Generalised Morrey smoothness spaces

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Outline

Preliminaries

Classical Morrey spaces

Definition (Morrey, 1938)

Let $0 < p \leq u < \infty$. The Morrey space $\mathcal{M}_{u,p}(\mathbb{R}^d)$ is the set of all $f \in L_p^{\mathrm{loc}}(\mathbb{R}^d)$ such that

$$\|f \mid \mathcal{M}_{u,p}(\mathbb{R}^d)\| := \sup_{x\in\mathbb{R}^d, r>0} r^{\frac{d}{u}} \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(y)|^p \,\mathrm{d}y \right)^{\frac{1}{p}} < \infty.$$

Remarks

- $L_u(\mathbb{R}^d) = \mathcal{M}_{u,u}(\mathbb{R}^d) \underset{\subsetneq}{\hookrightarrow} \mathcal{M}_{u,p}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{u,r}(\mathbb{R}^d), \quad 0 < r \leq p \underset{\neq}{\leq} u < \infty$

Generalised Morrey spaces

Definition (Mizuhara 1991, Nakai 1994)

Let $0 < p < \infty$ and $\varphi : (0, \infty) \to [0, \infty)$ be a function which does not satisfy $\varphi \equiv 0$. The generalised Morrey space $\mathcal{M}_{\varphi,p}(\mathbb{R}^d)$ is defined to be the set of all $f \in L_p^{\mathrm{loc}}(\mathbb{R}^d)$ such that

$$\|f \mid \mathcal{M}_{\varphi,p}(\mathbb{R}^d)\| := \sup_{x\in\mathbb{R}^d, r>0} \varphi(r) \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(y)|^p \, \mathrm{d}y \right)^{\frac{1}{p}} < \infty.$$

Remarks

- If $\varphi(t) = t^{\frac{d}{u}}$, $0 < p \le u < \infty$, then $\mathcal{M}_{\varphi,p}(\mathbb{R}^d) = \mathcal{M}_{u,p}(\mathbb{R}^d)$ classical Morrey spaces
  in particular, when $u = p$, then $\mathcal{M}_{\varphi,p}(\mathbb{R}^d) = L_p(\mathbb{R}^d)$ Lebesgue spaces
- If $\varphi(t) = 1$, then $\mathcal{M}_{\varphi,p}(\mathbb{R}^d) = L_\infty(\mathbb{R}^d)$
- If $\varphi(t) = t^{-\sigma} \chi_{(0,1)}(t)$, $-\frac{d}{p} \le \sigma < 0$ then $\mathcal{M}_{\varphi,p}(\mathbb{R}^d) = \mathcal{L}_p^\sigma(\mathbb{R}^d)$ local Morrey spaces
  in particular, when $\sigma = -\frac{d}{p}$, then $\mathcal{M}_{\varphi,p}(\mathbb{R}^d) = \mathcal{L}_p(\mathbb{R}^d)$ uniform Lebesgue spaces

The class $\mathcal{G}_p$

Definition

Let $0 < p < \infty$. By $\mathcal{G}_p$ is denoted the set of functions $\varphi : (0, \infty) \to [0, \infty)$ such that

$$\varphi \quad \text{is increasing} \quad \text{and} \quad \varphi(t)t^{-d/p} \quad \text{is decreasing.}$$

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Remarks

- $\mathcal{M}_{\varphi,p}(\mathbb{R}^d) \neq \{0\} \quad \Leftrightarrow \quad \sup_{t>0} \varphi(t)\min(t^{-\frac{d}{p}}, 1) < \infty.$

  If $\sup_{t>0} \varphi(t)\min(t^{-\frac{d}{p}}, 1) < \infty$, then there exists $\varphi^* \in \mathcal{G}_p$ such that

  $$\mathcal{M}_{\varphi,p}(\mathbb{R}^d) = \mathcal{M}_{\varphi^*,p}(\mathbb{R}^d).$$

- If $\varphi \in \mathcal{G}_p$, then φ satisfies the doubling condition: $\varphi(r) \le \varphi(2r) \le 2^{\frac{d}{p}}\varphi(r)$

$$\|f \mid \mathcal{M}_{\varphi,p}(\mathbb{R}^d)\| \sim \sup_{P\in\mathcal{Q}} \varphi(\ell(P)) \left(\frac{1}{|P|}\int_P |f(y)|^p \,\mathrm{d}y\right)^{\frac{1}{p}}.$$

$\mathcal{Q} := \{Q_{j,k} := 2^{-j}([0,1)^d + k) : j \in \mathbb{Z}, k \in \mathbb{Z}^d\}, \quad \ell(P) = |P|^{\frac{1}{d}}$ side length of P

The class $\mathcal{G}_p$

Remarks

- $\mathcal{G}_{p_1} \subset \mathcal{G}_{p_2}$ if $0 < p_2 \leq p_1 < \infty$
- $\mathcal{M}_{\varphi,p}(\mathbb{R}^d) \hookrightarrow L_\infty(\mathbb{R}^d) \quad \Leftrightarrow \quad \inf_{t>0} \varphi(t) > 0$
- $L_\infty(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{\varphi,p}(\mathbb{R}^d) \quad \Leftrightarrow \quad \sup_{t>0} \varphi(t) > 0$
- Let $\inf_{t>0} \varphi(t) > 0$. Then

$$\mathcal{M}_{\varphi_1,p_1}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{\varphi_2,p_2}(\mathbb{R}^d) \quad \Leftrightarrow \quad p_1 \leq p_2 \text{ and } \varphi_1(t) \geq C\varphi_2(t)$$

Sawano, Di Fazio, Hakim 2020

The class $\mathcal{G}_p$

Examples

- For $0 < u, v < \infty$,

$$\varphi_{u,v}(t) = \begin{cases} t^{d/u} & \text{if} \quad t \leq 1 \\ t^{d/v} & \text{if} \quad t > 1 \end{cases}$$

  belongs to $\mathcal{G}_p$ for $p = \min(u, v)$. In particular:

  $\varphi(t) = t^{d/u}$ belongs to $\mathcal{G}_p$ whenever $0 < p \leq u < \infty$,

  $\varphi(t) = \max(1, t^{d/v})$ and $\varphi(t) = \min(t^{d/u}, 1)$ belong to $\mathcal{G}_p$.

- For $0 < u < \infty$, $a \leq 0$ and $L >> 1$, $\varphi(t) = t^{d/u}(\log(L + t))^a$ belongs to $\mathcal{G}_u$.

- Let

$$\varphi(t) = \begin{cases} \frac{1}{\log 2} \log(1 + t), & 0 < t < 1, \\ t, & t \geq 1. \end{cases}$$

  Then $\varphi \in \mathcal{G}_d$.

