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Complex Interpolation of Closed Subspaces of Maximal Banach Function Spaces

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1 Some remarks on complex interpolation

Throughout this talk, for simplicity, we assume that the measure space (Ω, Σ, μ) is complete and σ-finite.

2 Review of the known notions

Definition 1. Let (Ω, Σ, μ) be a measure space.

(i) A Banach space $X \subset L^0(\mu)$ with norm $\|\cdot\|_X$ is called a *Banach lattice over* Ω if for $f, g \in L^0(\mu)$,

$$f \in X \text{ and } |g| \le |f| \ \mu\text{-a.e. on } \Omega$$

imply

$$g \in X \quad \text{and} \quad \|g\|_X \le \|f\|_X.$$

(ii) A Banach lattice X is called a *Banach function space over* Ω if there exists a function $u \in X$ such that $u > 0$ μ-a.e. on Ω.

(iii) Let X be a Banach function space over Ω. The *Köthe dual* X' of X is defined by

$$X' := \left\{ f \in L^0(\mu) : \|f\|_{X'} < \infty \right\},$$

where $\|\cdot\|_{X'}$ is called the *associate norm* of $\|\cdot\|_X$ and is given by

$$\|f\|_{X'} := \sup_{\substack{g \in X \\ \|g\|_X = 1}} \int_\Omega |fg| \, d\mu.$$

(iv) A Banach function space X over Ω is said to be *maximal* if

$$X'' = X$$

with equality of norms.

Definition 2. Let (Ω, Σ, μ) be a measure space and X be a Banach lattice over Ω.

(i) A linear subspace $U \subset L^0(\mu)$ is said to have the *lattice property* if $f, g \in L^0(\mu)$ satisfy $f \in U$ and $|g| \leq |f|$ μ-a.e. on Ω, then $g \in U$.

(ii) If $U \subset L^0(\mu)$ is a linear subspace, the *restriction subspace* UX *of* X *to* U is defined by setting $UX := \overline{U \cap X}^X$.

We define the complex interpolation functors as follows:

Definition 3 (Calderón's first complex interpolation space). Suppose that $\overline{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ is a compatible couple of complex Banach spaces.

(1) The space $\mathcal{F}(\mathcal{X}_0, \mathcal{X}_1)$ is defined as the set of all functions $F : \bar{S} \to \mathcal{X}_0 + \mathcal{X}_1$ such that

(i) F is continuous on $\bar{S}$ and

$$\sup_{z \in \bar{S}} \|F(z)\|_{\mathcal{X}_0+\mathcal{X}_1} < \infty,$$

(ii) F is holomorphic on S,

(iii) the function $t \in \mathbb{R} \mapsto F(j+it) \in \mathcal{X}_j$ is bounded and continuous on $\mathbb{R}$ for each $j = 0, 1$.

The space $\mathcal{F}(\mathcal{X}_0, \mathcal{X}_1)$ is equipped with the norm

$$\|F\|_{\mathcal{F}(\mathcal{X}_0,\mathcal{X}_1)} \equiv \max_{j=0,1} \left(\sup_{z \in j+i\mathbb{R}} \|F(z)\|_{\mathcal{X}_j} \right).$$

(2) Let $\theta \in (0,1)$. The *first complex interpolation space* $[\mathcal{X}_0, \mathcal{X}_1]_\theta$ with respect to $\overline{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ is defined as the set of all functions $f \in \mathcal{X}_0 + \mathcal{X}_1$ such that $f = F(\theta)$ for some $F \in \mathcal{F}(\mathcal{X}_0, \mathcal{X}_1)$. The norm on $[\mathcal{X}_0, \mathcal{X}_1]_\theta$ is defined by

$$\begin{aligned} &\|f\|_{[\mathcal{X}_0,\mathcal{X}_1]_\theta} \\ &\equiv \inf\{\|F\|_{\mathcal{F}(\mathcal{X}_0,\mathcal{X}_1)} : f = F(\theta) \\ &\text{for some } F \in \mathcal{F}(\mathcal{X}_0, \mathcal{X}_1)\}. \end{aligned}$$

The space $[\mathcal{X}_0, \mathcal{X}_1]_\theta$ is also called *Calderón's first complex interpolation space*, or the *lower complex interpolation space* of $(\mathcal{X}_0, \mathcal{X}_1)$.

