Multipliers for Besov-Lizorkin-Triebel spaces and Morrey Smoothness Spaces

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1. Introduction

Let X, Y be quasi-Banach spaces continuously embedded into $\mathcal{S}'(\mathbb{R}^d)$. We are interested in the bilinear mapping

$$(f, g) \mapsto f \cdot g\,, \qquad f \in X,\, g \in Y\,.$$

Here: $X = A^{s_1}_{p_1,q_1}(\mathbb{R}^d)$ and $Y = A^{s_2}_{p_2,q_2}(\mathbb{R}^d)$, $A \in \{B, F\}$.

Contributions by Peetre, Triebel, Zolesio, Maz'ya & Shaposhnikova, Netrusov, Gulisashvili, Valent, Hanouzet, Franke, Johnsen, Amann, Nguyen, Holst, . . . , Cobos, Fernández-Cabrera, Kühn.

Multiplier spaces

Definition 1

Let X, Y be quasi-Banach spaces continuously embedded into $\mathcal{S}'(\mathbb{R}^d)$. A pointwise multiplier for the pair (X, Y) is a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the (distributional) product $f \cdot g$ exists for all $g \in X$ in $\mathcal{S}'(\mathbb{R}^d)$, belong to Y for all $g \in X$ and

$$\sup_{\| g \, | X\| \le 1} \|f \cdot g|Y\| < \infty \, .$$

The collection of all pointwise multipliers of the pair (X, Y) is denoted by $M(X, Y)$. In case $X = Y$ we will write $M(X)$ instead of $M(X, X)$.

2. Besov and Lizorkin-Triebel spaces

A smooth dyadic decomposition of unity:
$\varphi_0 \in C_0^\infty(\mathbb{R}^d)$, $\varphi_0(\xi) = 1$ if $|\xi| \le 1$, $\varphi_0(\xi) = 0$ if $|\xi| \ge 3/2$.

$$\varphi_j(\xi) := \varphi_0(2^{-j-1}\xi) - \varphi_0(2^{-j}\xi)\,, \qquad \xi \in \mathbb{R}^d\,, \quad j \in \mathbb{N}\,.$$

$$\sum_{j=0}^{\infty} \varphi_j(\xi) = 1 \qquad \text{for all} \quad \xi \in \mathbb{R}^d\,.$$

We define

$$S_j f(\xi) := \mathcal{F}^{-1}[\varphi_j\, \mathcal{F} f](\xi) \quad \text{and} \quad S^j f := \sum_{\ell=0}^{j} S_j f\,, \quad j \in \mathbb{N}_0\,.$$

Definition 2

Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. Then $B^s_{p,q}(\mathbb{R}^d)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f|B^s_{p,q}(\mathbb{R}^d)\|_{\varphi_0} := \Big(\sum_{j=0}^{\infty} 2^{jsq} \|S_j f|L_p(\mathbb{R}^d)\|^q \Big)^{1/q} < \infty \, .$$

Definition 3

Let $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. Then the Lizorkin-Triebel space $F^s_{p,q}(\mathbb{R}^d)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f|F^s_{p,q}(\mathbb{R}^d)\|_{\varphi_0} := \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |S_j f|^q \Big)^{1/q} \Big| L_p(\mathbb{R}^d) \Big\| < \infty \, .$$

3. Some known characterizations

3.1 The case $A^s_{p,q}(\mathbb{R}^d) \hookrightarrow L_\infty(\mathbb{R}^d)$

Definition 4

Let $A \in \{B, F\}$. Let $\psi \in C_0^\infty(\mathbb{R}^d)$ be nontrivial and nonnegative. Then $A^s_{p,q,\mathrm{unif}}(\mathbb{R}^d)$ is the collection of all $f \in A^{s,\ell oc}_{p,q}(\mathbb{R}^d)$ satisfying

$$\| f \,|A^s_{p,q,\mathrm{unif}}(\mathbb{R}^d)\| := \sup_{\lambda \in \mathbb{R}^d} \|\psi(\,\cdot\, - \lambda)\,\cdot\, f(\,\cdot\,)\,|A^s_{p,q}(\mathbb{R}^d)\| < \infty\,.$$

Theorem 1

Let $0 < q \le \infty$. *Let either* $0 < p \le 1$ *and* $s \ge d/p$ *or* $1 < p < \infty$ *and* $s > d/p$. *Then it follows*

$$M(F^s_{p,q}(\mathbb{R}^d)) = F^s_{p,q,\mathrm{unif}}(\mathbb{R}^d)$$

in the sense of equivalent norms.