Generalised smoothness Morrey spaces

Generalised Besov-Morrey and Triebel-Lizorkin-Morrey spaces

Definition (Nakamura, Noi, Sawano 2016)

Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$ and $\varphi \in \mathcal{G}_p$.

(i) The generalised Besov-Morrey space $\mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f \mid \mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d)\| := \left(\sum_{j=0}^{\infty} 2^{jsq} \|(\eta_j \widehat{f})^\vee \mid \mathcal{M}_{\varphi,p}(\mathbb{R}^d)\|^q\right)^{\frac{1}{q}} < \infty.$$

(ii) When $0 < q < \infty$, assume further the existence of $C, \varepsilon > 0$ so that

$$\varphi(t)t^{-\varepsilon} \geq C\varphi(r)r^{-\varepsilon}, \quad t \geq r.$$

The generalised Triebel-Lizorkin-Morrey space $\mathcal{E}^s_{\varphi,p,q}(\mathbb{R}^d)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f \mid \mathcal{E}^s_{\varphi,p,q}(\mathbb{R}^d)\| := \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |(\eta_j \widehat{f})^\vee(\cdot)|^q\right)^{\frac{1}{q}} \mid \mathcal{M}_{\varphi,p}(\mathbb{R}^d) \right\| < \infty.$$

Remarks

- If $\varphi(t) = t^{\frac{d}{u}}$, $0 < p \le u < \infty$, then

$$\mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d) = \mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$$ classical Besov-Morrey spaces

$$\mathcal{E}^s_{\varphi,p,q}(\mathbb{R}^d) = \mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$$ classical Triebel-Lizorkin-Morrey spaces

  Moreover, if $u = p$, then

$$B^s_{p,q}(\mathbb{R}^d) = \mathcal{N}^s_{p,p,q}(\mathbb{R}^d) \quad \text{and} \quad F^s_{p,q}(\mathbb{R}^d) = \mathcal{E}^s_{p,p,q}(\mathbb{R}^d).$$

- If $\varphi(t) \sim 1$, then

$$\mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d) = B^s_{\infty,q}(\mathbb{R}^d) \quad \text{and} \quad \mathcal{E}^s_{\varphi,p,\infty}(\mathbb{R}^d) = B^s_{\infty,\infty}(\mathbb{R}^d) = F^s_{\infty,\infty}(\mathbb{R}^d)$$

- If $\varphi(t) = \min(t^{\frac{d}{u}}, 1)$, $\sigma = -\frac{d}{u}$, $0 < p \le u < \infty$, then

$$\mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d) = B^s_q(\mathcal{L}^\sigma_p, \mathbb{R}^d)$$ local Besov-Morrey spaces

- If $1 < p < \infty$, then

$$\mathcal{E}^0_{\varphi,p,2}(\mathbb{R}^d) = \mathcal{M}_{\varphi,p}(\mathbb{R}^d)$$

  assuming that $\varphi \in \mathcal{G}_p$ satisfies the ε-condition.

Generalised Besov-type and Triebel-Lizorkin-type spaces

Definition (Haroske, Liu 2023)

Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$ and $\varphi \in \mathcal{G}_p$.

(i) The generalised Besov-type space $B^{s,\varphi}_{p,q}(\mathbb{R}^d)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f \mid B^{s,\varphi}_{p,q}(\mathbb{R}^d)\| := \sup_{P \in \mathcal{Q}} \frac{\varphi(\ell(P))}{|P|^{\frac{1}{p}}} \left(\sum_{j=j_P \vee 0}^{\infty} 2^{jsq} \left(\int_P |(\theta_j \widehat{f})^{\vee}(x)|^p \,\mathrm{d}x \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty.$$

(ii) When $0 < q < \infty$, assume further the existence of $C, \varepsilon > 0$ so that

$$\varphi(t) t^{-\varepsilon} \ge C \varphi(r) r^{-\varepsilon}, \quad t \ge r.$$

The generalised Triebel-Lizorkin-type space $F^{s,\varphi}_{p,q}(\mathbb{R}^d)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f \mid F^{s,\varphi}_{p,q}(\mathbb{R}^d)\| := \sup_{P \in \mathcal{Q}} \frac{\varphi(\ell(P))}{|P|^{\frac{1}{p}}} \left(\int_P \left(\sum_{j=j_P \vee 0}^{\infty} 2^{jsq} |(\theta_j \widehat{f})^{\vee}(x)|^q \right)^{\frac{p}{q}} \mathrm{d}x \right)^{\frac{1}{p}} < \infty.$$

Remarks

- If $\varphi(t) := t^{d(\frac{1}{p}-\tau)}$ for $\tau \in [0, \frac{1}{p}]$, then

$$B^{s,\varphi}_{p,q}(\mathbb{R}^d) = B^{s,\tau}_{p,q}(\mathbb{R}^d)$$ classical Besov-type spaces

$$F^{s,\varphi}_{p,q}(\mathbb{R}^d) = F^{s,\tau}_{p,q}(\mathbb{R}^d)$$ classical Triebel-Lizorkin-type spaces

Moreover, if $\tau = 0$, then

$$B^{s}_{p,q}(\mathbb{R}^d) = B^{s,0}_{p,q}(\mathbb{R}^d) \quad \text{and} \quad F^{s}_{p,q}(\mathbb{R}^d) = F^{s,0}_{p,q}(\mathbb{R}^d).$$

and if $\tau = \frac{1}{p}$, then

$$F^{s}_{\infty,q}(\mathbb{R}^d) = F^{s,\frac{1}{p}}_{p,q}(\mathbb{R}^d) = B^{s,\frac{1}{p}}_{p,q}(\mathbb{R}^d)$$

In particular,

$$\mathrm{bmo}(\mathbb{R}^d) = F^{0}_{\infty,2}(\mathbb{R}^d) = F^{0,\frac{1}{p}}_{p,2}(\mathbb{R}^d)$$

Elementary embeddings

Standard notation: $\mathcal{A}^s_{\varphi,p,q}$ denotes either $\mathcal{N}^s_{\varphi,p,q}$ or $\mathcal{E}^s_{\varphi,p,q}$,

$A^{s,\varphi}_{p,q}$ denotes either $B^{s,\varphi}_{p,q}$ or $F^{s,\varphi}_{p,q}$, etc.

Assumption: When $q < \infty$ and $\mathcal{A}^s_{\varphi,p,q} = \mathcal{E}^s_{\varphi,p,q}$ or $A^{s,\varphi}_{p,q} = F^{s,\varphi}_{p,q}$, assume always that $\varphi \in \mathcal{G}_p$ satisfies

$$\varphi(t)t^{-\varepsilon} \geq C\varphi(r)r^{-\varepsilon}, \quad t \geq r.$$

for some $C, \varepsilon > 0$.