Let $\mathcal{X}$ be a complex Banach space. Then recall that the $\mathcal{X}$-valued Lipschitz space $\mathrm{Lip}(\mathbb{R},\mathcal{X})$ is defined as the set of all functions $F:\mathbb{R}\to\mathcal{X}$ for which the following quantity (semi-norm)

$$\|F\|_{\mathrm{Lip}(\mathbb{R},\mathcal{X})} \equiv \sup_{-\infty<s<t<\infty} \frac{\|F(t)-F(s)\|_{\mathcal{X}}}{|t-s|}$$ is

finite.

Definition 4 (Calderón's second complex interpolation space). Let $\overline{\mathcal{X}}=(\mathcal{X}_0,\mathcal{X}_1)$ be a compatible couple of complex Banach spaces.

(1) Define $\mathcal{G}(\mathcal{X}_0, \mathcal{X}_1)$ as the set of all functions $G : \bar{S} \to \mathcal{X}_0 + \mathcal{X}_1$ such that

(i) G is continuous on $\bar{S}$ and

$$\sup_{z \in \bar{S}} \left\| \frac{G(z)}{1+|z|} \right\|_{\mathcal{X}_0+\mathcal{X}_1} < \infty,$$

(ii) G is holomorphic on S,

(iii) the function $t \in \mathbb{R} \mapsto G(j+it) - G(j) \in \mathcal{X}_j$ is Lipschitz continuous on $\mathbb{R}$ for each $j = 0, 1$.

The space $\mathcal{G}(\mathcal{X}_0, \mathcal{X}_1)$ is equipped with the norm

$$\|G\|_{\mathcal{G}(\mathcal{X}_0,\mathcal{X}_1)} \equiv \max\left\{\|G(i\cdot)\|_{\mathrm{Lip}(\mathbb{R},\mathcal{X}_0)},\ \|G(1+i\cdot)\|_{\mathrm{Lip}(\mathbb{R},\mathcal{X}_1)}\right\}. \tag{2.1}$$

(2) Let $\theta \in (0,1)$. The *second complex interpolation space* $[\mathcal{X}_0,\mathcal{X}_1]^\theta$ with respect to $\overline{\mathcal{X}} = (\mathcal{X}_0,\mathcal{X}_1)$ is defined as the set of all functions $f \in \mathcal{X}_0 + \mathcal{X}_1$ such that $f = G'(\theta)$ for some $G \in \mathcal{G}(\mathcal{X}_0,\mathcal{X}_1)$. The norm on

$[\mathcal{X}_0, \mathcal{X}_1]^\theta$ is defined by

$$\|f\|_{[\mathcal{X}_0,\mathcal{X}_1]^\theta} \equiv \inf\{\|G\|_{\mathcal{G}(\mathcal{X}_0,\mathcal{X}_1)} : f = G'(\theta) \text{ for some } G \in \mathcal{G}(\mathcal{X}_0, \mathcal{X}_1)\}.$$

The space $[\mathcal{X}_0, \mathcal{X}_1]^\theta$ is also called *Calderón's second complex interpolation space*, or the *upper complex interpolation space* of $(\mathcal{X}_0, \mathcal{X}_1)$.

3 Calderón-product level sets

Let (Ω, Σ, μ) be a measure space and $\theta \in (0, 1)$. Suppose that X_0 and X_1 are Banach lattices over Ω.

Definition 5.

(i) For any $f, f_0, f_1 \in L^0(\mu)$, if

$$|f| \le |f_0|^{1-\theta} |f_1|^{\theta} \quad \mu\text{-a.e. on } \Omega, \tag{3.1}$$

then (f_0, f_1) is called a θ-*Calderón product domination* (for short, θ-CP-*domination*) of f.

