FUNCTION SPACES, INTERPOLATION THEORY AND RELATED TOPICS 2026

Tuesday, March 10; Wednesday, March 11; and Thursday, March 12, 2026.

Organized by Fernando Cobos and Luz M. Fernández-Cabrera

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Departamento de Analisis Matemático y Matemática Aplicada

Universidad Complutense de Madrid

Seminario 222

Tuesday 11:10 - 12:00

Mapping properties of Fourier transforms in function spaces, some recent results

Dorothee D. Haroske

Friedrich Schiller University Jena, Germany

We study continuous and compact mappings generated by the Fourier transform between distinguished Besov spaces $B^s_{p,p}(\mathbb{R}^n)$, $1 \le p \le \infty$, and between Sobolev spaces $H^s_p(\mathbb{R}^n)$, $1 < p < \infty$. Here we rely mainly on wavelet expansions, duality and interpolation of corresponding (unweighted) spaces, and (appropriately extended) Hausdorff-Young inequalities. The degree of compactness will be measured in terms of entropy numbers and approximation numbers, now using the symbiotic relationship to weighted spaces. We can also characterise the situation when the Fourier transform acts as a nuclear operator.

This is joint work with Leszek Skrzypczak (Poznań) and Hans Triebel (Jena).

Tuesday 12:10 - 13:00

Degrees of non-compactness for some operators in analysis

Jan Lang

The Ohio State University, Columbus, Ohio, USA

Many standard and typical operators in harmonic analysis and also embeddings between function spaces are bounded but non-compact at critical ("optimal") parameter choices. In this talk I will survey recent results that quantify *how* non-compact such operators are, using the operator-ideal notions of *strict singularity* and its quantitative strengthening via *Bernstein numbers* $b_n(T)$.

A guiding example is the classical Fourier transform $\mathcal{F}$ on $\mathbb{R}^n$. For $1 < p < 2$ (with $1/p + 1/p' = 1$) real interpolation improves the Hausdorff–Young inequality to $\mathcal{F} : L^p(\mathbb{R}^n) \to L^{p',p}(\mathbb{R}^n)$. We obtain a sharp trichotomy on the Lorentz scale:

$$\mathcal{F} : L^p(\mathbb{R}^n) \to L^{p',r}(\mathbb{R}^n)$$

- unbounded, $r < p$,
- non-compact and not strictly singular, $r = p$,
- non-compact but finitely strictly singular, $r > p$,

together with quantitative bounds $b_n(\mathcal{F}) \leq c(p)n^{\frac{1}{2r}-\frac{1}{2p}}$ for $p < r < p'$. Thus $L^{p',p}$ is an optimal target in a precise operator-theoretic sense. For $2 < p \leq \infty$ we prove an analogous trichotomy for $\mathcal{F} : L^p(\mathbb{R}^n) \to B^s_p(\mathbb{R}^n)$ at the critical smoothness $d^n_p = 2n\left(\frac{1}{p} - \frac{1}{2}\right) < 0$, and we discuss the corresponding dual statements for $\mathcal{F} : B^s_p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ when $1 < p < 2$. Beyond Fourier analysis, I will briefly connect these ideas to recent work on "optimal" non-compact Sobolev/Sobolev–Lorentz embeddings and to classification results for the degree of non-compactness of embeddings between Besov spaces and related sequence spaces. The common theme is that strict singularity and Bernstein numbers distinguish phenomena that are invisible to standard measures of non-compactness, and they provide a robust language for *optimality* in endpoint mapping problems.

References

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[3] J. Lang, A. Nekvinda, *Embeddings between Lorentz sequence spaces are strictly but not finitely strictly singular*, Studia Math. **272** (2023), no. 1, 35–57.

