On the Boundedness of Dilation Operators in the Context of Triebel-Lizorkin-Morrey Spaces

Marc Hovemann
(joint work with Markus Weimar)

March 2026

Deutsche Forschungsgemeinschaft

Triebel-Lizorkin-Morrey Spaces

DFG Deutsche Forschungsgemeinschaft

Morrey Spaces $\mathcal{M}_p^u(\mathbb{R}^d)$

Definition (Morrey Spaces $\mathcal{M}_p^u(\mathbb{R}^d)$)

Let $0 < p \le u < \infty$. Then the Morrey space $\mathcal{M}_p^u(\mathbb{R}^d)$ is defined to be the set of all functions $f \in L_p^{loc}(\mathbb{R}^d)$ such that

$$\|f|\mathcal{M}_p^u(\mathbb{R}^d)\| := \sup_B |B|^{\frac{1}{u}-\frac{1}{p}} \left(\int_B |f(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$

Here the supremum is taken over all balls $B \subset \mathbb{R}^d$.

- $L_u(\mathbb{R}^d) = \mathcal{M}_u^u(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{p_1}^u(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{p_2}^u(\mathbb{R}^d), \quad 0 < p_2 \le p_1 \le u < \infty$
- $L_u(\mathbb{R}^d) \subsetneq \mathcal{M}_p^u(\mathbb{R}^d), \quad 0 < p < u < \infty$
- Example: $f(x) = |x|^{-\frac{d}{u}} \cdot e^{\frac{-1}{(1-|x|^2)}} \cdot \chi_{B(0,1)}(x)$. Then $f \in \mathcal{M}_p^u(\mathbb{R}^d)$, but $f \notin L_u(\mathbb{R}^d)$.

DFG Deutsche Forschungsgemeinschaft

Morrey spaces are more than just "generalized Lebesgue spaces"!

For $1 < p < u < \infty$ for the spaces $\mathcal{M}_p^u(\mathbb{R}^d)$ we observe :

- they are not reflexive;
- they do not have $C_0^\infty(\mathbb{R}^d)$ as a dense subspace;
- they are not separable;
- they are not included in $L_1(\mathbb{R}^d) + L_\infty(\mathbb{R}^d)$.

DFG Deutsche Forschungsgemeinschaft

A smooth dyadic decomposition of the unity

Let $\psi \in C_0^\infty(\mathbb{R}^d)$ be a non-negative function such that:

$$\psi(x) = 1 \text{ for } |x| \leq 1 \qquad \text{and} \qquad \psi(x) = 0 \text{ for } |x| \geq \tfrac{3}{2}$$

Now we define:

- $\varphi_0(x) = \psi(x)$
- $\varphi_1(x) = \psi(\frac{x}{2}) - \psi(x)$
- $\varphi_j(x) = \varphi_1(2^{-j+1}x) \qquad \text{for} \qquad j \geq 2\,,\ j \in \mathbb{N}$

Some special properties of these functions are:

- $\varphi_j \in C_0^\infty(\mathbb{R}^d) \qquad \forall\, j \in \mathbb{N}_0$
- $supp(\varphi_j) \subset \{x \in \mathbb{R}^d : 2^{j-1} \leq |x| \leq 3 \cdot 2^{j-1}\} \qquad \text{for} \qquad j \geq 1$
- $\sum_{j=0}^\infty \varphi_j(x) = 1 \qquad \forall\, x \in \mathbb{R}^d$

The Triebel-Lizorkin-Morrey spaces

Definition

Let $0 < p \leq u < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. $(\varphi_k)_{k \in \mathbb{N}_0}$ is a smooth dyadic decomposition of the unity. Then the Triebel-Lizorkin-Morrey space $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ is defined to be the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\| := \Big\| \Big(\sum_{k=0}^{\infty} 2^{ksq} |\mathcal{F}^{-1}[\varphi_k \mathcal{F} f](x)|^q \Big)^{\frac{1}{q}} \Big| \mathcal{M}^u_p(\mathbb{R}^d) \Big\| < \infty.$$

- For $u = p$ we recover the original Triebel-Lizorkin spaces: $\mathcal{E}^s_{p,p,q}(\mathbb{R}^d) = F^s_{p,q}(\mathbb{R}^d)$.
- For $1 < p < \infty$ Lebesgue spaces are Triebel-Lizorkin-Morrey spaces: $L_p(\mathbb{R}^d) = \mathcal{E}^0_{p,p,2}(\mathbb{R}^d)$.
- For $1 < p < \infty$ and $m \in \mathbb{N}$ Sobolev spaces are Triebel-Lizorkin-Morrey spaces: $W^m_p(\mathbb{R}^d) = \mathcal{E}^m_{p,p,2}(\mathbb{R}^d)$.

DFG Deutsche Forschungsgemeinschaft

A piece of history

- The spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$:

- introduced by Kozono and Yamazaki in 1994

- The spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$:

- introduced by Tang and Xu in 2005

- The Triebel-Lizorkin-type spaces $F^{s,\tau}_{p,q}(\mathbb{R}^d)$:

- have been introduced by Yang and Yuan in 2008

- for $0 < p \leq u < \infty$ we have $F^{s,\frac{1}{p}-\frac{1}{u}}_{p,q}(\mathbb{R}^d) = \mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$

- The hybrid spaces $L^r F^s_{p,q}(\mathbb{R}^d)$:

- introduced by Triebel in 2014

- for $-\frac{d}{p} \leq r < \infty$ and $\tau = \frac{1}{p} + \frac{r}{d}$ we have $L^r F^s_{p,q}(\mathbb{R}^d) = F^{s,\tau}_{p,q}(\mathbb{R}^d)$

Elementary properties of $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$

Theorem

Let $0 < p \leq u < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. Then the following assertions are true.

(i) The spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ are independent of the chosen smooth dyadic decomposition of the unity in the sense of equivalent quasi-norms.

(ii) The spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ are quasi-Banach spaces. In the case $p, q \geq 1$ they are Banach spaces.

(iii) We have

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d).$$

(iv) For $q_1 \leq q_2$ we have

$$\mathcal{E}^s_{u,p,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^s_{u,p,q_2}(\mathbb{R}^d).$$

(v) For $\varepsilon > 0$ and $r \in (0, \infty]$ we have $\mathcal{E}^{s+\varepsilon}_{u,p,r}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$.

DFG Deutsche Forschungsgemeinschaft

Dilation Operators: Definitions and Basic Properties

Dilation Operators: A Definition

Definition

(i) *Let* $M(\mathbb{R}^d)$ *be the space of equivalence classes of measurable functions on* $\mathbb{R}^d$. *Then for* $\lambda > 0$ *and* $g \in M(\mathbb{R}^d)$ *we define the dilation operator* D_λ *by*

$$D_\lambda \colon M(\mathbb{R}^d) \to M(\mathbb{R}^d), \qquad g \mapsto D_\lambda g := g(\lambda\,\cdot).$$

(ii) *Let* $Y \in \{\mathcal{D}, \mathcal{S}\}$. *Then we define* D_λ *for the spaces of distributions* $Y'(\mathbb{R}^d)$ *by setting*

$$(D_\lambda f)(\varphi) := \lambda^{-d} f(D_{\lambda^{-1}}\varphi), \qquad f \in Y'(\mathbb{R}^d),\ \varphi \in Y(\mathbb{R}^d),$$

such that $D_\lambda \colon Y'(\mathbb{R}^d) \to Y'(\mathbb{R}^d)$.

DFG Deutsche Forschungsgemeinschaft

Dilation Operators: Basic Properties

Lemma

Let $0 < p < \infty$ *and* $X \in \{\mathcal{D}, \mathcal{S}, L_p^{loc}\}$. *Let* $Y \in \{\mathcal{D}, \mathcal{S}\}$ *and* $\lambda > 0$.

(i) D_λ *is bijective on* $X(\mathbb{R}^d)$ *resp.* $Y'(\mathbb{R}^d)$ *with* $(D_\lambda)^{-1} = D_{\lambda^{-1}}$.

