

Duality for interpolation spaces defined via
slowly varying functions:
the case $0 < q < 1$

Joint paper with Grover and Signes

Septiembre, 2025

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Outline

- 1 A brief reminder of some basic concepts
- 2 Main result: the details
- 3 Examples

$\bar{A} = (A_0, A_1)$ stands for a Banach interpolation couple.

Definition

For $0 < \theta < 1$ and $0 < q \leq \infty$

$$\|a\|_{\theta, q}^K = \left\| t^{-\theta} K(t, a) \right\|_{L_q((0, \infty), \frac{dt}{t})}$$

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For $\theta = 0, 1$ and $q = \infty$ the definition makes sense

$$(A_0, A_1)_{0, \infty}^K = A_0^{\sim}$$

$$(A_0, A_1)_{1, \infty}^K = A_1^{\sim}$$

For any other $0 < q < \infty$,

$$(A_0, A_1)_{0,\infty}^K = (A_0, A_1)_{1,\infty}^K = \{0\}$$

However there are situations in which interpolation with parameters $\theta = 0, 1$ is desirable.

Example

Consider the couple (L_1, L_∞) over a finite measure space.

$$\|f\|_{L \log L} = \int_0^1 f^*(t) \log\left(\frac{1}{t}\right) dt = \int_0^1 \frac{K(t, f)}{t} dt$$

$$\|f\|_{L_{exp}} = \sup_{0 < t < 1} \frac{K(t, f)}{t\ell(t)} = \|t^{-1}(\ell(t))^{-1}K(t, f)\|_{L_\infty(0,1)}$$

$$\begin{aligned} \|f\|_{L_{exp}} &\sim \|t^{-1}(\ell(t))^{-1}K(t, f)\|_{L_\infty(0,\infty)} \\ &= \|f\|_{1, L_\infty, \ell^{-1}}^K \end{aligned}$$

Definition

A measurable $b : (0, \infty) \rightarrow (0, \infty)$ is slowly varying if $\forall \varepsilon > 0$

$t \rightsquigarrow t^\varepsilon b(t)$ increases

$t \rightsquigarrow t^{-\varepsilon} b(t)$ decreases

Definition

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Example

Let $(\alpha_0, \alpha_1) \in \mathbb{R}^2$.

$$\ell^{(\alpha_0, \alpha_1)}(t) = \begin{cases} (1 + |\log(t)|)^{\alpha_0} & t \in (0, 1] \\ (1 + |\log(t)|)^{\alpha_1} & t \in (1, \infty) \end{cases}$$

Definition

Let $0 \leq \theta \leq 1$, $0 < q \leq \infty$ & b slowly varying.

The space

$$(A_0, A_1)_{\theta, q, b}^K$$

is defined by the norm

$$\|f\|_{\theta, q, b}^K = \left\| t^{-\theta} b(t) K(t, f; A_0, A_1) \right\|_{L_q\left(\frac{dt}{t}\right)}$$

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$(A_0, A_1)_{\theta, q, b}^K$ is an interpolation space for (A_0, A_1) if

- (i) $0 < \theta < 1$,
- (ii) $\theta = 0$ and $\|b(t)\|_{L_q((1, \infty), \frac{dt}{t})} < \infty$ or
- (iii) $\theta = 1$, $\|b(t)\|_{L_q((0, 1), \frac{dt}{t})} < \infty$.

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In the case $b = \ell^{(\alpha_0, \alpha_1)}$ we talk of logarithmic interpolation.