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Tuesday 15:30 - 16:20

Generalised Morrey smoothness spaces

Susana Moura

University of Coimbra, Portugal

Smoothness spaces of Besov and Triebel-Lizorkin type built over Morrey spaces $\mathcal{M}_{u,p}(\mathbb{R}^d)$, $0 < p \leq u < \infty$, form a well-established extension of the classical $B^s_{p,q}(\mathbb{R}^d)$ and $F^s_{p,q}(\mathbb{R}^d)$ scales and have proved useful in the study of PDEs. The latter are recovered from the former, denoted respectively by $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ and $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$, $s \in \mathbb{R}$, $0 < q \leq \infty$, when $p = u$. This is because $\mathcal{M}_{p,p}(\mathbb{R}^d) = L_p(\mathbb{R}^d)$ for $0 < p < \infty$.

The Besov-type and Triebel-Lizorkin-type spaces, $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ and $F^{s,\tau}_{p,q}(\mathbb{R}^d)$, $\tau \geq 0$, are closely connected scales of function spaces that have been studied extensively, from which the classical Besov and Triebel-Lizorkin spaces are recovered when $\tau = 0$.

Generalised Morrey spaces, denoted by $\mathcal{M}_{\varphi,p}(\mathbb{R}^d)$, were introduced by Mizuhara and Nakai. Rather than using a fixed Morrey index u, local control is described by a function φ belonging to a suitable class, which recovers the classical Morrey setting for power-type choices. Using this approach, generalised Besov-Morrey spaces denoted by $\mathcal{N}^s_{\varphi,p,q}\left(\mathbb{R}^d\right)$, and Triebel-Lizorkin-Morrey spaces denoted by $\mathcal{E}^s_{\varphi,p,q}\left(\mathbb{R}^d\right)$, were defined by Nakamura, Noi and Sawano. Similarly, Haroske and Liu recently considered $B^{s,\varphi}_{p,q}\left(\mathbb{R}^d\right)$ and $F^{s,\varphi}_{p,q}\left(\mathbb{R}^d\right)$ spaces.

We plan to provide an overview of recent findings on these generalised scales, with a focus on embeddings and analysing the influence of the qualitative behaviour of φ on the resulting function spaces.

This is a joint work with Dorothee Haroske (Friedrich Schiller Uni- versity Jena), Leszek Skrzypczak (Adam Mickiewicz University Poznań) and Zhen Liu (Beijing Normal University).

Tuesday 16:30 - 17:20

Spaces of integrable functions associated to vector measures and limiting real interpolation

Antonio Manzano

Universidad de Burgos, Spain

If $m : \Sigma \to X$ is a vector measure defined on a σ-algebra Σ of subsets (of a nonempty set) with values in a Banach space X, associated with m are (for $p \geq 1$) the Banach lattices $L^p(m)$ or $L^p_w(m)$ of (m-a.e. equivalence classes of) scalar functions that are p-integrable or, respectively, weakly p-integrable with respect to m. These spaces find application in important problems such as the representation of abstract Banach lattices as spaces of integrable functions or the study of the optimal domain of linear operators.
As a natural issue that concerns interpolation theory, it has been investigated the description of the spaces obtained by applying different interpolation methods to couples formed by these classes of spaces of scalar integrable functions with respect to a vector measure. In the case of real interpolation, this leads to naturally also considering Lorentz spaces $L^{p,q}(\|m\|)$ or, more generally, the Lorentz-Zygmund spaces $L^{p,q}(\log L)^{\alpha}(\|m\|)$, constructed by means of the semivariation $\|m\|$ of the vector measure m.
In this talk we are interested in continuing the study of this problem when considering limiting real interpolation. More precisely, we will show which spaces are obtained by applying certain limiting real interpolation methods to ordered Banach couples formed by spaces of scalar integrable functions associated to a vector measure. On the other hand, we will also establish results on extreme real interpolation for some classes of operators closely connected with this kind of spaces of functions, such as p-th power factorable operators or bidual (p, q)-power-concave operators.

The results we will present are part of a joint paper with Antonio Fernández (Universidad de Sevilla).