(ii) It holds $D_1 = \mathrm{id}$.

(iii) The set $\{D_\lambda : \lambda > 0\}$ *together with concatenation forms a group. We have*

$$D_{\lambda_1} \circ D_{\lambda_2} = D_{\lambda_1 \cdot \lambda_2}, \qquad \lambda_1, \lambda_2 > 0.$$

(iv) For $\varphi \in \mathcal{S}(\mathbb{R}^d)$ *and* $\lambda > 0$ *it holds*

$$\mathcal{F}^{\pm 1}(D_\lambda \varphi) = \lambda^{-d}\, D_{\lambda^{-1}}[\mathcal{F}^{\pm 1}\varphi] \in \mathcal{S}(\mathbb{R}^d).$$

For $f \in \mathcal{S}'(\mathbb{R}^d)$ *we have*

$$\mathcal{F}^{\pm 1}(D_\lambda f) = \lambda^{-d}\, D_{\lambda^{-1}}[\mathcal{F}^{\pm 1} f] \in \mathcal{S}'(\mathbb{R}^d).$$

(v) For $\eta \in \mathcal{D}(\mathbb{R}^d)$ *and* $f \in \mathcal{S}'(\mathbb{R}^d)$ *we have*

$$\mathcal{F}^{-1}\big(\eta\, \mathcal{F}[D_\lambda f]\big) = D_\lambda\big[\mathcal{F}^{-1}\big([D_\lambda \eta]\, \mathcal{F} f\big)\big].$$

DFG Deutsche Forschungsgemeinschaft

Dilation Operators and Triebel-Lizorkin-Morrey Spaces: The Main Result

Marc Hovemann (joint work with Markus Weimar)

DFG Deutsche Forschungsgemeinschaft

Question:

Find upper and lower bounds for the operator (quasi-)norm $\|D_\lambda|\mathcal{L}\big(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\big)\|$.

Important observation: The answer depends on the parameters s, u, p and q.

DFG Deutsche Forschungsgemeinschaft

The Operator Norm

$$
\begin{aligned}
\|D_\lambda|\mathcal{L}\big(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\big)\| \\
&= \inf\{C > 0 : \forall f \in \mathcal{E}^s_{u,p,q}(\mathbb{R}^d) : \\
&\qquad \|f(\lambda\cdot)|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\| \le C\|f|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\|\}.
\end{aligned}
$$

Question:

What is $C > 0$?

Very often C depends on λ.

Some Applications

Dilation operators D_λ have a lot of significant applications within the theory of function spaces, and also in related fields of research.

Some applications are:

(1) Dilation operators have been widely used in the regularity analysis of PDEs as an important tool in order to construct corresponding (appropriately weighted) function spaces.

(2) The operators D_λ play a crucial role in the theories of so-called refined localization spaces and tempered homogeneous spaces due to Triebel.

- Triebel, H.: Function Spaces and Wavelets on Domains. EMS Tracts in Mathematics, Vol. 7, EMS Publishing House, Zürich (2008)

(3) Sharp bounds of the operator norm $\|D_\lambda|\mathcal{L}(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d))\|$ can be used to disprove the equivalence of certain quasi-norms, at least if specific conditions on the parameters hold.

- Hovemann, M.: Triebel-Lizorkin-Morrey spaces and differences. Math. Nachr. **295**(4), 725–761 (2022)

DFG Deutsche Forschungsgemeinschaft

Bessel Potential Spaces (fractional Sobolev Spaces)

For the range of Bessel potential spaces (i.e. fractional L_p-Sobolev spaces), it is well-known since many years that with constants independent of $\lambda > 1$ with $1 < p < \infty$ and $s \in \mathbb{R}$ there holds

$$\left\| D_\lambda | \mathcal{L}\left(H^s_p(\mathbb{R}^d) \right) \right\| \sim \lambda^{\max\{s,0\} - \frac{d}{p}}.$$

Classical Triebel-Lizorkin Spaces $F^s_{p,q}(\mathbb{R}^d)$: Some References

For the special case of the Triebel-Lizorkin Spaces $F^s_{p,q}(\mathbb{R}^d)$ most of the answer is already known.

Some References:

(1) Triebel, H.: Theory of Function Spaces. Birkhäuser, Basel (1983)

- the case $s > \sigma_p := d \max\left\{0, \frac{1}{p} - 1\right\}$

(2) Edmunds, D.E. and Triebel, H.: Function Spaces, Entropy Numbers and Differential Operators. Cambridge university press, Cambridge (1996)

- improvements for the case $s > \sigma_p$

(3) Schneider, C. and Vybíral, J.: On Dilation Operators in Triebel-Lizorkin Spaces. Funct. Approx. Comment. Math. **41**(2), 139-162 (2009)

- the case $s = \sigma_p$ with $\lambda = 2^j$ and $j \in \mathbb{N}$
- for $0 < p \leq 1$ there remained some gaps between the upper and lower bounds

DFG Deutsche Forschungsgemeinschaft

Triebel-Lizorkin Spaces $F^s_{p,q}(\mathbb{R}^d)$ and Dilation Operators

Theorem

Let $0 < p < \infty$, $0 < q \le \infty$ *and* $s \in \mathbb{R}$. *For* $\frac{1}{2} < \lambda < 2$ *we have*

$$\|D_\lambda | \mathcal{L}\big(F^s_{p,q}(\mathbb{R}^d)\big)\| \sim 1,$$

while for $\lambda \ge 2$ *there holds*

$$\|D_\lambda | \mathcal{L}\big(F^s_{p,q}(\mathbb{R}^d)\big)\| \sim \lambda^{\max\{s,\sigma_p\}-\frac{d}{p}} \qquad \text{if} \qquad s \ne \sigma_p$$

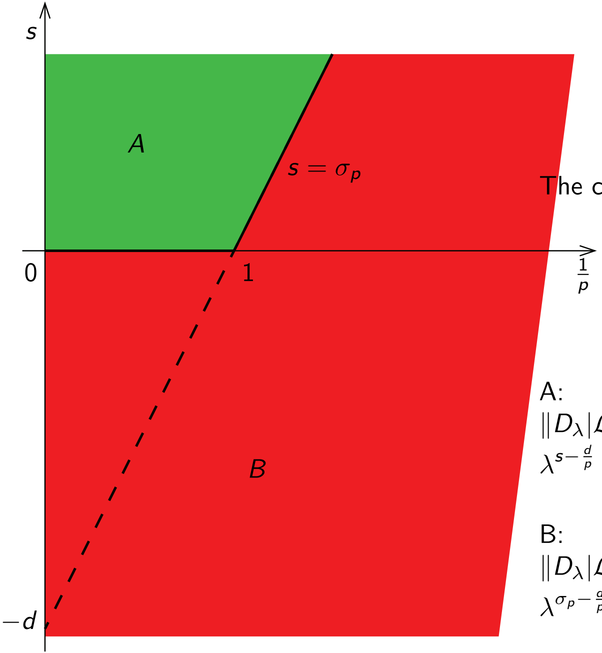
as well as

$$\|D_\lambda | \mathcal{L}\big(F^0_{p,q}(\mathbb{R}^d)\big)\| \sim \lambda^{0-\frac{d}{p}} (\log_2 \lambda)^{\max\{0,\frac{1}{q}-\frac{1}{2}\}} \quad \text{if} \quad p > 1, s = 0$$

and for $s = \sigma_p$, $p \le 1$

$$\lambda^{\sigma_p - \frac{d}{p}} (\log_2 \lambda)^{\max\{\frac{1}{p},\frac{1}{q}\}} \gtrsim \|D_\lambda | \mathcal{L}\big(F^{\sigma_p}_{p,q}(\mathbb{R}^d)\big)\| \gtrsim \lambda^{\sigma_p - \frac{d}{p}} \cdot \begin{cases} (\log_2 \lambda)^{\max\{\frac{1}{p},\frac{1}{q}-\frac{1}{2}\}}, & p = 1, \\ (\log_2 \lambda)^{\frac{1}{p}}, & p < 1. \end{cases}$$

All constants are independent of λ. *We use* $\frac{1}{\infty} = 0$.