- Strichartz (Bessel potential spaces $H^s_p(\mathbb{R}^d) = F^s_{p,2}(\mathbb{R}^d)$, 1967)
- Peetre ($B^s_{p,p}(\mathbb{R}^d) = F^s_{p,p}(\mathbb{R}^d)$, 1976)
- Maz'ya and Shaposhnikova ($W^m_p(\mathbb{R}^d)$, $H^s_p(\mathbb{R}^d)$, $B^s_{p,p}(\mathbb{R}^d)$, 1979)
- Franke (general F-case, 1986)
- Triebel (algebra property, 1978, 1983).

The case $s > d/p$ and $p \neq q$

Theorem 2

Let $0 < p \le q \le \infty$ and let either $s > d/p$ or $0 < p < \infty$, $s = d/p$ and $0 < q \le 1$. Then

$$M(B^s_{p,q}(\mathbb{R}^d)) = B^s_{p,q,\mathrm{unif}}(\mathbb{R}^d)$$

in the sense of equivalent norms.

- S. & Smirnov (1999), Nguyen & S. (2018).
- Bourdaud (1988): $M(B^s_{p,q}(\mathbb{R}^d)) \neq B^s_{p,q,\mathrm{unif}}(\mathbb{R}^d)$, $q < p$.
- For $q < p$ see Nguyen & S. (2018).

3.2 The case $0 < s < d/p$

The approach of Maz'ya and Shaposhnikova in case $1 < p < \infty$:
$e \subset \mathbb{R}^d$, compact:

$$C_{p,s}(e) := \inf\{\|(1-\Delta)^{s/2}u\,|L_p\|^p \ : \ u \in C_0^\infty(\mathbb{R}^d)\,, u \geq 1 \quad \text{on} \quad e\}.$$

$s := k + \alpha$, $k \in \mathbb{N}_0$, $0 < \alpha \leq 1$:

$$\nabla_k f(x) := (\partial_{x_1}^{\beta_1}, \ldots, \partial_{x_d}^{\beta_d})\,, \qquad \beta_1 + \ldots + \beta_d = k\,.$$

$$\mathcal{D}_{p,s,k} f(x) := \sum_{\beta:\ |\beta|=k} \Big(\int_{\mathbb{R}^d} |\Delta_h^2 \nabla_k f(x)|^p |h|^{-d-p\alpha} dh \Big)^{1/p}$$

Observe

$$\| f\,|L_p(\mathbb{R}^d)\| + \| \mathcal{D}_{p,s,k} f\,|L_p(\mathbb{R}^d)\| \sim \| f\,|B^s_{p,p}(\mathbb{R}^d)\|$$

Theorem 3

Let $1 < p < \infty$ and $0 < s < d/p$. Then $M(B^s_{p,p}(\mathbb{R}^d))$ is the collection of all $f \in L_1^{\ell oc}(\mathbb{R}^d)$ such that

$$\| f \,|M(B^s_{p,p}(\mathbb{R}^d))\| \asymp \| f \,|L_\infty(\mathbb{R}^d)\| + \Big(\sup_{\text{diam } e<1} \frac{\|\mathcal{D}_{p,s,k} f|L_p(e)\|}{(C_{p,s}(e))^{1/p}} \Big) < \infty \, .$$

- Maz'ya and Shaposhnikova 1979.

Instead of $\mathcal{D}_{p,s,k} f$ we shall work with the Strichartz function. Let $1 < p < \infty$. We define

$$S_{m,m} u(x) := |\nabla_m u(x)| \, , \qquad x \in \mathbb{R}^d \, , \quad m \in \mathbb{N} \, .$$

$s := k + \alpha$, $0 < \alpha < 1$:

$$S_{s,k} u(x) := \sum_{\beta :\ |\beta| = k} \Big(\int_0^\infty \Big[\int_{B(0,1)} |\nabla_k u(x+\theta y) - \nabla_k u(x)| d\theta \Big]^2 |y|^{-1-2\alpha} \, dy \Big)^{1/2}$$

$$\| \, f \, | L_p(\mathbb{R}^d) \| + \| \, S_{s,k} u \, | L_p(\mathbb{R}^d) \| \sim \| \, u \, | H^s_p(\mathbb{R}^d) \| \sim \| \, u \, | F^s_{p,2}(\mathbb{R}^d) \| \, .$$

- Strichartz 1967.

