Economic growth: Theory and numerical solution methods
Description of contents

A. Novales, E. Fernandez, J. Ruiz*

August 2007

1 Chapter 1. Introduction

1.1 A few time series concepts: Some simple stochastic processes/ Stationarity, mean reversion, impulse responses/ Numerical exercise: Simulating simple stochastic processes

An overview of some statistical concepts using simple time series models: Stationarity, mean reversion, autocorrelation, impulse responses, autoregressive processes, stability. A section on simulating white noise, random walk, autoregressive processes comments on results in file *Simple_ simul.xls.* Lack of stationarity is illustrated, and impulse response functions are computed for processes with different characteristics.

1.2 Structural macroeconomic models: Static structural models/ Dynamic structural models/ Dynamic behavior of endogenous variables/ Dynamic multipliers/ Stochastic, dynamic structural models/ Stochastic simulation/ Numerical exercise: Simulating dynamic, structural macroeconomic models.

We introduce some basic concepts: model solution, short- and long-run equilibrium, impact and dynamic multipliers, steady-state. Stochastic macroeconomic models are simulated in file *Dynamic responses.xls*, and some statistical properties of simulated time series for consumption, output, investment, government expenditures and real interest rates like relative volatility, cross-correlation functions, consumption/output regressions are estimated. Impulse responses to innovation shocks are obtained for different parameterizations.

*Correspondence to: Alfonso Novales, Departamento de Economía Cuantitativa, Universidad Complutense, Somosaguas, 28223 Madrid, Spain. e-mail: anovales@ccee.ucm.es

1.3 Why are Economic Growth models interesting?: Microeconomic foundations of Macroeconomics/ Lucas' critique on economic policy evaluation/ A brief overview of developments on Growth theory/ The use of Growth models for actual policy making

We emphasize the significance of the microeconomic foundations of Growth models, and how they help to be safe from Lucas' critique to the traditional approach to economic policy evaluation. We present a first example of a Growth model. We review the main developments in Growth theory, and the specific characteristics of Dynamic, Stochastic, General Equilibrium models and Endogenous Growth models, and how they are presented in the textbook A reference is made to the use of these types of models in actual policy-making.

1.4 Numerical solution methods: Why do we need to compute numerical solutions to Growth models?/ Stability/ Indeterminacy/ The type of questions we ask and the conclusions we reach

After explaining why numerical solutions are needed, we describe the relevance of stability and how it is handled by numerical solution methods. The possibility of equilibrium indeterminacy is explained. We discuss how the work with Growth models with microfoundations and rational expectations has changed the type of policy questions we ask, and the type of conclusions we reach from policy analysis.

1.5 Synopsis of the book

The content of the book is described in detail.

2 Chapter 2. The Neoclassical Growth Model Under a Constant Savings Rate

2.1 Introduction

2.2 Returns to scale and sustained growth

The relationship between the returns to scale of cumulative inputs and the possibilities for positive growth are discussed

2.3 The neoclassical growth model of Solow and Swan: Description of model/The dynamics of the economy/Steady-state/The transition towards steady-state/The duration of the transition to steady-state/The growth rate of output and consumption/Convergence in the neoclassical model/A special steady-state: the Golden Rule of capital accumulation

A detailed description of the structure of the model is provided, and the possibilities for growth are discussed using the results in the previous section. The steady-state is defined and analytically characterized. An analytical exercise is performed on the steady-state effects of changes in structural parameters. The transition towards steady-state is discussed using graphical and analytical methods. Special attention is paid to the stability of the economy in its transition towards steady-state, as well as to the duration of the transition process. A brief reference is made to the implications for convergence of actual economies. After introducing the concept of Golden Rule as an optimal steady-state, it is characterized for the Solow-Swan economy, suggesting that a trajectory converging to the Golden Rule from an initial condition will generally be suboptimal.

2.4 Solving the continuous-time Solow-Swan model: Solution to the exact model/The linear approximation to the Solow-Swan model/Changes in structural parameters/Dynamic inefficiency

An exact, analytical solution to the differential equation characterizing the dynamics of the model is initially provided. To introduce issues discussed in later chapters, the analytical linear approximation to the model is obtained and solved for the Cobb-Douglas technology. The effects of changes in savings rate or in structural parameters on steady-state and on the transition paths is discussed graphically and analytically. The concept of dynamic inefficiency is introduced. Dynamically inefficient steady-states in the Solow-Swan economy are then characterized.

2.5 The deterministic, discrete-time Solow-Swan model: The exact solution/Approximate solution to the discrete-time model/Numerical exercise: solving the deterministic Solow-Swan model/Numerical exercise: a permanent change in the savings rate/Numerical exercise: Dynamic inefficiency

We start computing the analytical exact solution to the discrete-time, deterministic version of the model. To advance issues discussed in subsequent chapters, an analytical solution to the linear approximation to the model around steady-state is also obtained. A numerical exercise analyzing the exact and the approximate solutions for given parameterizations is discussed [*Solow_ deterministic.xls*]. We also compute numerical effects of changes in savings rate [*Change_ savings.xls,* *Change_ savings.m*]. A last numerical exercise shows how to characterize dynamically inefficient steady-states [*Dynamic inefficiency.xls, Dynamic_ inefficiency.m*].

2.6 The stochastic, discrete-time version Solow-Swan model: **Numerical exercise: solving the stochastic Solow-Swan model**

The solution to the linear approximation to the stochastic, Solow-Swan model is explained [*Solow_ stochastic.xls, Solow_ stochastic.m*]. A discussion is made on the possibility of producing a large number of realizations (numerical solutions) to the stochastic model, opening the possibility of estimating the distribution of any statistic of interest.

