{k=0}^\infty c_k z^{2k} \,,$$

we define the lift $\mathcal{L}_d(f) : \mathbb{C}^d \to \mathbb{C}$ by

$$\mathcal{L}_d(f)(\boldsymbol{z}) := \sum_{k=0}^\infty c_k (z_1^2 + \ldots + z_d^2)^k.$$

For an entire function $F : \mathbb{C}^d \to \mathbb{C}$, with $d > 1$, we define its radial symmetrization $\widetilde{F} : \mathbb{C}^d \to \mathbb{C}$ by

$$\widetilde{F}(\boldsymbol{z}) := \int_{SO(d)} F(R\boldsymbol{z}) \, \mathrm{d}\sigma(R),$$

where $SO(d)$ denotes the compact topological group of real orthogonal $d \times d$ matrices with determinant 1, with associated Haar measure σ normalized so that $\sigma(SO(d)) = 1$. If $d = 1$ we simply set $\widetilde{F}(z) := \frac{1}{2}\{F(z) + F(-z)\}$. The next result is a compilation of [21, Lemmas 18 and 19].

Lemma 11. *The following propositions hold:*

(i) *Let $f : \mathbb{C} \to \mathbb{C}$ be an even entire function. Then f has exponential type if and only if $\mathcal{L}_d(f)$ has exponential type, and $\tau(f) = \tau(\mathcal{L}_d(f))$.*

(ii) *Let $F : \mathbb{C}^d \to \mathbb{C}$ be an entire function. Then $\widetilde{F} : \mathbb{C}^d \to \mathbb{C}$ is an entire function with power series expansion of the form*

$$\widetilde{F}(\boldsymbol{z}) = \sum_{k=0}^{\infty} c_k (z_1^2 + \ldots + z_d^2)^k.$$

Moreover, if F has exponential type then $\widetilde{F}$ has exponential type and $\tau(\widetilde{F}) \leq \tau(F)$.

2.4. **Krein's decomposition.** We now recall a suitable version of a classical result of Krein on the decomposition of entire functions of exponential type that are non-negative on $\mathbb{R}$; see [1, p. 154]. The version presented below is a particular case of a more general version for de Branges spaces; see [12, Lemma 14].

Lemma 12 (Krein's decomposition). *Let $\nu > -1$ and $\delta > 0$. Let f be a real entire function of exponential type at most 2δ, that is non-negative on $\mathbb{R}$ and belongs to $L^1(\mathbb{R}, |x|^{2\nu+1}\, \mathrm{d}x)$. Then there exists $g \in \mathcal{H}_\nu(1\,;\delta)$ such that*

$$f(z) = g(z)\, g^*(z)$$

for all $z \in \mathbb{C}$.

2.5. **Estimates for Bessel functions.** The following result is classical in the theory of Bessel functions. We include a short proof for convenience.

Lemma 13. *Let $\nu > -1$. For any $0 \leq x \leq 2\sqrt{(\nu + 1)}$ we have*

$$0 \leq 1 - \frac{x^2}{4(\nu + 1)} \leq A_\nu(x) \leq 1 - \frac{x^2}{4(\nu + 1)} + \frac{x^4}{32(\nu + 1)(\nu + 2)} \leq 1. \tag{2.10}$$

Also, for any $0 \leq x \leq 2\sqrt{(\nu + 2)}$ we have

$$0 \leq \frac{x}{2(\nu + 1)} - \frac{x^3}{8(\nu + 1)(\nu + 2)} \leq B_\nu(x) \leq \frac{x}{2(\nu + 1)}. \tag{2.11}$$

Proof. The proof is by comparing consecutive terms in the series expansions (1.7) and (1.8). For instance, in order to show (2.10) we compare the terms $n = 2k$ and $n = 2k + 1$ in the series (1.7). We observe that

$$\frac{(-1)^{2k}\left(\frac{1}{2}x\right)^{4k}}{(2k)!(\nu + 1)(\nu + 2) \ldots (\nu + 2k)} \geq -\frac{(-1)^{2k+1}\left(\frac{1}{2}x\right)^{4k+2}}{(2k + 1)!(\nu + 1)(\nu + 2) \ldots (\nu + 2k + 1)}$$

holds for $x \geq 0$ if and only if

$$1 \geq \frac{x^2}{4(2k + 1)(\nu + 2k + 1)}.$$

In the worst case scenario, i.e. when $k = 0$, this holds if $x \leq 2\sqrt{(\nu + 1)}$. This plainly leads us to the inequalities on the left-hand side of (2.10). The inequality on the right-hand side of (2.10) is proved

analogously, comparing the terms $n = 2k+1$ and $n = 2k+2$, for $k \geq 1$, in the series (1.7). Similarly, one arrives at (2.11) via the series (1.8). $\qquad\square$

The next result is a basic upper bound that suffices for our purposes in §7.3. For $\nu > -1$, let K_ν be the reproducing kernel of the Hilbert space $\mathcal{H}(E_\nu)$ defined in §2.2.

Lemma 14. *For $-1 < \nu \leq 0$ we have, for $0 \leq x \leq 2\sqrt{\nu+1}$,*

$$\pi K_\nu(x,x) \leq \frac{1}{2(\nu+1)} + \frac{x^2}{4(\nu+1)^2}. \tag{2.12}$$

Proof. From (2.7) and (1.10) we have

$$\pi K_\nu(x,x) = A_\nu(x)^2 + B_\nu(x)^2 - \frac{(2\nu+1)A_\nu(x)B_\nu(x)}{x}.$$

When $-1 < \nu \leq -\frac{1}{2}$, then $2\nu + 1 \leq 0$ and may use the upper bounds in (2.10) and (2.11) to get

$$\pi K_\nu(x,x) \leq 1 + \frac{x^2}{4(\nu+1)^2} - \frac{(2\nu+1)}{2(\nu+1)} = \frac{1}{2(\nu+1)} + \frac{x^2}{4(\nu+1)^2}.$$

If $-\frac{1}{2} < \nu \leq 0$, then $2\nu + 1 > 0$ and we may use again (2.10) and (2.11) (now both upper and lower bounds) to get

$$\pi K_\nu(x,x) \leq \left(1 - \frac{x^2}{4(\nu+1)} + \frac{x^4}{32(\nu+1)(\nu+2)}\right) + \frac{x^2}{4(\nu+1)^2}$$

$$- (2\nu+1)\left(1 - \frac{x^2}{4(\nu+1)}\right)\left(\frac{1}{2(\nu+1)} - \frac{x^2}{8(\nu+1)(\nu+2)}\right)$$

$$= \frac{1}{2(\nu+1)} + \frac{x^2}{4(\nu+1)^2} + \frac{x^2}{32(\nu+1)^2(\nu+2)}\left(8\nu^2 + 8\nu - 4 - \nu x^2\right).$$

Observe that in this range we have $8\nu^2 + 8\nu - 4 - \nu x^2 \leq 0$, which leads us to (2.12). $\qquad\square$

3. Dimension shifts: proof of Theorem 1

We now introduce an extremal problem that is closely related to our problem (EP1).

Extremal Problem 3 (EP3): Let $\alpha \geq \beta > -1$ and $\delta > 0$ be real parameters, and $d \in \mathbb{N}$. Let $\mathcal{W}_\alpha^+(d;2\delta)$ be the set of real entire functions $M : \mathbb{C}^d \to \mathbb{C}$ of exponential type at most 2δ that are non-negative on $\mathbb{R}^d$ and such that

$$\int_{\mathbb{R}^d} M(\boldsymbol{x})\,|\boldsymbol{x}|^{2\alpha+2-d}\,\mathrm{d}\boldsymbol{x} < \infty.$$

Find

$$(\mathbb{EP}3)(\alpha,\beta\,;d\,;\delta) := \inf_{0 \neq M \in \mathcal{W}_\alpha^+(d;2\delta)} \frac{\int_{\mathbb{R}^d} M(\boldsymbol{x})\,|\boldsymbol{x}|^{2\alpha+2-d}\,\mathrm{d}\boldsymbol{x}}{\int_{\mathbb{R}^d} M(\boldsymbol{x})\,|\boldsymbol{x}|^{2\beta+2-d}\,\mathrm{d}\boldsymbol{x}}. \tag{3.1}$$

We now proceed to the proof of Theorem 1.

3.0.1. *Step 1.* Observe first that if $F \in \mathcal{H}_\alpha(d;\delta)$ then $M(\boldsymbol{z}) = F(\boldsymbol{z})F^*(\boldsymbol{z}) \in \mathcal{W}_\alpha^+(d;2\delta)$. Hence

$$(\mathbb{EP}1)(\alpha,\beta\,;d\,;\delta) \geq (\mathbb{EP}3)(\alpha,\beta\,;d\,;\delta). \tag{3.2}$$

3.0.2. *Step 2.* If $M \in \mathcal{W}_\alpha^+(d; 2\delta)$, by Lemma 11 (ii) we have that $\widetilde{M} \in \mathcal{W}_\alpha^+(d; 2\delta)$ and an application of Fubini's theorem yields

$$\int_{\mathbb{R}^d} M(\boldsymbol{x}) \, |\boldsymbol{x}|^{2\gamma+2-d} \, \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} \widetilde{M}(\boldsymbol{x}) \, |\boldsymbol{x}|^{2\gamma+2-d} \, \mathrm{d}\boldsymbol{x}$$

for any $\gamma > -1$. In particular, we can restrict our search for the infimum in (3.1) to functions $M \in \mathcal{W}_\alpha^+(d; 2\delta)$ that are radial on $\mathbb{R}^d$. By Lemma 11, such M is the lift $\mathcal{L}_d(f)$ of an even entire function $f \in \mathcal{W}_\alpha^+(1; 2\delta)$, and conversely. Reducing to polar coordinates we have

$$\frac{\int_{\mathbb{R}^d} M(\boldsymbol{x}) \, |\boldsymbol{x}|^{2\alpha+2-d} \, \mathrm{d}\boldsymbol{x}}{\int_{\mathbb{R}^d} M(\boldsymbol{x}) \, |\boldsymbol{x}|^{2\beta+2-d} \, \mathrm{d}\boldsymbol{x}} = \frac{\int_{\mathbb{R}} f(x) \, |x|^{2\alpha+1} \, \mathrm{d}x}{\int_{\mathbb{R}} f(x) \, |x|^{2\beta+1} \, \mathrm{d}x} \,,$$

and we conclude that

$$(\mathbb{EP}3)(\alpha, \beta \,; d \,; \delta) = (\mathbb{EP}3)(\alpha, \beta \,; 1 \,; \delta). \tag{3.3}$$

3.0.3. *Step 3.* By Krein's decomposition (Lemma 12 above), every $f \in \mathcal{W}_\alpha^+(1; 2\delta)$ can be written as $f(z) = g(z) \, g^*(z)$ with $g \in \mathcal{H}_\alpha(1; \delta)$, and conversely. This implies that

$$(\mathbb{EP}3)(\alpha, \beta \,; 1 \,; \delta) = (\mathbb{EP}1)(\alpha, \beta \,; 1 \,; \delta). \tag{3.4}$$

3.0.4. *Step 4.* We now use a fact that will be proved independently in the next section: there exists an even extremizer $g \in \mathcal{H}_\alpha(1; \delta)$ for $(\mathbb{EP}1)(\alpha, \beta \,; 1 \,; \delta)$. By Lemma 11 (i), we may lift it to $\mathcal{L}_d(g) \in \mathcal{H}_\alpha(d; \delta)$, and we find that

$$(\mathbb{EP}1)(\alpha, \beta \,; 1 \,; \delta) = \frac{\int_{\mathbb{R}} |g(x)|^2 \, |x|^{2\alpha+1} \, \mathrm{d}x}{\int_{\mathbb{R}} |g(x)|^2 \, |x|^{2\beta+1} \, \mathrm{d}x} = \frac{\int_{\mathbb{R}^d} |\mathcal{L}_d(g)(\boldsymbol{x})|^2 \, |\boldsymbol{x}|^{2\alpha+2-d} \, \mathrm{d}\boldsymbol{x}}{\int_{\mathbb{R}^d} |\mathcal{L}_d(g)(\boldsymbol{x})|^2 \, |\boldsymbol{x}|^{2\beta+2-d} \, \mathrm{d}\boldsymbol{x}} \geq (\mathbb{EP}1)(\alpha, \beta \,; d \,; \delta). \tag{3.5}$$

3.0.5. *Conclusion.* Combining (3.2), (3.3), (3.4) and (3.5) we arrive at

$$(\mathbb{EP}1)(\alpha, \beta \,; d \,; \delta) \geq (\mathbb{EP}3)(\alpha, \beta \,; d \,; \delta) = (\mathbb{EP}3)(\alpha, \beta \,; 1 \,; \delta) = (\mathbb{EP}1)(\alpha, \beta \,; 1 \,; \delta) \geq (\mathbb{EP}1)(\alpha, \beta \,; d \,; \delta),$$

and hence we must have the equalities

$$(\mathbb{EP}1)(\alpha, \beta \,; d \,; \delta) = (\mathbb{EP}3)(\alpha, \beta \,; d \,; \delta) = (\mathbb{EP}3)(\alpha, \beta \,; 1 \,; \delta) = (\mathbb{EP}1)(\alpha, \beta \,; 1 \,; \delta).$$

4. Existence of extremizers: proof of Theorem 2

4.1. **Existence of extremizers in dimension** $d = 1$**.** We now want to establish the existence of extremizers for $(\mathbb{EP}1)(\alpha, \beta \,; 1 \,; \delta)$. By dilation, it suffices to consider $\delta = 1$. To shorten the notation, let us write $(\mathbb{EP}1)(\alpha, \beta \,; 1 \,; 1)$ in this subsection simply as $(\mathbb{EP}1)$. Assume also that $\alpha > \beta$.

Let $\{f_n\}_{n \geq 1} \subset \mathcal{H}_\alpha(1; 1)$ be an extremizing sequence, normalized so that $\|f_n\|_{\mathcal{H}_\alpha(1;1)} = 1$. This means that

$$\|f_n\|_{\mathcal{H}_\beta(1;1)}^2 \to (\mathbb{EP}1)^{-1}.$$

By reflexivity, up to a subsequence, we may assume that $f_n \rightharpoonup g$ for some $g \in \mathcal{H}_\alpha(1; 1)$. Note that

$$\|g\|_{\mathcal{H}_\alpha(1;1)} \leq 1. \tag{4.1}$$

Since $\mathcal{H}_\alpha(1\,;1) = \mathcal{H}(E_\alpha)$ as sets, with norms differing by a multiplicative constant (Lemma 10), this is a reproducing kernel Hilbert space; see §2.2. Then, for any $w \in \mathbb{C}$, we have the pointwise convergence

$$f_n(w) = \langle f_n, K_\alpha(w, \cdot) \rangle_{\mathcal{H}(E_\alpha)} \to \langle g, K_\alpha(w, \cdot) \rangle_{\mathcal{H}(E_\alpha)} = g(w). \tag{4.2}$$

By the Cauchy-Schwarz inequality, we also get that

$$|f_n(w)| = |\langle f_n, K_\alpha(w, \cdot) \rangle_{\mathcal{H}(E_\alpha)}| \leq \|f_n\|_{\mathcal{H}(E_\alpha)} \|K_\alpha(w, \cdot)\|_{\mathcal{H}(E_\alpha)} = c_\alpha^{1/2} K_\alpha(w, w)^{1/2}, \tag{4.3}$$

and since K_α is a continuous function of two variables, we get that f_n is uniformly bounded in compact subsets of $\mathbb{C}$.