The J spaces

Let (A_0, A_1) be an interpolation couple and $a \in A_0 \cap A_1$

The J -functional

$$J(t, a; A_0, A_1) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}$$

Definition

$f \in (A_0, A_1)_{\theta, q, b}^J$ iff

$$f = \sum_{k \in \mathbb{Z}} u_k \quad (\text{convergence in } A_0 + A_1) \quad (1.1)$$

where the sequence $(u_k)_{k \in \mathbb{Z}}$ is in $A_0 \cap A_1$ and

$$\| (2^{-\theta k} b(2^k) J(2^k, u_k)) \|_{\ell_q(\mathbb{Z})} < \infty. \quad (1.2)$$

Definition

$$\|f\|_{\theta,q,b}^J = \inf \left\{ \left(\sum_{k=-\infty}^{k=\infty} (2^{-\theta k} b(2^k) J(2^k, u_k))^q \right)^{1/q} \right\}.$$

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We will assume that b satisfies the condition

$$\sup_{m \in \mathbb{Z}} \left\{ \frac{\min\{1, 2^m\}}{b(2^m)} \right\} < \infty \quad (1.1)$$

which guarantees that $(A_0, A_1)_{1,q,b}^J$ is intermediate for the couple (A_0, A_1) .

Main result

$$\left((A_0, A_1)_{1,q,b}^K \right)' = (A'_0, A'_1)_{1,\infty,1/\bar{a}}^J.$$

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We need to work with regular couples for (A'_0, A'_1) to make sense

Definition

$$A_0 \cap A_1 \leftrightarrow A \leftrightarrow A_0 + A_1.$$

$$A^\circ = \overline{A_0 \cap A_1}^A$$

(A_0, A_1) is regular if $A_0^\circ = A_0$ and $A_1^\circ = A_1$.

Strategy to identify $\left[(A_0, A_1)_{1,q;b}^K \right]'$

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b)

$$\begin{aligned} \left[(A_0, A_1)_{1,q;b}^K \right]' &= \left[\left((A_0, A_1)_{1,q;b}^K \right)^\sharp \right]' && \text{Peetre's result} \\ &= \left[\left((A_0^\sim, A_1^\sim)_{1,q;B}^J \right)^\sharp \right]' && J\text{-}K \text{ equivalence} \\ &= \left[\left((A_0^\sim, A_1^\sim)_{1,1;B}^J \right)^\sharp \right]' && \text{Compute } \left((A_0, A_1)_{1,q;B}^J \right)^\sharp \\ &= \left[\left((A_0, A_1)_{1,1;a}^K \right)^\sharp \right]' && J\text{-}K \text{ equivalence} \\ &= (A_0', A_1')_{1,\infty;\frac{1}{a}}^J && \text{Duality result} \end{aligned}$$

Peetre's duality result

Let $(E, \|\cdot\|)$ be a quasi-Banach space.

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$$\|x\|^{\sharp} = \inf \left\{ \sum_{k=1}^n \|x_k\|_E; \text{ where } x = \sum_{k=1}^n x_k \right\}.$$

Definition

Put $N = \{x \in E, \text{ s.t. } \|x\|^{\sharp} = 0\}$.

E^{\sharp} is the completion of the quotient space E/N with the quotient norm induced by $\|\cdot\|^{\sharp}$.

Peetre's duality result

Theorem (Peetre, 1974)

$$(E^\#)' = E'. \quad (2.1)$$

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Example

For $0 < p < 1$

$$E = L_p \implies E^\sharp = \{0\} \ \& \ E^* = L_p^* = \{0\} \quad \text{Haaker 1970}$$

$$E = \ell_p, \implies E^\sharp = \ell_1 \ \& \ E^* = \ell_p^* = \ell_1^* = \ell_\infty \quad \text{Day 1940}$$

Peetre's duality result

Theorem (Peetre, 1974)

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A result for J -spaces

Lemma (Cobos-Besoy)

Let (A_0, A_1) be a quasi-Banach couple, $0 < q \leq 1$ and $b \in SV(0, \infty)$. Then

$$\left((A_0, A_1)_{1,q,b}^J \right)^\# = (A_0, A_1)_{1,1,b}^J.$$

The Gagliardo completion

Definition (Gagliardo)

$X \subset Y$ normed spaces.