2.7 Exercises

3 Chapter 3. Optimal Growth

Maintaining the same structure of the exogenous growth economy (Solow-Swan model) analyzed in the first chapter, we consider the role of a benevolent planner who cares about the well being of privates agents in the economy.

3.1 The continuous-time version of the Cass-Koopmans model: **Optimality conditions for the Cass-Koopmans model / The instantaneous elasticity of substitution of consumption (IES)/ Risk aversion and the intertemporal substitution of consumption / Keynes-Ramsey condition/The optimal steady-state/Numerical exercise: The sensitivity of steady-state levels to changes in structural parameters/Existence, uniqueness and stability of long-run equilibrium: a graphical discussion / Suboptimality of the Golden Rule**

The continuous-time version of the economy is introduced and Pontryagin's principle is used to characterize the optimality conditions for the planner's problem. The instantaneous elasticity of substitution of consumption is defined and interpreted. Special attention is paid to its relationship with the elasticity of the marginal utility of consumption, and with the degree of risk aversion under constant relative risk aversion preferences. Keynes-Ramsey optimality condition is obtained and discussed under this family of utility functions. The *Steady-state* spreadsheet in the *CK_ continuous_ time.xls* file shows how steady-state levels differ between economies with different structural parameters. The same analysis can be also done using the *CK_ c_ steady state.m* MATLAB file. A graphical discussion is made to show the existence and uniqueness of a steady-state to the solution to the planner's problem. The optimal steady-state is shown to have a stock of capital below that of the Golden Rule, showing that the latter is a suboptimal steady-state. We present the Short- and the long-run effects of a change in the output elasticity of physical capital, can be computed with the

CKcTRANS.m MATLAB file, and the *Change in output share of* k spreadsheet in the *CK_continuous_time.xls* file.

3.2 Stability and convergence: **The trajectory for income/Numerical exercise: Characterizing the transition after a change in a structural parameter. The speed of convergence to steady-state. A note of caution. Convergence to steady-state as a function of the degree of risk aversion/ A change in the output elasticity of capital: transition between steady-state**

A first characterization of stability conditions on the linear approximation to the optimality conditions to the model. This approach is extensively used throughout the book. The set of optimality conditions for the planner's problem is shown to have a saddle-path structure, the optimal steady-state being stable from any initial stock of capital, provided the initial level of consumption is chosen appropriately. The income trajectory consistent with the saddle path for consumption and capital is characterized. We compute different characteristics regarding the transition of the planner's economy to its optimal steady-state in *CK_ continuous_ time.xls.* We pay special attention to the speed of convergence, as well as how this depends on structural parameters, specifically the degree of risk aversion. An important point that emerges in this analysis is the difficulty of drawing inferences from a sample of equally spaced observations from a continuous-time variable. Below it is shown that the solution to the discrete-time version of the problem should rather be used for this type of inference. One of the spreadsheets in *CK_ continuous_ time.xls.* is devoted to characterizing the transition between steady-state following a change in the output elasticity of capital. Similar analysis could be done with other structural parameters, as mentioned in some exercises proposed at the end of the chapter.

3.3 Interpreting the central planners's model as a competitive equilibrium economy: **The efficiency of competitive equilibrium**

The benevolent planner's problem is shown to be equivalent to the competitive equilibrium in an economy without frictions or distortions, like those introduced by distortionary taxation. Welfare theorems are shown for such an economic structure.

3.4 A competitive equilibrium with government: The structure of the economy/Feasible stationary public expenditure and financing policies/Competitive equilibrium/Global constraint of resources/The representative agent problem

The model is extended to incorporate a government that consumes some resources, financed by debt issuing and lump-sum taxes. No externality is considered in preferences or technology. The competitive equilibrium allocation is characterized. The representative agent's problem, which will be extensively used in next chapters, is introduced

3.5 On the efficiency of equilibrium with government: On the efficiency of equilibrium under lump-sum taxes and debt/ The inefficiency of the competitive equilibrium allocation under distortionary taxes. The inefficiency of the competitive equilibrium allocation under consumption taxes. Leisure in the utility function. The inefficiency of the competitive equilibrium allocation under income taxes

This section contains a non-standard detailed discussion on the possible sources of inefficiency. After a general discussion on the role of government expenditures to produce inefficiencies, we discuss the inefficiencies introduced by distortionary taxation. On the contrary, constant consumption taxes are shown not to be the cause of additional inefficiencies, except when leisure appears as an argument in the utility function.

3.6 The Ricardian Doctrine: The Ricardian Doctrine under non-distorting taxes/Failure of the Ricardian Doctrine under distorting taxes

The section starts by showing that the Ricardian doctrine holds under non-distortionary taxation. The second part is devoted to an analytical proof of its failure under distortionary taxes.

3.7 Appendices: **Log-linear approximation to the continuous-time version of the Cass-Koopmans model/An alternative presentation of the equivalence between the planner's and the competitive equilibrium mechanisms in an economy without government**

4 Chapter 4. Discrete-time version of the deterministic Cass-Koopmans model:

4.1 Discrete time version of the optimal growth model: **The global constraint of resources/ Discrete-time formulation of the planner's problem/The optimal steady-state/The dynamics of the economy: The phase diagram/Transversality condition in discrete time/Competitive equilibrium with government**

The section starts with the discrete-time specification of the planner's problem. Optimality conditions are obtained and a graphical discussion of stability is provided. The role of the transversality condition is explained. The competitive equilibrium mechanism in an economy with government is analytically characterized in this setup.