(ii) The *Calderón product* $X_0^{1-\theta} X_1^{\theta}$ of X_0 and X_1 is defined to be the set of all $f \in L^0(\mu)$ s.t.

$$\|f\|_{X_0^{1-\theta} X_1^{\theta}} := \inf \Big\{ \|f_0\|_{X_0}^{1-\theta} \|f_1\|_{X_1}^{\theta} : (f_0, f_1) \text{ is a } \theta\text{-CP-domination of } f \text{ with } f_0 \in X_0 \text{ and } f_1 \in X_1 \Big\}$$

is finite, where the infimum is taken over all above θ-CP-dominations of f.

Definition 6. Let (Ω, Σ, μ) be a measure space and $\theta \in (0,1)$. For any $f \in L^0(\mu)$ and its any θ-CP-domination (f_0, f_1) and for any $\lambda \in (0,1)$,

$$\mathcal{C}_{f,\lambda}^{f_0,f_1,\theta} \tag{3.2}$$
$$:= \left\{x \in \Omega : \lambda |f_0(x)| \leq |f_1(x)| \leq \lambda^{-1} |f_0(x)|\right\}$$

is called the (f_0, f_1, θ)-*Calderón Product* [for short, (f_0, f_1, θ)-CP] *level set* of the function f at the level λ.

If $(\mathcal{X}_0$ and $\mathcal{X}_1)$ are maximal Banach function spaces, then $[\mathcal{X}_0, \mathcal{X}_1]^\theta$ and $X_0^{1-\theta} X_1^\theta$ are identical.

4 Main results

Theorem 4.1. *Let* (Ω, Σ, μ) *be a measure space and* $\theta \in (0, 1)$. *Suppose that* X_0 *and* X_1 *are Banach lattices over* Ω. *Let* $f \in X_0 + X_1$. *Then the following three statements are mutually equivalent.*

(i) $f \in [X_0, X_1]_\theta$.

(ii) $f \in X_0^{1-\theta} X_1^{\theta}$ *and, for any* θ-CP*-domination* (f_0, f_1) *of* f *with* $f_0 \in X_0$ *and* $f_1 \in X_1$*,*

$$f = \lim_{\lambda \to 0^+} f \mathbf{1}_{\mathcal{C}_{f,\lambda}^{f_0, f_1, \theta}} \tag{4.1}$$

in the norm of $X_0^{1-\theta} X_1^{\theta}$*, where* $\mathcal{C}_{f,\lambda}^{f_0, f_1, \theta}$ *is as in* (3.2)*.*

(iii) $f \in X_0^{1-\theta} X_1^{\theta}$ *and there exists a* θ-CP*-domination* (f_0, f_1) *of* f *with* $f_0 \in X_0$ *and* $f_1 \in X_1$ *satisfying* (4.1)*.*

Theorem 4.2. *Let (Ω, Σ, μ) be a measure space and $\theta \in (0,1)$. Suppose that X_0 and X_1 are maximal Banach function spaces over Ω. Assume that U and V are linear subspaces of $L^0(\mu)$ having the lattice property. The following six assertions are mutually equivalent.*

(i) $f \in [UX_0, VX_1]^\theta$.

(ii) $f \in [X_0, X_1]^\theta \cap \overline{UX_0 \cap VX_1}^{UX_0 + VX_1}$.

(iii) $f \in [X_0, X_1]^\theta \cap \overline{UX_0 \cap VX_1}^{X_0 + X_1}$.

(iv) $f \in [X_0, X_1]^\theta$ *and, for any* θ-CP*-domination* (f_0, f_1) *of* f *with* $0 \le f_0 \in X_0$ *and* $0 \le f_1 \in X_1$ *and for any* $\lambda \in (0,1)$, $f\mathbf{1}_{\mathcal{C}_{f,\lambda}^{f_0,f_1,\theta}} \in [UX_0, VX_1]_\theta$*, where* $\mathcal{C}_{f,\lambda}^{f_0,f_1,\theta}$ *is as in* (3.2).