The relative completion of X in Y is the space $X^{c,Y}$ whose unit ball is

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Notation

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$$A_0^\sim = A_0^{c, A_0 + A_1} \quad A_1^\sim = A_1^{c, A_0 + A_1}$$

$$\|a\|_{A_0^\sim} = \lim_{t \rightarrow \infty} K(t, f) \quad \|a\|_{A_1^\sim} = \lim_{t \rightarrow 0} \frac{K(t, f)}{t}$$

A_0^\sim and A_1^\sim are the Gagliardo completions of A_0 and A_1 .

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(A_0, A_1) is a Gagliardo couple (mutually closed) if

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Mutually close couples have a role in J - K equivalence theorems

Equivalence Theorems

$$A_0 \cap A_1 \xrightarrow[\text{J funct}]{\rightsquigarrow} (A_0, A_1)_{\theta, q, b}^J$$

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K funct

The question whether these families coincide has been studied recently.

Equivalence Theorems

We look for the following result

Theorem

Let $0 < q \leq 1$ and let $b \in SV(0, \infty)$ such that

$$\|b(t)\|_{L_q((0,1), \frac{dt}{t})} < \infty, \quad \|b(t)\|_{L_q((1,\infty), \frac{dt}{t})} = \infty.$$

Then

$$(A_0, A_1)_{1,q,b}^K = (A_0^\sim, A_1^\sim)_{1,q,B}^J$$

$$B(t) = \left(\int_0^t b(s)^q \frac{ds}{s} \right)^{1/q} = \|s^{-1/q} b(s)\|_{q,(0,t)}.$$

$$(A_0^{\sim}, A_1^{\sim})_{1,q,B}^J \hookrightarrow (A_0, A_1)_{1,q,b}^K$$

$$(A_0, A_1)_{1,q,b}^K \hookrightarrow (A_0^\sim, A_1^\sim)_{1,q,B}^J$$

For every $a \in (A_0, A_1)_{1,q,b}^K$ we must find

$$a = \sum_{m \in \mathbb{Z}} u_m$$
$$\left(\sum_{m \in \mathbb{Z}} (2^{-m} b(2^m) J(2^m, u_m))^q \right)^{1/q} < \infty$$

Theorem (Nilsson, 1982)

Let \bar{A} be a mutually closed pair of p -normed quasi-Banach spaces. Let $a \in \Sigma(\bar{A})^\circ$ and $0 < q < \infty$. Then, there exist a sequence $(a_\nu)_{\nu \in \mathbb{Z}} \in A_0 \cap A_1$ such that

$$a = \sum_{\nu \in \mathbb{Z}} a_\nu$$

$$\left(\sum_{\nu} \left(\min\left\{1, \frac{t}{2^\nu}\right\} J(2^\nu, a_\nu) \right)^q \right)^{\frac{1}{q}} \leq c_{p,q} K(t, a), \quad t > 0.$$

Here $c_{p,q}$ depends only on p and q .

$$(A_0, A_1)_{1,q,b}^K \hookrightarrow \Sigma(\bar{A})^\circ?$$

Proposition

$$\|b\|_{L_q((1,\infty), \frac{ds}{s})} = \infty \implies (A_0, A_1)_{1,q,b}^K \subset (A_0 + A_1)^\circ$$

This has force the hypothesis $\|b\|_{L_q((1,\infty), \frac{dt}{t})} = \infty$ in many results.

$$\|a\|_{(A_0^\sim, A_1^\sim)_{1,q,B}^{J,\diamond}} \lesssim \left(\sum_{k \in \mathbb{Z}} (2^{-k} B(2^k) J(2^k, u_k))^q \right)^{1/q}$$

.

$$\begin{aligned}
\|a\|_{(A_0^\sim, A_1^\sim)_{1,q,B}^{J,\diamond}} &\lesssim \left(\sum_{k \in \mathbb{Z}} (2^{-k} B(2^k) J(2^k, u_k))^q \right)^{1/q} \\
&= \left(\sum_{k \in \mathbb{Z}} \left(2^{-k} \int_0^{2^k} b(s)^q \frac{ds}{s} J(2^k, u_k) \right)^q \right)^{1/q} \\
&\sim \left(\sum_{k \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} (\min\{1, 2^{m-k}\} 2^{-m} b(2^m))^q J(2^k, u_k)^q \right)^{1/q}
\end{aligned}$$