Given any $a < (\mathbb{EP}1)^{-1}$, there exists an $N_0 = N_0(a)$ such that for $n \geq N_0$ we have

$$\begin{aligned}
a &\leq \int_{\mathbb{R}} |f_n(x)|^2 \, |x|^{2\beta+1} \, \mathrm{d}x \\
&= \int_{|x| \leq R} |f_n(x)|^2 \, |x|^{2\beta+1} \, \mathrm{d}x + \int_{|x| > R} |f_n(x)|^2 \, |x|^{2\beta+1} \, \mathrm{d}x \\
&\leq \int_{|x| \leq R} |f_n(x)|^2 \, |x|^{2\beta+1} \, \mathrm{d}x + R^{2\beta-2\alpha} \int_{|x| > R} |f_n(x)|^2 \, |x|^{2\alpha+1} \, \mathrm{d}x \\
&\leq \int_{|x| \leq R} |f_n(x)|^2 \, |x|^{2\beta+1} \, \mathrm{d}x + R^{2\beta-2\alpha},
\end{aligned} \tag{4.4}$$

for any fixed R, where we have used that $\alpha > \beta$ and $\|f_n\|_{\mathcal{H}_\alpha(1;1)} = 1$. Letting $n \to \infty$ and applying the dominated convergence theorem in (4.4) we get

$$a - R^{2\beta-2\alpha} \leq \int_{|x| \leq R} |g(x)|^2 \, |x|^{2\beta+1} \, \mathrm{d}x \leq \int_{\mathbb{R}} |g(x)|^2 \, |x|^{2\beta+1} \, \mathrm{d}x. \tag{4.5}$$

Letting $R \to \infty$ in (4.5) we arrive at

$$a \leq \int_{\mathbb{R}} |g(x)|^2 \, |x|^{2\beta+1} \, \mathrm{d}x.$$

This shows that $0 \neq g$ and, since $a < (\mathbb{EP}1)^{-1}$ is arbitrary, we conclude that

$$(\mathbb{EP}1)^{-1} \leq \int_{\mathbb{R}} |g(x)|^2 \, |x|^{2\beta+1} \, \mathrm{d}x. \tag{4.6}$$

In particular $(\mathbb{EP}1)^{-1}$ is finite (as we had already observed with a different reasoning in the introduction). From the setup of our problem, we must have equalities both in (4.1) and (4.6), and $g \in \mathcal{H}_\alpha(1\,;1)$ is therefore an extremizer.

4.2. **Even extremizers in dimension $d = 1$.** Having established the existence of extremizers in dimension $d = 1$ in §4.1, we now prove the following complementary result.

Proposition 15. *Let $\alpha > \beta > -1$ and $\delta > 0$ be real parameters. Any extremizer for $(\mathbb{EP}1)(\alpha, \beta\,;1\,;\delta)$ must be an even function.*

Proof. Let $g \in \mathcal{H}_\alpha(1\,;\delta)$ be an extremizer of $(\mathbb{EP}1)(\alpha, \beta\,;1\,;\delta)$. Write

$$g(z) = g_e(z) + g_o(z),$$

where $g_e(z) := \frac{1}{2}(g(z) + g(-z))$ and $g_o(z) := \frac{1}{2}(g(z) - g(-z))$ are the even and odd parts of g. Due to the orthogonality between g_e and g_o observe that

$$(\mathbb{EP}1)(\alpha, \beta\,;1\,;\delta) = \frac{\int_{\mathbb{R}} |g(x)|^2\,|x|^{2\alpha+1}\,\mathrm{d}x}{\int_{\mathbb{R}} |g(x)|^2\,|x|^{2\beta+1}\,\mathrm{d}x} = \frac{\int_{\mathbb{R}} |g_e(x)|^2\,|x|^{2\alpha+1}\,\mathrm{d}x + \int_{\mathbb{R}} |g_o(x)|^2\,|x|^{2\alpha+1}\,\mathrm{d}x}{\int_{\mathbb{R}} |g_e(x)|^2\,|x|^{2\beta+1}\,\mathrm{d}x + \int_{\mathbb{R}} |g_o(x)|^2\,|x|^{2\beta+1}\,\mathrm{d}x}. \qquad (4.7)$$

Assume g_o is not identically zero. Since g_e and g_o are both in $\mathcal{H}_\alpha(1\,;\delta)$, by (4.7) we observe that g_o must also be an extremizer. We may write $g_o(z) = z\,h(z)$ with $h \in \mathcal{H}_{\alpha+1}(1\,;\delta) \subset \mathcal{H}_\alpha(1\,;\delta)$. Then, by the setup of our problem,

$$\frac{\int_{\mathbb{R}} |h(x)|^2\,|x|^{2\alpha+3}\,\mathrm{d}x}{\int_{\mathbb{R}} |h(x)|^2\,|x|^{2\beta+3}\,\mathrm{d}x} = \frac{\int_{\mathbb{R}} |g_o(x)|^2\,|x|^{2\alpha+1}\,\mathrm{d}x}{\int_{\mathbb{R}} |g_o(x)|^2\,|x|^{2\beta+1}\,\mathrm{d}x} = (\mathbb{EP}1)(\alpha, \beta\,;1\,;\delta) \leq \frac{\int_{\mathbb{R}} |h(x)|^2\,|x|^{2\alpha+1}\,\mathrm{d}x}{\int_{\mathbb{R}} |h(x)|^2\,|x|^{2\beta+1}\,\mathrm{d}x}. \qquad (4.8)$$

Letting $\mathrm{d}\mu(x) = |h(x)|^2\,|x|^{2\beta+1}\,\mathrm{d}x$ and assuming without loss of generality that $\mathrm{d}\mu$ is normalized so that $\int_{\mathbb{R}} \mathrm{d}\mu(x) = 1$, the inequality in (4.8) can be rewritten as

$$\int_{\mathbb{R}} |x|^{2\gamma+2}\,\mathrm{d}\mu(x) \leq \left(\int_{\mathbb{R}} |x|^{2\gamma}\,\mathrm{d}\mu(x)\right)\left(\int_{\mathbb{R}} |x|^2\,\mathrm{d}\mu(x)\right), \qquad (4.9)$$

with $\gamma = \alpha - \beta > 0$. On the other hand, by Hölder's inequality we have

$$\int_{\mathbb{R}} |x|^{2\gamma}\,\mathrm{d}\mu(x) \leq \left(\int_{\mathbb{R}} |x|^{2\gamma+2}\,\mathrm{d}\mu(x)\right)^{\frac{2\gamma}{2\gamma+2}}\left(\int_{\mathbb{R}} \mathrm{d}\mu(x)\right)^{\frac{2}{2\gamma+2}} \qquad (4.10)$$

and

$$\int_{\mathbb{R}} |x|^2\,\mathrm{d}\mu(x) \leq \left(\int_{\mathbb{R}} |x|^{2\gamma+2}\,\mathrm{d}\mu(x)\right)^{\frac{2}{2\gamma+2}}\left(\int_{\mathbb{R}} \mathrm{d}\mu(x)\right)^{\frac{2\gamma}{2\gamma+2}}, \qquad (4.11)$$

which multiply out to yield

$$\left(\int_{\mathbb{R}} |x|^{2\gamma}\,\mathrm{d}\mu(x)\right)\left(\int_{\mathbb{R}} |x|^2\,\mathrm{d}\mu(x)\right) \leq \int_{\mathbb{R}} |x|^{2\gamma+2}\,\mathrm{d}\mu(x). \qquad (4.12)$$

We must then have equalities in (4.9),(4.10), (4.11) and (4.12). By the case of equality in Hölder's inequality, this implies that $|x|^{2\gamma+2}$ is constant in the support of $\mathrm{d}\mu$, which is a contradiction.

The conclusion is that our original extremizer g must be an even function. $\qquad \square$

4.3. A brief detour: some basic monotonicity properties.

It is worth mentioning the following consequence of Proposition 15.

Proposition 16. *For $\alpha > \beta > -1$ and $\delta > 0$ real parameters, and $d \in \mathbb{N}$, we have*

$$(\mathbb{EP}1)(\alpha, \beta\,;d\,;\delta) < (\mathbb{EP}1)(\alpha+1, \beta+1\,;d\,;\delta). \qquad (4.13)$$

Proof. From Theorem 1 it suffices to verify (4.13) for $d = 1$. For any $0 \neq f \in \mathcal{H}_{\alpha+1}(1\,;\delta)$ we have $f(z)\,z \in \mathcal{H}_\alpha(1\,;\delta)$ and hence

$$(\mathbb{EP}1)(\alpha, \beta\,;1\,;\delta) \leq \frac{\int_{\mathbb{R}} |f(x)\,x|^2\,|x|^{2\alpha+1}\,\mathrm{d}x}{\int_{\mathbb{R}} |f(x)\,x|^2\,|x|^{2\beta+1}\,\mathrm{d}x} = \frac{\int_{\mathbb{R}} |f(x)|^2\,|x|^{2\alpha+3}\,\mathrm{d}x}{\int_{\mathbb{R}} |f(x)|^2\,|x|^{2\beta+3}\,\mathrm{d}x}.$$

This plainly leads us to the inequality

$$(\mathbb{EP}1)(\alpha, \beta\,;1\,;\delta) \leq (\mathbb{EP}1)(\alpha+1, \beta+1\,;1\,;\delta). \tag{4.14}$$

If we had equality in (4.14), letting f be an extremizer for $(\mathbb{EP}1)(\alpha+1, \beta+1\,;1\,;\delta)$, which is even by Proposition 15, the function $f(z)\,z$ would be an odd extremizer to $(\mathbb{EP}1)(\alpha, \beta\,;1\,;\delta)$, a contradiction. $\qquad\square$

In sympathy with our asymptotics of Theorem 4, we remark that the following monotonicity property also holds.

Proposition 17. *For $\alpha > \beta > -1$ and $\delta > 0$ real parameters, and $d \in \mathbb{N}$, the map*

$$\alpha \mapsto \frac{1}{(\alpha-\beta)}\,\log\big((\mathbb{EP}1)(\alpha, \beta\,;d\,;\delta)\big)$$

is non-decreasing.

Proof. From Theorem 1 it suffices to prove these claims for $d=1$. Let $\alpha_2 > \alpha_1 > \beta$ and $f \in \mathcal{H}_{\alpha_2}(1;\delta)$ be normalized so that $\int_{\mathbb{R}} |f(x)|^2\,|x|^{2\beta+1}\,\mathrm{d}x = 1$. Write $\mathrm{d}\mu(x) = |f(x)|^2\,|x|^{2\beta+1}\,\mathrm{d}x$. Applying Hölder's inequality with exponent $p = (\alpha_2-\beta)/(\alpha_1-\beta)$ we get

$$(\mathbb{EP}1)(\alpha_1, \beta\,;1\,;\delta) \leq \int_{\mathbb{R}} |x|^{2(\alpha_1-\beta)}\mathrm{d}\mu(x) \leq \left(\int_{\mathbb{R}} |x|^{2(\alpha_2-\beta)}\mathrm{d}\mu(x)\right)^{1/p}.$$

This leads us to

$$(\mathbb{EP}1)(\alpha_1, \beta\,;1\,;\delta)^{1/(\alpha_1-\beta)} \leq (\mathbb{EP}1)(\alpha_2, \beta\,;1\,;\delta)^{1/(\alpha_2-\beta)}, \tag{4.15}$$

as desired. $\qquad\square$

4.4. **Existence of extremizers in higher dimensions.** Let g be an extremizer for $(\mathbb{EP}1)(\alpha, \beta\,;1\,;\delta)$. By Proposition 15, g is even. From Theorem 1 one plainly verifies that the lift $\mathcal{L}_d(g)$ is a radial extremizer for $(\mathbb{EP}1)(\alpha, \beta\,;d\,;\delta)$.

5. CONTINUITY: PROOF OF THEOREM 3

In light of Theorem 1 and the dilation relation (1.4), in order to prove Theorem 3 it suffices to show that $(\alpha, \beta) \mapsto (\mathbb{EP}1)(\alpha, \beta\,;1\,;1)$ is continuous in range $\alpha \geq \beta > -1$.

5.1. **Approximation by Schwartz functions.** Let $\mathcal{S}(\mathbb{R})$ be the class of Schwartz functions. We shall first prove the following lemma.

Lemma 18. *Let $\alpha \geq \beta > -1$. Then*

$$(\mathbb{EP}1)(\alpha, \beta\,;1\,;1) = \inf_{0 \neq f \in \mathcal{S}(\mathbb{R}) \cap \mathcal{H}_\alpha(1;1)} \frac{\int_{\mathbb{R}} |f(x)|^2\,|x|^{2\alpha+1}\,\mathrm{d}x}{\int_{\mathbb{R}} |f(x)|^2\,|x|^{2\beta+1}\,\mathrm{d}x}.$$

In other words, when finding $(\mathbb{EP}1)(\alpha, \beta\,;1\,;1)$, one can take the infimum over the subclass of the entire functions f of exponential type at most 1 whose restriction to $\mathbb{R}$ belongs to $\mathcal{S}(\mathbb{R})$ (which, by the classical Paley-Wiener theorem, is equivalent to saying that $f \in \mathcal{S}(\mathbb{R})$ and $\mathrm{supp}(\widehat{f}) \subset \big[-\frac{1}{2\pi}, \frac{1}{2\pi}\big]$).

Proof. Let $\varphi \in C^\infty(\mathbb{R})$ be an even function such that $\mathrm{supp}(\varphi) \subset [-1,1]$; $\varphi \geq 0$; $\widehat{\varphi} \geq 0$; and $\int_\mathbb{R} \varphi(y)\,\mathrm{d}y = 1$. As usual, for $\varepsilon > 0$, set $\varphi_\varepsilon(y) := \frac{1}{\varepsilon}\varphi\left(\frac{y}{\varepsilon}\right)$. Then $\widehat{\varphi_\varepsilon}(x) = \widehat{\varphi}(\varepsilon x)$, and we see that $0 \leq \widehat{\varphi_\varepsilon}(x) \leq \widehat{\varphi_\varepsilon}(0) = 1$ and $\lim_{\varepsilon \to 0} \widehat{\varphi_\varepsilon}(x) = 1$ for all $x \in \mathbb{R}$.