Theorem 4

Let $1 < p < \infty$ *and* $0 < s < d/p$. *Then* $M(H^s_p(\mathbb{R}^d))$ *is the collection of all* $f \in L_1^{\ell oc}(\mathbb{R}^d)$ *such that*

$$\| f \,|M(H^s_p(\mathbb{R}^d))\| \asymp \| f \,|L_\infty(\mathbb{R}^d)\| + \Big(\sup_{\mathrm{diam}\; e<1} \frac{\| S_{s,k} f \,|L_p(e)\|}{(C_{p,s}(e))^{1/p}} \Big) < \infty \,.$$

- Maz'ya and Shaposhnikova 1979.

The case $p = 1$ and $0 < s < d$

Theorem 5

Let $0 < s < 1$. Then $M(B^s_{1,1})$ is the collection of all $f \in L_\infty(\mathbb{R}^d)$ such that

$$\sup_{x\in\mathbb{R}^d} \sup_{0<r<1} r^{s-d} \left(\int_{B_r(x)} \int_{B_r(x)} \frac{|f(y) - f(z)|}{|y - z|^{d+s}} \, dy \, dz \right)$$

is finite.

- $M(B^s_{1,1}(\mathbb{R}^d)) = L_\infty(\mathbb{R}^d) \cap B^{s,1-s/d}_{1,1,unif}(\mathbb{R}^d)$.
- Maz'ya, Shaposhnikova 1979 (for all $s > 0$).

3.3 Triebel's characterization in case $0 < p \leq 1$

Let $\psi \in C_0^\infty(\mathbb{R}^d)$ be a non-negative function with $\operatorname{supp} \psi \subset \{y \in \mathbb{R}^d : \ |y| \leq d\}$ and

$$\sum_{m \in \mathbb{Z}^d} \psi(x-m) = 1 \qquad \text{for all} \quad x \in \mathbb{R}^d \, .$$

Definition 5

A tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ belongs to $B^s_{p,p,selfs}(\mathbb{R}^d)$ if

$$\| \, f \, \|_{B^s_{p,p,selfs}(\mathbb{R}^d)} := \sup_{m \in \mathbb{Z}^d} \sup_{j \in \mathbb{N}_0} \| \, f(2^{-j} \cdot) \psi(\, \cdot \, - m) \, \|_{B^s_{p,p}(\mathbb{R}^d)} < \infty \, .$$

Triebel 2003:

Theorem 6

Let $0 < p \le 1$ and $s > d\left(\frac{1}{p} - 1\right)$. Then $M(B^s_{p,p}(\mathbb{R}^d)) = B^s_{p,p,selfs}(\mathbb{R}^d)$ in the sense of equivalent quasi-norms.

Further characterizations by:

- Netrusov 1992, 1997 (F-case, $0 < p \le 1, \ \sigma_{p,q} < s < d/p$)
- S. 1999 (F-case, $0 < p < \infty, \ \sigma_{p,q} < s < d/p$, use of capacities).

4. A characterization of $M(B^0_{1,1}(\mathbb{R}^d))$ by Y. Meyer

Theorem 7

Under some mild restrictions on the wavelet system $(\psi_{i,j,k})_{i,j,k}$ we have the following characterization. The pointwise multiplier space $M(B^0_{1,1}(\mathbb{R}^d))$ is the collection of all functions $f \in L_\infty(\mathbb{R}^d)$ such that

$$\| f \| := \sup_{i=1,\ldots,2^d-1} \sup_{j\in\mathbb{N}_{-1}} \sup_{k\in\mathbb{Z}^d} 2^{jd/2} \| f \cdot \psi_{i,j,k} \, | B^0_{1,1}(\mathbb{R}^d)\| < \infty \, .$$

Furthermore, $\|f\|$ is equivalent to $\| f \, | M(B^0_{1,1}(\mathbb{R}^d))\|$.

- Y. Meyer (much more general result, 1992);
- Balazs, Gröchenig, Speckbacher (2019): extension to $\dot{B}^s_{1,1}(\mathbb{R}^d)$ for all $s \in \mathbb{R}$.

The extension to $0 < p \leq 1$ and arbitrary s

Theorem 8

Let $0 < p \leq 1$ and $s \in \mathbb{R}$. Under some mild restrictions on the wavelet system $(\psi_{i,j,k})_{i,j,k}$ we have the following characterization. The pointwise multiplier space $M(B^s_{p,p}(\mathbb{R}^d))$ is the collection of all functions $f \in L_\infty(\mathbb{R}^d)$ such that

$$\|f\| := \sup_{i,j,k} 2^{-j(s+d(\frac{1}{2}-\frac{1}{p}))} \| f \cdot \psi_{i,j,k} |B^s_{p,p}(\mathbb{R}^d)\| .$$

Furthermore, $\|f\|$ is equivalent to $\| f |M(B^s_{p,p}(\mathbb{R}^d)\|$.