Elementary embeddings:

- $\mathcal{A}^s_{\varphi,p,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{A}^s_{\varphi,p,q_2}(\mathbb{R}^d)$ and $A^{s,\varphi}_{p,q_1}(\mathbb{R}^d) \hookrightarrow A^{s,\varphi}_{p,q_2}(\mathbb{R}^d), \quad q_1 \leq q_2$
- $\mathcal{A}^{s+\varepsilon}_{\varphi,p,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{A}^s_{\varphi,p,q_2}(\mathbb{R}^d)$ and $A^{s+\varepsilon,\varphi}_{p,q_1}(\mathbb{R}^d) \hookrightarrow A^{s,\varphi}_{p,q_2}(\mathbb{R}^d), \quad \varepsilon > 0$

Atomic and wavelet decompositions

Wavelet decompositions

- For $\mathcal{A}^s_{u,p,q}(\mathbb{R}^d)$ — Sawano 2008, Rosenthal 2013
- For $A^{s,\tau}_{p,q}(\mathbb{R}^d)$ — Yuan, Sickel, Yang 2010; Liang, Sawano, Ullrich, Yang, Yuan 2012
- For $\mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d)$ — Haroske, M., Skrzypczak 2022

Atomic decompositions $\rightsquigarrow$ Wavelet decompositions

(Haroske, Skandera, Triebel 2018)

$\mathcal{A}^s_{\varphi,p,q}(\mathbb{R}^d)$ $\rightsquigarrow$

Nakamura, Noi, Sawano 2016 — Haroske, Liu, M., Skrzypczak 2025

$A^{s,\varphi}_{p,q}(\mathbb{R}^d)$ $\rightsquigarrow$

Haroske, Liu 2023

Relations between the different scales

Comparison between $A^{s,\tau}_{p,q}$ and $\mathcal{A}^s_{u,p,q}$

Let $0 < p \leq u < \infty$.

- $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d) = F^{s,\tau}_{p,q}(\mathbb{R}^d)$ with $\tau = \frac{1}{p} - \frac{1}{u} \in [0, \frac{1}{p}[$
- $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow B^{s,\tau}_{p,q}(\mathbb{R}^d)$ with $\tau = \frac{1}{p} - \frac{1}{u}$

  proper embedding if $\tau > 0$ and $q < \infty$, while for $\tau = 0$ or $q = \infty$, both spaces coincide:

$$\mathcal{N}^s_{u,p,\infty}(\mathbb{R}^d) = B^{s,\frac{1}{p}-\frac{1}{u}}_{p,\infty}(\mathbb{R}^d)$$

  and

$$\mathcal{N}^s_{p,p,q}(\mathbb{R}^d) = B^s_{p,q}(\mathbb{R}^d) = B^{s,0}_{p,q}(\mathbb{R}^d)$$

Yuan, Sickel, Yang 2010
Sawano, Yang, Yuan 2010

Comparison between $A^{s,\varphi}_{p,q}$ and $\mathcal{A}^s_{\varphi,p,q}$

Theorem (Haroske, Liu, M., Skrzypczak 2025)

Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$ *and* $\varphi \in \mathcal{G}_p$.

(i) *It holds* $F^{s,\varphi}_{p,q}(\mathbb{R}^d) = \mathcal{E}^s_{\varphi,p,q}(\mathbb{R}^d)$ *with equivalent norms.*

(ii) *It holds*

$$\mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d) \hookrightarrow B^{s,\varphi}_{p,q}(\mathbb{R}^d).$$

When $q = \infty$,

$$\mathcal{N}^s_{\varphi,p,\infty}(\mathbb{R}^d) = B^{s,\varphi}_{p,\infty}(\mathbb{R}^d),$$

and when $q < \infty$ *and*

$$\lim_{t\to 0^+} \varphi(t)t^{-\frac{d}{p}} = \infty \qquad \text{or} \qquad \lim_{t\to +\infty} \varphi(t)t^{-\frac{d}{p}} = 0 \qquad \boxed{\bigstar}$$

then $\mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d)$ *is a proper subspace of* $B^{s,\varphi}_{p,q}(\mathbb{R}^d)$.

(i) in Nakamura, Noi, Sawano 2016

Remarks

- If $\boxed{\bigstar}$ does not hold, then

$$0 < \lim_{t\to+\infty} \varphi(t)t^{-\frac{d}{p}} \leq \lim_{t\to 0^+} \varphi(t)t^{-\frac{d}{p}} < \infty$$

  hence $\varphi(t) \sim t^{\frac{d}{p}}$, $t > 0$, and

$$\mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d) = B^{s,\varphi}_{p,q}(\mathbb{R}^d) = B^s_{p,q}(\mathbb{R}^d).$$

- When $\varphi(t) := t^{\frac{d}{u}}$, $0 < p \leq u$, previous results are recovered. Indeed, condition $\boxed{\bigstar}$ is equivalent to

$$\lim_{t\to 0^+} \varphi(t)t^{-\frac{d}{p}} = \infty \quad \Leftrightarrow \quad \lim_{t\to+\infty} \varphi(t)t^{-\frac{d}{p}} = 0 \quad \Leftrightarrow \quad p < u$$

Examples

Both, the behaviour of φ near 0 and near ∞ influence the embedding behaviour of $\mathcal{N}^s_{\varphi,p,q}(\mathbb{R}^d) \hookrightarrow B^{s,\varphi}_{p,q}(\mathbb{R}^d)$.

- Let $0 < u,\, v \leq \infty$ with $u \neq v$. Let $p := \min(u, v)$. Then

  $$\varphi_{u,v}(t) = \begin{cases} t^{\frac{d}{u}}, & \text{if} \quad t \leq 1 \\ t^{\frac{d}{v}}, & \text{if} \quad t > 1 \end{cases}$$

  belongs to $\mathcal{G}_p$ and it holds

  $$\lim_{t \to 0^+} \varphi(t) t^{-\frac{d}{p}} = \infty \qquad \text{or} \qquad \lim_{t \to +\infty} \varphi(t) t^{-\frac{d}{p}} = 0. \qquad \boxed{\star}$$

- Let $0 < p < \infty$, $a < 0$ and $L >> 1$. Then

  $$\varphi(t) = t^{\frac{d}{p}} (\log(L + t))^a$$

  belongs to $\mathcal{G}_p$, satisfies the second condition in $\boxed{\star}$ but not the first one.