(v) $f \in [X_0, X_1]^\theta$ *and, for any* θ-CP*-domination* (f_0, f_1) *of* f *with* $0 \le f_0 \in X_0$ *and* $0 \le f_1 \in X_1$ *and for any* $\varepsilon \in (0, \infty)$, *there exist a* θ-CP*-domination* (G, H) *of* f *with* $0 \le G \in X_0$ *and* $0 \le H \in X_1$ *satisfying*

$$\max\{\|G\|_{X_0},\ \|H\|_{X_1}\} \le (1+\varepsilon)\|f\|_{[X_0,X_1]^\theta}$$

and, for any $\lambda \in (0, 1)$,

$$G\mathbf{1}_{\mathcal{C}_{f,\lambda}^{f_0,f_1,\theta}} \in UX_0 \text{ and } H\mathbf{1}_{\mathcal{C}_{f,\lambda}^{f_0,f_1,\theta}} \in VX_1. \tag{4.2}$$

(vi) $f \in [X_0, X_1]^\theta$ *and there exist two* θ-CP-*dominations* (f_0, f_1) *and* (G, H) *of* f *with* $0 \leq f_0, G \in X_0$ *and* $0 \leq f_1, H \in X_1$ *such that* (4.2) *holds.*

If one of the above assertions holds, then $\|f\|_{[UX_0,VX_1]^\theta} = \|f\|_{[X_0,X_1]^\theta}$.
In particular,

$$[UX_0, VX_1]^\theta$$
$$= [X_0, X_1]^\theta \cap \overline{UX_0 \cap VX_1}^{UX_0+VX_1}$$
$$= [X_0, X_1]^\theta \cap \overline{UX_0 \cap VX_1}^{X_0+X_1}$$

with equal norms.

Theorem 4.3. *Let* (Ω, Σ, μ) *be a measure space and* $\theta \in (0,1)$. *Suppose that* X_0 *and* X_1 *are Banach lattices over* Ω. *Assume that* U *and* V *are linear subspaces of* $L^0(\mu)$ *having the lattice property. Then*

$$
\begin{aligned}
&[U X_0, V X_1]_\theta \\
&= \overline{U \cap V \cap X_0 \cap X_1}^{X_0^{1-\theta} X_1^{\theta}} \\
&= \left[(U \cap V)\left(X_0^{1-\theta} X_1^{\theta}\right)\right] \cap [X_0, X_1]_\theta
\end{aligned}
$$

hold with equal norms.

Remark 1. $[UX_0, UX_1]^\theta = U[X_0, X_1]^\theta$ is false in general.

Proposition 4.4. *Let* $\theta \in (0,1)$. *Then* $[c_0(\mathbb{N}), \ell^\infty(\mathbb{N})]_\theta = c_0(\mathbb{N}) = [c_0(\mathbb{N}), \ell^\infty(\mathbb{N})]^\theta$ *with equal norms.*

We define the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ with $1 \le q \le p < \infty$. Define the Morrey norm $\| \cdot \|_{\mathcal{M}^p_q}$ by

$$\|f\|_{\mathcal{M}^p_q} \equiv \sup \left\{ |Q|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(Q)} \ : \ Q \text{ is a cube} \right\}$$

for a measurable function f. The *Morrey space* $\mathcal{M}^p_q(\mathbb{R}^n)$ is the set of all the measurable functions f for which $\|f\|_{\mathcal{M}^p_q}$ is finite. The parameter p describes the global integrability of functions, while q describes the local one. We remark that our result extends some existing results since $\mathcal{M}^p_p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ with coincidence of norms for $1 \leq p < \infty$.

Here we consider a quite general situation. Let $(X, \mathcal{B}, \mu)$ be a measure space. Suppose that we have a countable covering $\mathcal{U} \subset \mathcal{B}$ of X and two complex Banach spaces E_U^0 and E_U^1 contained continuously in $L^0(\mu)$ for each $U \in \mathcal{U}$. Assume that any function in E_U^0 and E_U^1 vanishes μ-almost everywhere outside of U.

Write $\mathcal{E}^k \equiv \{E^k_U\}_{U \in \mathcal{U}}$ for each $k = 0, 1$. Define

$$E(\mathcal{E}^k) \equiv \{f \in L^0(\mu) \ :$$

$$f\chi_U \in E^k_U \text{ for each } U \in \mathcal{U} \text{ and } \|f\|_{E(\mathcal{E}^k)} < \infty\},$$

where $\|f\|_{E(\mathcal{E}^k)} \equiv \sup\limits_{U \in \mathcal{U}} \|f\chi_U\|_{E^k_U}$ for $f \in L^0(\mu)$.