- **One condition for all p $\leq$ 1 and all s !**
- Li, Sickel, Yang, Yuan (2024).

Theorem 9

Let $p \in (0,1]$ and $s, r \in \mathbb{R}$ with $s > r$. Under some mild restrictions on the wavelet system $(\psi_{i,j,k})_{i,j,k}$ we have the following characterization. Then $f \in M(B^s_{p,p}(\mathbb{R}^d), B^r_{p,p}(\mathbb{R}^d))$ if and only if $f \in B^{r,\ell oc}_{p,p}(\mathbb{R}^d) \cap B^{\sigma_p - s,\ell oc}_{\infty,\infty}(\mathbb{R}^d)$ and

$$\|f\|^* := \sup_{i,j,k} 2^{-j[s+d(\frac{1}{2}-\frac{1}{p})]} \|f \cdot \psi_{i,j,k} | B^r_{p,p}(\mathbb{R}^d)\| < \infty.$$

Moreover, $\|f\|^*$ is equivalent to $\|f \,| M(B^s_{p,p}(\mathbb{R}^d), B^r_{p,p}(\mathbb{R}^d))\|$ with the positive equivalence constants independent of f.

- Even in the Hilbert space case

$$M(B^s_{2,2}(\mathbb{R}^d), B^r_{2,2}(\mathbb{R}^d)) = M(H^s_2(\mathbb{R}^d), H^r_2(\mathbb{R}^d)), \qquad s > r,$$

one has a partial knowledge only, see Maz'ya, Shaposnikova (2009), Lemarie-Rieusset et al (2007-2013), Gala, Sawano (2010) and Belyaev, Shkalikov (2017).

5. Besov-type spaces

$$S_k f(\xi) := \mathcal{F}^{-1}[\varphi_k \, \mathcal{F} f](x) \, , \qquad x \in \mathbb{R}^d \, , \; k \in \mathbb{N}_0 \, .$$

Definition

Let $s \in \mathbb{R}$, $p, q \in (0, \infty]$ and $\tau \in [0, \infty)$. The Besov-type space $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f \,|B^{s,\tau}_{p,q}(\mathbb{R}^d)\| := \sup_{P \in \mathcal{Q}} |P|^{-\tau} \left\{ \sum_{k=\max\{0,-\log_2 l(P)\}}^{\infty} \Big[2^{ks} \, \|S_k f \,|L_p(P)\|\Big]^q \right\}^{1/q} < \infty.$$

Remark

These Morrey-type smoothness spaces have been introduced by El Baraka around 2002. Later Sawano, Yang, Yuan, Triebel, Haroske, Moura, Skrzypczak, Hovemann, Weimar and many others contributed to the theory of these spaces.

Spaces of logarithmic smoothness

To describe the outcome we need one more space.

Definition

Let $b \in \mathbb{R}$. Then the Besov space with some type of logarithmic smoothness $\mathcal{B}^{0,b}_{\infty,\infty}(\mathbb{R}^d)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\,|\mathcal{B}^{0,b}_{\infty,\infty}(\mathbb{R}^d)\| := \sup_{k\in\mathbb{N}_0} \left\{(1+k)^b\, \|S_k f\|_{L_\infty(\mathbb{R}^d)}\right\} < \infty.$$

Compare with

$$\|f\,|B^{s}_{\infty,\infty}(\mathbb{R}^d)\| := \sup_{k\in\mathbb{N}_0} \left\{2^{ks}\ \|S_k f\|_{L_\infty(\mathbb{R}^d)}\right\}$$

Theorem 10

Let $p \in (0,1]$ *and* $s \in (-\infty, \frac{d}{p})$. *We put* $\sigma_p := d\Big(\frac{1}{p} - 1\Big)$.