The case $F^s_{p,q}(\mathbb{R}^d)$.

A: $\|D_\lambda|\mathcal{L}(F^s_{p,q}(\mathbb{R}^d))\| \sim \lambda^{s-\frac{d}{p}}$

B: $\|D_\lambda|\mathcal{L}(F^s_{p,q}(\mathbb{R}^d))\| \sim \lambda^{\sigma_p-\frac{d}{p}}$

DFG Deutsche Forschungsgemeinschaft

Triebel-Lizorkin Spaces on Domains

Definition (Triebel-Lizorkin space $F^s_{p,q}(\Omega)$)

For $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, *and domains* $\Omega \subsetneq \mathbb{R}^d$ *let*

$$F^s_{p,q}(\Omega) := \left\{ f \in \mathcal{D}'(\Omega) : f = g|_\Omega \ \textit{ for some } \ g \in F^s_{p,q}(\mathbb{R}^d) \right\}$$

be endowed with the quasi-norm

$$\|f|F^s_{p,q}(\Omega)\| := \inf \left\{ \|g|F^s_{p,q}(\mathbb{R}^d)\| : f = g|_\Omega \ \textit{ for some } \ g \in F^s_{p,q}(\mathbb{R}^d) \right\}.$$

Definition (Triebel-Lizorkin space $\tilde{F}^s_{p,q}(\Omega)$)

For $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, *and domains* $\Omega \subsetneq \mathbb{R}^d$ *let* $\tilde{F}^s_{p,q}(\overline{\Omega}) := \{ f \in F^s_{p,q}(\mathbb{R}^d) : \operatorname{supp} f \subset \overline{\Omega} \}$. *Then we put*

$$\tilde{F}^s_{p,q}(\Omega) := \left\{ f \in \mathcal{D}'(\Omega) : f = g|_\Omega \ \textit{ for some } \ g \in \tilde{F}^s_{p,q}(\overline{\Omega}) \right\}$$

endowed with the quasi-norm

$$\|f|\tilde{F}^s_{p,q}(\Omega)\| := \inf \left\{ \|g|F^s_{p,q}(\mathbb{R}^d)\| : f = g|_\Omega \ \textit{ for some } \ g \in \tilde{F}^s_{p,q}(\overline{\Omega}) \right\}.$$

DFG Deutsche Forschungsgemeinschaft

The Homogeneity Property

Define $U_\lambda := \{x \in \mathbb{R}^d : |x| < \lambda\}$ for $0 < \lambda < \infty$. Put

$$\overline{F}^s_{p,q}(U_\lambda) := \begin{cases} \tilde{F}^s_{p,q}(U_\lambda), & 0 < p < \infty, 0 < q \leq \infty, s > \sigma_{p,q} \\ F^0_{p,q}(U_\lambda), & 1 < p < \infty, 1 \leq q < \infty, s = 0 \\ F^s_{p,q}(U_\lambda), & 0 < p < \infty, 0 < q \leq \infty, s < 0 \end{cases}$$

Then it holds:

Theorem (Triebel, 2008)

Let $0 < \lambda \leq 1$ and p, q, s as above. Let $f \in \overline{F}^s_{p,q}(U_\lambda)$. Then it holds

$$\|f(\lambda\cdot)|\overline{F}^s_{p,q}(U_1)\| \sim \lambda^{s-\frac{d}{p}}\|f|\overline{F}^s_{p,q}(U_\lambda)\|.$$

There exists a counterpart for the Besov spaces $\overline{B}^s_{p,q}(U_\lambda)$. It has been proved by Caetano, Lopes and Triebel in 2007.

DFG Deutsche Forschungsgemeinschaft

From Lebesgue Spaces to Morrey Spaces

Lemma

Let $0 < p \le u < \infty$ *and* $\lambda > 0$*. Then we have* $f \in \mathcal{M}^u_p(\mathbb{R}^d)$ *if and only if* $D_\lambda f \in \mathcal{M}^u_p(\mathbb{R}^d)$*. In this case there holds*

$$\|D_\lambda f|\mathcal{M}^u_p(\mathbb{R}^d)\| \sim \lambda^{-\frac{d}{u}}\|f|\mathcal{M}^u_p(\mathbb{R}^d)\|$$

with constants that do not depend on λ *and* f*.*

Recall,

$$\|f(\lambda\cdot)|\mathcal{M}^u_p(\mathbb{R}^d)\| := \sup_B |B|^{\frac{1}{u}-\frac{1}{p}} \left(\int_B |f(\lambda x)|^p dx\right)^{\frac{1}{p}}.$$

Now, use a transformation of the coordinates.

DFG Deutsche Forschungsgemeinschaft

Differences of Order N

Let $f : \mathbb{R}^d \to \mathbb{C}$ be a function. Then for $x, h \in \mathbb{R}^d$ we define the difference of the first order in the following way:

$$\Delta_h^1 f(x) = f(x+h) - f(x).$$

Now let $N \in \mathbb{N}$ with $N > 1$. Then we define the difference of order N by

$$\Delta_h^N f(x) = \left(\Delta_h^1 \left(\Delta_h^{N-1} f\right)\right)(x).$$

DFG Deutsche Forschungsgemeinschaft

A Characterization of $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ in Terms of Differences

Theorem

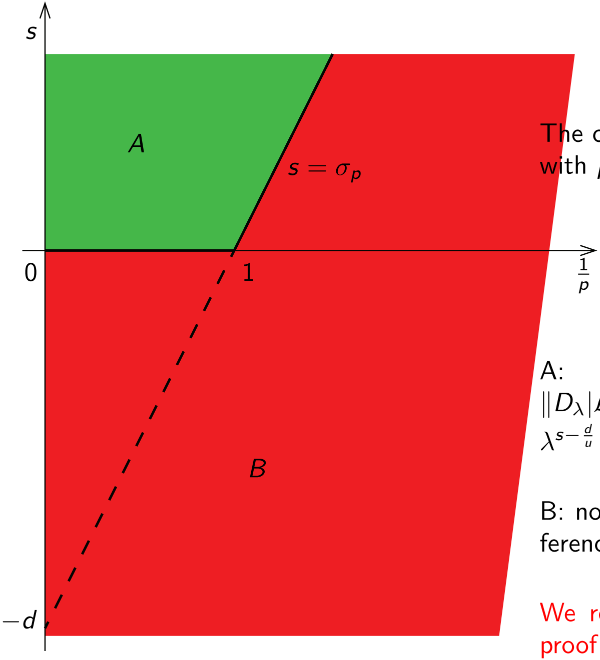
Let $0 < p \le u < \infty$ *and* $0 < q \le \infty$. *Let* $s > d \max\left(0, \frac{1}{p} - 1, \frac{1}{q} - 1\right)$. *Let* $N \in \mathbb{N}$ *with* $N > s$. *Then* $f \in L^{loc}_{\min(p,q)}(\mathbb{R}^d)$ *belongs to* $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ *if and only if* $f \in L^{loc}_1(\mathbb{R}^d)$ *and*

$$\underbrace{\|f|\mathcal{M}^u_p(\mathbb{R}^d)\| + \left\| \left(\int_0^\infty t^{-sq} \left(t^{-d} \int_{B(0,t)} |\Delta^N_h f(x)| dh \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \middle| \mathcal{M}^u_p(\mathbb{R}^d) \right\|}_{\|f|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(1)} :=}$$

is finite. The quasi-norms $\| f \,|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\|$ *and* $\| f \,|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\|^{(1)}$ *are equivalent for* $f \in L^{loc}_{\min(p,q)}(\mathbb{R}^d)$. *In the case* $q = \infty$ *the usual modifications have to be made.*

Marc Hovemann (joint work with Markus Weimar)

DFG Deutsche Forschungsgemeinschaft



The case $\mathcal{E}^s_{u,p,p}(\mathbb{R}^d)$ with $p = q$.