Embeddings between $A^{s,\tau}_{p,q}$ and $\mathcal{A}^{s}_{u,p,q}$

$$B^s_{p,\min(p,q)}(\mathbb{R}^d) \hookrightarrow F^s_{p,q}(\mathbb{R}^d) \hookrightarrow B^s_{p,\max(p,q)}(\mathbb{R}^d)$$

Sickel, Triebel 1995

Extends to $A^{s,\tau}_{p,q}(\mathbb{R}^d)$ as :

$$B^{s,\tau}_{p,\min(p,q)}(\mathbb{R}^d) \hookrightarrow F^{s,\tau}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{s,\tau}_{p,\max(p,q)}(\mathbb{R}^d)$$

Yuan, Sickel, Yang 2010

As for $\mathcal{A}^s_{u,p,q}(\mathbb{R}^d)$:

$$\mathcal{N}^s_{u,p,\min(p,q)}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^s_{u,p,q_0}(\mathbb{R}^d)$$

$$\Updownarrow$$

$$q_0 = \infty \quad \text{or} \quad p = u \ \wedge \ q_0 \geq \max(p,q)$$

Sawano 2008

Extension to $A^{s,\varphi}_{p,q}$ and $\mathcal{A}^s_{\varphi,p,q}$

Theorem (Haroske, Liu 2023)

Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$ *and* $\varphi \in \mathcal{G}_p$. *It holds*

$$B^{s,\varphi}_{p,\min(p,q)}(\mathbb{R}^d) \hookrightarrow F^{s,\varphi}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{s,\varphi}_{p,\max(p,q)}(\mathbb{R}^d).$$

Theorem (Haroske, Liu, M., Skrzypczak 2025)

Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q, q_0 \le \infty$ *and* $\varphi \in \mathcal{G}_p$.

(i) *It holds*

$$\mathcal{N}^s_{\varphi,p,\min(p,q)}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^s_{\varphi,p,q}(\mathbb{R}^d).$$

(ii) *Assume*

$$\lim_{t\to 0^+} \varphi(t)t^{-\frac{d}{p}} < \infty \quad \textit{or} \quad \lim_{t\to +\infty} \varphi(t)t^{-\frac{d}{p}} > 0. \qquad \boxed{\bullet}$$

Then

$$\mathcal{E}^s_{\varphi,p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^s_{\varphi,p,q_0}(\mathbb{R}^d) \iff q_0 \ge \max(p,q).$$

(iii) *If* $\boxed{\bullet}$ *is not satisfied, then*

$$\mathcal{E}^s_{\varphi,p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^s_{\varphi,p,q_0}(\mathbb{R}^d) \iff q_0 = \infty.$$

Embeddings between generalised Morrey smoothness spaces

Embeddings between generalised Morrey smoothness spaces

Embeddings to be considered:

- $\mathcal{N}^{s_1}_{\varphi_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^{s_2}_{\varphi_2,p_2,q_2}(\mathbb{R}^d)$
- $\mathcal{E}^{s_1}_{\varphi_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^{s_2}_{\varphi_2,p_2,q_2}(\mathbb{R}^d)$: case $p_1 \geq p_2$
- $\mathcal{E}^{s_1}_{\varphi_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^{s_2}_{\varphi_2,p_2,q_2}(\mathbb{R}^d)$: case $p_1 < p_2$
- $B^{s_1,\varphi_1}_{p_1,q_1}(\mathbb{R}^d) \hookrightarrow B^{s_2,\varphi_2}_{p_2,q_2}(\mathbb{R}^d)$

Embeddings between generalised Morrey smoothness spaces

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- $\mathcal{E}^{s_1}_{\varphi_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^{s_2}_{\varphi_2,p_2,q_2}(\mathbb{R}^d)$: case $p_1 \geq p_2$
- $\mathcal{E}^{s_1}_{\varphi_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^{s_2}_{\varphi_2,p_2,q_2}(\mathbb{R}^d)$: case $p_1 < p_2$
- $B^{s_1,\varphi_1}_{p_1,q_1}(\mathbb{R}^d) \hookrightarrow B^{s_2,\varphi_2}_{p_2,q_2}(\mathbb{R}^d)$

Assumptions:

- In case of $\mathcal{E}^{s}_{\varphi,p,q}(\mathbb{R}^d)$ and when $q < \infty$, it is always assumed, even if not stated explicitly, that $\varphi \in \mathcal{G}_p$ satisfies
$$\varphi(t)t^{-\varepsilon} \geq C\varphi(r)r^{-\varepsilon}, \quad t \geq r.$$
for some $C, \varepsilon > 0$.
- We will always assume, without loss of generality, that $\varphi_1(1) = \varphi_2(1) = 1$.

Embedding $\mathcal{N}^{s_1}_{\varphi_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^{s_2}_{\varphi_2,p_2,q_2}(\mathbb{R}^d)$

Theorem (Haroske, M., Skrzypczak 2022)

Let $s_i \in \mathbb{R}$, $0 < p_i < \infty$, $0 < q_i \leq \infty$ *and* $\varphi_i \in \mathcal{G}_{p_i}$, $i = 1, 2$. *Let* $\varrho = \min(1, \frac{p_1}{p_2})$. *It holds*

$$\mathcal{N}^{s_1}_{\varphi_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^{s_2}_{\varphi_2,p_2,q_2}(\mathbb{R}^d)$$

if, and only if,

$$\sup_{\nu \leq 0} \frac{\varphi_2(2^{-\nu})}{\varphi_1(2^{-\nu})^{\varrho}} < \infty \qquad \boxed{\mathbf{1}}$$

and

$$\left\{ 2^{j(s_2-s_1)}\, \alpha_j\, \varphi_1(2^{-j})^{\varrho-1} \right\}_{j\in\mathbb{N}} \in \ell_{q^*} \qquad \boxed{\mathbf{2}}$$

where

$$\frac{1}{q^*} = \left(\frac{1}{q_2} - \frac{1}{q_1}\right)_+ \quad \text{and} \quad \alpha_j = \sup_{\nu \leq j} \frac{\varphi_2(2^{-\nu})}{\varphi_1(2^{-\nu})^{\varrho}}, \quad j \in \mathbb{N}_0.$$

The embedding is never compact.

Example

Let $\varphi_i(t) = t^{\frac{d}{u_i}}$, $0 < p_i \le u_i < \infty$, $i = 1, 2$. Then $\mathcal{N}^{s_i}_{\varphi_i, p_i, q_i}(\mathbb{R}^d) = \mathcal{N}^{s_i}_{u_i, p_i, q_i}(\mathbb{R}^d)$.