(i) *If* $s \in (\sigma_p, \frac{d}{p})$, *then*

$$M\left(B^s_{p,p}(\mathbb{R}^d)\right) = L_\infty(\mathbb{R}^d) \cap B^{s,\frac{1}{p}-\frac{s}{d}}_{p,p,\mathrm{unif}}(\mathbb{R}^d).$$

(ii) *If* $s = \sigma_p$, *then*

$$M\left(B^s_{p,p}(\mathbb{R}^d)\right) = L_\infty(\mathbb{R}^d) \cap B^{s,\frac{1}{p}-\frac{s}{d}}_{p,p,\mathrm{unif}}(\mathbb{R}^d) \cap \mathcal{B}^{0,\frac{1}{p}}_{\infty,\infty}(\mathbb{R}^d).$$

(iii) *If* $p = 1$ *and* $s \in (-\infty, 0)$, *then*

$$M\left(B^s_{1,1}(\mathbb{R}^d)\right) = B^{-s}_{\infty,\infty}(\mathbb{R}^d).$$

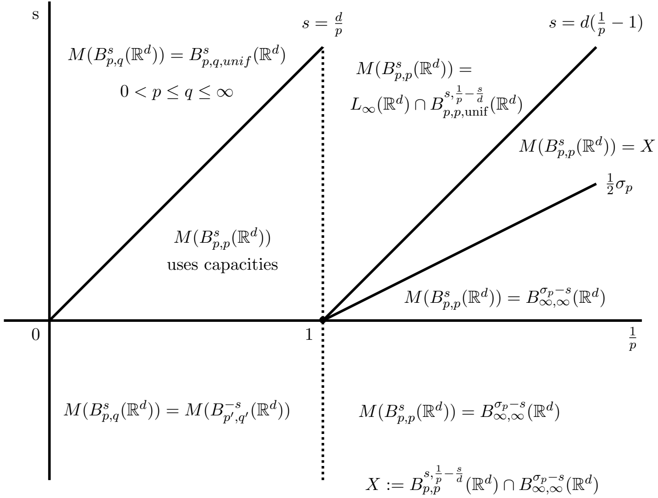
(iv) If $p \in (0,1)$ and $s \in [\frac{1}{2}\sigma_p, \sigma_p)$, then

$$M\left(B^s_{p,p}(\mathbb{R}^d)\right) = B^{s,\frac{1}{p}-\frac{s}{d}}_{p,p,\mathrm{unif}}(\mathbb{R}^d) \cap B^{\sigma_p - s}_{\infty,\infty}(\mathbb{R}^d).$$

(v) If $p \in (0,1)$ and $s \in (-\infty, \frac{1}{2}\sigma_p)$, then

$$M\left(B^s_{p,p}(\mathbb{R}^d)\right) = B^{\sigma_p - s}_{\infty,\infty}(\mathbb{R}^d).$$

- Li, Sickel, Yang, Yuan (2024).


s
$s = \frac{d}{p}$
$s = d(\frac{1}{p} - 1)$
$M(B^s_{p,q}(\mathbb{R}^d)) = B^s_{p,q,unif}(\mathbb{R}^d)$
$0 < p \le q \le \infty$
$M(B^s_{p,p}(\mathbb{R}^d)) =$
$L_\infty(\mathbb{R}^d) \cap B^{s,\frac{1}{p}-\frac{s}{d}}_{p,p,\mathrm{unif}}(\mathbb{R}^d)$
$M(B^s_{p,p}(\mathbb{R}^d)) = X$
$\frac{1}{2}\sigma_p$
$M(B^s_{p,p}(\mathbb{R}^d))$
uses capacities
$M(B^s_{p,p}(\mathbb{R}^d)) = B^{\sigma_p - s}_{\infty,\infty}(\mathbb{R}^d)$
0
1
$\frac{1}{p}$
$M(B^s_{p,q}(\mathbb{R}^d)) = M(B^{-s}_{p',q'}(\mathbb{R}^d))$
$M(B^s_{p,p}(\mathbb{R}^d)) = B^{\sigma_p - s}_{\infty,\infty}(\mathbb{R}^d)$
$X := B^{s,\frac{1}{p}-\frac{s}{d}}_{p,p}(\mathbb{R}^d) \cap B^{\sigma_p - s}_{\infty,\infty}(\mathbb{R}^d)$

6. Proofs: paraproducts

$f, g \in \mathcal{S}'(\mathbb{R}^d)$, $\operatorname{supp} \mathcal{F} S_j f,\ \operatorname{supp} \mathcal{F} S_j g \subset B(0, 2^{j+1}) \setminus B(0, 2^{j-1})$, $j \in \mathbb{N}$,

$$f = \sum_{j=0}^{\infty} S_j f \qquad \text{and} \qquad g = \sum_{k=0}^{\infty} S_k g$$

Formally we have

$$\begin{aligned} f \cdot g &= \sum_{j,k=0}^{\infty} S_j f\, S_k g \\ &= \sum_{k=2}^{\infty} \Big(\sum_{j=0}^{k-2} S_j f\, S_k g \Big) + \sum_{j=2}^{\infty} \Big(\sum_{k=0}^{j-2} S_j f\, S_k g \Big) + \sum_{j,k \ge 0,\ |j-k| \le 1} S_j f\, S_k g \\ &= \Pi_1(f,g) + \Pi_3(f,g) + \Pi_2(f,g) \end{aligned}$$

- Peetre (1976), Triebel (1977), Bony (1981), Y. Meyer (1981) ...