A: $\|D_\lambda|\mathcal{L}(\mathcal{E}^s_{u,p,p}(\mathbb{R}^d))\| \lesssim \lambda^{s-\frac{d}{u}}$ for $\lambda \geq 2$

B: no result via differences

We require another proof technique!

DFG Deutsche Forschungsgemeinschaft

Triebel-Lizorkin-Morrey Spaces: The Main Result

Theorem

Let $0 < p \leq u < \infty$, $0 < q \leq \infty$ *and* $s \in \mathbb{R}$. *For* $\frac{1}{2} < \lambda < 2$ *we have*

$$\| D_\lambda | \mathcal{L}\big(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\big) \| \sim 1.$$

For $\lambda \geq 2$ *the following estimates hold:*

(1) If $s > \sigma_p$, *then* $\| D_\lambda | \mathcal{L}\big(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\big) \| \sim \lambda^{s - \frac{d}{u}}$.

(2) If $s = \sigma_p$, *then for* $p > 1$ *it is*

$$\| D_\lambda | \mathcal{L}\big(\mathcal{E}^0_{u,p,q}(\mathbb{R}^d)\big) \| \sim \lambda^{0 - \frac{d}{u}} (\log_2 \lambda)^{\max\{0, \frac{1}{q} - \frac{1}{2}\}}$$

while for $p = 1$ *there holds*

$$\lambda^{-\frac{d}{u}} (\log_2 \lambda)^{\max\{\frac{1}{p}, \frac{1}{q}\}} \gtrsim \| D_\lambda | \mathcal{L}\Big(\mathcal{E}^0_{u,p,q}(\mathbb{R}^d)\Big) \| \gtrsim \lambda^{-\frac{d}{u}} \cdot \begin{cases} (\log_2 \lambda)^{\max\{\frac{1}{p}, \frac{1}{q} - \frac{1}{2}\}}, & p = u, \\ (\log_2 \lambda)^{\max\{0, \frac{1}{q} - \frac{1}{2}\}}, & p < u, \end{cases}$$

and for $p < 1$ *we have*

$$\lambda^{\sigma_p - \frac{d}{u}} (\log_2 \lambda)^{\max\{\frac{1}{p}, \frac{1}{q}\}} \gtrsim \| D_\lambda | \mathcal{L}\Big(\mathcal{E}^{\sigma_p}_{u,p,q}(\mathbb{R}^d)\Big) \| \gtrsim \lambda^{\sigma_p - \frac{d}{u}} \cdot \begin{cases} (\log_2 \lambda)^{\frac{1}{p}}, & p = u, \\ 1, & p < u. \end{cases}$$

DFG Deutsche Forschungsgemeinschaft

Triebel-Lizorkin-Morrey Spaces: The Main Result (continued)

Theorem

(3) If $s < \sigma_p$, then for $p \geq 1$ it holds

$$\| D_\lambda | \mathcal{L}\big(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\big) \| \sim \lambda^{\sigma_p - \frac{d}{u}}$$

while for $p < 1$ we have

$$\begin{aligned} \lambda^{\sigma_p - \frac{d}{u}} &\gtrsim \| D_\lambda | \mathcal{L}\big(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\big) \| \\ &\gtrsim \lambda^{\max\{s,\sigma_u\} - \frac{d}{u}} \cdot \begin{cases} (\log_2 \lambda)^{\max\{0, \frac{1}{q} - \frac{1}{2}\}}, & s = 0 \text{ and } 1 \leq u, \\ 1, & \text{else.} \end{cases} \end{aligned}$$

Therein, all constants are independent of λ. To incorporate the case $q = \infty$ we use the convention $\frac{1}{\infty} = 0$.

Reference: M. Hovemann, M. Weimar: On the boundedness of dilation operators in the context of Triebel-Lizorkin-Morrey spaces, arXiv:2510.11439

DFG Deutsche Forschungsgemeinschaft

The Proof of the Main Result: Upper Bounds

Deutsche Forschungsgemeinschaft

Upper Bounds:

We have to find estimates of the form $\|D_\lambda|\mathcal{L}\big(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\big)\| \leq C(\lambda)$. How does $C(\lambda)$ depend on λ ?

To study this question, we need some auxiliary results.

Tool 1: A Fourier Multiplier Theorem

Theorem

Let $0 < p \le u < \infty$ *and* $m, \ell > 0$. *Then for all* $M \in \mathcal{S}(\mathbb{R}^d)$ *with* $\operatorname{supp} M \subseteq B(0,m)$ *and every* $f \in \mathcal{M}^u_p(\mathbb{R}^d)$ *with* $\operatorname{supp}(\mathcal{F}f) \subseteq B(0,\ell)$ *there holds*

$$\|\mathcal{F}^{-1}[M\,\mathcal{F}f]|\mathcal{M}^u_p(\mathbb{R}^d)\| \lesssim (m+\ell)^{\sigma_p} \|\mathcal{F}^{-1}M|L_{\min\{1,p\}}(\mathbb{R}^d)\| \|f|\mathcal{M}^u_p(\mathbb{R}^d)\|$$

with an implicit constant that does not depend on f, M, m *or* ℓ.

DFG Deutsche Forschungsgemeinschaft

Tool 2: The Case $\lambda \approx 1$

Theorem

Let $0 < p \leq u < \infty$, $0 < q \leq \infty$ *and* $s \in \mathbb{R}$ *as well as* $2^{-1} < \mu < 2$ *and* $f \in \mathcal{S}'(\mathbb{R}^d)$. *Then* $D_\mu f \in \mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ *if and only if* $f \in \mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$. *In this case there holds*

$$\|D_\mu f|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\| \sim \|f|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\|$$

with implicit constants independent of f *and* μ. *In particular,*

$$\|D_\mu|\mathcal{L}\big(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\big)\| \sim 1.$$

Tool 3: It is enough to deal with $\lambda_j := 2^j$

Tool 2 allows to reduce (quasi-)norm estimates of the form $\lambda^\alpha (\log_2 \lambda)^\beta$ for dilation operators D_λ, where $\lambda \geq 2$ and $\alpha, \beta \in \mathbb{R}$, to the dyadic case $\lambda_j := 2^j$ with $j \in \mathbb{N}$. Indeed, for $\lambda > 2$ there exists a unique $j \in \mathbb{N} \setminus \{1\}$ such that

$$\mu := \lambda_j^{-1} \lambda \in (2^{-1}, 1].$$

Then we have

$$\lambda_j^\alpha (\log_2 \lambda_j)^\beta \sim \lambda^\alpha (\log_2 \lambda)^\beta, \qquad \alpha, \beta \in \mathbb{R}$$

and observe $D_{\lambda_j} = D_\lambda \circ D_{\mu^{-1}}$ as well as $D_\lambda = D_{\lambda_j} \circ D_\mu$ such that Tool 2 yields that for all $0 < p \leq u < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$

$$\begin{aligned} \|D_{\lambda_j}|\mathcal{L}(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d))\| &\leq \|D_\lambda|\mathcal{L}(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d))\| \|D_{\mu^{-1}}|\mathcal{L}(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d))\| \\ &\sim \|D_\lambda|\mathcal{L}(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d))\| \\ &\lesssim \|D_{\lambda_j}|\mathcal{L}(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d))\|, \end{aligned}$$

i.e., $\|D_{\lambda_j}|\mathcal{L}(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d))\| \sim \|D_\lambda|\mathcal{L}(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d))\|$.

DFG Deutsche Forschungsgemeinschaft

Upper Bounds

Theorem

Let $0 < p \leq u < \infty$, $0 < q \leq \infty$ as well as $s \in \mathbb{R}$ and $\lambda \geq 2$. Then for the restriction of D_λ to $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ there holds

(i) $\|D_\lambda|\mathcal{L}(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d))\| \lesssim \lambda^{s-\frac{d}{u}}$ if $s > \sigma_p$.