4.2 Fiscal policy in the Cass-Koopmans model: **The deterministic case. Solving the representative agent problem. Stability/Numerical exercise: solving the deterministic competitive equilibrium with taxes/Numerical exercise: fiscal policy evaluation. Measuring welfare effects. Long-run effects of a tax reform. Short- and long-run effects of a tax reform experiment.**

A central section in this chapter, devoted to the evaluation of alternative fiscal policies, designed to finance an exogenous sequence of government expenditures. The representative problem is presented and solved, paying special attention to the characterization of stability. This type of analysis is extensively used throughout the book. A numerical exercise is devoted to solve for the deterministic competitive equilibrium with taxes [*CK_ taxes_ deterministic.xls, CK_ d_ transition.m*]. A second numerical exercise computes the short- and long-run effects of changes in fiscal policy on the time paths followed by real variables, as well as the final effect on the level of welfare. A raise in either consumption taxes or in income taxes is considered. We also consider a simultaneous increase in both tax rates keeping tax revenues constant, or keeping constant the revenues to income ratio Steady-state effects [*Steady-state* spreadsheet in *CK_ solution_ changes_ in_ tax.xls*, *CK_ d_ long run tax changes_ L.m*] and short- and long-run effects [*Transition* spreadsheet in *CK_ solution_ changes_ in_ tax.xls*, *CK_ d_ long_ short run tax changes.m*] are computed.

4.3 A reformulation of the stability condition for the deterministic version of the model

A comment on the interpretation of the stability condition characterized in previous sections.

4.4 Appendices: The intertemporal government budget constraint. Sustainable steady-state expenditures and financing policies/The Ricardian proposition under non-distortionary taxes in discrete time

4.5 Exercises

5 Chapter 5. Numerical Solution Methods

5.1 Introduction: Numerical solutions and simulation analysis

- Part I. The basic stochastic growth model

5.2 Analytical solutions to simple growth models: A model with full depreciation/A model with leisure in the utility function/Numerical solutions to the growth model under full depreciation

We follow McCallum to obtain the analytical solutions to planner problems in two simple, stochastic economies. Full depreciation is shown to be crucial for the analytical solution to exist. An EXCEL file is provided with numerical simulations for each of these two economies [*Simple_ models.xls*].

5.3 Solving a simple, stochastic version of the planner's problem: Solving the linear-quadratic approximation to the planner's problem/The log-linear approximation to the model/The Blanchard-Kahn solution method for the stochastic planners' problem. Log-linear approximation/Uhlig's undetermined coefficients approach. Log-linear approximation/Sims' eigenvalue-eigenvector decomposition method using a linear approximation to the model. Numerical exercise: Solving the stochastic representative agent's model through the eingenvector-eigenvalue decomposition approach. Solving the planner's problem with the eigenvalue-eigenvector decomposition under a log-linear approximation

Optimality conditions for the planner's problem in the standard Brock-Mirman stochastic growth economy are obtained. This model is used to illustrate application of the numerical solution methods discussed below. We start by showing the analytical details needed to obtain the solution to the linear approximation to the stochastic growth model. It is shown how to guarantee stability of the implied solution. *Simple_planner_problem.xls* computes a single realization of this solution. Additional realizations could be obtained by drawing a different realization of the productivity shock. Tax rates can be set to zero in the *lq.m* file to obtain a single numerical realization.

The log-linear approximation to the model is obtained in detail. It is used by the solution methods described in the sections that follow.

The Blanchard-Kahn method is described in detail, explaining how to guarantee stability. Tax rates can be set to zero in *CK_solution_BK.xls* to compute a single realization for the stochastic growth economy using this solution approach. The same can be done with the *Blanchard Kahn.m* MATLAB file. Analytical details for the implementation of Uhlig's undetermined coefficients solution approach are provided for the stochastic growth economy. Special attention is paid to guarantee stability of the implied solution. *Simple_planner_problem.xls* computes a single realization using this solution approach. The same can be done using the *uhlig.m* MATLAB file.

The eigenvalue-eigenvector solution approach is illustrated on a linear approximation to the stochastic growth economy. Stability is implied through the use of the eigenvectors associated to unstable eigenvalues, to define stability relationships between control and state variables. A single realization of the numerical solution obtained through this solution approach is obtained in *CK_Stochastic.xls* for different parameterizations. The same exercise can be done with the *Sims.m* program. Besides carefully dealing with stability, the obtained solution is tested for rationality by examining the autocorrelation function of the expectations error and its cross-correlation functions with variables know at the time the conditional expectation was formed. Details are provided in an additional paragraph to implement the solution method on the log-linear approximation to the stochastic growth economy. This exercise can also be done with the *Lsims.m* program.

Details of the application of the parametrized expectations method to the stochastic growth economy are discussed in detail. Special attention is paid to the initial conditions for the solution method, as well as to the stability of the implied solution. MATLAB program *marcet.m* estimates the parameters in the exponential polynomial used to approximate the conditional expectation in the model, *marcet1.m* solves the planer's problem using the *fminunc.m* MATLAB minimization routine, while *marcet2.m* does the same using a Gauss-Newton algorithm.