Let $0 \neq f \in \mathcal{H}_\alpha(1;1)$ be an extremizer for $(\mathbb{EP}1)(\alpha, \beta\,;1\,;1)$ (being an extremizer here is not really crucial, it would suffice to take a near-extremizer in this argument). Set

$$f_\varepsilon(x) := f(x)\,\widehat{\varphi_\varepsilon}(x).$$

We claim that $f_\varepsilon \in \mathcal{S}(\mathbb{R})$. In fact, note that $f_\varepsilon^{(n)} = \sum_{k=0}^n \binom{n}{k} f^{(k)} \widehat{\varphi_\varepsilon}^{(n-k)}$ and recall the fact that the de Branges space $\mathcal{H}(E_\alpha)$ (equal to $\mathcal{H}_\alpha(1;1)$ as sets, with norms differing by a multiplicative constant) is closed under differentiation; see [12, Theorem 20]. To verify our claim, it then suffices to prove that any $h \in \mathcal{H}(E_\alpha)$ has at most polynomial growth. For this, recall the pointwise bound (4.3)

$$|h(x)| \leq \|h\|_{\mathcal{H}(E_\alpha)}\, K_\alpha(x,x)^{1/2}\,,$$

and the fact that $x \mapsto K_\alpha(x,x)$ is even. For $x > 0$, using (2.7), (1.10) and (1.9), we have

$$\pi K_\alpha(x,x) = A_\alpha(x)^2 + B_\alpha(x)^2 - \frac{(2\alpha+1)A_\alpha(x)B_\alpha(x)}{x}$$
$$= \frac{2^{2\alpha}\,\Gamma(\alpha+1)^2}{x^{2\alpha+1}}\left(x\,J_\alpha(x)^2 + x\,J_{\alpha+1}(x)^2 - (2\alpha+1)J_\alpha(x)\,J_{\alpha+1}(x)\right).$$

From the well-known asymptotic

$$J_\alpha(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O_\alpha\left(x^{-3/2}\right),$$

one gets that $K_\alpha(x,x) \sim x^{-2\alpha-1}$ for x large.

We now observe that $\widehat{f_\varepsilon} = \widehat{f} * \varphi_\varepsilon$. If $\alpha < -\frac{1}{2}$, note that $\widehat{f}$ is in principle only a tempered distribution, but nevertheless $\widehat{f} * \varphi_\varepsilon$ is a function; see [35, Theorem 3.13, Chapter I]. From the Paley-Wiener Theorem (one needs the version for distributions in [22, Theorem 1.7.5 and Theorem 1.7.7] if $\alpha < -\frac{1}{2}$) we have $\mathrm{supp}(\widehat{f}) \subset \left[-\frac{1}{2\pi}, \frac{1}{2\pi}\right]$ and this leads to $\mathrm{supp}(\widehat{f_\varepsilon}) \subset \left[-\frac{1}{2\pi} - \varepsilon, \frac{1}{2\pi} + \varepsilon\right]$. We now have just to dilate back by a factor that tends to 1, by considering

$$g^\varepsilon(x) := f_\varepsilon\left(\left(\frac{\frac{1}{2\pi}}{\frac{1}{2\pi} + \varepsilon}\right)x\right).$$

Then $\mathrm{supp}(\widehat{g^\varepsilon}) \subset \left[-\frac{1}{2\pi}, \frac{1}{2\pi}\right]$. By dominated convergence (both in the numerator and denominator) note that

$$\lim_{\varepsilon \to 0} \frac{\int_\mathbb{R} |g^\varepsilon(x)|^2\,|x|^{2\alpha+1}\,\mathrm{d}x}{\int_\mathbb{R} |g^\varepsilon(x)|^2\,|x|^{2\beta+1}\,\mathrm{d}x} = \frac{\int_\mathbb{R} |f(x)|^2\,|x|^{2\alpha+1}\,\mathrm{d}x}{\int_\mathbb{R} |f(x)|^2\,|x|^{2\beta+1}\,\mathrm{d}x}\,,$$

as desired. $\qquad\square$

5.2. **Proof of Theorem 3.** Let $\{\alpha_n\}_{n\geq 1}$ and $\{\beta_n\}_{n\geq 1}$ be sequences such that $\alpha_n \geq \beta_n > -1$ for all n, $\alpha_n \to \alpha$ and $\beta_n \to \beta$ as $n \to \infty$. We want to show that $(\mathbb{EP}1)(\alpha_n, \beta_n\,;1\,;1) \to (\mathbb{EP}1)(\alpha, \beta\,;1\,;1)$ as $n \to \infty$. The case $\alpha = \beta$ will follow from Theorem 4 and we exclude this case from the discussion below. Let us henceforth assume that $\alpha > \beta$.

5.2.1. *Upper bound.* Given $\varepsilon > 0$, by Lemma 18 there exists $\psi \in \mathcal{S}(\mathbb{R})$ of exponential type at most 1 with

$$\Phi_{\alpha,\beta}(\psi) := \frac{\int_{\mathbb{R}} |\psi(x)|^2 \, |x|^{2\alpha+1} \, \mathrm{d}x}{\int_{\mathbb{R}} |\psi(x)|^2 \, |x|^{2\beta+1} \, \mathrm{d}x} \leq (\mathbb{EP}1)(\alpha,\beta\,;1\,;1) + \varepsilon.$$

For such a ψ we have $\Phi_{\alpha_n,\beta_n}(\psi) \to \Phi_{\alpha,\beta}(\psi)$ as $n \to \infty$ by the dominated convergence theorem. This plainly leads us to

$$\limsup_{n\to\infty} (\mathbb{EP}1)(\alpha_n,\beta_n\,;1\,;1) \leq (\mathbb{EP}1)(\alpha,\beta\,;1\,;1). \tag{5.1}$$

5.2.2. *Lower bound.* From Lemma 18, let $\{\psi_n\}_{n\geq 1} \subset \mathcal{S}(\mathbb{R})$ be a sequence of functions of exponential type at most 1 such that

$$\Phi_{\alpha_n,\beta_n}(\psi_n) \leq (\mathbb{EP}1)(\alpha_n,\beta_n\,;1\,;1) + \tfrac{1}{n}. \tag{5.2}$$

Let us normalize ψ_n so that

$$\int_{\mathbb{R}} |\psi_n(x)|^2 \, |x|^{2\beta+1} \, \mathrm{d}x = 1. \tag{5.3}$$

The sequence $\{\psi_n\}_{n\geq 1}$ is then bounded in $\mathcal{H}_\beta(1;1)$ and hence, passing to a subsequence if necessary, we may assume that $\psi_n \rightharpoonup \psi$ for some $\psi \in \mathcal{H}_\beta(1\,;1)$. Since $\mathcal{H}_\beta(1\,;1)$ is a reproducing kernel Hilbert space, we have already seen in (4.2) and (4.3) that this implies that $\psi_n \to \psi$ pointwise everywhere, and that ψ_n is uniformly bounded in compact subsets of $\mathbb{R}$. By (5.3) and Fatou's lemma we get

$$\int_{\mathbb{R}} |\psi(x)|^2 \, |x|^{2\beta+1} \, \mathrm{d}x \leq 1. \tag{5.4}$$

We now state the following auxiliary result to this proof.

Lemma 19. *Let $\alpha > \beta$. Given any $\varepsilon > 0$, there exists $R = R(\varepsilon)$ such that*

$$\int_{[-R,R]^c} |\psi_n(x)|^2 \, |x|^{2\beta+1} \, \mathrm{d}x < \varepsilon, \tag{5.5}$$

and

$$\int_{[-R,R]^c} |\psi_n(x)|^2 \, |x|^{2\beta_n+1} \, \mathrm{d}x < \varepsilon, \tag{5.6}$$

for $n \geq n_0(\varepsilon)$.

Let us assume the validity of Lemma 19 in order to conclude. From (5.3) and (5.5) we get

$$\int_{[-R,R]} |\psi_n(x)|^2 \, |x|^{2\beta+1} \, \mathrm{d}x > 1 - \varepsilon.$$

$n \geq n_0(\varepsilon)$, and by the dominated convergence theorem we have

$$\int_{[-R,R]} |\psi(x)|^2 \, |x|^{2\beta+1} \, \mathrm{d}x \geq 1 - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, in light of (5.4) we conclude that

$$\int_{\mathbb{R}} |\psi(x)|^2 \, |x|^{2\beta+1} \, \mathrm{d}x = 1. \tag{5.7}$$

Using (5.6), (5.7), dominated convergence and Fatou's lemma, one can also conclude that

$$\lim_{n\to\infty} \int_{\mathbb{R}} |\psi_n(x)|^2 |x|^{2\beta_n+1} \,\mathrm{d}x = 1. \tag{5.8}$$

From (5.8), Fatou's lemma and (5.7) we conclude that

$$\liminf_{n\to\infty} (\mathbb{EP}1)(\alpha_n, \beta_n\,;1\,;1) = \liminf_{n\to\infty} \int_{\mathbb{R}} |\psi_n(x)|^2 |x|^{2\alpha_n+1} \,\mathrm{d}x$$
$$\geq \int_{\mathbb{R}} |\psi(x)|^2 |x|^{2\alpha+1} \,\mathrm{d}x \geq (\mathbb{EP}1)(\alpha, \beta\,;1\,;1). \tag{5.9}$$

The desired continuity then follows from (5.1) and (5.9).

5.3. **Proof of Lemma 19.** We first prove (5.6). Assume the result is not true, i.e. that there exists $\varepsilon > 0$ for which no R works. Then, for any $R > 1$, we have

$$\limsup_{n\to\infty} \int_{[-R,R]^c} |\psi_n(x)|^2 |x|^{2\beta_n+1} \,\mathrm{d}x \geq \varepsilon. \tag{5.10}$$

Since $\alpha > \beta$, we may take $\gamma = \frac{\alpha-\beta}{2} > 0$ and assume without loss of generality that $\alpha_n - \beta_m \geq \gamma$ for any n, m. Therefore

$$\int_{[-R,R]^c} |\psi_n(x)|^2 |x|^{2\alpha_n+1} \,\mathrm{d}x \geq R^{2\gamma} \int_{[-R,R]^c} |\psi_n(x)|^2 |x|^{2\beta_n+1} \,\mathrm{d}x. \tag{5.11}$$

Since ψ_n is uniformly bounded in compact subsets of $\mathbb{R}$, we have

$$\left| \int_{[-R,R]} |\psi_n(x)|^2 |x|^{2\beta_n+1} \,\mathrm{d}x - \int_{[-R,R]} |\psi_n(x)|^2 |x|^{2\beta+1} \,\mathrm{d}x \right| \leq \varepsilon$$

for n large. In light of (5.3) this gives us

$$\int_{[-R,R]} |\psi_n(x)|^2 |x|^{2\beta_n+1} \,\mathrm{d}x \leq 1 + \varepsilon \tag{5.12}$$

for n large. From (5.1), (5.2), (5.11) and (5.12), and then (5.10), we have

$$(\mathbb{EP}1)(\alpha, \beta\,;1\,;1) \geq \limsup_{n\to\infty} (\mathbb{EP}1)(\alpha_n, \beta_n\,;1\,;1) = \limsup_{n\to\infty} \Phi_{\alpha_n,\beta_n}(\psi_n)$$
$$\geq \limsup_{n\to\infty} \frac{R^{2\gamma} \int_{[-R,R]^c} |\psi_n(x)|^2 |x|^{2\beta_n+1} \,\mathrm{d}x}{(1+\varepsilon) + \int_{[-R,R]^c} |\psi_n(x)|^2 |x|^{2\beta_n+1} \,\mathrm{d}x}$$
$$\geq \frac{R^{2\gamma}}{\frac{1+\varepsilon}{\varepsilon} + 1}. \tag{5.13}$$

This is a contradiction when R is large.

The proof of (5.5) is in the same spirit. Assume it is not true, i.e. that there exists $\varepsilon > 0$ such that for any $R > 1$ we have

$$\limsup_{n\to\infty} \int_{[-R,R]^c} |\psi_n(x)|^2 |x|^{2\beta+1} \,\mathrm{d}x \geq \varepsilon. \tag{5.14}$$

In particular, let us also only consider $R > R(\varepsilon)$, with $R(\varepsilon)$ being such that (5.6) holds (that we already proved). In replacement of (5.11) we use

$$\int_{[-R,R]^c} |\psi_n(x)|^2 \, |x|^{2\alpha_n+1} \, \mathrm{d}x \geq R^{2\gamma} \int_{[-R,R]^c} |\psi_n(x)|^2 \, |x|^{2\beta+1} \, \mathrm{d}x. \tag{5.15}$$

The bound (5.12) continues to hold for n large. Together with (5.6) this leads us to

$$\int_{\mathbb{R}} |\psi_n(x)|^2 \, |x|^{2\beta_n+1} \, \mathrm{d}x \leq 1 + 2\varepsilon \tag{5.16}$$

for n large. Similarly to (5.13), now using (5.14), (5.15) and (5.16), we obtain

$$\begin{aligned}
(\mathbb{EP}1)(\alpha,\beta\,;1\,;1) &\geq \limsup_{n\to\infty} \, (\mathbb{EP}1)(\alpha_n,\beta_n\,;1\,;1) = \limsup_{n\to\infty} \Phi_{\alpha_n,\beta_n}(\psi_n) \\
&\geq \limsup_{n\to\infty} \frac{R^{2\gamma} \int_{[-R,R]^c} |\psi_n(x)|^2 \, |x|^{2\beta+1} \, \mathrm{d}x}{1 + 2\varepsilon} \\
&\geq \frac{R^{2\gamma} \, \varepsilon}{1 + 2\varepsilon} \,,
\end{aligned}$$

which is again a contradiction when R is large. This concludes the proof of Lemma 19.