(ii)
$$\|D_\lambda|\mathcal{L}(\mathcal{E}^{\sigma_p}_{u,p,q}(\mathbb{R}^d))\| \lesssim \lambda^{\sigma_p-\frac{d}{u}}\,(\log_2\lambda)^{\max\{\frac{1}{p},\frac{1}{q}\}}.$$

If we additionally assume that $p > 1$ (such that $\sigma_p = 0$), then we even have

$$\|D_\lambda|\mathcal{L}(\mathcal{E}^{0}_{u,p,q}(\mathbb{R}^d))\| \lesssim \lambda^{-\frac{d}{u}}\,(\log_2\lambda)^{\max\{0,\frac{1}{q}-\frac{1}{2}\}}.$$

(iii) $\|D_\lambda|\mathcal{L}(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d))\| \lesssim \lambda^{\sigma_p-\frac{d}{u}}$ if $s < \sigma_p$.

Therein the implicit constants do not depend on λ. To incorporate the case $q = \infty$ we use the convention $\frac{1}{\infty} = 0$.

DFG Deutsche Forschungsgemeinschaft

Upper Bounds: Sketch of the Proof

In view of Tool 3 it suffices to prove the claim for $\lambda = 2^j$ with $j \in \mathbb{N}$. Let $j \in \mathbb{N}$ and $f \in \mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ be fixed. Define

$$\psi(x) := \varphi_0(x) - (D_2\varphi_0)(x) = \varphi_0(x) - \varphi_0(2x), \qquad x \in \mathbb{R}^d,$$

as well as its dyadic dilates

$$\psi_\ell(x) := (D_{2^{-\ell}}\psi)(x) = \psi(2^{-\ell}x), \qquad \ell \in \mathbb{Z},\ x \in \mathbb{R}^d.$$

Then $\varphi_k = \psi_k$ for $k \in \mathbb{N}$. Hence, $D_{2^j}\varphi_k = D_{2^{-(k-j)}}\psi = \psi_{k-j}$ for all $k \in \mathbb{N}$ and

$$u_k := \mathcal{F}^{-1}\big(\varphi_k\, \mathcal{F}[D_{2^j} f]\big) = D_{2^j}\big[\mathcal{F}^{-1}\big([D_{2^j}\varphi_k]\,\mathcal{F}f\big)\big]$$
$$= \begin{cases} D_{2^j}\big[\mathcal{F}^{-1}\big([D_{2^j}\varphi_0]\,\mathcal{F}f\big)\big], & k = 0, \\ D_{2^j}\big[\mathcal{F}^{-1}\big([D_{2^{-(k-j)}}\psi]\,\mathcal{F}f\big)\big], & k = 1, \ldots, j \\ D_{2^j}\big[\mathcal{F}^{-1}\big(\varphi_{k-j}\,\mathcal{F}f\big)\big], & k \geq j+1. \end{cases}$$

Therefore, in $\mathcal{S}'(\mathbb{R}^d)$ we can decompose $D_{2^j} f$ into

$$D_{2^j} f = \sum_{k=0}^{\infty} \mathcal{F}^{-1}\big(\varphi_k\, \mathcal{F}[D_{2^j} f]\big) = u_0 + \sum_{k=1}^{j} u_k + \sum_{k=j+1}^{\infty} u_k =: U_1 + U_2 + U_3$$

with convergence of the series in $\mathcal{S}'(\mathbb{R}^d)$.

DFG Deutsche Forschungsgemeinschaft

Upper Bounds: Sketch of the Proof (Part 2)

The idea of the proof is to show that $U_n \in \mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ for $n = 1, 2, 3$, where

$$\|U_1|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\| \lesssim 2^{j(\sigma_p - \frac{d}{u})}\|f|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\|,$$

$$\|U_2|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\| \lesssim \begin{cases} 2^{j(s-\frac{d}{u})}\|f|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\|, & s > \sigma_p, \\ 2^{j(\sigma_p-\frac{d}{u})} j^{\max\{\frac{1}{p},\frac{1}{q}\}}\|f|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\|, & s = \sigma_p, \\ 2^{j(\sigma_p-\frac{d}{u})} j^{\max\{0,\frac{1}{q}-\frac{1}{2}\}}\|f|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\|, & s = \sigma_p, p > 1, \\ 2^{j(\sigma_p-\frac{d}{u})}\|f|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\|, & s < \sigma_p, \end{cases}$$

$$\|U_3|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\| \lesssim 2^{j(s-\frac{d}{u})}\|f|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\|.$$

So the claim for $\lambda = 2^j$ easily follows.

Investigate U_2: Use the dyadic annuli criterion

Lemma (Yuan, Sickel, Yang (2010))

Let $0 < p \leq u < \infty$, $0 < q \leq \infty$ *and* $s \in \mathbb{R}$. *Further, let* $(\widetilde{u}_k)_{k\in\mathbb{N}_0} \subseteq \mathcal{S}'(\mathbb{R}^d)$ *satisfy* $\operatorname{supp}(\mathcal{F}\widetilde{u}_0) \subseteq B(0,4)$,

$$\operatorname{supp}(\mathcal{F}\widetilde{u}_k) \subseteq B(0,2^{k+2}) \setminus B(0,2^{k-3}), \qquad k \in \mathbb{N},$$

and

$$A := \left\| \left(\sum_{k=0}^{\infty} 2^{ksq} |\widetilde{u}_k(\cdot)|^q \right)^{\frac{1}{q}} \Big| \mathcal{M}_p^u(\mathbb{R}^d) \right\| < \infty$$

(with the usual modification if $q = \infty$*). Then* $\sum_{k=0}^{\infty} \widetilde{u}_k$ *converges in* $\mathcal{S}'(\mathbb{R}^d)$ *to some* $U \in \mathcal{S}'(\mathbb{R}^d)$ *and there holds*

$$U \in \mathcal{E}^s_{u,p,q}(\mathbb{R}^d) \qquad \text{with} \qquad \|U|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\| \lesssim A,$$

where the implied constant does not depend on $(\widetilde{u}_k)_{k\in\mathbb{N}_0}$ *or* A*. If* $q < \infty$*, the convergence takes place in* $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ *and otherwise in* $\mathcal{E}^{s-\varepsilon}_{u,p,1}(\mathbb{R}^d)$ *for all* $\varepsilon > 0$.

DFG Deutsche Forschungsgemeinschaft

Investigate U_2: Let us start

In order to analyze $U_2 = \sum_{k=1}^{j} u_k$ we employ the dyadic annuli criterion with $(\widetilde{u}_k)_{k\in\mathbb{N}_0}$ given by $\widetilde{u}_k := u_k$ for $k = 1, \ldots, j$, and zero otherwise. Thus, we need to bound

$$\begin{aligned} A &:= \left\| \left(\sum_{k=0}^{\infty} 2^{ksq} \left| \widetilde{u}_k(\cdot) \right|^q \right)^{\frac{1}{q}} \middle| \mathcal{M}_p^u(\mathbb{R}^d) \right\| \\ &= \left\| \left(\sum_{k=1}^{j} 2^{ksq} \left| u_k(\cdot) \right|^q \right)^{\frac{1}{q}} \middle| \mathcal{M}_p^u(\mathbb{R}^d) \right\| \\ &= 2^{js} \left\| D_{2^j} \left(\sum_{k=1}^{j} 2^{(k-j)sq} \left| \left[\mathcal{F}^{-1} \left(\left[D_{2^{-(k-j)}} \psi \right] \mathcal{F} f \right) \right] (\cdot) \right|^q \right)^{\frac{1}{q}} \middle| \mathcal{M}_p^u(\mathbb{R}^d) \right\| . \end{aligned}$$

Setting $\ell := j - k$ we find

$$A \sim 2^{j(s-\frac{d}{u})} \left\| \left(\sum_{\ell=0}^{j-1} 2^{-\ell sq} \left| \left[\mathcal{F}^{-1} \left(\left[D_{2^\ell} \psi \right] \mathcal{F} f \right) \right] (\cdot) \right|^q \right)^{\frac{1}{q}} \middle| \mathcal{M}_p^u(\mathbb{R}^d) \right\| .$$

Here for $f \in \mathcal{M}_p^u(\mathbb{R}^d)$ we used

$$\left\| D_\lambda f \middle| \mathcal{M}_p^u(\mathbb{R}^d) \right\| \sim \lambda^{-\frac{d}{u}} \left\| f \middle| \mathcal{M}_p^u(\mathbb{R}^d) \right\| .$$