Finally, MATLAB program *methods.m* is a main program solving the planner's problem by different solution methods: the linear-quadratic approximation, Uhlig's undetermined coefficients method, Blanchard-Kahn's method, the eigenvalue-eigenvector decomposition implemented on either the linear or the log-linear approximation to the model, and the parameterized expectations method.

- # Part II. The stochastic growth economy with taxes

5.4 Solving the stochastic representative agent's problem with taxes: The log-linear approximation/Numerical exercise: Solving the stochastic representative agent's model with taxes through Blanchard and Kahn's approach. Log-linear approximation/Numerical exercise: Computing impulse responses to a technology shock. Log-linear approximation/Numerical exercise: Solving the stochastic representative agent's model with taxes through the eingenvector and eigenvalue decomposition approach. Linear approximation.

The analytical solution to the representative agent's problem in the discrete-time version of the stochastic growth economy subject to consumption and income taxes is characterized. In parallel with the analysis in the first part of the chapter, the log-linear approximation to the model is obtained, to be used by the solution methods in sections below. The *Stochastic-BK* spreadsheet in *CK_solution_BK.xls* presents a numerical simulation of the solution to the representative agent's problem using the Blanchard-Kahn approach. The solution is obtained with the same sample realization for the productivity innovation, but up to 7 different vectors of structural parameters. In each case, basic statistics like sample mean, variance, coefficient of variation, and cross-correlation functions with output are obtained for each variable.

The *BK-impulse* spreadsheet in the *CK-solution_BK.xls* file presents impulse responses to a transitory, one period technology shock for the benchmark parameterization used in the previous numerical exercise, that illustrated the Blanchard-Kahn solution approach. Qualitative and quantitative characteristics of the obtained responses are discussed.

The eigenvalue-eigenvector decomposition is used to solve the linear approximation to the representative agent's problem with taxes. A single realization of the numerical solution to this model following this procedure is described in *CK_stochastic_taxes.xls*. Different parameterizations regarding tax rates and preference and technology parameters are used in *CK_stochastic_taxes_structural_parameters.xls* to obtain different realizations of the solution, using the same sample realization for the productivity innovation.

- # Part III: Nonlinear numerical solution methods

5.5 Parameterized expectations: **Solution to the planner's problem**

5.6 Projection methods: **Solution to the deterministic optimal growth model. Solution to the stochastic optimal growth model. Implmentation of Galerkin's method.**

We present the method of Parameterized Expectations (already included in Chapter 4 in the first draft of the book) and different versions of Projection methods. We describe the use of basis functions, and different approaches to choose the polynomial function that approximates bets the decision rules as function of the current states. We discuss the solution to deterministic and stochastic models separately. MATLAB program *coll_ cheb.m,* together with some additional functions written for this book, solves the deterministic version of the optimal growth model using the Collocation approach, while *g_ cheb_ s_ 3.m*, *g_ cheb _s_ 4.m* and *g_ cheb_ s_ 5.m* and several auxiliary MATLAB functions solve the stochastic optimal growth model following Galerkin's approach, for different orders of the polynomial approximating function. The reader is instructed on how to modify the programs for different choices of the function approximating the optimal decision rule.

5.7 Appendix: Solving the planner's model under full depreciation

5.8 Exercises

6 Chapter 6. Endogenous Growth models

6.1 The AK model: **Balanced growth path/Transitional dynamics/Boundedness of time-aggregate utility**

The analytical, continuous-time version of the AK model is introduced and the planner's problem is solved. A steady-state is seen to exist in which per capita variables grow at a positive rate. Growth rates are shown to be the same for all variables. We show the model lacks any transitional dynamics, the economy adjusting instantaneously to any change in steady-state. Parametric restrictions imposed by the transversality conditions are obtained. The relationship between parametric conditions needed for the transversality condition to hold and for boundedness of time aggregate utility is shown.

6.2 The discrete time version of the AK model: The transversality condition and bounded utility/Absence of transitional dynamics. Relationship between the stock of physical capital and consumption

The representative agent's problem is solved in the discrete-time version of the AK economy. The same properties shown for the continuous time model are shown in this case. We again discuss the structure and implications of the transversality condition and interpret it as a condition for stability.

6.3 Stability in the AK model

In a deterministic, endogenous growth model, a stable solution displays constant ratios of per capita variables. In a stochastic model, a stable solution would imply ratios fluctuating around their mean values. As in previous chapters, a linear approximation is used to characterize stability conditions. The transition matrix in the AK model is shown to have a unit eigenvalue, characteristic of endogenous growth models. The existence of a second, unstable eigenvalue, allows us to characterize the link between control and state variables needed for stability.

6.4 Effects from transitory changes in policy parameters: A policy intervention/A comparison with the Cass Koopmans economy

A transitory policy intervention is shown to have long-lasting effects in the AK economy, as opposed to what happens in the Cass-Koopmans exogenous growth economy.

6.5 Dynamic Laffer curves: Numerical exercise on dynamic Laffer curves

Following Ireland (1994), we show that in an endogenous growth economy it may be possible to increase revenues by lowering distortionary tax rates. We characterize conditions under which the result will arise in the AK economy. A numerical exercise is performed computing the possibility of a feasible reduction in taxes while financing the same time sequence of government expenditures [*DynamicLaffer.xls*].

6.6 Solving the stochastic, discrete time version of the AK model: A linear approximation to the stochastic AK model/Numerical exercise: solving the stochastic AK model

The stochastic, discrete-time version of the AK model is solved. To compute the numerical solution, we exploit the eigenvalue structure of the linear approximation to the model. A stable solution can be computed for the consumption-capital ratio from which per-capita variables can be then readily obtained. Growth rates are time varying, but they fluctuate around their common value in the deterministic version of the model [*AK_ Stochastic.m* for a single realization, *mAKStochastic.m* for multiple realizations].