6. Radial non-increasing delta majorant: proof of Theorem 8

Let $M \in \mathcal{R}^+(d\,;2\delta)$ and assume that $\int_{\mathbb{R}^d} M(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} < \infty$. By Lemma 11 we know that M is the lift $\mathcal{L}_d(f)$ of a real entire function $f : \mathbb{C} \to \mathbb{C}$ of exponential type at most 2δ that is also radial (i.e. even) non-increasing with

$$\int_{\mathbb{R}^d} M(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \tfrac{1}{2} \, \omega_{d-1} \int_{\mathbb{R}} f(x) \, |x|^{d-1} \, \mathrm{d}x. \tag{6.1}$$

The fact that $M(\boldsymbol{0}) \geq 1$ implies that $f(0) \geq 1$, and of course we may assume that $f(0) = 1$ (otherwise we could dilate down and get a smaller integral). Let $g = f'$. Since f has exponential type at most 2δ and belongs to $L^1(\mathbb{R})$ by (6.1), by a classical result of Plancherel and Pólya [33], so does g. We may then write

$$f(x) = \int_{-\infty}^{x} g(u) \, \mathrm{d}u. \tag{6.2}$$

Using (6.2) and Fubini's theorem we arrive at

$$\int_{\mathbb{R}} f(x) \, |x|^{d-1} \, \mathrm{d}x = \frac{1}{d} \int_{\mathbb{R}} |g(u)| \, |u|^d \, \mathrm{d}u. \tag{6.3}$$

Note that $H(u) = -u \, g(u)$ is a function of exponential type at most 2δ, that is non-negative on $\mathbb{R}$ and belongs to $L^1(\mathbb{R})$. By Krein's decomposition in Lemma 12 we may write $H(z) = z^2 \, h(z) \, h^*(z)$, where h has exponential type at most δ. This implies that

$$g(z) = -z \, h(z) \, h^*(z). \tag{6.4}$$

Since g is odd, the condition that $f(0) = 1$ means that

$$\int_{\mathbb{R}} |g(u)| \, \mathrm{d}u = 2 \, ,$$

which, in light of (6.4), implies

$$\int_{\mathbb{R}} |h(u)|^2 \, |u| \, \mathrm{d}u = 2. \tag{6.5}$$

From (6.1), (6.3) and (6.4) the quantity we want to minimize is

$$\frac{\omega_{d-1}}{2d} \int_{\mathbb{R}} |h(u)|^2 \, |u|^{d+1} \, \mathrm{d}u \, ,$$

and we plainly conclude that

$$(\mathbb{EP}2)(d \, ; \delta) \geq \frac{\omega_{d-1}}{d} \, (\mathbb{EP}1)(\tfrac{d}{2}, 0 \, ; 1 \, ; \delta).$$

Conversely, we have seen that $(\mathbb{EP}1)(\tfrac{d}{2}, 0 \, ; 1 \, ; \delta)$ admits an extremizer $h \in \mathcal{H}_{\frac{d}{2}}(1 \, ; \delta)$ that is even. With the normalization (6.5), we can reverse the steps above by defining g as in (6.4), then f as in (6.2), and then $M = \mathcal{L}_d(f) \in \mathcal{R}^+(d \, ; 2\delta)$ with the help of Lemma 11. This shows that

$$(\mathbb{EP}2)(d \, ; \delta) \leq \frac{\omega_{d-1}}{d} \, (\mathbb{EP}1)(\tfrac{d}{2}, 0 \, ; 1 \, ; \delta).$$

We therefore conclude that

$$(\mathbb{EP}2)(d \, ; \delta) = \frac{\omega_{d-1}}{d} \, (\mathbb{EP}1)(\tfrac{d}{2}, 0 \, ; 1 \, ; \delta) \, ,$$

and that extremizers exist for $(\mathbb{EP}2)(d \, ; \delta)$.

7. Asymptotics: proof of Theorem 4

Throughout this section we work in dimension $d = 1$ and exponential type $\delta = 1$. We may also assume that $\alpha > \beta$.

7.1. Upper bound.
Recall the functions A_α and B_α defined in (1.7) and (1.8), respectively. Note that $A_\alpha(0) = 1$ and $B_\alpha(0) = 0$. Within the framework of §2.2, let us consider the test function

$$f(z) = K_\alpha(0, z) = \frac{B_\alpha(z)}{\pi z}.$$

From the rightmost identity in (1.10), as $z \to 0$, we get $B'_\alpha(0) = 1/(2\alpha + 2)$. Then, by (2.9),

$$\int_{\mathbb{R}} |f(x)|^2 \, |x|^{2\alpha+1} \, \mathrm{d}x = (c_\alpha)^{-1} \, \|f\|^2_{\mathcal{H}(E_\alpha)} = (c_\alpha)^{-1} \, K_\alpha(0,0) = \frac{B'_\alpha(0)}{\pi \, c_\alpha} = \frac{2^{2\alpha+1} \, \Gamma(\alpha+1)^2}{\pi^2 (2\alpha+2)}.$$

Also, from [20, p. 683, §6.574, Eq. 2]

$$\int_{\mathbb{R}} |f(x)|^2 \, |x|^{2\beta+1} \, \mathrm{d}x = \frac{2^{2\alpha+1} \, \Gamma(\alpha+1)^2}{\pi^2} \int_0^\infty |J_{\alpha+1}(x)|^2 \, |x|^{2\beta-2\alpha-1} \, \mathrm{d}x$$

$$= \frac{2^{2\alpha+1} \, \Gamma(\alpha+1)^2}{\pi^2} \, \frac{\Gamma(2\alpha - 2\beta + 1) \, \Gamma(\beta+1)}{2^{2\alpha-2\beta+1} \, \Gamma(\alpha - \beta + 1)^2 \, \Gamma(2\alpha - \beta + 2)}.$$

Therefore we find

$$\frac{\int_{\mathbb{R}} |f(x)|^2\, |x|^{2\alpha+1}\, \mathrm{d}x}{\int_{\mathbb{R}} |f(x)|^2\, |x|^{2\beta+1}\, \mathrm{d}x} = \frac{2^{2\alpha-2\beta}\,\Gamma(\alpha-\beta+1)^2\,\Gamma(2\alpha-\beta+2)}{(\alpha+1)\,\Gamma(2\alpha-2\beta+1)\,\Gamma(\beta+1)}$$

$$= 2^{2\alpha-2\beta}\left(\frac{\beta+1}{\alpha+1}\right)\frac{\Gamma(\alpha-\beta+1)^2\,\Gamma(2\alpha-\beta+2)}{\Gamma(2\alpha-2\beta+1)\,\Gamma(\beta+2)},$$

and hence

$$(\mathbb{EP}1)(\alpha,\beta\,;1\,;1) \le 2^{2\alpha-2\beta}\left(\frac{\beta+1}{\alpha+1}\right)\frac{\Gamma(\alpha-\beta+1)^2\,\Gamma(2\alpha-\beta+2)}{\Gamma(2\alpha-2\beta+1)\,\Gamma(\beta+2)}.$$

This leads us to

$$\log(\mathbb{EP}1)(\alpha,\beta\,;1\,;1) \le 2(\alpha-\beta)\log 2 + \log\left(\frac{\beta+1}{\alpha+1}\right) + \log\left(\frac{\Gamma(\alpha-\beta+1)^2\,\Gamma(2\alpha-\beta+2)}{\Gamma(2\alpha-2\beta+1)\,\Gamma(\beta+2)}\right). \quad (7.1)$$

Let us denote momentarily by $U_0(\alpha,\beta)$ the function appearing on the right-hand side of (7.1). We now verify the following proposition.

Proposition 20. *For $\alpha > \beta > -1$ we have*

$$U_0(\alpha,\beta) = 2(\alpha-\beta)\log(\alpha+2) + \log\left(\frac{\beta+1}{\alpha+1}\right) + O(\alpha-\beta), \quad (7.2)$$

where the implied constant is universal.

Proof. In order to verify (7.2), if suffices to verify that

$$U_1(\alpha,\beta) := \log\left(\frac{\Gamma(\alpha-\beta+1)^2\,\Gamma(2\alpha-\beta+2)}{\Gamma(2\alpha-2\beta+1)\,\Gamma(\beta+2)}\right) - 2(\alpha-\beta)\log(\alpha+2) = O(\alpha-\beta).$$

Note that

$$U_1(\alpha,\beta) = 2\log\Gamma(\alpha-\beta+1) + \log\Gamma(2\alpha-\beta+2)$$
$$- \log\Gamma(2\alpha-2\beta+1) - \log\Gamma(\beta+2) - 2(\alpha-\beta)\log(\alpha+2).$$

Letting $x = \alpha - \beta$ and $y = \alpha + 1$, we note that $y > x > 0$ and rewrite $U_1(\alpha,\beta) = U_2(x,y)$ with

$$U_2(x,y) = 2\log\Gamma(x+1) + \log\Gamma(x+y+1) - \log\Gamma(2x+1) - \log\Gamma(y-x+1) - 2x\log(y+1).$$

We want to show that $U_2(x,y) = O(x)$, and towards this goal we split our analysis into two cases. Note that $U_2(0,y) = 0$.

Case 1: $0 < x \le 1$. We differentiate with respect to x and show that this partial derivative is uniformly bounded. The claim then follows by the mean value inequality. We have

$$\frac{\partial}{\partial x}U_2(x,y) = 2\frac{\Gamma'}{\Gamma}(x+1) + \frac{\Gamma'}{\Gamma}(x+y+1) - 2\frac{\Gamma'}{\Gamma}(2x+1) + \frac{\Gamma'}{\Gamma}(y-x+1) - 2\log(y+1).$$

We now use Stirling's formula

$$\frac{\Gamma'}{\Gamma}(s) = \log s + O(|s|^{-1}),$$

that holds uniformly in the range $-\pi + \varepsilon < \arg(s) < \pi - \varepsilon$, for any fixed $\varepsilon > 0$ (which is the case; we are always in the real line and at least 1). Hence

$$\frac{\partial}{\partial x} U_2(x,y) = 2\log(x+1) + \log(x+y+1) - 2\log(2x+1) + \log(y-x+1) - 2\log(y+1) + O(1)$$

$$= \log\left(1 + \frac{x}{y+1}\right) + \log\left(1 - \frac{x}{y+1}\right) + 2\log(x+1) - 2\log(2x+1) + O(1)$$

$$= O\left(\frac{x}{y+1}\right) + O(x) + O(1)$$

$$= O(1),$$

for $0 < x \le 1$, independently of $y > x$.

Case 2: $x > 1$. In this case, we use Stirling's formula in the following form

$$\log \Gamma(s) = (s - \tfrac{1}{2})\log s - s + O(1),$$

that also holds uniformly in the range $-\pi + \varepsilon < \arg(s) < \pi - \varepsilon$, for any fixed $\varepsilon > 0$. We get

$$U_2(x,y) = (2x+1)\log(x+1) - (2x+2) + (x+y+\tfrac{1}{2})\log(x+y+1) - (x+y+1)$$

$$- (2x+\tfrac{1}{2})\log(2x+1) + (2x+1) - (y-x+\tfrac{1}{2})\log(y-x+1) + (y-x+1)$$

$$- 2x\log(y+1) + O(1)$$

$$= (2x+1)\log(x+1) + (x+y+\tfrac{1}{2})\log(x+y+1)$$

$$- (2x+\tfrac{1}{2})\log(2x+1) - (y-x+\tfrac{1}{2})\log(y-x+1) - 2x\log(y+1) + O(x)$$

$$= -(2x+1)\log\left(2 - \frac{1}{x+1}\right) + \tfrac{1}{2}\log(2x+1) + (x+y+\tfrac{1}{2})\log(x+y+1)$$

$$- (y-x+\tfrac{1}{2})\log(y-x+1) - 2x\log(y+1) + O(x)$$

$$= (x+y+\tfrac{1}{2})\log(x+y+1) - (y-x+\tfrac{1}{2})\log(y-x+1) - 2x\log(y+1) + O(x).$$

If $x \le (y+1)/2$ we rewrite this last expression as

$$= -(y+\tfrac{1}{2})\log\left(1 - \frac{2x}{x+y+1}\right) + x\log\left(1 + \frac{x}{y+1}\right) + x\log\left(1 - \frac{x}{y+1}\right) + O(x)$$

$$= (y+\tfrac{1}{2})O\left(\frac{x}{y+1}\right) + xO\left(\frac{x}{y+1}\right) + O(x)$$

$$= O(x).$$

If $x \ge (y+1)/2$, we rewrite it as

$$(x+y+1)\log(x+y+1) - (y-x+1)\log(y-x+1) - 2x\log(y+1) + O(x).$$

We then add and subtract $2x\log x$ to have

$$= (x+y+1)\log\left(\frac{y+x+1}{x}\right) - (y-x+1)\log\left(\frac{y-x+1}{x}\right) + O(x)$$

$$= (x + y + 1) \log \left(\frac{y + x + 1}{x} \right) - x \left(\frac{y - x + 1}{x} \right) \log \left(\frac{y - x + 1}{x} \right) + O(x).$$

Noting that $a \mapsto a \log a$ is bounded for $0 < a \leq 1$ we conclude that the above is $O(x)$. This concludes the proof of the proposition. $\qquad \square$

7.2. **Lower bound I: the case $\alpha > \beta \geq -\frac{1}{2}$.** As we saw in §7.1, the upper bound is proved via a suitable example. The proof of the desired lower bound is more convoluted and we split our analysis into different scenarios; a major hurdle here is the singular range when α and β are close to the endpoint -1, where the classical Fourier transform is not available.

We first treat the situation where $\alpha > \beta \geq -\frac{1}{2}$, where we can use uncertainty tools for the classical Fourier transform. In this range, our strategy is to use Lemma 9 and optimize our choice of parameters.

7.2.1. *Strategy.* If $f \in \mathcal{H}_\alpha(1\,;1)$, notice that $f \in L^2(\mathbb{R})$ and it has a Fourier transform in the classical sense. Then, using Lemma 9 and the fact that $\beta \geq -\frac{1}{2}$, we find

$$\int_{\mathbb{R}} |f(x)|^2 \, |x|^{2\beta+1} \, \mathrm{d}x = \int_{[-s,s]} |f(x)|^2 \, |x|^{2\beta+1} \, \mathrm{d}x + \int_{[-s,s]^c} |f(x)|^2 \, |x|^{2\beta+1} \, \mathrm{d}x$$

$$\leq s^{2\beta+1} \int_{[-s,s]} |f(x)|^2 \, \mathrm{d}x + s^{2(\beta-\alpha)} \int_{[-s,s]^c} |f(x)|^2 \, |x|^{2\alpha+1} \, \mathrm{d}x$$

$$\leq s^{2\beta+1} \, e^{Cr} \left(1 - e^{-Cs} \right) \int_{[-r,r]^c} |f(x)|^2 \, \mathrm{d}x + s^{2(\beta-\alpha)} \int_{\mathbb{R}} |f(x)|^2 \, |x|^{2\alpha+1} \, \mathrm{d}x$$

$$\leq \frac{s^{2\beta+1}}{r^{2\alpha+1}} \, e^{Cr} \left(1 - e^{-Cs} \right) \int_{[-r,r]^c} |f(x)|^2 \, |x|^{2\alpha+1} \, \mathrm{d}x + s^{2(\beta-\alpha)} \int_{\mathbb{R}} |f(x)|^2 \, |x|^{2\alpha+1} \, \mathrm{d}x$$

$$\leq \left(\frac{s^{2\beta+1}}{r^{2\alpha+1}} \, e^{Cr} \left(1 - e^{-Cs} \right) + s^{2(\beta-\alpha)} \right) \int_{\mathbb{R}} |f(x)|^2 \, |x|^{2\alpha+1} \, \mathrm{d}x.$$

Hence

$$(\mathbb{EP}1)(\alpha, \beta\,;1\,;1) \geq \left(\frac{s^{2\beta+1}}{r^{2\alpha+1}} \, e^{Cr} \left(1 - e^{-Cs} \right) + s^{2(\beta-\alpha)} \right)^{-1} \tag{7.3}$$

for any $r, s > 0$. Our task is now to optimize the right-hand side.