DFG Deutsche Forschungsgemeinschaft

Investigate U_2: Use the Fourier Multiplier Theorem

Let us fix $\mu := \min\{p,q\}$. Then

$$A \lesssim 2^{j(s-\frac{d}{u})} \left(\sum_{\ell=0}^{j-1} 2^{-\ell s \mu} \left\| \mathcal{F}^{-1}\big([D_{2^\ell}\psi]\,\mathcal{F}f\big) \,\middle|\, \mathcal{M}^u_p(\mathbb{R}^d) \right\|^\mu \right)^{\frac{1}{\mu}}.$$

Next we apply the Fourier Multiplier Theorem. Note that

$$\operatorname{supp}(D_{2^\ell}\psi) \subseteq B(0,2^{1-\ell}) \subseteq B(0,2), \qquad \ell = 0,\ldots,j-1.$$

Moreover, for $\widetilde{\varphi}_0 := D_{2^{-1}}\varphi_0 = \varphi_0 + \varphi_1$ we have $\widetilde{\varphi}_0(x) = 1$ if $|x| < 2$, while $\operatorname{supp}\widetilde{\varphi}_0 \subseteq B(0,4)$. Setting $\widetilde{f}_0 := \mathcal{F}^{-1}[\widetilde{\varphi}_0\,\mathcal{F}f]$ such that $\operatorname{supp}(\mathcal{F}\widetilde{f}_0) \subseteq B(0,4)$ we obtain

$$\begin{aligned}
&\left\| \mathcal{F}^{-1}\big([D_{2^\ell}\psi]\,\mathcal{F}f\big) \,\middle|\, \mathcal{M}^u_p(\mathbb{R}^d) \right\| \\
&\quad = \left\| \mathcal{F}^{-1}\big([D_{2^\ell}\psi]\,\mathcal{F}\widetilde{f}_0\big) \,\middle|\, \mathcal{M}^u_p(\mathbb{R}^d) \right\| \\
&\quad \lesssim (2+4)^{\sigma_p} \left\| \mathcal{F}^{-1}(D_{2^\ell}\psi) \,\middle|\, L_{\min\{1,p\}}(\mathbb{R}^d) \right\| \left\| \widetilde{f}_0 \,\middle|\, \mathcal{M}^u_p(\mathbb{R}^d) \right\| \\
&\quad \sim 2^{\ell\sigma_p} \left\| \widetilde{f}_0 \,\middle|\, \mathcal{M}^u_p(\mathbb{R}^d) \right\|
\end{aligned}$$

for all $\ell = 0,\ldots,j-1$.

DFG Deutsche Forschungsgemeinschaft

Investigate U_2: A Distinction of Cases

For this term we find

$$\left\| \widetilde{f_0} \,\middle|\, \mathcal{M}^u_p(\mathbb{R}^d) \right\| = \left\| \mathcal{F}^{-1}[(\varphi_0 + \varphi_1)\,\mathcal{F}f] \,\middle|\, \mathcal{M}^u_p(\mathbb{R}^d) \right\| \lesssim \left\| f \,\middle|\, \mathcal{E}^s_{u,p,q}(\mathbb{R}^d) \right\| .$$

Combining these estimates we conclude

$$\left\| U_2 \,\middle|\, \mathcal{E}^s_{u,p,q}(\mathbb{R}^d) \right\| \lesssim A \lesssim 2^{j(s-\frac{d}{u})} \left(\sum_{\ell=0}^{j-1} 2^{-\ell(s-\sigma_p)\mu} \right)^{\frac{1}{\mu}} \left\| f \,\middle|\, \mathcal{E}^s_{u,p,q}(\mathbb{R}^d) \right\| .$$

If $s > \sigma_p$, then we can estimate

$$\left(\sum_{\ell=0}^{j-1} 2^{-\ell(s-\sigma_p)\mu} \right)^{\frac{1}{\mu}} \leq \left(\sum_{\ell=0}^{\infty} \left(2^{-(s-\sigma_p)\mu}\right)^{\ell} \right)^{\frac{1}{\mu}} < \infty$$

such that $\left\| U_2 \,\middle|\, \mathcal{E}^s_{u,p,q}(\mathbb{R}^d) \right\| \lesssim 2^{j(s-\frac{d}{u})} \left\| f \,\middle|\, \mathcal{E}^s_{u,p,q}(\mathbb{R}^d) \right\|$ as claimed.
In case $s = \sigma_p$, we derive

$$\left(\sum_{\ell=0}^{j-1} 2^{-\ell(s-\sigma_p)\mu} \right)^{\frac{1}{\mu}} = j^{\frac{1}{\mu}} = j^{\max\{\frac{1}{p},\frac{1}{q}\}}$$

and hence $\left\| U_2 \,\middle|\, \mathcal{E}^{\sigma_p}_{u,p,q}(\mathbb{R}^d) \right\| \lesssim 2^{j(\sigma_p-\frac{d}{u})} j^{\max\{\frac{1}{p},\frac{1}{q}\}} \left\| f \,\middle|\, \mathcal{E}^{\sigma_p}_{u,p,q}(\mathbb{R}^d) \right\|$.

DFG Deutsche Forschungsgemeinschaft

Investigate U_2: A Distinction of Cases (Part 2)

If $s < \sigma_p$, we obtain

$$\left(\sum_{\ell=0}^{j-1} 2^{-\ell(s-\sigma_p)\mu}\right)^{\frac{1}{\mu}} = \left(\sum_{\ell=0}^{j-1} \left(2^{(\sigma_p-s)\mu}\right)^{\ell}\right)^{\frac{1}{\mu}} \lesssim \left(\left(2^{(\sigma_p-s)\mu}\right)^{j}\right)^{\frac{1}{\mu}} = 2^{j(\sigma_p-s)}$$

such that $\left\| U_2 \,\middle|\, \mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\right\| \lesssim 2^{j(\sigma_p-\frac{d}{u})} \left\| f \,\middle|\, \mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\right\|$.

Investigate U_2: Refinement for $p > 1$ and $s = \sigma_p = 0$

For $s = \sigma_p = 0$ we have

$$A \sim 2^{j(\sigma_p - \frac{d}{u})} \left\| \left(\sum_{\ell=0}^{j-1} \left| \left[\mathcal{F}^{-1} \left([D_{2^\ell} \psi] \, \mathcal{F} f \right) \right] (\cdot) \right|^q \right)^{\frac{1}{q}} \middle| \mathcal{M}_p^u(\mathbb{R}^d) \right\| .$$

As before replace f by $\widetilde{f}_0 = \mathcal{F}^{-1}[\widetilde{\varphi}_0 \, \mathcal{F} f]$. If $q < 2$, Hölder's inequality with $r := \frac{2}{q}$ (i.e. $\frac{1}{r'} = 1 - \frac{q}{2} = q \max\{0, \frac{1}{q} - \frac{1}{2}\}$) yields that for a.e. $x \in \mathbb{R}^d$ there holds

$$\left(\sum_{\ell=0}^{j-1} \left| \left[\mathcal{F}^{-1} \left([D_{2^\ell} \psi] \, \mathcal{F} \widetilde{f}_0 \right) \right] (x) \right|^q \right)^{\frac{1}{q}} \leq j^{\max\{0, \frac{1}{q} - \frac{1}{2}\}} \left(\sum_{\ell=0}^{j-1} \left| \left[\mathcal{F}^{-1} \left([D_{2^\ell} \psi] \, \mathcal{F} \widetilde{f}_0 \right) \right] (x) \right|^2 \right)^{\frac{1}{2}} .$$

If otherwise $q \geq 2$, then the same estimate holds true since $\frac{2}{q} \leq 1$ and $\max\{0, \frac{1}{q} - \frac{1}{2}\} = 0$.