6.7 An endogenous model with productive public expenditures. Barro's model

The endogenous growth model of Barro, that includes public capital as an input in the production function is presented, and the optimal paths for consumption and capital are characterized.

6.8 Transitional dynamics in endogenous growth. The Jones and Manuelli model: Steady-state/Solving the deterministic version of Jones and Manuelli's model through a linear approximation

An endogenous growth allowing for a non-trivial transition to steady-state is presented, and the optimal paths are characterized. A linear approximation to the model around steady-state is used to obtain a numerical solution.

6.9 The stochastic version of Jones and Manuelli's model: Deterministic balanced growth path/Transforming the model in stationary ratios/The phase diagram of the deterministic version of the Jones-Manuelli model. Transitional dynamics/Computing the dynamics. Log-linear approximation/The stochastic AK model as a special case.

Optimality conditions for the stochastic version of the Jones-Manuelli endogenous growth model are obtained and interpreted. A transformation of the system of optimality conditions in ratios is provided that allows for a numerical stationary solution to be obtained. The phase diagram of the model is presented and explained. The dynamics are characterized through a linear approximation to the system of optimality conditions around the Balanced Growth Path. The stochastic AK model is obtained as a special case.

7 Chapter 7. Additional Endogenous Growth Models

7.1 A variety of producer products: **The economy/ The equilibrium/ The inefficiency of the equilibrium allocation/ Time varying productivity/ A stochastic version of the varieties model. Analogy with the AK model. Numerical solution. Steady-state. Log-linear approximation.**

We describe endogenous growth in an economy with a variety of intermediate goods produced by monopolistic firms, as in Romer (1987) and (1990). Lack of transitional dynamics, the existence of a balanced growth path steady-state and the inefficiency of the equilibrium solution are shown. An algorithm to solve the stochastic version of the model using a log-linear approximation is presented.

7.2 Technological diffusion and growth: **The problem of the follower country/ Deterministic steady-state/ Numerical solution. The log-linear approximation and numerical derivatives.**

The two-country technological diffusion of Barro and Sala-i-Martin (1997) is described. The country that adapts technology is shown to have a nontrivial transition to steady-state. An algorithm to solve the stochastic version of the model is presented. The *simul_ diffus.m* program computes numerical solutions for this and the previous model.

7.3 Schumpeterian growth: **The economy/ Computing equilibrium trajectories/ Deterministic steady-state**

Endogenous growth is shown to arise in a model by Aghion and Howitt (2005) that incorporates Schumpeter's suggestion of creative destruction. Firms can decide to spend more resources on research and development which, with some probability, may end up improving the quality of the intermediate good they would produce as monopolists. Equilibrium is characterized and an algorithm is provided to generate time series from the solution to the model. The economy is shown to experience a nontrivial transition between steady-states, which are shown to take the form of a balanced growth path.

7.4 Endogenous growth with accumulation of human capital: **The competitive equilibrium/ Analyzing the deterministic steady-state. Conditions for endogenous growth/ Numerical exercise: steady-state effects of fiscal policy/ The steady-state takes the form of a balanced growth path. Computing equilibrium trajectories in a stochastic setup under the assumption of rational expectations. Log-linear approximation/ Indeterminacy of equilibria/ Simulating the human capital accumulation model under indeterminacy of equilibria. Numerical exercise: the correlation between productivity and hours worked in the capital accumulation model.**

The endogenous growth model with human capital accumulation is presented in this section. We characterize the equilibrium, and show the existence of non-trivial transition between steady-states, which again take the form of a balanced growth path. MATLAB files *lucas_ss_c.m, lucas_ss_r.m* and *lucas_ss_w.m* compute steady-state effects of changes in either consumption, capital income or labour income tax rates. We provide algorithms to compute numerical solutions to the stochastic version of the model. We also show the possibility of equilibrium indeterminacy in a version of the model that includes an externality in the production of the final good, discuss and interpret that possibility, and explain how to solve the model in that situation. Matlab programs *lucas_sim1.m* and *lucas_sim2.m* compute single time series realizations for the solution under a well-determined equilibrium and indeterminacy of equilibrium, respectively. These programs allow for a possible aggregate human capital externality in the production of the final good. MATLAB program *Mlucas_sim.m* computes multiple realizations under different situations regarding determinacy of equilibrium and human capital externalities, and a variety of statistics are estimated. Special attention is paid to the correlation between productivity and hours worked, which has been traditional object of discussion in the Business Cycle literature.

7.5 Exercises

8 Chapter 8. Growth in Monetary Economies: Theoretical models and steady-state analysis of monetary policy

8.1 Introduction

The chapter is structured in two parts, which address a variety of issues regarding monetary policy, as well as the interaction between fiscal and monetary policy. The focus of the first part of the chapter is on steady-state comparisons, while the second part of the chapter analyzes monetary policy outside steady-state. This type of analysis needs of methods to obtain numerical solutions and compute equilibrium time series under alternative choices of control variables for the implementation of monetary policy.