7. A result of Gulisashvili

Gulisashvili 1984:

Proposition 1

Let $p \in (1,\infty)$ *and* $s \in (0, \frac{1}{p}]$*. Then* $f \in M(B^s_{p,1}(\mathbb{R}^d), B^s_{p,\infty}(\mathbb{R}^d))$ *if and only if* $f \in L_\infty(\mathbb{R}^d)$ *and*

$$A(f) := \sup_{h\in\mathbb{R}^d\setminus\{\mathbf{0}\}} |h|^{-sp} \sup_{r\in(0,1)} r^{ps-d} \sup_{x\in\mathbb{R}^d} \int_{B(x,r)} |f(y+h)-f(y)|^p dy < \infty\, .$$

Moreover,

$$\|f\|_{M\left(B^s_{p,1}(\mathbb{R}^d),B^s_{p,\infty}(\mathbb{R}^d)\right)} \sim \|f\|_{L^\infty(\mathbb{R}^d)} + [A(f)]^{\frac{1}{p}}\, ,$$

where the positive equivalent constants are independent of f*.*

A reformulation

Proposition 2

Let $p \in (1, \infty)$ *and* $s \in (0, \frac{1}{p}]$*. Let* $\tau := \frac{1}{p} - \frac{s}{d}$ *and* $u := d/s$*. Then* $f \in M(B^s_{p,1}(\mathbb{R}^d), B^s_{p,\infty}(\mathbb{R}^d))$ *if and only if* $f \in L_\infty(\mathbb{R}^d) \cap B^{s,\tau}_{p,\infty,\mathrm{unif}}(\mathbb{R}^d)$*. Moreover,*

$$\|f\|_{M(B^s_{p,1}(\mathbb{R}^d), B^s_{p,\infty}(\mathbb{R}^d))} \sim \|f\|_{L_\infty(\mathbb{R}^d)} + \sup_{0<|h|<1} |h|^{-s} \, \|f(\,\cdot + h) - f(\,\cdot\,)\|_{\mathcal{M}^u_p(\mathbb{R}^d)} \,,$$

where the positive equivalent constants are independent of f*.*

- $B^{s,\tau}_{p,\infty}(\mathbb{R}^d) = \mathcal{N}^s_{u,p,\infty}(\mathbb{R}^d)$, where $\tau := \frac{1}{p} - \frac{s}{d}$ and $u := d/s$.

Besov-Morrey spaces

Definition

Let $s \in \mathbb{R}$, $0 < p \le u < \infty$ and $0 < q \le \infty$. The Besov-Morrey space $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\| f \, |\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\| := \left\{ \sum_{k=0}^{\infty} \left[2^{ks} \, \| S_k f | \mathcal{M}^u_p(\mathbb{R}^d) \| \right]^q \right\}^{1/q} < \infty.$$

These Morrey-type smoothness spaces have been introduced by Kozono and Yamazaki 1994.
Let $\tau := \frac{1}{p} - \frac{1}{u}$. Then it holds that

$$B^{s,\tau}_{p,\infty}(\mathbb{R}^d) = \mathcal{N}^s_{u,p,\infty}(\mathbb{R}^d)$$

and

$$\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow B^{s,\tau}_{p,q}(\mathbb{R}^d) \, .$$

Theorem 11

Let $1 < p < \infty$, $0 < s < \frac{d}{p}$ *and* $u = d/s$.

(i) *Let* $0 < q \le 1$. *Then we have*

$$\mathcal{N}^{s}_{u,p,q,unif}(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d) \hookrightarrow M(B^s_{p,q}(\mathbb{R}^d)) \, .$$

(ii) *Let* $1 < q \le \infty$. *Then we have*

$$\mathcal{N}^{s}_{u,p,q,unif}(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d) \hookrightarrow M(B^s_{p,1}(\mathbb{R}^d), B^s_{p,q}(\mathbb{R}^d)) \, .$$

(iii) *Let* $0 < q \le \infty$ *and* $s' > s$. *Then we have*

$$B^{s', \frac{1}{p} - \frac{s}{d}}_{p,\infty,unif}(\mathbb{R}^d) = \mathcal{N}^{s'}_{u,p,\infty,unif}(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d) \hookrightarrow M(B^s_{p,q}(\mathbb{R}^d)) \, .$$

Thank you very much for your attention !