Wednesday 11:10 - 12:00

Limiting K- and J-spaces in the real interpolation, their relationship and duals

Bohumír Opic

Charles University, Prague, Czech Republic

In the paper [1] we have establish conditions under which the limiting K-space $(X_0, X_1)_{0,q,b;K}$, where $1 \leq q \leq \infty$ and b is a slowly varying function on the interval $(0, \infty)$, can be described by means of the J-space $(X_0, X_1)_{0,q,a;J}$, with a convenient slowly varying function a, and we have also solved the reverse problem. It has been shown that if these conditions are not satisfied that the given problem may not have a solution.
Assume now that these conditions are not satisfied. Nevertheless, our aim is to express the limiting K-space $(X_0, X_1)_{0,q,b;K}$ as some limiting J-space $(Y_0, Y_1)_{0,q,A;J}$, and, similarly, to express the limiting J-space $(X_0, X_1)_{0,q,a;J}$ as a convenient limiting K-space $(Z_0, Z_1)_{0,q,B;K}$. To be more precise, we show that

$$(X_0, X_1)_{0,q,b;K} = (X_0, X_0 + X_1)_{0,q,A;J} = X_0 + (X_0, X_1)_{0,q,A;J}$$

and

$$(X_0, X_1)_{0,q,a;J} = (X_0, X_0 \cap X_1)_{0,q,B;K} = X_0 \cap (X_0, X_1)_{0,q,B;K},$$

where A and B are convenient weights. Moreover, we establish equivalent norms in the above mentioned spaces. The obtained results are applied to get density theorems for spaces in question.
We also establish duals of the limiting real interpolation K- and J-spaces $(X_0, X_1)_{0,q,v;K}$ and $(X_0, X_1)_{0,q,v;J}$, where $1 \leq q < \infty$ and v is a slowly varying function on the interval $(0, \infty)$. In the case of the classical real interpolation method $(X_0, X_1)_{\theta,q}$, with $\theta \in (0, 1)$ and $1 \leq q < \infty$, this problem was solved by Lions and Peetre.

References

[1] B. Opic and M. Grover, *Description of K-spaces by means of J-spaces and the reverse problem in the limiting interpolation*, Math. Nachr. **296** (2023), 4002–4031.

Wednesday 12:10 - 13:00

Complex Interpolation of Closed Subspaces of Maximal Banach Function Spaces

Yoshihiro Sawano

Chuo University, Tokyo, Japan

In this talk, we introduce the notion of *Calderón product level sets* and use it to characterize the complex interpolation spaces of closed subspaces with the lattice property for arbitrary couples of maximal Banach function spaces (Banach lattices). As applications, we obtain several new results on the complex interpolation of closed subspaces of Morrey spaces and Lorentz spaces. Furthermore, for smooth function spaces that do not possess the lattice property, we investigate various closed subspaces and establish corresponding complex interpolation results by means of their real-variable characterizations.

This is joint work with Dachun Yang, Wen Yuan, and Mingdong Zhang (Beijing Normal University); see [1].

References

[1] Complex Interpolation of Closed Subspaces of Maximal Banach Function Spaces: Calderón Product Level Set Characterizations and Their Applications, in preparation.

Wednesday 15:30 - 16:20

On the Boundedness of Dilation Operators in the Context of Triebel-Lizorkin-Morrey Spaces

Marc Hovemann

Friedrich-Schiller-Universität Jena, Germany

In this talk we study the behavior of dilation operators $D_\lambda \colon f \mapsto f(\lambda\,\cdot)$ with $\lambda > 1$ in the context of Triebel-Lizorkin-Morrey spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$. For that purpose we prove upper and lower bounds for the operator (quasi-)norm $\|D_\lambda|\mathcal{L}(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d))\|$. We show that for $s > \sigma_p$ the operator (quasi-)norm $\|D_\lambda|\mathcal{L}(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d))\|$ up to constants behaves as $\lambda^{s-\frac{d}{u}}$. For the borderline case $s = \sigma_p$ we observe a behavior of the form $\lambda^{\sigma_p-\frac{d}{u}}$, multiplied with logarithmic terms of λ that also depend on the fine index q. For $s < \sigma_p$ and $p \geq 1$ we find the relation $\|D_\lambda|\mathcal{L}(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d))\| \sim \lambda^{-\frac{d}{u}}$. The case $s < \sigma_p$ and $p < 1$ is investigated as well. The proofs are mainly based on the Fourier analytic approach to Triebel-Lizorkin-Morrey spaces. As byproducts we show an advanced Fourier multiplier theorem for band-limited functions in the context of Morrey spaces and derive some new equivalent (quasi-)norms and characterizations of $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$.