8.2 Optimal growth in a monetary economy: The Sidrauski model: **The representative agent's problem/Steady-state in the monetary growth economy/Golden Rule**

The theoretical discussion starts with a version of Sidrauski (1967) model, to characterize optimal growth in a monetary economy. After describing the economy, the representative agent's problem is solved, and analytical expressions for steady-state levels are derived. These are used to analyze alternative designs for monetary policy. The inflation tax is defined. Optimality conditions for the representative agent's problem are derived are interpreted. The demand for money function as an optimality relationship. The Golden Rule in the monetary economy is characterized.

8.3 Steady-state policy analysis: **Optimal steady-state rate of inflation/The welfare cost of inflation**

Steady-state expressions above are used for a detailed discussion of alternative combinations of fiscal and monetary policy. Special attention is paid to the interaction between monetary and fiscal policy, and to the conditions for a given combination to be feasible. Definition and characterization of optimal inflation. After introducing measures of the welfare cost of inflation.

8.4 Two modelling issues: nominal bonds and the timing of real balances: **Nominal bonds: the relationship between real and nominal interest rates/Real balances in the utility function: ¿at the beginning or at the end of the period?/Numerical exercise: optimal rate of inflation under alternative assumptions on preferences**

By considering nominal as well as real bonds in the model, an equilibrium relationship is obtained to relate the rates of return on the two type of bonds. We also illustrate the difference between considering real balances at the beginning or at the end of the period as an argument in the utility function. Qualitative results associated to either of these formulations are the same when the analysis is made in steady-state. However, there will generally be some differences among numerical values of the main variables. Furthermore, results may be different when analyzed along the transition between steady-states.

We compare in this section the steady-state welfare cost of inflation in two economies, differing only in the specification of preferences. In the first case, the consumer is supposed to get utility from real balances at the beginning of the period, while in the second case, real balances at the end of the period enter as the argument in the utility function. In both economies, the welfare cost of inflation is increasing in the level of the inflation rate, so the lowest feasible inflation rate turns out to be optimum.

Timing real balances.xls file presents steady-state computations for both economies for a wide range of values of the inflation rate. A numerical exercise is presented to evaluate the welfare implications of alternative monetary policies.

8.5 Monetary policy analysis under consumption and income taxes: Steady-state/Numerical computation of steady-state levels under alternative policy choices

We now review the model incorporating the assumption that, in addition to printing money and issuing bonds and maybe collecting lump-sum taxes, the government also levies proportional taxes on consumption and income. Optimality conditions for the representative consumer problem are obtained and interpreted. Analytical expressions determining steady-state levels are obtained and discussed. The non-neutrality of fiscal policy is shown. The neutrality of monetary policy is shown to depend on the specific combination of fiscal and monetary policy chosen. Numerical computation of steady-state levels for the main variables in the economy under alternative policy choices. We use again steady-state optimality conditions to characterize the feasible combinations of monetary and fiscal policies. A Laffer curve is shown to arise in some cases [*Steady state fiscal policy.xls*].

8.6 Monetary policy under endogenous labor supply: The neutrality of monetary policy under endogenous labor supply/Numerical evaluation of steady-state policies with an endogenous labour supply/

The analysis in previous sections is extended to consider an endogenous supply of labor. The neutrality of monetary policy is shown to depend on the separability between consumption, real balances and leisure in the utility function. Two specific examples of preferences leading to non-neutral monetary policy are considered. Numerical values of steady-state levels for the main variables in the economy are obtained under alternative specifications of preferences. The optimal rate of inflation is calculated numerically, and the validity of Friedman's rule is discussed. The non-neutrality of monetary policy is discussed [*SS inflation endogenous leisure.xls* file].

8.7 Optimal monetary policy under distortionary taxation and endogenous labor: The model/Implementability condition/The Ramsey problem

To characterize the optimal inflation tax, we compute a *Ramsey equilibrium.* Ramsey's problem is solved under the assumption that there is some sort of institutional commitment forcing the government to actually implement in the future the policy that is chosen at each point in time. For the first time in this chapter, we analytically characterize the transitional dynamics.

9 Chapter 9. Transitional dynamics in monetary economies. Numerical solutions to analyze optimal monetary policy.

9.1 Introduction

Up to this point, we have characterized dynamic optimality conditions for the monetary economies we have considered, but we have only performed some numerical evaluations of steady-state characteristics. It is however important to learn about the transitional dynamics of this monetary growth model, as it moves from the initial condition to the steady-state. Three questions are particularly interesting to discuss: i) how the dynamics of the economy restrict the set of feasible policies, ii) whether the monetary authority should control the interest rate or the money supply, iii) even though monetary policy may be neutral in the long-run, it may be non-neutral in the short-run. Furthermore, monetary policy effects may be larger if the government changes policy gradually, relative to the possibility of a drastic policy change.

9.2 Stability of public debt

The backfeeding character of the stock of public debt implies that when the government implements an active monetary policy like controlling the nominal rate of interest, it is forced to use a passive fiscal policy regarding the stock of public debt and the amount of the transfers to the private sector, with the only objective that the government budget constraint holds each period. We compute numerically the short- and long-run effects produced by a permanent increase in the nominal rate of interest together with an income tax rate cut, implemented in such a way that the government budget constraint holds in steady-state before and after the change in nominal rates. Special attention is paid to the way how to link the size of the lump-sum transfer to the private sector to the stock of public debt each period, to avoid instability of the path for the public debt variable not only when monetary policy is designed to control the nominal rate of interest, but also when the control variable is the money supply [*Change in nominal interest rates* spreadsheet in *Short-run nonneutrality.xls*].