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&\sim \left(\sum_{k \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} (\min\{1, 2^{m-k}\} 2^{-m} b(2^m))^q J(2^k, u_k)^q \right)^{1/q} \\
&= \left(\sum_{m \in \mathbb{Z}} 2^{-mq} b(2^m)^q \sum_{k \in \mathbb{Z}} (\min\{1, 2^{m-k}\} J(2^k, u_k))^q \right)^{1/q} \\
&\leq \left(\sum_{m \in \mathbb{Z}} 2^{-mq} b(2^m)^q K(2^m, a)^q \right)^{1/q} \\
&= \|a\|_{(A_0^\sim, A_1^\sim)_{1,q,b}^{K,\diamond}}.
\end{aligned}$$

Theorem

Let $0 < q \leq 1$ and let $b \in SV(0, \infty)$ such that

$$\|b(t)\|_{L_q((0,1), \frac{dt}{t})} < \infty \quad \|b(t)\|_{L_q((1,\infty), \frac{dt}{t})} = \infty.$$

Then

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Remark

In case $\|b(t)\|_{L_q((1,\infty), \frac{dt}{t})} < \infty$, Cobos and Segurado proved by means of a counterexample that, **the space $(A_0, A_1)_{1,q;b}^K$ does not have a J -representation.**

They used the non-regular couple (ℓ_1, ℓ_∞) to build their counterexample.

Remark

In case $\|b(t)\|_{L_q((1,\infty), \frac{dt}{t})} < \infty$, Cobos and Segurado proved by means of a counterexample that, in general, **the space $(A_0, A_1)_{1,q;b}^K$ does not have a J -representation.**

They used the non-regular couple (ℓ_1, ℓ_∞) to build their counterexample.

Reason for this obstruction If $\|b(t)\|_{L_q((1,\infty), \frac{dt}{t})} < \infty$ it cannot be assure that

$$(A_0, A_1)_{1,q;b}^K \subset (A_0^\sim + A_1^\sim)^\circ.$$

Therefore, Nilsson's theorem cannot be applied.

Remark

However, if as an additional hypothesis we ask the couple (A_0, A_1) to be regular, we have the following facts:

- 1 (A_0^\sim, A_1^\sim) is a regular couple too.
- 2 $(A_0^\sim, A_1^\sim)_{1,q,b}^K \subset A_0 + A_1 = (A_0 + A_1)^\circ$.

In these conditions, we can apply Nilsson's theorem and repeat the arguments of Thm. 2.8 to obtain the following result.

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In these conditions, we can apply Nilsson's theorem and repeat the arguments of Thm. 2.8 to obtain the following result.

Theorem

Let (A_0, A_1) be a regular Banach couple, $0 < q \leq 1$ and let $b \in SV(0, \infty)$ such that $\|t^{-1/q}b(t)\|_{q,(0,1)} < \infty$. Then, we have with equivalence of quasi-norms

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A different insight

Theorem

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$$(A_0, A_1)_{1,q, \left(t \frac{d}{dt} B^q(t)\right)^{1/q}}^K = (A_0^\sim, A_1^\sim)_{1,q,B}^J$$

Lemma

Let (A_0, A_1) be a regular couple and let $b \in SV(0, \infty)$ a slowly varying function.

$$(\overline{A}_{1,1,b}^K)' = \overline{A}'_{1,\infty,1/\bar{b}}^J.$$

Here, $\bar{b}(t) = b(\frac{1}{t})$ for $t \in (0, \infty)$.

Duality results

Main result

Theorem

Let (A_0, A_1) be a regular couple, $0 < q \leq \infty$ and $b \in SV(0, \infty)$ satisfying that, for $t > 0$, $\int_0^t b^q(s) \frac{ds}{s} < \infty$. Then,

$$\left((A_0, A_1)_{1,q,b}^K \right)' = (A'_0, A'_1)_{1,\infty,1/\bar{a}}^J.$$

$a(t) = t \frac{d}{dt} \left(\int_0^t b(s)^q \frac{ds}{s} \right)^{1/q}$, $t > 0$, is in $SV(0, \infty)$.