Example 1. We work in Euclidean space $\mathbb{R}^n$. Let $\mathcal{U} = \mathcal{D}(\mathbb{R}^n)$, $1 \le q_0 \le p_0 < \infty$ and $1 \le q_1 \le p_1 < \infty$. If we set

$$E^k_Q \equiv L^{q_k}(Q),$$

$$\|f\|_{E^k_Q} \equiv |Q|^{\frac{1}{p_k}-\frac{1}{q_k}}\|\chi_Q f\|_{L^{q_k}} \quad (Q \in \mathcal{D}),$$

then $E(\mathcal{E}^k) = \mathcal{M}^{p_k}_{q_k}(\mathbb{R}^n)$.

We know that the Calderón product $E(\mathcal{E}^0)^{1-\theta}E(\mathcal{E}^1)^\theta$ is nothing but the second complex interpolation $[E(\mathcal{E}^0), E(\mathcal{E}^1)]^\theta$. We have the following description of the Calderón product.

Theorem 4.5. *Let $f \in L^0(\mu)$. Let $\mathcal{E}^0 = \{E_U^0\}_{U \in \mathcal{U}}$ and $\mathcal{E}^1 = \{E_U^1\}_{U \in \mathcal{U}}$ be families of complex Banach spaces contained in $L^0(\mu)$ such that any function in $E_U^0 \cup E_U^1$ vanishes μ-almost everywhere outside U. Then $f \in E(\mathcal{E}^0)^{1-\theta} E(\mathcal{E}^1)^\theta$ if and only if there exist $\{f_U^0\}_{U \in \mathcal{U}}$ and $\{f_U^1\}_{U \in \mathcal{U}}$ such that $\chi_{U_0 \cap U_1}|f| \le |f_{U_0}^0|^{1-\theta}|f_{U_1}^1|^\theta$ for μ-almost everywhere for any $U_0, U_1 \in \mathcal{U}$ and that*

$$\sup_{U \in \mathcal{U}} (\|f_U^0\|_{E_U^0} + \|f_U^1\|_{E_U^1}) < \infty.$$

Theorem 4.6. *Let* $\theta \in (0,1)$ *and* $1 \le q_j \le p_j < \infty$ *for* $j = 0, 1$. *Assume that* $\frac{p_0}{q_0} = \frac{p_1}{q_1}$. *Define* p *and* q *by*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \text{ and } \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then, with coincidence of norms we have $[\mathcal{M}^{p_0}_{q_0}(\mathbb{R}^n), \mathcal{M}^{p_1}_{q_1}(\mathbb{R}^n)]^\theta = \mathcal{M}^p_q(\mathbb{R}^n)$ *and* $[\mathcal{M}^{p_0}_{q_0}(\mathbb{R}^n), \mathcal{M}^{p_1}_{q_1}(\mathbb{R}^n)]_\theta = \overline{\mathcal{M}^{p_0}_{q_0}(\mathbb{R}^n) \cap \mathcal{M}^{p_1}_{q_1}(\mathbb{R}^n)}^{\mathcal{M}^p_q(\mathbb{R}^n)}$.

Theorem 4.7.

$$\begin{aligned}&[U\mathcal{M}^{p_0}_{q_0}(\mathbb{R}^n), U\mathcal{M}^{p_1}_{q_1}(\mathbb{R}^n)]_\theta\\ &= \Big\{ f \in U\mathcal{M}^p_q(\mathbb{R}^n) \cap \overline{\mathcal{M}}^p_q(\mathbb{R}^n) \ : \\ &\qquad \lim_{a \downarrow 0} \|\chi_{[0,a)}(|f|)f\|_{\mathcal{M}^p_q} = 0 \Big\}.\end{aligned}$$

and

$$[U\mathcal{M}^{p_0}_{q_0}(\mathbb{R}^n), U\mathcal{M}^{p_1}_{q_1}(\mathbb{R}^n)]^\theta$$

$$= \{ f \in \mathcal{M}^p_q(\mathbb{R}^n) \ : \ \chi_{[a,a^{-1}]}(|f|)f \in U\mathcal{M}^p_q(\mathbb{R}^n) \}$$