7.2.2. *Optimization and asymptotic analysis.* First we minimize a function of the type $r \mapsto e^{Cr} r^{-2\alpha-1}$. Taking logarithms we must minimize

$$g(r) := Cr - (2\alpha + 1) \log r.$$

Such minimum occurs at $r_0 := (2\alpha + 1)/C$. Plugging $r = r_0$ in our original expression (7.3), we must then minimize the function

$$h(s) := \frac{s^{2\beta+1}}{(2\alpha + 1)^{2\alpha+1}} (Ce)^{2\alpha+1} \left(1 - e^{-Cs} \right) + s^{2(\beta-\alpha)}.$$

This seems a bit harder to explicitly optimize in s. We make a simplification: using the fact that $1 - e^{-Cs} \leq Cs$ for any $s > 0$ we find that

$$h_1(s) := \frac{C \, s^{2\beta+2}}{(2\alpha+1)^{2\alpha+1}}(Ce)^{2\alpha+1} + s^{2(\beta-\alpha)} \geq h(s).$$

From (7.3) note that $(\mathbb{EP}1)(\alpha,\beta\,;1\,;1) \geq h(s)^{-1} \geq h_1(s)^{-1}$, and we can minimize $h_1(s)$ instead. Letting $A := \frac{C}{(2\alpha+1)^{2\alpha+1}}(Ce)^{2\alpha+1}$, we have $h_1(s) = As^{2\beta+2} + s^{2(\beta-\alpha)}$, and the minimum occurs at

$$s_0 = \left(\frac{\alpha-\beta}{A(\beta+1)}\right)^{1/(2\alpha+2)}.$$

We then get

$$h_1(s_0)^{-1} = \frac{(\beta+1)^{\frac{\beta+1}{\alpha+1}}(\alpha-\beta)^{\frac{\alpha-\beta}{\alpha+1}}}{A^{\frac{\alpha-\beta}{\alpha+1}}(\alpha+1)},$$

and

$$
\begin{aligned}
\log\big((\mathbb{EP}1)(\alpha,\beta\,;1\,;1)\big) &\geq \log h_1(s_0)^{-1} \\
&= \frac{\beta+1}{\alpha+1}\log(\beta+1) + \frac{\alpha-\beta}{\alpha+1}\log(\alpha-\beta) - \frac{\alpha-\beta}{\alpha+1}\log A - \log(\alpha+1) \\
&= \log\left(\frac{\beta+1}{\alpha+1}\right) - \frac{\alpha-\beta}{\alpha+1}\log(\beta+1) + \frac{\alpha-\beta}{\alpha+1}\log(\alpha-\beta) - 2(\alpha-\beta)\log C \qquad (7.4) \\
&\quad - \frac{(\alpha-\beta)(2\alpha+1)}{\alpha+1} + \frac{(\alpha-\beta)(2\alpha+1)}{\alpha+1}\log(2\alpha+1).
\end{aligned}
$$

Let us denote momentarily by $L_0(\alpha,\beta)$ the function appearing on the right-hand side of (7.4). We now verify the following proposition.

Proposition 21. *For $\alpha > \beta \geq -\frac{1}{2}$ we have*

$$L_0(\alpha,\beta) = 2(\alpha-\beta)\log(\alpha+2) + O\left((\alpha-\beta)\log\left(\frac{2(\alpha-\beta+1)}{(\alpha-\beta)}\right)\right),$$

where the implied constant is universal.

Proof. If $\alpha \geq \beta \geq -\frac{1}{2}$, we have $\log\left(\frac{\beta+1}{\alpha+1}\right) = O(\alpha-\beta)$. Hence

$$L_0(\alpha,\beta) = 2(\alpha-\beta)\log(\alpha+2) + L_1(\alpha,\beta) + O(\alpha-\beta),$$

where $L_1(\alpha,\beta)$ is given by

$$L_1(\alpha,\beta) := \frac{\alpha-\beta}{\alpha+1}\log(\alpha-\beta) + \frac{(\alpha-\beta)(2\alpha+1)}{\alpha+1}\log(2\alpha+1) - 2(\alpha-\beta)\log(\alpha+2).$$

Letting $x = \alpha-\beta$ and $y = \alpha+1$, we note that $y > x > 0$ and rewrite $L_1(\alpha,\beta) = L_2(x,y)$ with

$$L_2(x,y) := \frac{x}{y}\log x + \frac{x(2y-1)}{y}\log(2y-1) - 2x\log(y+1).$$

We need to show that

$$L_2(x, y) = O\left(x \log\left(\frac{2(x+1)}{x}\right)\right).$$

Since $y \geq \frac{1}{2}$ we have $\frac{x}{y} \log x = O\big(x \log\big(2(x+1)/x\big)\big)$. The remaining part has the form $x \, L_3(y)$, with

$$L_3(y) := \frac{(2y-1)}{y} \log(2y-1) - 2 \log(y+1),$$

and we claim that this quantity is uniformly bounded in y. In fact, for $\frac{1}{2} \leq y \leq 1$, this is the case (recall that $a \log a \to 0$ as $a \to 0^+$), and for $y \geq 1$ we rewrite it as

$$L_3(y) = 2 \log\left(2 - \frac{3}{y+1}\right) - \frac{\log(2y-1)}{y} = O(1).$$

$\square$

7.3. Lower bounds II: the case $-1 < \beta < \alpha \leq 0$.

The exact choice of 0 as the right endpoint for this case is not particularly important, as long as we are at a positive distance from $-1/2$; this will be suitable for §7.4. In any case, to treat this singular range where both α and β can be close to the lower endpoint -1, a different strategy is required. We use the reproducing kernel of the space.

7.3.1. *An alternative strategy.* Let $f \in \mathcal{H}_\alpha(1\,;1)$, and recall that $\mathcal{H}_\alpha(1\,;1) = \mathcal{H}(E_\alpha)$ as sets, with norms differing by a multiplicative constant (Lemma 10). From the reproducing kernel identity

$$f(x) = \langle f, K_\alpha(x, \cdot) \rangle_{\mathcal{H}(E_\alpha)} \tag{7.5}$$

we get, after an application of the Cauchy-Schwarz inequality,

$$|f(x)|^2 \leq \|f\|^2_{\mathcal{H}(E_\alpha)} \langle K_\alpha(x, \cdot), K_\alpha(x, \cdot) \rangle_{\mathcal{H}(E_\alpha)} = \|f\|^2_{\mathcal{H}(E_\alpha)} K_\alpha(x, x)$$
$$= c_\alpha \left(\int_{-\infty}^{\infty} |f(y)|^2 \, |y|^{2\alpha+1} \, \mathrm{d}y\right) K_\alpha(x, x). \tag{7.6}$$

From the pointwise bound (7.6) we obtain

$$\int_{\mathbb{R}} |f(x)|^2 \, |x|^{2\beta+1} \, \mathrm{d}x = \int_{[-s,s]} |f(x)|^2 \, |x|^{2\beta+1} \, \mathrm{d}x + \int_{[-s,s]^c} |f(x)|^2 \, |x|^{2\beta+1} \, \mathrm{d}x$$

$$\leq c_\alpha \left(\int_{-\infty}^{\infty} |f(x)|^2 \, |x|^{2\alpha+1} \, \mathrm{d}x\right) \int_{[-s,s]} K_\alpha(x, x) \, |x|^{2\beta+1} \, \mathrm{d}x + s^{2(\beta-\alpha)} \int_{[-s,s]^c} |f(x)|^2 \, |x|^{2\alpha+1} \, \mathrm{d}x \tag{7.7}$$

$$\leq \left(c_\alpha \, \|K_\alpha(x, x)\|_{L^\infty[-s,s]} \, \frac{2s^{2\beta+2}}{2\beta+2} + s^{2(\beta-\alpha)}\right) \left(\int_{-\infty}^{\infty} |f(x)|^2 \, |x|^{2\alpha+1} \, \mathrm{d}x\right),$$

valid for any $s > 0$.

As long as we are inside the range

$$s \leq 2\sqrt{\alpha+1}, \tag{7.8}$$

we can use Lemma 14 to get

$$\|K_\alpha(x, x)\|_{L^\infty[-s,s]} \leq \frac{1}{2\pi(\alpha+1)} + \frac{s^2}{4\pi(\alpha+1)^2} \leq \frac{3}{2\pi(\alpha+1)}. \tag{7.9}$$

In this situation, from (7.7) and (7.9) we get

$$(\mathbb{EP}1)(\alpha, \beta \,; 1 \,; 1) \geq \left(\frac{\frac{3}{2\pi} c_\alpha}{(\alpha + 1)(\beta + 1)} \, s^{2\beta + 2} + s^{2(\beta - \alpha)} \right)^{-1}$$

and we now seek to optimize the right-hand side in the admissible range (7.8).

7.3.2. *Optimization and asymptotic analysis.* The global minimum of the function

$$h(s) := \frac{\frac{3}{2\pi} c_\alpha}{(\alpha + 1)(\beta + 1)} \, s^{2\beta + 2} + s^{2(\beta - \alpha)}$$

for $s > 0$ occurs at

$$s_0 = \left(\frac{(\alpha - \beta)(\alpha + 1)}{\frac{3}{2\pi} c_\alpha} \right)^{\frac{1}{2\alpha + 2}}.$$

First observe that such s_0 belongs to our admissible range (7.8). In fact, recalling that $c_\alpha = \pi \, 2^{-2\alpha - 1} \Gamma(\alpha + 1)^{-2}$, this claim is equivalent to the fact that

$$(\alpha - \beta) \leq \frac{3(\alpha + 1)^\alpha}{\Gamma(\alpha + 1)^2} = \frac{3(\alpha + 1)^{\alpha + 2}}{\Gamma(\alpha + 2)^2}. \tag{7.10}$$

Since $(\alpha - \beta) \leq (\alpha + 1)$, it suffices to verify that

$$1 \leq \frac{3(\alpha + 1)^{\alpha + 1}}{\Gamma(\alpha + 2)^2},$$

which is true in our range $-1 < \alpha \leq 0$ (note, however, that this would fail for large α).

Proceeding as in §7.2.2 we have

$$h(s_0)^{-1} = \frac{(\beta + 1)^{\frac{\beta + 1}{\alpha + 1}} (\alpha - \beta)^{\frac{\alpha - \beta}{\alpha + 1}}}{A^{\frac{\alpha - \beta}{\alpha + 1}} (\alpha + 1)},$$

with

$$A = \frac{\frac{3}{2\pi} c_\alpha}{(\alpha + 1)(\beta + 1)} = \frac{3}{2^{2\alpha + 2} \Gamma(\alpha + 1)^2 (\alpha + 1)(\beta + 1)} = \frac{3(\alpha + 1)}{2^{2\alpha + 2} \Gamma(\alpha + 2)^2 (\beta + 1)}.$$

This yields

$$\begin{aligned}
\log(\mathbb{EP}1)(\alpha, \beta \,; 1 \,; 1) &\geq \log h(s_0)^{-1} \\
&= \frac{\beta + 1}{\alpha + 1} \log(\beta + 1) + \frac{\alpha - \beta}{\alpha + 1} \log(\alpha - \beta) - \log(\alpha + 1) - \frac{\alpha - \beta}{\alpha + 1} \log A \\
&= \log\left(\frac{\beta + 1}{\alpha + 1} \right) - \frac{\alpha - \beta}{\alpha + 1} \log(\beta + 1) + \frac{\alpha - \beta}{\alpha + 1} \log(\alpha - \beta) - \frac{\alpha - \beta}{\alpha + 1} \log 3 \\
&\quad + 2(\alpha - \beta) \log 2 - \frac{\alpha - \beta}{\alpha + 1} \log\left(\frac{\alpha + 1}{\beta + 1} \right) + \frac{2(\alpha - \beta)}{\alpha + 1} \log \Gamma(\alpha + 2) \\
&= \log\left(\frac{\beta + 1}{\alpha + 1} \right) - \frac{\alpha - \beta}{\alpha + 1} \log\left(\frac{3(\alpha + 1)}{\alpha - \beta} \right) + 2(\alpha - \beta) \log 2 + \frac{2(\alpha - \beta)}{\alpha + 1} \log \Gamma(\alpha + 2).
\end{aligned} \tag{7.11}$$

Let us denote momentarily by $L_4(\alpha, \beta)$ the function appearing on the right-hand side of (7.11). We now verify the following proposition.

Proposition 22. *For* $-1 < \beta < \alpha \leq 0$ *we have*

$$L_4(\alpha, \beta) = \log\left(\frac{\beta + 1}{\alpha + 1}\right) + O\left(\left(\frac{\alpha - \beta}{\alpha + 1}\right)\log\left(\frac{2(\alpha + 1)}{\alpha - \beta}\right)\right),$$

where the implied constant is universal.

Proof. The last line of (7.11) is written in such a way that we are essentially done. Just observe that in our range we have $\log\Gamma(\alpha + 2) = O(1)$ and

$$\frac{\alpha - \beta}{\alpha + 1} = O\left(\left(\frac{\alpha - \beta}{\alpha + 1}\right)\log\left(\frac{2(\alpha + 1)}{\alpha - \beta}\right)\right).$$

$\square$

7.4. **Lower bounds III: the case** $-1 < \beta < -\frac{1}{2}$ **and** $0 < \alpha$. In this remaining case, note that $\alpha - \beta \geq \frac{1}{2}$. We argue by introducing a test point, say $-\frac{1}{4}$ and using the previous two cases. In fact, from the definition of our problem we have the inequality

$$(\mathbb{EP}1)(\alpha, \beta\,;1\,;1) \geq (\mathbb{EP}1)(\alpha, -\tfrac{1}{4}\,;1\,;1)\,(\mathbb{EP}1)(-\tfrac{1}{4}, \beta\,;1\,;1). \tag{7.12}$$

Hence, from (7.12) and our work in §7.2 and §7.3, we obtain

$$\log\left((\mathbb{EP}1)(\alpha, \beta\,;1\,;1)\right) \geq \log\left((\mathbb{EP}1)(\alpha, -\tfrac{1}{4}\,;1\,;1)\right) + \log\left((\mathbb{EP}1)(-\tfrac{1}{4}, \beta\,;1\,;1)\right)$$

$$\geq \left(2(\alpha + \tfrac{1}{4})\log(\alpha + 2) + O(\alpha + \tfrac{1}{4})\right) + \left(\log\left(\frac{\beta + 1}{\frac{3}{4}}\right) + O(1)\right)$$

$$= 2(\alpha - \beta)\log(\alpha + 2) + \log\left(\frac{\beta + 1}{\alpha + 1}\right) + O(\alpha - \beta),$$

as desired.