9.3 Alternative strategies for monetary policy: control of nominal rates versus money growth control

We consider the deterministic version of the discrete-time version of Sidrauski's monetary economy under two different designs for monetary policy, depending on whether the monetary authority uses the rate of growth of money supply or nominal interest rates as a control policy variable. The concepts of exogeneity and causality are explained using the description of the numerical solution approach. We also explain the nominal indeterminacy that may arise in these economies.

9.4 Deterministic monetary model with the monetary authority choosing money growth: Steady-state/Solution through a log-linear approximation/Complex eigenvalues

Optimality conditions for the representative agent economy are obtained under this policy design. Analytical expressions for the log-linear approximation to the model are obtained, and a detailed discussion of how to obtain numerical solutions following the Blanchard-Kahn approach are provided [MATLAB program *money_M_d_c.m*].

9.5 Deterministic monetary model with the monetary authority choosing nominal interest rates

We assume in this section that the monetary authority chooses nominal rates of interest each period. We obtain the optimality conditions and describe in detail how to obtain numerical solutions to the model following Blanchard and Kahn's approach [MATLAB program *money_i_d.m*].

9.6 Transitional effects of policy interventions: Solving the model with nominal interest rates as control variable, using a linear approximation/Numerical exercise: Changes in nominal interest rates/Solving the model with money growth as control variable, using a linear approximation/Numerical exercise: Gradual versus drastic changes in money growth

We analyze numerically the effects of different monetary policy experiments [*Short-run nonneutrality.xls* file]. The first one considers a government that finances a transfer to consumers by printing money, issuing debt and raising proportional taxes on output. We first compute the steady-state rate of inflation associated to each level of the tax rate. We then assume the government changes monetary and fiscal policy in a consistent way. The spreadsheet displays the transitional dynamics for the main variables in the economy, as well as steady-state effects.

We analyze the effects of a change in the rate of money growth. We show in this section that a monetary policy intervention can be non-neutral if it is implemented gradually, while being neutral if the new policy target is achieved immediately. We maintain a setup similar to those in the previous sections, but we now assume that the government controls the rate of growth of the money supply at each point in time. A detailed discussion on how to obtain numerical solutions to the model under this policy design using a linear approximation are provided. We compare the effects of a gradual versus a drastic policy change [*Gradual money change* and *Once and for all money change* spreadsheets in *Short-run nonneutrality.xls*]. In the *money_M_d.m* MATLAB file, the reader can perform the same exercises by setting $\rho_x = 0$.

9.7 The stochastic version of the monetary model

We will now consider an economy where the time evolution of the general level of productivity follows a given stochastic process.

9.8 The monetary authority chooses nominal interest rates

To gain generality, the government is supposed to follow a Taylor's rule to implement policy. This is a central section in which two cases are considered separately, depending on the specific structure of the Taylor's rule. In each case, optimality conditions for the representative agent's problem are obtained, and a detailed discussion on how to obtain numerical solutions following Blanchard and Kahn's approach is provided. In each case, a log-linearization to the model is used to characterize stability. Using these conditions, the original nonlinear model is used to generate the numerical solutions. We provide an analytical proof of the *indeterminacy* of the price level under this policy design. Being a stochastic economy, we can now compute as many realizations as desired, since they will all be different from each other. These realizations can be used to compute the value of any statistic, whose probability distribution will be estimated over a large set of simulations. Two cases are considered, depending on whether the rate of inflation is included in the specification of Taylor's rule [*S_i_npi_s.m, S_i_pi_s.m* MATLAB files]. Multiple realizations for both specifications of Taylor's rule can be obtained using program *mS_i.m*.

9.9 The monetary authority chooses money supply growth

We consider the case when the monetary authority uses money growth as control variable for monetary policy implementation. A detailed discussion of stability is provided, and of the sequence of steps needed to compute a full numerical solution [*S_M_s.m* MATLAB file for a single realization, *mS_M.m* for multiple realizations].

9.10 A new Keynesian monetary model: A model without capital accumulation: Ireland's (2004)/A new Keynesian monetary model with capital accumulation

9.11 Appendices: The stability condition in the state-space formulation

9.12 Exercises

10 Chapter 10. Mathematical Appendix

A variety of mathematical issues which are used in the previous chapter is gathered in this Appendix.. The first section reviews the continuous-time control problem, paying special attention to the transversality condition. The discounted control problem is also considered, and the calculus of variations is shown as a special case of the more general control problem. The second section briefly reviews first order differential equations, while the last section is devoted to some results from matrix algebra.

10.1 The deterministic control problem: Transversality condition / The discounted problem / Calculus of variations

10.2 First order differential equations

10.3 Matrix algebra: Systems with a saddle path property

10.4 Some notes on complex numbers

10.5 Solving a dynamic two-equation system with complex roots

Economic growth: Theory and numerical solution methods
Description of contents

A.Novales, E. Fernández, J. Ruiz

June 2021

Third Edition

The third edition contains two new chapters. These are chapter 10, entitled: Empirical Methods 1: Frequentist Estimation, and chapter 11, entitled: Empirical Methods 2: Bayesian Estimation. Chapter 10 of the first and second editions becomes chapter 12 in the third edition.

Chapter 10. Empirical Methods 1: Frequentist Estimation

Abstract

The chapter starts with the Generalized Method of Moments estimator, describing its main properties, and applying it to the estimation of an equilibrium asset pricing model. After that, we explain the implementation of the Maximum Likelihood estimator. The Kalman filter, the main tool for the numerical evaluation of the likelihood on the state-space representation of the model, is discussed in detail. We estimate cyclical and trend components in US GDP and the unemployment rate. Finally, we compute the ML estimator to the Hansen (1985) model of indivisible labor. We explain Matlab programs provided to estimate structural parameters and generate some interesting properties of Growth models, as impulse responses to supply and demand shocks, and the decomposition of the variance of forecast errors.