This talk is based on a joint work with Markus Weimar.

Wednesday 16:30 - 17:20

Duality for limiting K interpolation spaces. The case $0 < q \leq 1$

Pedro Fernández-Martínez

Universidad de Murcia, Spain

We identify the duals of limiting K-method spaces $(A_0, A_1)^K_{\theta,q;b}$ in the endpoint cases $\theta = 0, 1$ and $0 < q \leq 1$; here b is a slowly varying function. Duality for the limiting cases $\theta = 0, 1$ has been previously investigated in two settings: first for logarithmic interpolation (when b is a power of the logarithm), see [1–3], and more recently for general slowly varying functions in the range $1 \leq q < \infty$; see [7].
We present results for the duals of the spaces $(A_0, A_1)^K_{\theta,q;b}$ with $0 < q \leq 1$. Thus, some results in [7] extend to the case $0 < q < 1$ using techniques from [1]. Also, several restrictions on the logarithmic exponents appearing in [1] and [3] can be replaced by more natural assumptions on the slowly varying function b, yielding conditions that are easier to interpret.

References

[1] B.F. Besoy and F. Cobos, *Duality for logarithmic interpolation spaces when $0 < q < 1$ and applications*, J. Math. Anal. Appl. **466** (2018), 373–399.

[2] F. Cobos, L.M. Fernández-Cabrera, T. Kühn, and T. Ullrich, *On an extreme class of real interpolation spaces*, J. Funct. Anal. **256** (2009), no. 7, 2321–2366.

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Thursday 11:10 - 12:00

Multipliers for Besov and Lizorkin-Triebel Spaces and Morrey Smoothness Spaces

Winfried Sickel

Jena, Germany

In my talk I will speak about the multiplier classes associated to Besov and Lizorkin-Triebel Spaces and the role of Morrey smoothness spaces inside this theory. Therefore I will recall various old results from the last century including those of Maz'ya, Shaposhnikova, Peetre, Triebel, Franke and Netrusov. This will be complemented by more recent results due to Li, Yang, Yuan and myself. Here the so-called Besov-type spaces $B^{s,\tau}_{p,p}(\mathbb{R}^d)$ come into play. Besov-type spaces represent one version of smoothness spaces built on Morrey spaces. Finally I will talk about a generalization of an old result of Gulisashvili. Here Besov-Morrey spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ will be used, which represent another variant of smoothness spaces built on Morrey spaces.

References

[1] Y. Li, W. Sickel, D. Yang and W. Yuan, *C*haracterizations of pointwise multipliers of Besov spaces in endpoint cases with an application to the duality principle, J. Func. Anal. **286** (2024) 110198.

[2] Y. Li, W. Sickel, D. Yang and W. Yuan, *W*avelet and Fourier analytic characterizations of pointwise multipliers of Besov spaces $B^s_{p,p}(\mathbb{R}^d)$ with $0 < p \leq 1$, J. Func. Anal. **287** (2024) 110654.

Thursday 12:10 - 13:00

Some new results on function spaces of Lorentz-Sobolev type

Fernando Cobos

Universidad Complutense de Madrid, Spain

Function spaces of Lorentz-Sobolev type $F^s_q L_{p,r}(\mathbb{R}^n)$, $B^s_q L_{p,r}(\mathbb{R}^n)$ arise by real interpolation of couples of Triebel-Lizorkin spaces and of Besov spaces, respectively. They have been considered in the literature since the early 1960s.
In this talk we describe sufficient conditions on parameters for $F^s_q L_{p,r}(\mathbb{R}^n)$ and $B^s_q L_{p,r}(\mathbb{R}^n)$ to be multiplication algebras. We also study duality for spaces $B^s_q L_{p,r}(\mathbb{R}^n)$.

Results are taken from joint papers with Blanca F. Besoy (Madrid), Luz M. Fernández-Cabrera (Madrid), Thomas Kühn (Leipzig) and Hans Triebel (Jena).

Thursday 15:30 - ...

Open Problems