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The definition of the product

Let $0 < r < R < \infty$. Let $\varphi_0 \in C_0^\infty(\mathbb{R}^d)$, $\varphi_0(\xi) = 1$ if $|\xi| \le r$, $\varphi_0(\xi) = 0$ if $|\xi| \ge R$. Then, for $f \in \mathcal{S}'(\mathbb{R}^d)$, we put

$$S^j f(x) := \mathcal{F}^{-1}[\varphi_0(2^{-j}\,\cdot\,)\mathcal{F}f(\cdot)](x)\,, \qquad x \in \mathbb{R}^d, j \in \mathbb{N}_0,$$

For all $f \in \mathcal{S}'(\mathbb{R}^d)$ it follows

$$f = \lim_{j\to\infty} S^j f \qquad \text{(weak convergence)}\,.$$

Definition 6

Let φ_0 and $\widetilde{\varphi}_0$ be as above. Let $f, g \in \mathcal{S}'(\mathbb{R}^d)$. Then we put

$$f \,\cdot\, g := \lim_{j\to\infty} S^j f \,\cdot\, \widetilde{S}^j g\,,$$

provided the limit exists in $\mathcal{S}'(\mathbb{R}^d)$ for all admissible pairs $(\varphi_0,\ \widetilde{\varphi}_0)$ and is independent of the pair $(\varphi_0,\ \widetilde{\varphi}_0)$ chosen.

The algebra property

Theorem 12

(i) $F^s_{p,q}(\mathbb{R}^d)$ *is an algebra if and only if either* $s > d/p$ *or* $s = d/p$ *and* $0 < p \le 1$.
(ii) $B^s_{p,q}(\mathbb{R}^d)$ *is an algebra if and only if either* $s > d/p$ *or* $0 < p < \infty$, $s = d/p$ *and* $0 < q \le 1$.

- $1 < p < \infty$, $q = 2$: $F^s_{p,2}(\mathbb{R}^d) = H^s_p(\mathbb{R}^d)$ Strichartz 1967.
- $0 < p, q \le \infty$: $B^s_{p,q}(\mathbb{R}^d)$ Peetre 1970.
- $0 < p, q \le \infty$: B-case and some results for F-spaces Triebel 1978.
- $1 < p, q < \infty$: F-case Kalyabin 1980.
- F-case: Franke 1986.

$$\ell_p^+ := \Big\{ \{C_\mu\}_{\mu\in\mathbb{Z}^d} : \; C_\mu \ge 0 \quad \text{and} \sum_{\mu\in\mathbb{Z}^d} C_\mu^p \le 1 \Big\}.$$

$$\psi_\mu(x) := \psi(x-\mu), \;\; x \in \mathbb{R}^d.$$

Definition 7

Let $0 < p, q \le \infty$ and $s > d\max(0, \frac{1}{p} - 1)$. Let $m > s$, $m \in \mathbb{N}$. The class $M^s_{p,q}(\mathbb{R}^d)$ is the collection of all $g \in L_1^{\ell oc}$ such that

$$\|g|M^s_{p,q}(\mathbb{R}^d)\| := \sup_{\{C_\mu\}\in\ell_p^+} \Bigg\{ \Big\| g \sum_{\mu\in\mathbb{Z}^d} C_\mu \psi_\mu \Big| L_p \Big\| + \Bigg[\sum_{k=0}^\infty \Big(2^{ksp} \sup_{|h|<2^{-k}} \sum_{\mu\in\mathbb{Z}^d} C_\mu^p \|\Delta_h^m(\psi_\mu g)\,|L_p\|^p \Big)^{q/p} \Bigg]^{1/q} \Bigg\} < \infty.$$

Theorem 13

Let $0 < q < p < \infty$ and let $s > d/p$. Then

$$M(B^s_{p,q}(\mathbb{R}^d)) = M^s_{p,q}(\mathbb{R}^d)$$

in the sense of equivalent norms.

- Nguyen & S. (2018).