Idea of the proof.

$$\begin{aligned} \left((A_0, A_1)_{1,q,b}^K \right)^\# &= \left((A_0^\sim, A_1^\sim)_{1,q,B}^J \right)^\# \\ &= (A_0^\sim, A_1^\sim)_{1,1,B}^J \\ &= (A_0^\sim, A_1^\sim)_{1,1,a}^K \\ &= (A_0, A_1)_{1,1,a}^K. \end{aligned}$$

Idea of the proof.

$$\begin{aligned} \left((A_0, A_1)_{1,q,b}^K \right)^\# &= \left((A_0^\sim, A_1^\sim)_{1,q,B}^J \right)^\# \\ &= (A_0^\sim, A_1^\sim)_{1,1,B}^J \\ &= (A_0^\sim, A_1^\sim)_{1,1,a}^K \\ &= (A_0, A_1)_{1,1,a}^K. \end{aligned}$$

Now,

$$\begin{aligned} \left[(A_0, A_1)_{1,q,b}^K \right]' &= \left[\left((A_0, A_1)_{1,q,b}^K \right)^\# \right]' \\ &= \left[(A_0, A_1)_{1,1,a}^K \right]' \\ &= (A_0', A_1')_{1,\infty,1/\bar{a}}^J. \end{aligned}$$

$\left[(A_0, A_1)_{1,q,\ell^{(\alpha_0,\alpha_1)}}^K \right]'$ Theorem 4.3, Cobos-Besoy

The case $\alpha_\infty q + 1 > 0$.

$$a(t) = t \frac{d}{dt} \left(\int_0^t \ell^{(\alpha_0,\alpha_\infty)q}(s) \frac{ds}{s} \right)^{1/q} \sim \begin{cases} \ell^{\alpha_0+1/q-1}(t) & \text{in } (0, 1) \\ \ell^{\alpha_\infty+1/q-1}(t) & \text{in } (1, \infty). \end{cases}$$

Then,

$$\begin{aligned} \left[(A_0, A_1)_{1,q,\ell^{(\alpha_0,\alpha_1)}}^K \right]' &= (A'_0, A'_1)_{1,\infty,\ell^{(-\alpha_\infty-1/q+1,\alpha_0-1/q+1)}}^J \\ &= (A'_0, A'_1)_{1,\infty,\ell^{(-\alpha_\infty-1/q,\alpha_0-1/q)}}^K \quad [4, \text{Thm. 3.5}]. \end{aligned}$$

This recovers the case i) of [1, Thm. 4.2].

$\left[(A_0, A_1)_{1,q,\ell^{(\alpha_0,\alpha_1)}}^K \right]'$ Theorem 4.3, Cobos-Besoy

The case $\alpha_\infty + 1/q = 0$

In this case $b(t) = \ell^{(\alpha_0, -1/q)}(t)$. Then, after coefficients adjustment

$$B(t) = \left(\int_0^t b^q(s) \frac{ds}{s} \right)^{1/q} \sim \begin{cases} \ell^{\alpha_0+1/q}(t) & \text{if } t \in (0, 1) \\ \ell \ell^{1/q}(t) & \text{if } t \in (1, \infty). \end{cases}$$

$$a(t) = t \frac{d}{dt} B(t) \sim \begin{cases} \ell^{\alpha_0+1/q-1}(t) & \text{if } t \in (0, 1) \\ \ell \ell^{1/q-1}(t) \ell(t)^{-1} & \text{if } t \in (1, \infty). \end{cases}$$

$$\begin{aligned} \left[(A_0, A_1)_{1,q,b}^K \right]' &= (A'_0, A'_1)_{1,\infty,\ell^{(1,1-\alpha_0-1/q)} \ell \ell^{(1-1/q,0)}}^J \\ &= (A'_0, A'_1)_{1,\infty,\ell^{(0,-\alpha_0-1/q)} \ell \ell^{(-1/q,0)}}^K \end{aligned}$$

This is the case ii) of [1, Thm. 4.2].