Keywords

Generalized Method of Moments, Maximum Likelihood, State-space representation, Kalman filter

10.1 Introduction

10.2. The Generalized Method of Moments

We start by providing in section 10.2.1 a general description of the GMM estimation approach. The implementation and properties of the method are described in section 10.2.2, while 10.2.3 contains an application to the estimation of an equilibrium asset pricing model

10.2.1. Introduction

10.2.2. Estimation procedure

We describe in this section the general approach to the implementation of the GMM estimator, using a simple representative agent, utility maximization framework as illustration.

10.2.3. Application to the Consumption-based Capital Asset Pricing Model

We are going to apply the GMM estimation procedure described in the previous section to a dynamic and stochastic standard model under the assumption of Rational Expectations, the portfolio decision-making model of Hansen and Singleton (1982).

10.3. Maximum Likelihood

Unlike GMM estimation, maximum likelihood estimation allows us to evaluate a dynamic general equilibrium model using the full set of statistical properties of the model, subject to an assumption about the probability distribution of the structural shocks of the model.

10.3.1. State-Space Representation

To estimate the structural parameters of the theoretical model, given observed data, we are going to take advantage of the fact that any of the models seen in the previous chapters can be written using the state-space representation of dynamic, stochastic systems

10.3.2. Kalman Filter

Description of Kalman Filter.

10.3.3. Using Kalman Filter to evaluate the likelihood function

10.3.4. Estimation of Hansen's (1985) model with indivisible labor

In this section we are going to estimate the structural parameters of a Dynamic, Stochastic General Equilibrium (DSGE) model. As an illustration, we use Hansen (1985) model with indivisible work, in which we are going to introduce two types of shocks: a supply shock (a shock in productivity) and a demand shock (a shock in preferences). The objective will be to estimate the structural parameters of the model as well as the parameters that characterize the stochastic process for the two shocks.

Chapter 11. Empirical Methods 2: Bayesian Estimation

Abstract

The chapter starts with an introduction to Bayesian inference, and two applications examples in the context of regression models. After that, we introduce Markov Chain Monte Carlo Methods, and provide a theoretical discussion of two families of such methods: Gibbs-sampling and Metropolis-Hastings algorithms. We estimate the parameters of a linear regression model using the Gibbs-sampling algorithm. Three applications of the Metropolis-Hastings algorithm are considered: random number generation from a Cauchy distribution; estimation of a GARCH(1,1) model, and estimation of a DSGE model which has been already estimated in chapter 10 under a frequentist approach, so that the reader can compare the two different methodologies for the estimation of Growth models.

Keywords
Markov Chain Monte Carlo Methods; Gibbs-sampling; Metropolis-Hastings algorithms, GARCH(1,1)

11.1 Introduction

The chapter starts with a brief introduction to Bayesian inference, and some application to the estimation of parameters in a linear regression model.

The structure of this chapter is as follows. Section 2 provides an introduction to Bayesian inference, including two examples estimating different parameters of a general linear regression model. Section 3 provides an introduction to Markov Chain Monte Carlo Methods. Sections 4 and 5 contain the theoretical discussion of two families of these methods: the Gibbs-sampling and Metropolis-Hastings algorithms. In section 4, parameters of a linear regression model are estimated with the Gibbs-sampling algorithm. Three applications of the Metropolis-Hastings algorithm are studied: in section 5 the method is applied to generate random numbers from a Cauchy distribution; in section 6 a GARCH(1,1) model is estimated; finally, in Section 7 we estimate the same DSGE model that was estimated in the previous chapter using the Metropolis Hastings algorithm.

11.2 Introduction to Bayesian Inference

11.2.1. Examples of Bayesian estimation

We now present two examples of Bayesian estimation. They are characterized by the fact that we can determine the posterior distribution of the parameters analytically, and we can use their mean and variance to provide numerical estimates

11.3 Markov Chain Monte Carlo Methods (MCMC Methods)

In many empirical modeling applications, the data analyst would like to use complex models, but it is often forced to use simplified models in order to use the frequentist and Bayesian techniques available, in which posterior distributions can be obtained analytically. To avoid that limitation, Markov Chain Monte Carlo (MCMC) methods are based on simulation and help the researcher to examine the data using more realistic and complex statistical models, such as dynamic, stochastic, general equilibrium models (DSGE).

11.4 The Gibbs-sampling algorithm

11.5 Metropolis-Hastings Algorithm

11.5.1 Metropolis-Hastings Sampler

11.5.2 Metropolis Sampler

11.5.3 Random-walk Metropolis Sampler

11.5.4 Independence Sampler

11.6 Application: Estimating a GARCH(1,1) model

In this section we are going to get Bayesian estimates of a GARCH (1, 1) model using the Metropolis-Hastings algorithm with random walk (*Random Walk Metropolis-Hastings*). This example will give us all the necessary keys to understand how to get Bayesian estimates of a DSGE model.

11.7. Bayesian Estimation of a DSGE Model by the MCMC method: Hansen's (1985) model with indivisible labor

In this section we apply everything we learned in the previous section, to estimate the structural parameters of the dynamic, stochastic general equilibrium model Hansen (1985), which was already estimated in Chapter 10 from a frequentist point of view.