Investigate U_2: Refinement for $p > 1$ and $s = \sigma_p = 0$ (Part 2)

In both cases we obtain

$$
\begin{aligned}
&\left\| U_2 \,\middle|\, \mathcal{E}^{\sigma_p}_{u,p,q}(\mathbb{R}^d) \right\| \\
&\quad \lesssim A \\
&\quad \lesssim 2^{j(\sigma_p - \frac{d}{u})} j^{\max\{0, \frac{1}{q} - \frac{1}{2}\}} \left\| \left(\sum_{\ell=0}^{j-1} \left| \left[\mathcal{F}^{-1} \left([D_{2^\ell} \psi] \, \mathcal{F} \widetilde{f_0} \right) \right] (\cdot) \right|^2 \right)^{\frac{1}{2}} \,\middle|\, \mathcal{M}^u_p(\mathbb{R}^d) \right\| \\
&\quad \leq 2^{j(\sigma_p - \frac{d}{u})} j^{\max\{0, \frac{1}{q} - \frac{1}{2}\}} \left\| \left(\sum_{\ell=-\infty}^{\infty} \left| \left[\mathcal{F}^{-1} \left([D_{2^\ell} \psi] \, \mathcal{F} \widetilde{f_0} \right) \right] (\cdot) \right|^2 \right)^{\frac{1}{2}} \,\middle|\, \mathcal{M}^u_p(\mathbb{R}^d) \right\| \\
&\quad \sim 2^{j(\sigma_p - \frac{d}{u})} j^{\max\{0, \frac{1}{q} - \frac{1}{2}\}} \left\| \widetilde{f_0} \,\middle|\, \mathcal{M}^u_p(\mathbb{R}^d) \right\| \\
&\quad \lesssim 2^{j(\sigma_p - \frac{d}{u})} j^{\max\{0, \frac{1}{q} - \frac{1}{2}\}} \left\| f \,\middle|\, \mathcal{E}^{\sigma_p}_{u,p,q}(\mathbb{R}^d) \right\| ,
\end{aligned}
$$

where we used a Littlewood-Paley Characterization of $\mathcal{M}^u_p(\mathbb{R}^d)$.

DFG Deutsche Forschungsgemeinschaft

Investigate U_1 and U_3

Recall

$$D_{2^j} f = \sum_{k=0}^{\infty} \mathcal{F}^{-1}\big(\varphi_k\, \mathcal{F}[D_{2^j} f]\big) = u_0 + \sum_{k=1}^{j} u_k + \sum_{k=j+1}^{\infty} u_k =: U_1 + U_2 + U_3$$

We can use similar methods as for U_2 to obtain

$$\|U_1|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\| \lesssim 2^{j(\sigma_p - \frac{d}{u})}\|f|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\|,$$
$$\|U_3|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\| \lesssim 2^{j(s - \frac{d}{u})}\|f|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\|.$$

In particular, we use the tools:

(i) the dyadic annuli criterion

(ii) the Fourier Multiplier Theorem

(iii)
$$\left\|D_\lambda f \,\middle|\, \mathcal{M}^u_p(\mathbb{R}^d)\right\| \sim \lambda^{-\frac{d}{u}} \left\|f \,\middle|\, \mathcal{M}^u_p(\mathbb{R}^d)\right\|.$$

DFG Deutsche Forschungsgemeinschaft

The Proof of the Main Result: Lower Bounds

DFG Deutsche Forschungsgemeinschaft

Lower Bounds:

We have to find estimates of the form $\| D_\lambda | \mathcal{L}\big(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\big)\| \geq C(\lambda)$. How does $C(\lambda)$ depend on λ ?

Here we have to deal with certain test functions.

DFG Deutsche Forschungsgemeinschaft

Lower Bounds

Theorem

Let $0 < p \leq u < \infty$, $0 < q \leq \infty$ *and* $s \in \mathbb{R}$. *Then for* $\lambda \geq 2$ *we have*

$$\|D_\lambda | \mathcal{L}(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d))\| \gtrsim \lambda^{\max\{s,0\} - \frac{d}{u}}$$

and

$$\|D_\lambda | \mathcal{L}(\mathcal{E}^0_{u,p,q}(\mathbb{R}^d))\| \gtrsim \lambda^{-\frac{d}{u}} \cdot \begin{cases} (\log_2 \lambda)^{\frac{1}{q} - 1}, & 0 < u < 1, \\ (\log_2 \lambda)^{\frac{1}{q} - \frac{1}{2}}, & 1 \leq u < \infty, \end{cases}$$

with implied constants that do not depend on λ. *To incorporate the case* $q = \infty$ *we use the convention* $\frac{1}{\infty} = 0$.

Lower Bounds: Sketch of the Proof (Part 1)

For some $\varepsilon < \min\{a-1, 2-b\}$, let us choose $\eta \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ with $\operatorname{supp} \eta \subseteq B(0,\varepsilon)$ and let $\eta_m := c_m \eta(\,\cdot - 2^m e_1)$, $m = 1, \ldots, j$, where $c := (c_m)_{m=1}^j \in \mathbb{C}^j \setminus \{0\}$ and e_1 denotes the first unit vector. Thus, we have $\operatorname{supp} \eta_m \subseteq B(2^m e_1, \varepsilon)$ for all m. Based on these functions we define $f, \zeta \in \mathcal{S}(\mathbb{R}^d)$ by

$$f := \sum_{m=1}^{j} \mathcal{F}^{-1}\eta_m \qquad \text{and} \qquad \zeta := D_{2^{-j}} f.$$

Then, by the support properties and definitions of the functions

$$\|D_{2^j}\zeta|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\| \sim \left(\sum_{m=1}^{j} \left(2^{ms}\,|c_m|\right)^q \right)^{1/q}.$$

On the other hand, for $k \in \mathbb{N}$ the equation $(D_{2^{-j}}\varphi_k)\,\eta_m = 0$ for $m = 1, \ldots, j$ yields

$$\|\zeta|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\| = \|\mathcal{F}^{-1}[\varphi_0\,\mathcal{F}\zeta]|\mathcal{M}^u_p(\mathbb{R}^d)\| \sim 2^{j\frac{d}{u}}\,\|f|\mathcal{M}^u_p(\mathbb{R}^d)\|.$$

DFG Deutsche Forschungsgemeinschaft

Lower Bounds: Sketch of the Proof (Part 2)

For $u < 1$ by the definition of f we get

$$\|f|\mathcal{M}_p^u(\mathbb{R}^d)\| \lesssim \sum_{m=1}^{j} |c_m|$$

and for $u \geq 1$ (see Part 1 of the proof),

$$\|f|\mathcal{M}_p^u(\mathbb{R}^d)\| \lesssim \left(\sum_{m=1}^{j} |c_m|^2\right)^{\frac{1}{2}}.$$

In conclusion,

$$\|D_{2^j}|\mathcal{L}\big(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\big)\| \geq \frac{\|D_{2^j}\zeta|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\|}{\|\zeta|\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\|}$$

$$\gtrsim 2^{-j\frac{d}{u}} \left(\sum_{m=1}^{j} \left(2^{ms}|c_m|\right)^q\right)^{\frac{1}{q}} \begin{cases} \left(\sum_{m=1}^{j} |c_m|\right)^{-1}, & 0 < u < 1, \\ \left(\sum_{m=1}^{j} |c_m|^2\right)^{-\frac{1}{2}}, & 1 \leq u < \infty. \end{cases}$$

Setting $c = (c_m)_{m=1}^{j} := e_j$ yields $\|D_{2^j}|\mathcal{L}\big(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\big)\| \gtrsim 2^{j(s-\frac{d}{u})}$,
while $c := e_1$ gives $\|D_{2^j}|\mathcal{L}\big(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\big)\| \gtrsim 2^{j(0-\frac{d}{u})}$.