- $\boxed{1}$ $\quad \Leftrightarrow \quad \sup_{\nu \le 0} \frac{\varphi_2(2^{-\nu})}{\varphi_1(2^{-\nu})^\varrho} < \infty \quad \Leftrightarrow \quad \sup_{\nu \le 0} 2^{-\nu\left(\frac{1}{u_2} - \frac{\varrho}{u_1}\right)} < \infty$

$$\Leftrightarrow \quad \frac{u_1}{u_2} \le \varrho = \min(1, \frac{p_1}{p_2}) \quad \Leftrightarrow \quad \boxed{u_1 \le u_2} \quad \wedge \quad \boxed{\frac{u_1}{p_1} \le \frac{u_2}{p_2}}$$

$$\rightsquigarrow \quad \alpha_j = \sup_{\nu \le j} \frac{\varphi_2(2^{-\nu})}{\varphi_1(2^{-\nu})^\varrho} = \sup_{\nu \le j} 2^{\nu d\left(\frac{\varrho}{u_1} - \frac{1}{u_2}\right)} = 2^{jd\left(\frac{\varrho}{u_1} - \frac{1}{u_2}\right)}$$

- $\boxed{2}$ $\quad \Leftrightarrow \quad \left\{2^{j(s_2 - s_1)} \alpha_j \, \varphi_1(2^{-j})^{\varrho - 1}\right\}_j \in \ell_{q^*} \quad \Leftrightarrow \quad \left\{2^{j(s_2 - s_1 + \frac{d}{u_1} - \frac{d}{u_2})}\right\}_j \in \ell_{q^*}$

$$\Leftrightarrow \quad \boxed{\frac{s_1 - s_2}{d} > \frac{1}{u_1} - \frac{1}{u_2}} \quad \text{or} \quad \boxed{\frac{s_1 - s_2}{d} = \frac{1}{u_1} - \frac{1}{u_2}} \quad \wedge \quad \boxed{q_1 \le q_2}$$

Haroske, Skrzypczak 2012

Embedding $\mathcal{E}^{s_1}_{\varphi_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^{s_2}_{\varphi_2,p_2,q_2}(\mathbb{R}^d)$

$$\mathcal{N}^{s_1}_{\varphi_1,p_1,\min(p_1,q_1)}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^{s_1}_{\varphi_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^{s_2}_{\varphi_2,p_2,q_2}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^{s_2}_{\varphi_2,p_2,\infty}(\mathbb{R}^d)$$

$$\mathcal{E}^{s_1}_{\varphi_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^{s_2}_{\varphi_2,p_2,q_2}(\mathbb{R}^d) \quad \Rightarrow \quad \begin{cases} \sup\limits_{\nu\le 0} \dfrac{\varphi_2(2^{-\nu})}{\varphi_1(2^{-\nu})^{\varrho}} < \infty \\ \left\{2^{j(s_2-s_1)}\,\alpha_j\,\varphi_1(2^{-j})^{\varrho-1}\right\}_{j\in\mathbb{N}} \in \ell_\infty \end{cases}$$

If $p_1 \ge p_2$: $\varrho = \min(1, \frac{p_1}{p_2}) = 1, \quad \alpha_j = \sup\limits_{\nu\le j} \dfrac{\varphi_2(2^{-\nu})}{\varphi_1(2^{-\nu})^{\varrho}} \ge \dfrac{\varphi_2(2^{-j})}{\varphi_1(2^{-j})}$

then

$$\mathcal{E}^{s_1}_{\varphi_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^{s_2}_{\varphi_2,p_2,q_2}(\mathbb{R}^d) \quad \Rightarrow \quad \begin{cases} \sup\limits_{\nu\le 0} \dfrac{\varphi_2(2^{-\nu})}{\varphi_1(2^{-\nu})} < \infty \\ \left\{2^{j(s_2-s_1)}\,\dfrac{\varphi_2(2^{-j})}{\varphi_1(2^{-j})}\right\}_{j\in\mathbb{N}} \in \ell_\infty \end{cases}$$

Embedding $\mathcal{E}^{s_1}_{\varphi_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^{s_2}_{\varphi_2,p_2,q_2}(\mathbb{R}^d)$: case $p_1 \geq p_2$

Theorem (Haroske, Liu, M., Skrzypczak 2025)

Let $0 < p_2 \leq p_1 < \infty$, $s_i \in \mathbb{R}$, $0 < q_i \leq \infty$, $\varphi_i \in \mathcal{G}_{p_i}$, *for* $i = 1, 2$. *It holds*

$$\mathcal{E}^{s_1}_{\varphi_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^{s_2}_{\varphi_2,p_2,q_2}(\mathbb{R}^d)$$

if, and only if,

$$\sup_{\nu \leq 0} \frac{\varphi_2(2^{-\nu})}{\varphi_1(2^{-\nu})} < \infty \qquad \boxed{1}$$

and

$$\left\{ 2^{j(s_2-s_1)} \frac{\varphi_2(2^{-j})}{\varphi_1(2^{-j})} \right\}_{j \in \mathbb{N}} \in \ell_\infty \qquad \boxed{3}$$

with

$$s_1 > s_2 \qquad \text{or} \qquad s_1 = s_2 \quad \wedge \quad q_1 \leq q_2. \qquad \boxed{4}$$

The embedding is never compact.

Example

Let $\varphi_i(t) = t^{\frac{d}{u_i}}$, $0 < p_i \le u_i < \infty$, $i = 1, 2$. Then $\mathcal{E}^{s_i}_{\varphi_i,p_i,q_i}(\mathbb{R}^d) = \mathcal{E}^{s_i}_{u_i,p_i,q_i}(\mathbb{R}^d)$.

Case $p_2 \le p_1$

- $\boxed{1} \quad \Leftrightarrow \quad \frac{u_1}{u_2} \le \varrho = \min(1, \frac{p_1}{p_2}) = 1 \quad \Leftrightarrow \quad u_1 \le u_2$
- $\boxed{3} \quad \Leftrightarrow \quad \left\{2^{j(s_2 - s_1 + \frac{d}{u_1} - \frac{d}{u_2})}\right\}_j \in \ell_\infty \quad \Leftrightarrow \quad \frac{s_1 - s_2}{d} \ge \frac{1}{u_1} - \frac{1}{u_2}$

$\boxed{1}$ & $\boxed{3}$ & $\boxed{4}$ $\quad \Leftrightarrow \quad \frac{s_1 - s_2}{d} > \frac{1}{u_1} - \frac{1}{u_2}$ or

$\frac{s_1 - s_2}{d} = \frac{1}{u_1} - \frac{1}{u_2} \quad \wedge \quad u_1 < u_2$ or

$s_1 = s_2 \quad \wedge \quad u_1 = u_2 \quad \wedge \quad q_1 \le q_2$

Haroske, Skrzypczak 2014

Embedding $\mathcal{E}^{s_1}_{\varphi_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^{s_2}_{\varphi_2,p_2,q_2}(\mathbb{R}^d)$: case $p_1 \geq p_2$

Corollary (Haroske, Liu, M., Skrzypczak 2025)