$\left[(A_0, A_1)_{1, q, \ell^{(\alpha_0, \alpha_1)}}^K \right]'$ Theorem 4.3, Cobos-Besoy

The case $\alpha_\infty q + 1 < 0$.

Finally, for the case $\alpha_\infty + 1/q < 0$ we have the equivalence

$$B(t) \sim \begin{cases} \ell^{\alpha_0 + 1/q}(t) & \text{if } t \in (0, 1) \\ 1 & \text{if } t \in (1, \infty) \end{cases}.$$

$\left[(A_0, A_1)_{1, q, \ell^{(\alpha_0, \alpha_1)}}^K \right]'$ Theorem 4.3, Cobos-Besoy

The case $\alpha_\infty q + 1 < 0$.

Hence,

$$a(t) = t \frac{d}{dt} B(t) \sim \begin{cases} \ell^{\alpha_0 + 1/q - 1}(t) & \text{if } t \in (0, 1) \\ 0 & \text{if } t \in (1, \infty) \end{cases}.$$

$\left[(A_0, A_1)_{1, q, \ell^{(\alpha_0, \alpha_1)}}^K \right]'$ Theorem 4.3, Cobos-Besoy

The case $\alpha_\infty q + 1 < 0$.

Thus, using that $K(t, f; A_0 + A_1, A_1) = K(t, a; A_0, A_1)$ for $t \in (0, 1]$

$$\begin{aligned} (A_0, A_1)_{1, 1, a}^K &= (A_0, A_1)_{1, L_1(0,1), a} \\ &= (A_0 + A_1, A_1)_{1, L_1(0,1), \ell^{\alpha_0 + 1/q - 1}} \\ &= (A_0 + A_1, A_1)_{1, L_1, \ell^{(\alpha_0 + 1/q - 1, \alpha)}} \end{aligned}$$

for any $\alpha \in \mathbb{R}$, $\alpha \neq 0$.

$\left[(A_0, A_1)_{1,q,\ell(\alpha_0,\alpha_1)}^K \right]'$ Theorem 4.3, Cobos-Besoy

The case $\alpha_\infty q + 1 < 0$.

Therefore, choosing $\alpha + 1 < 0$ to apply [4, Thm. 5.10], we obtain that

$$\begin{aligned} \left[(A_0, A_1)_{1,1,a}^K \right]' &= \left[(A_0 + A_1, A_1)_{1,1,\ell(\alpha_0+1/q-1,\alpha)}^K \right]' \\ &= A'_1 \cap (A'_0 \cap A'_1, A'_1)_{1,L_\infty,(-1,-\alpha_0-1/q)}^K \quad [4, \text{Thm. 5.10}]. \end{aligned}$$

This recovers case iii) of [1, Thm. 4.2].

Thm. 3.1 from Opic and Grover 2025

If we assume the additional hypothesis







$$\|b\|_{L_q((0,\infty), \frac{dt}{t})} = \infty,$$

we can identify the dual of $(A_0, A_1)_{1,q,b}^K$ as a K -space for the couple (A'_0, A'_1) . Actually,

$$\begin{aligned} \left[(A_0, A_1)_{1,q,b}^K \right]' &= \left[(A_0, A_1)_{1,1,a}^K \right]' \\ &= \left[(A_1, A_0)_{0,1,\bar{a}}^K \right]' \\ &= (A'_1, A'_0)_{0,\infty,1/a}^J \quad [6, \text{Thm. 3.1}] \quad (3.1) \\ &= (A'_1, A'_0)_{0,\infty,1/B}^K \\ &= (A'_0, A'_1)_{1,\infty,1/\bar{B}}^K. \end{aligned}$$

This recovers Theorem 3.1 in [6] for the case $q = 1$.

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Thank you