7.5. **Conclusion.** The combination of our work in §7.1, §7.2, §7.3 and §7.4 leads to the unified formulation proposed in Theorem 4.

8. THE SHARP CONSTANT SAGA: MULTIPLICATION BY z^k IN DE BRANGES SPACES

8.1. **An extremal problem for general de Branges spaces.** Let $E(z) = A(z) - iB(z)$ be a Hermite-Biehler function, with A and B real entire functions, and $\mathcal{H}(E)$ be the associated de Branges space as discussed in §2.2.1. Throughout this section we assume that our function E verifies the following conditions:

(C1) E has no real zeros.

(C2) The function $z \mapsto E(iz)$ is real entire (or, equivalently, A is even and B is odd).

(C3) $A \notin \mathcal{H}(E)$.

We consider in the following extremal problem.

Extremal Problem 4 (EP4): Let $E = A - iB$ be a Hermite-Biehler function verifying the conditions (C1) - (C3) above, and let $k \in \mathbb{N}$. Assume, in addition, that A has at least $k + 1$ zeros. Find

$$(\mathbb{EP}4)(E; k) := \inf_{0 \neq f \in \mathcal{H}(E)} \frac{\|z^k f\|_{\mathcal{H}(E)}^2}{\|f\|_{\mathcal{H}(E)}^2}.$$

Recall that the Hermite-Biehler condition (2.5) implies that all the zeros of A and B are real. Since E has no real zeros, the sets of zeros of A and B are disjoint; in particular, $A(0) \neq 0$. Moreover, the zeros of A and B are all simple. In fact, if A had a double zero ξ, then by (2.7) one would have $K(\xi, \xi) = \|K(\xi, \cdot)\|_{\mathcal{H}(E)}^2 = 0$, and then $K(\xi, \cdot)$ would be identically zero and (2.6) would imply that $B(\xi) = 0$, a contradiction. Condition (C3), that $A \notin \mathcal{H}(E)$, is generic and allows for the use of the interpolation formulas in (2.8). The minor technical assumption that A has at least $k + 1$ zeros guarantees that the subspace

$$\mathcal{X}_k(E) := \{f \in \mathcal{H}(E); \, z^k f \in \mathcal{H}(E)\}$$

is non-empty. In fact, if $\xi_1, \ldots, \xi_{k+1}$ are zeros of A, the functions $K(\xi, z) = A(z)/\pi(z - \xi_j)$ all belong to $\mathcal{H}(E)$ and a suitable linear combination of them will decay like $|A(x)/x^{k+1}|$ as $|x| \to \infty$, and hence will be in $\mathcal{X}_k(E)$.

We first establish a few qualitative properties about the extremal problem (EP4), in the spirit of similar results that have already been discussed in this paper (hence, we shall be brief in the proof).

Proposition 23. *The following statements hold:*

(i) $(\mathbb{EP}4)(E; k) > 0$.

(ii) *There exist extremizers for* $(\mathbb{EP}4)(E; k)$.

(iii) *Any extremizer for* $(\mathbb{EP}4)(E; k)$ *must be an even function.*

Proof. (i) Let $f \in \mathcal{X}_k(E)$ be such that $\|f\|_{\mathcal{H}(E)}^2 = 1$. Recall, from the reproducing kernel identity and an application of the Cauchy-Schwarz inequality (like in (7.5) - (7.6)), that

$$|f(z)|^2 \leq \|f\|_{\mathcal{H}(E)}^2 K(z, z) \tag{8.1}$$

for all $z \in \mathbb{C}$. Since $x \mapsto K(x, x)$ is continuous, we see that f cannot concentrate at the origin. Letting $M = \max\{K(x, x); -1 \leq x \leq 1\}$ and $d\mu(x) = |E(x)|^{-2}dx$, choose $\eta \leq 1$ such that $M \int_{[-\eta, \eta]} d\mu(x) \leq \frac{1}{2}$. Then $\int_{[-\eta, \eta]} |f(x)|^2 \, d\mu(x) \leq \frac{1}{2}$ and

$$\|z^k f\|_{\mathcal{H}(E)}^2 \geq \int_{[-\eta, \eta]^c} |f(x)|^2 |x|^{2k} \, d\mu(x) \geq \eta^{2k} \int_{[-\eta, \eta]^c} |f(x)|^2 \, d\mu(x) \geq \frac{\eta^{2k}}{2} > 0.$$

(ii) The proof is similar to §4.1. Let $\{f_n\}_{n \geq 1} \subset \mathcal{X}_k(E)$ be an extremizing sequence, normalized so that $\|z^k f_n\|_{\mathcal{H}(E)} = 1$. This means that $\|f_n\|_{\mathcal{H}(E)}^2 \to (\mathbb{EP}4)(E; k)^{-1}$. By reflexivity, up to a subsequence, we may assume that $f_n \rightharpoonup g$ for some $g \in \mathcal{H}(E)$. The reproducing kernel identity implies that $f_n \to g$ pointwise everywhere, and by Fatou's lemma we have $\|z^k g\|_{\mathcal{H}(E)} \leq 1$. In particular $g \in \mathcal{X}_k(E)$. Using (8.1), note also that f_n is uniformly bounded in compact subsets of $\mathbb{C}$.

The proof that $\|g\|^2_{\mathcal{H}(E)} \geq (\mathbb{EP}4)(E;k)^{-1}$ (and hence g is the desired extremizer) follows the argument of (4.4), (4.5) and (4.6).

(iii) The proof follows the same reasoning of the proof of Proposition 15 (recall that $x \mapsto |E(x)|^{-2}$ is an even function in our case) and we omit the details. $\qquad\square$

Our main result in this section is a complete solution for the extremal problem (EP4). Let

$$0 < \xi_1 < \xi_2 < \xi_3 < \ldots$$

denote the sequence of positive zeros of A and define the meromorphic function

$$C(z) := \frac{B(z)}{A(z)}.$$

Theorem 24 (Sharp constants - general version). *Let $k \in \mathbb{N}$ and set $\lambda_0 := \left((\mathbb{EP}4)(E;k)\right)^{1/2k}$.*

(i) *If $k = 1$ we have $\lambda_0 = \xi_1$.*

(ii) *If $k \geq 2$, set $\ell := \lfloor k/2 \rfloor$. Then λ_0 is the smallest positive solution of the equation*

$$A(\lambda) \det \mathcal{V}(\lambda) = 0,$$

where $\mathcal{V}(\lambda)$ is the $\ell \times \ell$ matrix with entries

$$\left(\mathcal{V}(\lambda)\right)_{mj} = \sum_{r=0}^{k-1} \omega^{r(4\ell-2m-2j+3)} C\left(\omega^r \lambda\right) \qquad (1 \leq m,j \leq \ell);$$

and $\omega := e^{\pi i/k}$.

REMARK: For $k \geq 2$, we shall verify in our proof that $0 < \lambda_0 < \xi_{\ell+1}$. Note that $\lambda \mapsto \det \mathcal{V}(\lambda)$ is, in principle, a meromorphic function of the variable λ that is real-valued on $\mathbb{R}$, but we verify in our proof that the function $\lambda \mapsto A(\lambda) \det \mathcal{V}(\lambda)$ is in fact continuous on the interval $(0, \xi_{\ell+1})$.

We are also able to classify the extremizers for $(\mathbb{EP}4)(E;k)$. In what follows, let

$$c_n := \frac{-A'(\xi_n)}{B(\xi_n)}. \tag{8.2}$$

From (2.7) note that $c_n > 0$. Also, for $\ell = \lfloor k/2 \rfloor$ (when $k \geq 2$), let $\mathcal{T}$ be the $\ell \times \ell$ Vandermonde matrix with entries

$$\mathcal{T}_{mj} = \xi_m^{2\ell-2j+1} \qquad (1 \leq m,j \leq \ell), \tag{8.3}$$

with $\mathcal{T}^{-1}$ denoting its inverse, and let $\mathcal{Q}(\lambda)$ be the $\ell \times \ell$ matrix with entries

$$\left(\mathcal{Q}(\lambda)\right)_{mj} = \frac{\left(\mathcal{V}(\lambda)\right)_{mj}}{(2k)\, \lambda^{2k-4\ell+2m+2j-3}} \qquad (1 \leq m,j \leq \ell).$$

Theorem 25 (Classification of extremizers). *Let $k \in \mathbb{N}$ and $\lambda_0 = \left((\mathbb{EP}4)(E;k)\right)^{1/2k}$.*

(i) *If $k = 1$, the extremizers for $(\mathbb{EP}4)(E;k)$ are spanned over $\mathbb{C}$ by the real entire function*

$$f(z) = \frac{A(z)}{(z^2 - \xi_1^2)}.$$

(ii) *If $k \geq 2$, set $\ell := \lfloor k/2 \rfloor$. The extremizers for $(\mathbb{EP}4)(E; k)$ are spanned over $\mathbb{C}$ by the real entire functions*

$$f(z) = \sum_{n=1}^{\infty} a_n \frac{\xi_n A(z)}{(z^2 - \xi_n^2)}, \tag{8.4}$$

where

$$a_n = \frac{\sum_{i=1}^{\ell} c_i a_i \left(\sum_{r=1}^{\ell} \xi_n^{2\ell - 2r + 1} (\mathcal{T}^{-1})_{ri} \right) \left(\xi_i^{2k} - \lambda_0^{2k} \right)}{c_n \left(\xi_n^{2k} - \lambda_0^{2k} \right)} \quad ; \quad \text{for } n > \ell,$$

and $(a_1, a_2, \ldots, a_\ell) \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ belongs to $\ker \mathcal{W}(\lambda_0)^3$, where $\mathcal{W}(\lambda_0)$ is the $\ell \times \ell$ matrix with entries

$$\left(\mathcal{W}(\lambda_0) \right)_{ij} = c_i \left(\xi_i^{2k} - \lambda_0^{2k} \right) \left(\left(\mathcal{T}^{-1} \right)^t \mathcal{Q}(\lambda_0) \right)_{ij} \qquad (1 \leq i, j \leq \ell).$$

REMARK: If the even function A has only finitely many zeros, say $2N$, the sum in (8.4) naturally goes only from 1 to N. Throughout the proof below we slightly abuse the notation and keep writing the sums from 1 to ∞, with this understanding. In addition, note that $\det \mathcal{W}(\lambda_0) = 0$ since $A(\lambda_0) \det \mathcal{V}(\lambda_0) = 0$ and $\lambda_0 \in (0, \xi_{\ell+1})$; this will be further detailed in our proof.

In the case $k = 1$, Theorems 24 and 25 have already been established in a previous work of the first author with Chirre and Milinovich [10, Theorem 14], with applications to number theory, namely, bounding the height of the first non-trivial zero in families of L-functions. We include this case here for completeness (and also since our argument below is slightly different from that of [10]).

8.2. **Proof of Theorem 5 and characterization of its extremizers.** Note that Theorem 5 is a specialization of Theorem 24 in the case where $E(z) = E_\beta(z) = A_\beta(z) - i B_\beta(z)$ discussed in §2.2.2. If $\alpha = \beta + k$, note that $f \in \mathcal{H}_\alpha(1; 1)$ if and only if $f \in \mathcal{X}_k(E_\beta)$, and from Lemma 10 we plainly have

$$(\mathbb{EP}1)(\beta + k, \beta \,; 1 \,; 1) = (\mathbb{EP}4)(E_\beta; k).$$

The extremizers in Theorem 5 can be directly obtained from Theorem 25 with $E = E_\beta$, and hence $\xi_n = \mathsf{j}_{\beta,n}$. In this case, note the additional simplification, due to (1.10), that $c_n = 1$ for all $n \geq 1$.

8.3. **Preliminaries for the proofs of Theorems 24 and 25.** Before diving into the proofs, we present a few useful remarks.

We have seen in Proposition 23 that any extremizer for $(\mathbb{EP}4)(E; k)$ must be an even function. If f is an extremizer and we write $f(z) = g(z) - ih(z)$, where g and h are real entire functions (i.e. $g = (f + f^*)/2$ and $h = i(f - f^*)/2$), one has $|f(x)|^2 = |g(x)|^2 + |h(x)|^2$ for $x \in \mathbb{R}$ and, similarly to (4.7), one readily sees that g and h must also be extremizers (if they are not identically zero). We shall see in our proof below that the set of real entire extremizers is a finite dimensional vector space over $\mathbb{R}$. Hence, the full space of extremizers is the span over $\mathbb{C}$ of the real entire extremizers.

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[3]Here we mean right multiplication, i.e. $(a_1, a_2, \ldots, a_\ell) \mathcal{W}(\lambda_0) = (0, 0, \ldots, 0)$.

Recall from (2.6) and (2.7) that, for $A(\xi) = 0$, we have

$$K(\xi, z) = \frac{-A(z)B(\xi)}{\pi(z - \xi)} \quad \text{and} \quad K(\xi, \xi) = \frac{-A'(\xi)B(\xi)}{\pi}. \tag{8.5}$$

If $f \in \mathcal{H}(E)$ is an even function, using (8.5), we can group the zeros $+\xi$ and $-\xi$ in the interpolation formula (2.8) to get the representation

$$f(z) = \sum_{n=1}^{\infty} \frac{2\,\xi_n f(\xi_n)}{A'(\xi_n)} \frac{A(z)}{(z^2 - \xi_n^2)}. \tag{8.6}$$

If $f \in \mathcal{H}(E)$ is odd, grouping the zeros $+\xi$ and $-\xi$ in the interpolation formula (2.8) we get

$$f(z) = \sum_{n=1}^{\infty} \frac{2f(\xi_n)}{A'(\xi_n)} \frac{z\,A(z)}{(z^2 - \xi_n^2)}. \tag{8.7}$$

Both representations (8.6) and (8.7) are uniformly convergent in compact subsets of $\mathbb{C}$.

If $f \in \mathcal{X}_k(E)$, then $g(z) := z^k f(z) \in \mathcal{H}(E)$ and we have the following constraints

$$g(0) = g'(0) = \ldots = g^{(k-1)}(0) = 0. \tag{8.8}$$

Reciprocally, if $g \in \mathcal{H}(E)$ is a function satisfying (8.8), then $g(z) = z^k f(z)$ with $f \in \mathcal{X}_k(E)$.