The work of Y.V. Netrusov 1992, 1997

Theorem 14

Let $0 < p \le 1$, $0 < q \le \infty$ and $s > d \max\left(\frac{1}{p} - 1, \frac{1}{q} - 1\right)$. Then $M(F^s_{p,q}(\mathbb{R}^d))$ is the collection of all $f \in L_\infty(\mathbb{R}^d)$ such that there exists a representation

$$f = \sum_{j=0}^{\infty} f_j \, , \quad \mathrm{supp}\, \mathcal{F} f_0 \subset B(0, 2\, ,)$$

$\mathrm{supp}\, \mathcal{F} f_j \subset B(0, 2^{j+1}) \setminus B(0, 2^{j-1})$, *$j \in \mathbb{N}$, convergence in $\mathcal{S}'(\mathbb{R}^d)$ and*

$$\sup_{P \in \mathcal{Q}, |P| \le 1} \frac{1}{|P|^{\frac{1}{p} - \frac{s}{d}}} \Big\| \Big(\sum_{j=l(P)}^{\infty} 2^{jsq} |f_j|^q \Big)^{1/q} \Big| L_p(P) \Big\| < \infty \, .$$

The work of Y.V. Netrusov

Theorem 15

Let $0 < p \le 1$ *and* $d\left(\frac{1}{p}-1\right) < s < d/p$*. Then* $M(B^s_{p,\infty}(\mathbb{R}^d))$ *is the collection of all* $f \in L_\infty(\mathbb{R}^d)$ *such that there exists a representation*

$$f = \sum_{j=0}^{\infty} f_j\,, \quad \operatorname{supp} \mathcal{F} f_0 \subset B(0,2\,,)$$

$\operatorname{supp} \mathcal{F} f_j \subset B(0,2^{j+1}) \setminus B(0,2^{j-1})$, $j \in \mathbb{N}$, *convergence in* $\mathcal{S}'(\mathbb{R}^d)$ *and*

$$\sup_{i\in\mathbb{N}_0} \left(\int_{2^{-i-1}}^{1} 2^{isp}\left[\sup_{x\in\mathbb{R}^d}\int_{B(x,t)} |f_j(x)|^p dx\right] t^{sp-d}\,\frac{dt}{t}\right)^{1/p} < \infty\,.$$

The work of Y.V. Netrusov

Theorem 16

Let $0 < p \leq 1$ and $0 < s = d/p$. Then $M(B^s_{p,\infty}(\mathbb{R}^d))$ is the collection of all $f \in L_\infty(\mathbb{R}^d)$ such that there exists a representation

$$f = \sum_{j=0}^{\infty} f_j\,, \quad \mathrm{supp}\, \mathcal{F} f_0 \subset B(0, 2\,,)$$

$\mathrm{supp}\, \mathcal{F} f_j \subset B(0, 2^{j+1}) \setminus B(0, 2^{j-1})$, $j \in \mathbb{N}$, *convergence in* $\mathcal{S}'(\mathbb{R}^d)$ *and*

$$\sup_{i \in \mathbb{N}_0} \Big(\int_{2^{-i-2}}^{1} 2^{isp} \big(\sup_{x \in \mathbb{R}^d} \int_{B(x,t)} |f_j(x)|^p dx \big) \ln^p t \frac{dt}{t} \Big)^{1/p} < \infty\,.$$

S. 1999

Theorem 17

Let $0 < p < \infty$, $0 < q \le \infty$ and $s > d \max\left(\frac{1}{p} - 1, \frac{1}{q} - 1\right)$. Then $M(F^s_{p,q}(\mathbb{R}^d))$ is the collection of all $f \in L_\infty(\mathbb{R}^d)$ such that

$$\|f\| := \sup_{\substack{A \subset \mathbb{R}^d, open \\ \operatorname{diam} A \le 1}} \frac{\left(\int_A \left(\sum_{j=0}^{\infty} 2^{jsq} |S_j f(x)|^q dx \right)^{p/q} dx \right)^{1/p}}{\overline{cap}(A, F^s_{p,q}(\mathbb{R}^d))^{1/p}} < \infty\,.$$

Furthermore, $\| f \,|M(F^s_{p,q}(\mathbb{R}^d))\|$ is equivalent to $\|f\| + \| f \,|L_\infty(\mathbb{R}^d)\|$.

- Assume that $\sigma_{p,q} < s < d/p$. Then

$$M(F^s_{p,q}(\mathbb{R}^d)) \hookrightarrow F^{s, \frac{1}{p} - \frac{s}{d}}_{p,q,unif}(\mathbb{R}^d)$$