DFG Deutsche Forschungsgemeinschaft

Lower Bounds: Sketch of the Proof (Part 3)

For $s = 0$ we can also set $c := (1, \ldots, 1)$ to derive

$$\|D_{2^j}|\mathcal{L}\big(\mathcal{E}^{0}_{u,p,q}(\mathbb{R}^d)\big)\| \gtrsim 2^{j(0-\frac{d}{u})} \begin{cases} j^{\frac{1}{q}-1}, & 0 < u < 1, \\ j^{\frac{1}{q}-\frac{1}{2}}, & 1 \leq u < \infty. \end{cases}$$

DFG Deutsche Forschungsgemeinschaft

Lower Bounds: Improved Estimates

Theorem

Let $0 < p \leq u < \infty$, $0 < q \leq \infty$ *as well as* $s \in \mathbb{R}$ *and* $\lambda \geq 2$.

(i) If $0 < p \leq 1$ *and* $s = \sigma_p$, *then there holds*

$$\| D_\lambda | \mathcal{L}\big(\mathcal{E}^{\sigma_p}_{u,p,q}(\mathbb{R}^d)\big) \| \gtrsim \lambda^{\sigma_p - \frac{d}{u}} \, \lambda^{d(\frac{1}{u} - \frac{1}{p})} \, (\log_2 \lambda)^{\frac{1}{p}}.$$

(ii) If $0 < p < 1$ *and* $s < \sigma_p$, *then*

$$\| D_\lambda | \mathcal{L}\big(\mathcal{E}^{s}_{u,p,q}(\mathbb{R}^d)\big) \| \gtrsim \lambda^{d(\frac{1}{u} - 1) - \frac{d}{u}}.$$

DFG Deutsche Forschungsgemeinschaft

Lower Bounds: Sketch of the Proof

Let us fix $\zeta \in \mathcal{D}(\mathbb{R}^d)$ with

$$\zeta \geq 0, \quad \operatorname{supp} \zeta \subseteq S := B\Big(0, \frac{1}{8}\Big) \subseteq \mathbb{R}^d \qquad \text{and} \qquad \int_{\mathbb{R}^d} \zeta(x) dx = 1.$$

Then for all parameters there holds $\zeta, D_{2^j}\zeta \in \mathcal{S}(\mathbb{R}^d) \subseteq \mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ and

$$\|D_{2^j} | \mathcal{L}(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d))\| \geq \frac{\|D_{2^j}\zeta | \mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\|}{\|\zeta | \mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\|} \sim \|D_{2^j}\zeta | \mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\|.$$

By a local mean characterization due to Sawano and Tanaka (2007) we can show that

$$\|D_{2^j}\zeta | \mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\| \gtrsim 2^{j(s-\frac{d}{p})} \cdot \begin{cases} \left(\sum_{\ell=0}^{j-1} 2^{\ell(d\,[\frac{1}{p}-1]-s)p} \right)^{\frac{1}{p}}, & 0 < q < \infty, \\ \sup_{\ell=0,\ldots,j-1} 2^{\ell(d\,[\frac{1}{p}-1]-s)}, & q = \infty. \end{cases}$$

Lower Bounds: Sketch of the Proof (Part 2)

Now for $p < 1$ and $s < \sigma_p = d(\frac{1}{p} - 1)$ we can conclude

$$\| D_{2^j} | \mathcal{L}(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)) \| \gtrsim 2^{j(\sigma_p - \frac{d}{p})} = 2^{-jd} = 2^{j[d(\frac{1}{u}-1) - \frac{d}{u}]}.$$

For $p \leq 1$ and $s = \sigma_p$ we find

$$\| D_{2^j} \zeta | \mathcal{E}^{\sigma_p}_{u,p,q}(\mathbb{R}^d) \| \gtrsim 2^{j(\sigma_p - \frac{d}{p})} j^{\frac{1}{p}} = 2^{-jd} j^{\frac{1}{p}} \qquad \text{if} \quad q < \infty.$$

For $q = \infty$ we can use the Gagliardo-Nierenberg inequality.

DFG Deutsche Forschungsgemeinschaft

The Gagliardo-Nierenberg inequality

Theorem

For $i = 0, 1$ let $0 < p_i \le u_i < \infty$ and $0 < q_i, r \le \infty$ as well as $s_i \in \mathbb{R}$, where $s_0 < s_1$. For $0 < \Theta < 1$ define

$$s_0 < s_\Theta := (1 - \Theta)\, s_0 + \Theta\, s_1 < s_1$$

and

$$\frac{1}{u_\Theta} := \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1} \le \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1} =: \frac{1}{p_\Theta}.$$

Then for all $f \in \mathcal{S}'(\mathbb{R}^d)$ there holds (with a constant independent of f)

$$\|f|\mathcal{E}^{s_\Theta}_{u_\Theta,p_\Theta,r}(\mathbb{R}^d)\| \lesssim \|f|\mathcal{E}^{s_0}_{u_0,p_0,q_0}(\mathbb{R}^d)\|^{1-\Theta} \|f|\mathcal{E}^{s_1}_{u_1,p_1,q_1}(\mathbb{R}^d)\|^{\Theta}.$$

DFG Deutsche Forschungsgemeinschaft

Summary and outlook

Theorem

Let $0 < p \leq u < \infty$, $0 < q \leq \infty$ *and* $s \in \mathbb{R}$. *For* $\frac{1}{2} < \lambda < 2$ *we have*

$$\| D_\lambda | \mathcal{L}(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)) \| \sim 1.$$

For $\lambda \geq 2$ *the following estimates hold:*

(1) If $s > \sigma_p$, *then* $\| D_\lambda | \mathcal{L}(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)) \| \sim \lambda^{s-\frac{d}{u}}$.

(2) If $s = \sigma_p$, *then for* $p > 1$ *it is*

$$\| D_\lambda | \mathcal{L}(\mathcal{E}^0_{u,p,q}(\mathbb{R}^d)) \| \sim \lambda^{0-\frac{d}{u}} (\log_2 \lambda)^{\max\{0, \frac{1}{q} - \frac{1}{2}\}}$$

while for $p = 1$ *there holds*

$$\lambda^{-\frac{d}{u}} (\log_2 \lambda)^{\max\{\frac{1}{p}, \frac{1}{q}\}} \gtrsim \| D_\lambda | \mathcal{L}\left(\mathcal{E}^0_{u,p,q}(\mathbb{R}^d)\right) \| \gtrsim \lambda^{-\frac{d}{u}} \cdot \begin{cases} (\log_2 \lambda)^{\max\{\frac{1}{p}, \frac{1}{q} - \frac{1}{2}\}}, & p = u, \\ (\log_2 \lambda)^{\max\{0, \frac{1}{q} - \frac{1}{2}\}}, & p < u, \end{cases}$$

and for $p < 1$ *we have*

$$\lambda^{\sigma_p - \frac{d}{u}} (\log_2 \lambda)^{\max\{\frac{1}{p}, \frac{1}{q}\}} \gtrsim \| D_\lambda | \mathcal{L}\left(\mathcal{E}^{\sigma_p}_{u,p,q}(\mathbb{R}^d)\right) \| \gtrsim \lambda^{\sigma_p - \frac{d}{u}} \cdot \begin{cases} (\log_2 \lambda)^{\frac{1}{p}}, & p = u, \\ 1, & p < u. \end{cases}$$

DFG Deutsche Forschungsgemeinschaft

Summary and outlook

Theorem

(3) If $s < \sigma_p$*, then for* $p \geq 1$ *it holds*

$$\|D_\lambda | \mathcal{L}(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d))\| \sim \lambda^{\sigma_p - \frac{d}{u}}$$

while for $p < 1$ *we have*

$$\lambda^{\sigma_p - \frac{d}{u}} \gtrsim \|D_\lambda | \mathcal{L}(\mathcal{E}^s_{u,p,q}(\mathbb{R}^d))\| \gtrsim \lambda^{\max\{s,\sigma_u\} - \frac{d}{u}} \cdot \begin{cases} (\log_2 \lambda)^{\max\{0, \frac{1}{q} - \frac{1}{2}\}}, & s = 0 \text{ and } 1 \leq u, \\ 1, & \text{else.} \end{cases}$$

Therein, all constants are independent of λ*. To incorporate the case* $q = \infty$ *we use the convention* $\frac{1}{\infty} = 0$*.*

Thank you very much for your attention!

DFG Deutsche Forschungsgemeinschaft