8.4. **Proofs of Theorems 24 and 25: the case k even.**

8.4.1. *Sequential formulation.* Let $k = 2\ell$ with $\ell \in \mathbb{N}$. Let $f \in \mathcal{X}_k(E)$ be an even and real entire function. In this case, since $g(z) := z^k f(z)$ is even, half of the constraints in (8.8) are already taken care of and we only have to worry about

$$g(0) = g^{(2)}(0) = \ldots = g^{2(\ell-1)}(0) = 0. \tag{8.9}$$

From the representation (8.6) (for the function g) we have

$$g(z) = \sum_{n=1}^{\infty} \frac{2\,\xi_n^{k+1} f(\xi_n)}{A'(\xi_n)} \frac{A(z)}{(z^2 - \xi_n^2)}, \tag{8.10}$$

and then

$$g(0) = 0 \iff \sum_{n=1}^{\infty} \frac{\xi_n^{k-1} f(\xi_n)}{A'(\xi_n)} = 0. \tag{8.11}$$

Proceeding inductively by differentiating (8.10) and plugging in $z = 0$ (the initial case being (8.11)), we come to the conclusion that (8.9) is equivalent to the set of identities[4]

$$\sum_{n=1}^{\infty} \frac{\xi_n^{2\ell-2j+1} f(\xi_n)}{A'(\xi_n)} = 0 \qquad (j = 1, 2, \ldots, \ell). \tag{8.12}$$

---

[4]Note that in the rest of the proof we sometimes write our exponent with 2ℓ instead of k. This is simply to facilitate the analogy for the upcoming proof when k is odd in §8.5.

Let us set $a_n := f(\xi_n)/A'(\xi_n)$. Note that a_n is a real number since we are assuming that f is real entire. Recall the definition of the positive quantity c_n in (8.2). From (8.5) and (2.8) we get

$$\|f\|^2_{\mathcal{H}(E)} = 2\pi \sum_{n=1}^{\infty} c_n a_n^2 \qquad \text{and} \qquad \|z^k f(z)\|^2_{\mathcal{H}(E)} = 2\pi \sum_{n=1}^{\infty} c_n a_n^2 \, \xi_n^{2k}.$$

Letting $\lambda_0 := \big((\mathbb{EP}4)(E;k)\big)^{1/2k}$, an equivalent formulation of our problem is then

$$\lambda_0^{2k} = \inf_{\{a_n\} \in \mathcal{A}} \frac{\sum_{n=1}^{\infty} c_n a_n^2 \, \xi_n^{2k}}{\sum_{n=1}^{\infty} c_n a_n^2},$$

where the infimum is taken over the class $\mathcal{A}$ of real-valued sequences $\{a_n\}_{n \geq 1}$, non-identically zero, that verify:

- $\sum_{n=1}^{\infty} c_n a_n^2 \, \xi_n^{2k} < \infty$;
- and the ℓ constraints given by (8.12), i.e.

$$\sum_{n=1}^{\infty} a_n \, \xi_n^{2\ell-2j+1} = 0 \qquad (j = 1, 2, \dots, \ell). \tag{8.13}$$

REMARK: The function $K(0,z) = A(0)B(z)/(\pi z)$ belongs to $\mathcal{H}(E)$, and by (2.8) we have

$$\sum_{n=1}^{\infty} \frac{1}{c_n \, \xi_n^2} < \infty.$$

From this and the condition $\sum_{n=1}^{\infty} c_n a_n^2 \, \xi_n^{2k} < \infty$, we have an alternative way to see that (8.13) is absolutely summable by applying Cauchy-Schwarz appropriately.

8.4.2. *A basic upper bound.* Recall the definition of the $\ell \times \ell$ Vandermonde matrix $\mathcal{T}$ in (8.3). The system of equations (8.13) essentially says that we can express the variables $a_1, a_2, \dots, a_\ell$ in terms of the variables $\{a_n\}_{n > \ell}$. In fact, letting

$$S_j := \sum_{n > \ell} a_n \, \xi_n^{2\ell-2j+1} \qquad (j = 1, 2, \dots, \ell), \tag{8.14}$$

from (8.13) we have

$$(a_1, \dots, a_\ell) \, \mathcal{T} = (-S_1, -S_2, \dots, -S_\ell), \tag{8.15}$$

and hence

$$a_i = -\sum_{r=1}^{\ell} S_r \, (\mathcal{T}^{-1})_{ri} \qquad (i = 1, 2, \dots, \ell). \tag{8.16}$$

An easy choice of $\{a_n\}_{n > \ell}$ is $a_{\ell+1} = 1$ and $a_n = 0$ for $n \geq \ell + 2$. Then $\{a_i\}_{i=1}^{\ell}$ are given by (8.16) (they are not all zero since $\mathcal{T}$ is invertible) and with this test sequence we find that

$$\lambda_0^{2k} \leq \frac{\sum_{n=1}^{\ell+1} c_n a_n^2 \, \xi_n^{2k}}{\sum_{n=1}^{\ell+1} c_n a_n^2} < \xi_{\ell+1}^{2k},$$

and hence

$$\lambda_0 < \xi_{\ell+1}.$$

8.4.3. *Solution via variational methods.* From (8.14) and (8.16), letting

$$F\big(\{a_n\}_{n>\ell}\big) = \sum_{i=1}^{\ell} c_i \left(\sum_{r=1}^{\ell} \left(\sum_{n>\ell} a_n\, \xi_n^{2\ell-2r+1} \right) (\mathcal{T}^{-1})_{ri} \right)^2 \xi_i^{2k} + \sum_{n>\ell} c_n a_n^2\, \xi_n^{2k}$$

and

$$G\big(\{a_n\}_{n>\ell}\big) = \sum_{i=1}^{\ell} c_i \left(\sum_{r=1}^{\ell} \left(\sum_{n>\ell} a_n\, \xi_n^{2\ell-2r+1} \right) (\mathcal{T}^{-1})_{ri} \right)^2 + \sum_{n>\ell} c_n a_n^2,$$

our problem is to minimize F/G over all the class $\mathcal{A}_1$ of real-valued sequences $\{a_n\}_{n>\ell}$, non-identically zero, that verify $\sum_{n>\ell} c_n a_n^2\, \xi_n^{2k} < \infty$.

From Proposition 23 we know that there exists an extremal sequence, that we henceforth denote by $\{a_n^*\}_{n>\ell}$. This means that

$$\lambda_0^{2k} = \frac{F\big(\{a_n^*\}_{n>\ell}\big)}{G\big(\{a_n^*\}_{n>\ell}\big)},$$

and if we perturb this sequence by considering $\{a_n^* + \varepsilon b_n\}_{n>\ell}$, where $\{b_n\}_{n>\ell} \in \mathcal{A}_1$ and ε is small, we get that the function

$$\varphi(\varepsilon) := \frac{F\big(\{a_n^* + \varepsilon b_n\}_{n>\ell}\big)}{G\big(\{a_n^* + \varepsilon b_n\}_{n>\ell}\big)}$$

is differentiable and must verify

$$\varphi'(0) = 0.$$

In particular, with the perturbation $\{b_n\}_{n>\ell}$ given by $b_n = 1$ if $n = \ell + m$ and $b_n = 0$ otherwise (for any $m \in \mathbb{N}$), we arrive at the Lagrange multipliers

$$\frac{\partial F}{\partial a_n}\big(\{a_n^*\}_{n>\ell}\big) = \lambda_0^{2k} \frac{\partial G}{\partial a_n}\big(\{a_n^*\}_{n>\ell}\big) \tag{8.17}$$

for all $n > \ell$.

Using (8.16), note that

$$\frac{\partial F}{\partial a_n}\big(\{a_n\}_{n>\ell}\big) = \sum_{i=1}^{\ell} (-2c_i a_i) \left(\sum_{r=1}^{\ell} \xi_n^{2\ell-2r+1} (\mathcal{T}^{-1})_{ri} \right) \xi_i^{2k} + 2c_n a_n\, \xi_n^{2k}$$

and

$$\frac{\partial G}{\partial a_n}\big(\{a_n\}_{n>\ell}\big) = \sum_{i=1}^{\ell} (-2c_i a_i) \left(\sum_{r=1}^{\ell} \xi_n^{2\ell-2r+1} (\mathcal{T}^{-1})_{ri} \right) + 2c_n a_n.$$

Dividing by 2 we find that (8.17) is equivalent to

$$c_n a_n^* \big(\xi_n^{2k} - \lambda_0^{2k}\big) = \sum_{i=1}^{\ell} c_i a_i^* \left(\sum_{r=1}^{\ell} \xi_n^{2\ell-2r+1} (\mathcal{T}^{-1})_{ri} \right) \big(\xi_i^{2k} - \lambda_0^{2k}\big) \tag{8.18}$$

for all $n > \ell$. Since we have already established in §8.4.2 that $\lambda_0 < \xi_{\ell+1}$ we find that

$$a_n^* = \frac{\sum_{i=1}^{\ell} c_i a_i^* \left(\sum_{r=1}^{\ell} \xi_n^{2\ell-2r+1} (\mathcal{T}^{-1})_{ri} \right) \left(\xi_i^{2k} - \lambda_0^{2k} \right)}{c_n \left(\xi_n^{2k} - \lambda_0^{2k} \right)} \tag{8.19}$$

for all $n > \ell$. In particular, note that $(a_1^*, a_2^*, \ldots, a_\ell^*) \neq \mathbf{0}$.

At this point, if we multiply both sides of (8.19) by $\xi_n^{2\ell-2j+1}$ and add over $n > \ell$, using (8.14) (call it S_j^* the corresponding quantity for the sequence $\{a_n^*\}_{n>\ell}$) we get

$$S_j^* = \sum_{i=1}^{\ell} c_i a_i^* \sum_{n>\ell} \frac{\left(\sum_{r=1}^{\ell} \xi_n^{2\ell-2r+1} (\mathcal{T}^{-1})_{ri} \right) \left(\xi_i^{2k} - \lambda_0^{2k} \right) \xi_n^{2\ell-2j+1}}{c_n \left(\xi_n^{2k} - \lambda_0^{2k} \right)}.$$

Recall from (8.15) that

$$S_j^* = -\sum_{i=1}^{\ell} a_i^* \, \mathcal{T}_{ij} = -\sum_{i=1}^{\ell} a_i^* \, \xi_i^{2\ell-2j+1}.$$

If we subtract these two equations we get

$$0 = \sum_{i=1}^{\ell} a_i^* \left[\sum_{n>\ell} \frac{c_i \left(\sum_{r=1}^{\ell} \xi_n^{2\ell-2r+1} (\mathcal{T}^{-1})_{ri} \right) \left(\xi_i^{2k} - \lambda_0^{2k} \right) \xi_n^{2\ell-2j+1}}{c_n \left(\xi_n^{2k} - \lambda_0^{2k} \right)} + \xi_i^{2\ell-2j+1} \right] \tag{8.20}$$

for each $j = 1, 2, \ldots, \ell$.

Denoting by $\mathcal{W}(\lambda)$ the $\ell \times \ell$ matrix with entries

$$\big(\mathcal{W}(\lambda) \big)_{ij} = \left(\sum_{n>\ell} \frac{c_i \left(\sum_{r=1}^{\ell} \xi_n^{4\ell-2r-2j+2} (\mathcal{T}^{-1})_{ri} \right)}{c_n \left(\xi_n^{2k} - \lambda^{2k} \right)} \right) \left(\xi_i^{2k} - \lambda^{2k} \right) + \xi_i^{2\ell-2j+1}, \tag{8.21}$$

what we have seen from (8.20) is that we must have $\det \mathcal{W}(\lambda_0) = 0$ and the vector $(a_1^*, a_2^*, \ldots, a_\ell^*) \in \mathbb{R}^\ell \setminus \{\mathbf{0}\}$ belongs the $\ker \mathcal{W}(\lambda_0)$.

Conversely, if $0 < \lambda_0 < \xi_{\ell+1}$ is such that $\det \mathcal{W}(\lambda_0) = 0$, and we let $\{\mathbf{0}\} \neq (a_1^*, a_2^*, \ldots, a_\ell^*) \in \ker W(\lambda_0)$, we may define a_n^* for $n > \ell$ by (8.19). Then the constraints (8.13) are simply (8.20), and one can check directly that $\sum_{n \geq 1} c_n (a_n^*)^2 \, \xi_n^{2k} < \infty$. Working backwards, (8.18) follows from (8.19). Multiplying (8.18) by a_n^* and summing over $n > \ell$ we get (note the use of (8.14) and (8.16) below)

$$\sum_{n>\ell} c_n (a_n^*)^2 \big(\xi_n^{2k} - \lambda_0^{2k} \big) = \sum_{n>\ell} a_n^* \sum_{i=1}^{\ell} c_i a_i^* \left(\sum_{r=1}^{\ell} \xi_n^{2\ell-2r+1} (\mathcal{T}^{-1})_{ri} \right) \big(\xi_i^{2k} - \lambda_0^{2k} \big)$$

$$= \sum_{i=1}^{\ell} c_i a_i^* \left(\sum_{r=1}^{\ell} S_r^* \, (\mathcal{T}^{-1})_{ri} \right) \big(\xi_i^{2k} - \lambda_0^{2k} \big)$$

$$= -\sum_{i=1}^{\ell} c_i (a_i^*)^2 \big(\xi_i^{2k} - \lambda_0^{2k} \big).$$

The last expression can be rewritten as

$$\frac{\sum_{n=1}^{\infty} c_n(a_n^*)^2\, \xi_n^{2k}}{\sum_{n=1}^{\infty} c_n(a_n^*)^2} = \lambda_0^{2k}.$$

The conclusion is that the desired value λ_0 is the smallest positive solution of the equation $\det \mathcal{W}(\lambda) = 0$ and all the real-valued extremal sequences are given by (8.19) with $\{\mathbf{0}\} \neq (a_1^*, a_2^*, \ldots, a_\ell^*) \in \ker \mathcal{W}(\lambda_0)$.

8.4.4. *Simplifying the determinant.* We now work a bit on the determinant of the matrix $\mathcal{W}(\lambda)$, given by (8.21), in order to describe it in terms of the companion functions A and B that define the de Branges space $\mathcal{H}(E)$; this is better suited for computational purposes. Note that $\lambda \mapsto \det \mathcal{W}(\lambda)$ is a continuous function on the interval $(0, \xi_{\ell+1})$.