Let $0 < p_2 \leq p_1 < \infty$, $s_i \in \mathbb{R}$, $0 < q_i \leq \infty$, $\varphi_i \in \mathcal{G}_{p_i}$, *for* $i = 1, 2$. *It holds*

$$\mathcal{E}^{s_1}_{\varphi_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^{s_2}_{\varphi_2,p_2,q_2}(\mathbb{R}^d)$$

if, and only if,

$$\sup_{\nu \leq 0} \frac{\varphi_2(2^{-\nu})}{\varphi_1(2^{-\nu})} < \infty \qquad \boxed{1}$$

and

$$\left\{2^{j(s_2-s_1)}\alpha_j\right\}_{j\in\mathbb{N}} \in \ell_\infty$$

with

$$q_1 \leq q_2 \qquad \text{if} \qquad s_1 = s_2.$$

Definition

Let $0 < p < \infty$. By $\mathcal{G}_p^{\mathrm{loc}}$ is denoted the set of the increasing functions $\varphi : (0,\infty) \to [0,\infty)$ such that

$$\varphi(t)t^{-d/p} \quad \text{is decreasing} \quad \text{in} \quad (0,1).$$

Let

$$\mathcal{G}^{\mathrm{loc}} = \bigcup_{p>0} \mathcal{G}_p^{\mathrm{loc}} \qquad \text{and} \qquad \mathcal{G} = \bigcup_{p>0} \mathcal{G}_p.$$

For $\varphi \in \mathcal{G}_p^{\mathrm{loc}}$, let

$$\mathsf{r}_\varphi = \sup\{p : \ \varphi \in \mathcal{G}_p^{\mathrm{loc}}\}.$$

Remarks

- $\mathcal{G}_p \subset \mathcal{G}_p^{\mathrm{loc}}$, $p > 0$
- $0 < \mathsf{r}_\varphi < \infty$ or $\mathsf{r}_\varphi = \infty$

Case $p_1 < p_2$

Lemma

Let $\varphi \in \mathcal{G}^{\mathrm{loc}}$.

(i) *If* $\mathsf{r}_\varphi < \infty$, *then* $\varphi \in \mathcal{G}^{\mathrm{loc}}_{\mathsf{r}_\varphi}$, *i.e.* $\mathsf{r}_\varphi = \max\{p : \ \varphi \in \mathcal{G}^{\mathrm{loc}}_p\}$.

(ii) $\mathsf{r}_\varphi = \infty$ *if, and only if,* $\varphi = \mathrm{const}$ *on* $(0, 1)$.

Examples

- For $0 < u, v < \infty$,

$$\varphi_{u,v}(t) = \begin{cases} t^{d/u} & \text{if} \quad t \leq 1 \\ t^{d/v} & \text{if} \quad t > 1 \end{cases}$$

  belongs to $\mathcal{G}_p$ for $p = \min(u, v)$ and $\mathsf{r}_{\varphi_{u,v}} = u$.

- Let

$$\varphi(t) = \begin{cases} \frac{1}{\log 2} \log(1 + t), & 0 < t < 1, \\ t, & t \geq 1. \end{cases}$$

  Then $\varphi \in \mathcal{G}_d$ and $\mathsf{r}_\varphi = d$.

Embedding $\mathcal{E}^{s_1}_{\varphi_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^{s_2}_{\varphi_2,p_2,q_2}(\mathbb{R}^d)$: case $p_1 < p_2$

Theorem (Haroske, Liu, M., Skrzypczak 2025)

Let $0 < p_1 < p_2 < \infty$, $s_i \in \mathbb{R}$, $0 < q_i \le \infty$, $\varphi_i \in \mathcal{G}_{p_i}$, *for* $i = 1, 2$. *Let* $\varrho = \frac{p_1}{p_2} < 1$.

(i) *It holds*

$$\mathcal{E}^{s_1}_{\varphi_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^{s_2}_{\varphi_2,p_2,q_2}(\mathbb{R}^d) \qquad \boxed{\star}$$

if

$$\sup_{\nu \in \mathbb{N}_0} \frac{\varphi_2(2^\nu)}{\varphi_1(2^\nu)^\varrho} < \infty \qquad \boxed{1}$$

and

$$\left\{ 2^{j(s_2 - s_1 + \frac{d}{\mathfrak{r}_{\varphi_1}}(1-\varrho))} \alpha_j \right\}_{j \in \mathbb{N}} \in \ell_\infty. \qquad \boxed{5}$$

(ii) *If* $\boxed{\star}$ *holds, then* $\boxed{1}$ *is satisfied and*

$$\left\{ 2^{j(s_2 - s_1)} \alpha_j \varphi_1(2^{-j})^{\varrho - 1} \right\}_{j \in \mathbb{N}} \in \ell_\infty. \qquad \boxed{6}$$

Embedding $\mathcal{E}^{s_1}_{\varphi_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^{s_2}_{\varphi_2,p_2,q_2}(\mathbb{R}^d)$: case $p_1 < p_2$

Theorem (cont.)

(iii) *Assume that*

$$\lim_{t\to 0} \varphi_1(t) t^{-\frac{d}{r_{\varphi_1}}} \le c < \infty, \qquad \boxed{7}$$

then $\boxed{5}$ *and* $\boxed{6}$ *are equivalent, that is, the embedding* $\boxed{\star}$ *holds if, and only if,*

$$\sup_{\nu\in\mathbb{N}_0} \frac{\varphi_2(2^\nu)}{\varphi_1(2^\nu)^\varrho} < \infty \qquad \boxed{1}$$

and

$$\left\{ 2^{j(s_2 - s_1 + \frac{d}{r_{\varphi_1}}(1-\varrho))} \alpha_j \right\}_{j\in\mathbb{N}} \in \ell_\infty. \qquad \boxed{5}$$

(iv) *The embedding* $\boxed{\star}$ *is never compact.*

Embedding $\mathcal{E}^{s_1}_{\varphi_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^{s_2}_{\varphi_2,p_2,q_2}(\mathbb{R}^d)$: Case $p_1 < p_2$

The following examples satisfy condition **7** :

- For $0 < u, v < \infty$,
$$\varphi_{u,v}(t) = \begin{cases} t^{d/u} & \text{if} \quad t \le 1 \\ t^{d/v} & \text{if} \quad t > 1 \end{cases}$$
  which belongs to $\mathcal{G}_p$ for $p = \min(u,v)$ and such that $\mathrm{r}_{\varphi_{u,v}} = u$.
- Belonging to $\mathcal{G}_d$ and with $\mathrm{r}_\varphi = d$:
$$\varphi(t) = \begin{cases} \frac{1}{\log 2} \log(1+t), & 0 < t < 1, \\ t, & t \ge 1. \end{cases}$$

Example

Let $\varphi_i(t) = t^{\frac{d}{u_i}}$, $0 < p_i \le u_i < \infty$, $i = 1, 2$. Then $\mathcal{E}^{s_i}_{\varphi_i,p_i,q_i}(\mathbb{R}^d) = \mathcal{E}^{s_i}_{u_i,p_i,q_i}(\mathbb{R}^d)$.