Observe first that

$$
\begin{aligned}
\left(\mathcal{W}(\lambda)\right)_{ij} &= c_i\left(\xi_i^{2k} - \lambda^{2k}\right) \left[\left(\sum_{n>\ell} \frac{\left(\sum_{r=1}^{\ell} \xi_n^{4\ell-2r-2j+2}\, (\mathcal{T}^{-1})_{ri}\right)}{c_n\left(\xi_n^{2k} - \lambda^{2k}\right)}\right) + \frac{\xi_i^{2\ell-2j+1}}{c_i\left(\xi_i^{2k} - \lambda^{2k}\right)}\right] \\
&= c_i\left(\xi_i^{2k} - \lambda^{2k}\right) \sum_{n=1}^{\infty} \frac{\left(\sum_{r=1}^{\ell} \xi_n^{4\ell-2r-2j+2}\, (\mathcal{T}^{-1})_{ri}\right)}{c_n\left(\xi_n^{2k} - \lambda^{2k}\right)} \\
&= c_i\left(\xi_i^{2k} - \lambda^{2k}\right) \left(\left(\mathcal{T}^{-1}\right)^t \mathcal{Q}(\lambda)\right)_{ij},
\end{aligned}
\tag{8.22}
$$

where $\mathcal{Q}(\lambda)$ is the $\ell \times \ell$ matrix with entries given by

$$\left(\mathcal{Q}(\lambda)\right)_{mj} = \sum_{n=1}^{\infty} \frac{\xi_n^{4\ell-2m-2j+2}}{c_n\left(\xi_n^{2k} - \lambda^{2k}\right)} \qquad (1 \leq m, j \leq \ell).$$

Note that the second equality in (8.22) is a consequence of the identity $\sum_{r=1}^{\ell} \xi_n^{2\ell-2r+1}(\mathcal{T}^{-1})_{ri} = \delta_{ni}$, for $1 \leq n \leq \ell$. From (8.22) we arrive at

$$\det \mathcal{W}(\lambda) = \left(\prod_{i=1}^{\ell} c_i\left(\xi_i^{2k} - \lambda^{2k}\right)\right) \left(\det \mathcal{T}\right)^{-1} \det \mathcal{Q}(\lambda),
\tag{8.23}$$

and our task now is to understand this last determinant. The next lemma is helpful to rewrite each entry $\left(\mathcal{Q}(\lambda)\right)_{mj}$ in terms of the companion functions A and B.

Lemma 26. *Let $k \in \mathbb{N}$ and set $\zeta := e^{2\pi i/k}$. For $s \in \mathbb{Z}$ with $0 \leq s \leq k-1$ we have*

$$\sum_{r=0}^{k-1} \frac{\zeta^{-rs}}{x - \zeta^r y} = \frac{k\, x^{k-s-1}\, y^s}{x^k - y^k},$$

as rational functions of the variables x and y.

Proof. We argue briefly via series expansions. Observe that

$$\sum_{r=0}^{k-1} \frac{\zeta^{-rs}}{x - \zeta^r y} = \frac{1}{x}\sum_{r=0}^{k-1} \frac{\zeta^{-rs}}{1 - \zeta^r\left(\frac{y}{x}\right)} = \frac{1}{x}\sum_{r=0}^{k-1} \zeta^{-rs} \sum_{m=0}^{\infty} \zeta^{rm}\left(\frac{y}{x}\right)^m = \frac{1}{x}\sum_{m=0}^{\infty} \left(\frac{y}{x}\right)^m \sum_{r=0}^{k-1} \zeta^{r(m-s)}$$

$$= \frac{k}{x} \sum_{m \geq 0; \, m \equiv s \, (\mathrm{mod} \, k)} \left(\tfrac{y}{x}\right)^m = \frac{k \, y^s}{x \, x^s} \left(\frac{1}{1 - \left(\tfrac{y}{x}\right)^k}\right) = \frac{k \, x^{k-s-1} \, y^s}{x^k - y^k}.$$

$\square$

Recall that the even function $\pi K(0, z) = A(0) B(z)/z$ belongs to $\mathcal{H}(E)$. From (8.6) we get

$$\frac{B(z)}{2z} = \sum_{n=1}^{\infty} \frac{A(z)}{c_n (\xi_n^2 - z^2)}.$$

Then, with $C(z) = B(z)/A(z)$, we have

$$\frac{C(z)}{2z} = \sum_{n=1}^{\infty} \frac{1}{c_n (\xi_n^2 - z^2)}. \tag{8.24}$$

With $\zeta = e^{2\pi i/k}$, by Lemma 26 with $x = \xi_n^2$ and $y = \lambda^2$, and (8.24), we have

$$
\begin{aligned}
\left(\mathcal{Q}(\lambda)\right)_{mj} &= \sum_{n=1}^{\infty} \frac{\xi_n^{4\ell - 2m - 2j + 2}}{c_n \left(\xi_n^{2k} - \lambda^{2k}\right)} \\
&= \sum_{n=1}^{\infty} \frac{1}{k \, \lambda^{2k - 4\ell + 2m + 2j - 4}} \frac{k \, \xi_n^{4\ell - 2m - 2j + 2} \, \lambda^{2k - 4\ell + 2m + 2j - 4}}{c_n \left(\xi_n^{2k} - \lambda^{2k}\right)} \\
&= \sum_{n=1}^{\infty} \frac{1}{k \, \lambda^{2k - 4\ell + 2m + 2j - 4}} \sum_{r=0}^{k-1} \frac{\zeta^{-r(k - 2\ell + m + j - 2)}}{c_n \left(\xi_n^2 - \zeta^r \lambda^2\right)} \\
&= \sum_{r=0}^{k-1} \frac{\zeta^{-r(k - 2\ell + m + j - 2)}}{k \, \lambda^{2k - 4\ell + 2m + 2j - 4}} \sum_{n=1}^{\infty} \frac{1}{c_n \left(\xi_n^2 - \zeta^r \lambda^2\right)} \\
&= \sum_{r=0}^{k-1} \frac{\zeta^{-r(k - 2\ell + m + j - 2)}}{k \, \lambda^{2k - 4\ell + 2m + 2j - 4}} \frac{C\left(\zeta^{r/2} \lambda\right)}{2 \, \zeta^{r/2} \lambda} \\
&= \frac{\left(\mathcal{V}(\lambda)\right)_{mj}}{(2k) \, \lambda^{2k - 4\ell + 2m + 2j - 3}}, \tag{8.25}
\end{aligned}
$$

where $\mathcal{V}(\lambda)$ is the $\ell \times \ell$ matrix with entries

$$\left(\mathcal{V}(\lambda)\right)_{mj} = \sum_{r=0}^{k-1} \omega^{r(4\ell - 2m - 2j + 3)} C\left(\omega^r \lambda\right) \qquad (1 \leq m, j \leq \ell),$$

and $\omega := e^{\pi i/k}$. Recall that we are only interested in the roots of our original determinant in the open interval $(0, \xi_{\ell+1})$. Since the power of λ in the denominator of $\left(\mathcal{Q}(\lambda)\right)_{mj}$ in (8.25) depends only on $m + j$, when the determinant is multiplied out, there will be a common power of λ that can be factored out. Also, the expression $\prod_{i=1}^{\ell} c_i \left(\xi_i^{2k} - \lambda^{2k}\right)$ appearing in (8.23) can be replaced by $A(\lambda)$ (that has the same zeros in the interval $(0, \xi_{\ell+1})$). We conclude that, in the range $(0, \xi_{\ell+1})$, one has $\det \mathcal{W}(\lambda) = 0$ if and only if $A(\lambda) \det \mathcal{V}(\lambda) = 0$. This completes the proof in this case.

8.5. **Proof of Theorem 24: the case k odd.** Now let $k = 2\ell + 1$ with $\ell \in \mathbb{N} \cup \{0\}$. Let $f \in \mathcal{X}_k(E)$ be an even and real entire function. In this case, note that $g(z) := z^k f(z)$ is odd.

If $k = 1$, there are no non-trivial constraints in (8.8), and we plainly get

$$\lambda_0^2 = \inf_{\{a_n\} \in \mathcal{A}} \frac{\sum_{n=1}^{\infty} c_n a_n^2 \, \xi_n^2}{\sum_{n=1}^{\infty} c_n a_n^2} = \xi_1^2,$$

with the only extremal sequences being the ones with $a_1 \neq 0$ and $a_n = 0$ for $n \geq 2$.

If $k \geq 3$ (i.e. $\ell \geq 1$), then more than half of the constraints in (8.8) are already taken care of and we only have to worry about

$$g'(0) = g^{(3)}(0) = \ldots = g^{(2\ell-1)}(0) = 0. \tag{8.26}$$

From the representation (8.7) (for the function g) we have

$$g(z) = \sum_{n=1}^{\infty} \frac{2 \, \xi_n^k f(\xi_n)}{A'(\xi_n)} \, \frac{z \, A(z)}{(z^2 - \xi_n^2)}, \tag{8.27}$$

and then

$$g'(0) = 0 \iff \sum_{n=1}^{\infty} \frac{\xi_n^{k-2} f(\xi_n)}{A'(\xi_n)} = 0. \tag{8.28}$$

Proceeding inductively by differentiating (8.27) and plugging in $z = 0$ (the initial case being (8.28)), we come to the conclusion that (8.26) is equivalent to the set of identities

$$\sum_{n=1}^{\infty} \frac{\xi_n^{2\ell-2j+1} f(\xi_n)}{A'(\xi_n)} = 0 \qquad (j = 1, 2, \ldots, \ell),$$

just like we had in (8.12). The rest of the proof is completely identical to what was done in §8.4 (and that is the reason we kept track of the variables k and ℓ independently in the previous argument, so that it could be applied line by line to this case as well).

Acknowledgements

References

[1] N. I. Achieser, *Theory of approximation*, Frederick Ungar Publishing, New York, 1956.

[2] W. O. Amrein and A. M. Berthier, On support properties of L^p-functions and their Fourier transforms, J. Funct. Anal. 24 (1977), 258–267.

[3] A. Bonami and B. Demange, A survey on uncertainty principles related to quadratic forms, Collect. Math. 57 (2006), 1–36.

[4] L. de Branges, Homogeneous and periodic spaces of entire functions, Duke Math. Journal 29 (1962), 203–224.

[5] L. de Branges, *Hilbert spaces of entire functions*, Prentice-Hall, 1968.

[6] O. F. Brevig, A. Chirre, J. Ortega-Cerdà and K. Seip, Point evaluation in Paley-Wiener spaces, preprint at https://arxiv.org/abs/2210.13922.

[7] E. Carneiro, V. Chandee, F. Littmann and M. B. Milinovich, Hilbert spaces and the pair correlation of zeros of the Riemann zeta-function, J. Reine Angew. Math. 725 (2017), 143–182.

[8] E. Carneiro, V. Chandee and M. Milinovich, Bounding $S(t)$ and $S_1(t)$ on the Riemann hypothesis, Math. Ann. 356 (2013), no. 3, 939–968.

[9] E. Carneiro and A. Chirre, Bounding $S_n(t)$ on the Riemann hypothesis, Math. Proc. Cambridge Philos. Soc. 164 (2018), no. 2, 259–283.

[10] E. Carneiro, A. Chirre and M. Milinovich, Hilbert spaces and low-lying zeros of L-functions, Adv. Math. 410 (2022), 108748.

[11] E. Carneiro and F. Gonçalves, Extremal problems in de Branges spaces: the case of truncated and odd functions, Math. Z. 280 (2015), no. 1-2, 17–45.

[12] E. Carneiro and F. Littmann, Extremal functions in de Branges and Euclidean spaces, Adv. Math. 260 (2014), 281–349.

[13] E. Carneiro and F. Littmann, Extremal functions in de Branges and Euclidean spaces, II, Amer. J. Math. 139 (2017), no. 2, 525–566.

[14] E. Carneiro and F. Littmann, Monotone extremal functions and the weighted Hilbert's inequality, preprint at https://arxiv.org/abs/2302.14658.

[15] E. Carneiro, F. Littmann and J. D. Vaaler, Gaussian subordination for the Beurling-Selberg extremal problem, Trans. Amer. Math. Soc. 365 (2013), no. 7, 3493–3534.

[16] V. Chandee and K. Soundararajan, Bounding $|\zeta(\frac{1}{2} + it)|$ on the Riemann hypothesis, Bull. London Math. Soc. 43 (2) (2011), 243–250.

[17] A. Chirre, D. Dimitrov and E. Quesada-Herrera, An extremal problem and inequalities for entire functions of exponential type, preprint.

[18] D. L. Donoho and P. Stark, Uncertainty principles and signal recovery, SIAM J. Appl. Math 49 (1989), no.3, 906-931.

[19] G. B. Folland and A. Sitaram, The uncertainty principle: a mathematical survey, J. Fourier Anal. Appl. 3 (1997), 207–238.

[20] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, 7th ed., Academic Press, Elsevier, 2007.

[21] J. Holt and J. D. Vaaler, The Beurling-Selberg extremal functions for a ball in the Euclidean space, Duke Math. Journal 83 (1996), 203–247.

[22] L. Hörmander, *Linear Partial Differential Operators*, Springer-Verlag, 1964.

[23] P. Jaming, Nazarov's uncertainty principles in higher dimension, J. Approx. Theory 149 (2007), 30–41.

[24] M. Janet, Les valeurs moyennes des carres de deux dérivées d'ordres consecutifs, et le développement en fraction continue de tang x, Bull. Sci. Math., II. Ser., 55 (1931), 11–23.

[25] F. Littmann, Quadrature and extremal bandlimited functions, SIAM J. Math. Anal. 45 (2013), no. 2, 732–747.

[26] F. Littmann and M. Spanier, Extremal functions with vanishing condition, Constr. Approx. 42 (2015), no. 2, 209–229.

[27] F. Littmann and M. Spanier, Extremal signatures, Constr. Approx. 47 (2018), no. 2, 339–356.

[28] H. L. Montgomery and R. C. Vaughan, Hilbert's inequality, J. London Math. Soc. (2) 8 (1974), 73–81.

[29] F. L. Nazarov, Local estimates for exponential polynomials and their applications to inequalities of the uncertainty principle type (Russian), Algebra i Analiz 5 (1993) 3–66; translation in St. Petersburg Math. J. 5 (1994) 663–717.

[30] A. I. Nazarov and A. N. Petrova, On exact constants in some embedding theorems of high order, Vestnik St. Petersburg University: Mathematics 41 (2008), 298–302.

[31] A. I. Nazarov and A. P. Shcheglova, Steklov-type 1D inequalities (a survey), preprint at https://arxiv.org/abs/2101.10752.

[32] Yu. P. Petrova, Spectral asymptotics for problems with integral constraints, Math. Notes, 102 (2017), 369–377.

[33] M. Plancherel and G. Pólya, Fonctions entières et intégrales de Fourier multiples, (Seconde partie), Comment. Math. Helv. 10 (1938), 110–163.

[34] V. A. Steklov, The problem of cooling of an heterogeneous rigid, Communs Kharkov Math. Soc., Ser. 2, no. 5 (1896), 136–181 (in Russian).

[35] E. M. Stein and G. Weiss, *Fourier Analysis on Euclidean spaces*, Princeton Univ. Press, 1971.

[36] J. D. Vaaler, Some extremal functions in Fourier analysis, Bull. Amer. Math. Soc. 12 (1985), 183–215.

[37] J. D. Vaaler, On the number of lattice points in a ball preprint at https://arxiv.org/abs/2303.14905.

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