Case $p_1 < p_2$ $\curvearrowright$ $\mathsf{r}_{\varphi_1} = u_1 \curvearrowright$ $\boxed{7}$ ✓

- $\boxed{1}$ $\Leftrightarrow$ $\dfrac{u_1}{u_2} \le \varrho = \dfrac{p_1}{p_2}$ $\Leftrightarrow$ $\dfrac{u_1}{u_2} \le \dfrac{p_1}{p_2}$

$$\rightsquigarrow \quad \alpha_j = \sup_{\nu \le j} \frac{\varphi_2(2^{-\nu})}{\varphi_1(2^{-\nu})^{\varrho}} = \sup_{\nu \le j} 2^{\nu d\left(\frac{\varrho}{u_1} - \frac{1}{u_2}\right)} = 2^{jd\left(\frac{\varrho}{u_1} - \frac{1}{u_2}\right)}$$

- $\boxed{5}$ $\Leftrightarrow$ $\left\{2^{j(s_2 - s_1 + \frac{d}{\mathsf{r}_{\varphi_1}}(1-\varrho))}\alpha_j\right\}_{j \in \mathbb{N}} \in \ell_\infty$ $\Leftrightarrow$ $\left\{2^{-j(s_1 - s_2 - \frac{d}{u_1} + \frac{d}{u_2})}\right\}_{j \in \mathbb{N}} \in \ell_\infty$

$$\Leftrightarrow \quad \frac{s_1 - s_2}{d} \ge \frac{1}{u_1} - \frac{1}{u_2}$$

$\varrho < 1 \curvearrowright$ $\dfrac{s_1 - s_2}{d} > \dfrac{1}{u_1} - \dfrac{1}{u_2}$ or $\dfrac{s_1 - s_2}{d} = \dfrac{1}{u_1} - \dfrac{1}{u_2}$ $\wedge$ $u_1 < u_2$

Haroske, Skrzypczak 2014

Proof of (i): suff. condition for the embedding

Proposition

Let $-\infty < s_1 < s_0 < \infty$, $0 < \theta < 1$, $0 < q_i \leq \infty$, $\varphi_i \in \mathcal{G}_{p_i}$, *for* $i = 0, 1$.

(i) *Let* $0 < p_0, p_1 < \infty$ *and put*

$$\frac{1}{p_2} := \frac{1-\theta}{p_1} + \frac{\theta}{p_0}, \quad s_2 := s_1(1-\theta) + s_0\theta \quad \textit{and} \quad \varphi_2 := \varphi_1^{1-\theta}\varphi_0^{\theta}.$$

Then for any $f \in \mathcal{S}'(\mathbb{R}^d)$ *and any* $0 < q < \infty$,

$$\|f|F^{s_2,\varphi_2}_{p_2,q}(\mathbb{R}^d)\| \leq c\|f|F^{s_1,\varphi_1}_{p_1,q_1}(\mathbb{R}^d)\|^{1-\theta}\|f|F^{s_0,\varphi_0}_{p_0,q_0}(\mathbb{R}^d)\|^{\theta}.$$

Sickel 2013

Proof of (i): suff. condition for the embedding

Proposition

Let $-\infty < s_1 < s_0 < \infty$, $0 < \theta < 1$, $0 < q_i \le \infty$, $\varphi_i \in \mathcal{G}_{p_i}$, *for* $i = 0, 1$.

(ii) *Let* $0 < p_1 < \infty$ *and put*

$$\frac{1}{p_2} := \frac{1-\theta}{p_1}, \quad s_2 := (1-\theta)s_1 + \theta s_0, \quad \varphi_2 = \varphi_1^{1-\theta}\varphi_0^{\theta}.$$

Then for any $f \in \mathcal{S}'(\mathbb{R}^d)$ *and* $0 < q < \infty$,

$$\|f|F^{s_2,\varphi_2}_{p_2,q}(\mathbb{R}^d)\| \le c\|f|F^{s_1,\varphi_1}_{p_1,q_1}(\mathbb{R}^d)\|^{1-\theta}\|f|F^{s_0,\varphi_0}_{\infty,\infty}(\mathbb{R}^d)\|^{\theta}.$$

Proof of (i): suff. condition for the embedding

Proposition

Let $-\infty < s_1 < s_0 < \infty$, $0 < \theta < 1$, $0 < q_i \leq \infty$, $\varphi_i \in \mathcal{G}_{p_i}$, *for* $i = 0, 1$.

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$$\frac{1}{p_2} := \frac{1-\theta}{p_1}, \quad s_2 := (1-\theta)s_1 + \theta s_0, \quad \varphi_2 = \varphi_1^{1-\theta}\varphi_0^{\theta}.$$

Then for any $f \in \mathcal{S}'(\mathbb{R}^d)$ *and* $0 < q < \infty$,

$$\|f|F^{s_2,\varphi_2}_{p_2,q}(\mathbb{R}^d)\| \leq c\|f|F^{s_1,\varphi_1}_{p_1,q_1}(\mathbb{R}^d)\|^{1-\theta}\|f|F^{s_0,\varphi_0}_{\infty,\infty}(\mathbb{R}^d)\|^{\theta}.$$

Lemma

Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$ *and* $\varphi \in \mathcal{G}_p$ *with* $\mathsf{r}_\varphi < \infty$. *Then*

$$\mathcal{E}^s_{\varphi,p,q}(\mathbb{R}^d) = F^{s,\varphi}_{p,q}(\mathbb{R}^d) \hookrightarrow F^{s-\frac{d}{\mathsf{r}_\varphi}}_{\infty,\infty}(\mathbb{R}^d).$$

On the proof of (i)

Assume the following conditions hold

$$\underbrace{\sup_{\nu\in\mathbb{N}_0} \frac{\varphi_2(2^\nu)}{\varphi_1(2^\nu)^\varrho} < \infty}_{\boxed{1}} \quad \text{and} \quad \underbrace{\left\{2^{j(s_2-s_1+\frac{d}{\mathsf{r}_{\varphi_1}}(1-\varrho))}\alpha_j\right\}_{j\in\mathbb{N}} \in \ell_\infty}_{\boxed{5}} .$$

Clearly $\boxed{5}$ implies

$$s_2 \le s_1 - \frac{d}{\mathsf{r}_{\varphi_1}}(1-\varrho) \quad \text{and} \quad \sup_{j\in\mathbb{N}} 2^{j(s_2-s_1+\frac{d}{\mathsf{r}_{\varphi_1}}(1-\varrho))} \frac{\varphi_2(2^{-j})}{\varphi_1(2^{-j})^\varrho} < \infty.$$

First we assume that $\varphi_2 = \varphi_1^\varrho$. Then $\mathsf{r}_{\varphi_1} = \varrho \mathsf{r}_{\varphi_2}$ and taking $1-\theta := \varrho < 1$, $s_0 := s_1 - \frac{d}{\mathsf{r}_{\varphi_1}}$, it holds

$$\frac{1}{p_2} := \frac{1-\theta}{p_1}, \qquad (1-\theta)s_1 + \theta s_0 = s_1 - \frac{d}{\mathsf{r}_{\varphi_1}}(1-\varrho)$$

The proposition and the Lemma yields,

$$\|f|F^{s_2,\varphi_2}_{p_2,q_2}(\mathbb{R}^d)\| \le c\|f|F^{s_1,\varphi_1}_{p_1,q_1}(\mathbb{R}^d)\|^{1-\theta}\|f|F^{s_0}_{\infty,\infty}(\mathbb{R}^d)\|^\theta \le c'\|f|F^{s_1,\varphi_1}_{p_1,q_1}(\mathbb{R}^d)\|$$

Proof of (i): suff. condition for the embedding

In the general case we take $s_3 = s_1 - \frac{d}{\mathsf{r}_{\varphi_1}}(1-\varrho)$ and use the following factorisation

$$F^{s_1,\varphi_1}_{p_1,q_1}(\mathbb{R}^d) \hookrightarrow F^{s_3,\varphi_1^{\varrho}}_{p_2,q_2}(\mathbb{R}^d) \hookrightarrow F^{s_2,\varphi_2}_{p_2,q_2}(\mathbb{R}^d)$$

Embedding $\mathcal{E}^{s_1}_{\varphi_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^{s_2}_{\varphi_2,p_2,q_2}(\mathbb{R}^d)$

Corollary (Haroske, Liu, M., Skrzypczak 2025)

Let $0 < p_i < \infty$, $s_i \in \mathbb{R}$, $0 < q_i \le \infty$, $\varphi_i \in \mathcal{G}_{p_i}$, *for* $i = 1, 2$. *Let* $\varrho = \min(1, \frac{p_1}{p_2})$ *and* $\alpha_j = \sup\limits_{\nu \le j} \frac{\varphi_2(2^{-\nu})}{\varphi_1(2^{-\nu})^{\varrho}}$, $j \in \mathbb{N}_0$. *Then*

$$\mathcal{E}^{s_1}_{\varphi_1,p_1,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^{s_2}_{\varphi_2,p_2,q_2}(\mathbb{R}^d)$$

if, and only if,

$$\sup_{\nu \in \mathbb{N}_0} \frac{\varphi_2(2^{\nu})}{\varphi_1(2^{\nu})^{\varrho}} < \infty,$$

and

$$\left\{ 2^{j(s_2 - s_1 + \frac{d}{\mathfrak{r}_{\varphi_1}}(1-\varrho))} \alpha_j \right\}_{j \in \mathbb{N}} \in \ell_\infty,$$

where in case of $p_1 < p_2$ *we assume, in addition,*

$$\lim_{t \to 0} \varphi_1(t) t^{-\frac{d}{\mathfrak{r}_{\varphi_1}}} \le c < \infty,$$

and in the case $p_1 \ge p_2$ *we assume* $q_1 \le q_2$ *when* $s_1 = s_2$.

Corollary (Haroske, Liu, M., Skrzypczak 2025)

Let $0 < p_i < \infty$, $s_i \in \mathbb{R}$, $0 < q_i \leq \infty$ *and* $\varphi_i \in \mathcal{G}_{p_i}$, $i = 1, 2$. *Let* $\alpha_j = \sup_{\nu \leq j} \frac{\varphi_2(2^{-\nu})}{\varphi_1(2^{-\nu})^{\varrho}}$, $j \in \mathbb{N}_0$, *and* $\varrho = \min\left(1, \frac{p_1}{p_2}\right)$.

(i) *It holds*

$$B^{s_1,\varphi_1}_{p_1,q_1}(\mathbb{R}^d) \hookrightarrow B^{s_2,\varphi_2}_{p_2,q_2}(\mathbb{R}^d) \qquad \boxed{\star}$$

if

$$\sup_{\nu \in \mathbb{N}_0} \frac{\varphi_2(2^{\nu})}{\varphi_1(2^{\nu})^{\varrho}} < \infty \qquad \boxed{1}$$

and

$$\left\{2^{j(s_2-s_1)}\alpha_j \varphi_1(2^{-j})^{\varrho-1}\right\}_{j \in \mathbb{N}} \in \ell_{q_2}.$$

(ii) *If* $\boxed{\star}$ *holds, then* $\boxed{1}$ *is satisfied and*

$$\left\{2^{j(s_2-s_1)}\alpha_j \varphi_1(2^{-j})^{\varrho-1}\right\}_{j \in \mathbb{N}} \in \ell_{\infty}.$$

(iii) *The embedding* $\boxed{\star}$ *is never compact.*

On bounded smooth domains?

Theorem (Haroske, M., Skrzypczak 2026)

Let $s_i \in \mathbb{R}$, $0 < p_i < \infty$, $0 < q_i \leq \infty$, *and* $\varphi_i \in \mathcal{G}_{p_i}$, *for* $i = 1, 2$. *Let* $\varrho = \min(1, \frac{p_1}{p_2})$ *and* $\alpha_j = \sup_{0 \leq \nu \leq j} \frac{\varphi_2(2^{-\nu})}{\varphi_1(2^{-\nu})^{\varrho}}$, $j \in \mathbb{N}_0$. *It holds*

$$\mathcal{N}^{s_1}_{u_1,p_1,q_1}(\Omega) \hookrightarrow \mathcal{N}^{s_2}_{u_2,p_2,q_2}(\Omega) \tag{1}$$

if, and only if,

$$\left\{2^{j(s_2-s_1)}\alpha_j \varphi_1(2^{-j})^{\varrho-1}\right\}_{j\in\mathbb{N}_0} \in \ell_{q^*} \qquad \text{where} \quad \frac{1}{q^*} = \left(\frac{1}{q_2} - \frac{1}{q_1}\right)_+ . \tag{2}$$

Moreover, (1) *is compact if, and only if,* (2) *holds with* ℓ_∞ *replaced by* c_0 *if* $q^* = \infty$, *which means that*

$$2^{j(s_2-s_1)}\alpha_j \frac{\varphi_1(2^{-j})^{\varrho}}{\varphi_1(2^{-j})} \to 0 \qquad \text{if} \qquad q_1 \leq q_2.$$

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Thank you very much for your attention !