Lorentz-Sobolev spaces that are multiplication algebras

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Abstract

We verify a conjecture on Triebel-Lizorkin-Lorentz spaces $\boldsymbol{F^s_q L_{p,r}(\mathbb{R}^n)}$ proving that they are multiplication algebras if $\boldsymbol{s > n/p}$. We also prove the corresponding conjecture on Besov-Lorentz spaces $\boldsymbol{B^s_q L_{p,r}(\mathbb{R}^n)}$. Moreover, we show that in a certain range of parameters the condition $\boldsymbol{s > n/p}$ is necessary for these spaces to be multiplication algebras.

Keywords: Triebel-Lizorkin-Lorentz spaces, Besov-Lorentz spaces, multiplication algebras, real interpolation spaces

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Dedicated to Professor Hans Triebel on the occasion of his 90th birthday.

1 Introduction

Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ and Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ play a central role in the theory of function spaces. They include many well-known spaces of functions and distributions such as Sobolev spaces, classical Besov spaces, Hölder-Zygmund spaces, inhomogeneous Hardy spaces or Lebesgue spaces. We refer to the books by Triebel [39–41, 44] for the theory of these spaces. The two scales are important in the theory of partial differential equations.

Spaces $F^s_{p,q}(\mathbb{R}^n)$ and $B^s_{p,q}(\mathbb{R}^n)$ have many useful properties. In particular, they are multiplication algebras provided $s > n/p$. That is, the product of two elements of $F^s_{p,q}(\mathbb{R}^n)$ is again in $F^s_{p,q}(\mathbb{R}^n)$ if $s > n/p$, and the same happens with $B^s_{p,q}(\mathbb{R}^n)$. This property is important for applications to nonlinear heat equations and to Navier-Stokes equations (see the monographs by Triebel [42, 43]). Necessary and sufficient conditions on Triebel-Lizorkin and Besov spaces have been shown in order to be multiplication algebras. Contributions to this research are due to Triebel [36, 37], Kalyabin [22, 23], Nilsson and Franke [17] (see also [39, Remark 2.8.3/3] for references to earlier contributions).

But the scales of spaces $F^s_{p,q}(\mathbb{R}^n)$ and $B^s_{p,q}(\mathbb{R}^n)$ have a weak point: they are not closed by real interpolation. This is the reason for introducing Lorentz-Sobolev spaces $F^s_q L_{p,r}(\mathbb{R}^n), B^s_q L_{p,r}(\mathbb{R}^n)$, which are defined by replacing the role of the Lebesgue space $L_p(\mathbb{R}^n)$ by the more general Lorentz space $L_{p,r}(\mathbb{R}^n)$ in the Fourier-analytical definition of Triebel-Lizorkin and Besov spaces. These spaces with Lorentz smoothness have been used since the early 1960s in different contexts. See, for example, the papers by Peetre [26, 27], Fefferman, Riviere and Sagher [15], Stein [34], Edmunds and Triebel [14], Caetano [9], Yang, Cheng and Peng [48], Cianchi and Pick [10], Xiang and Yan [46, 47], Almeida and Caetano [1, 2], Wadade [45], and the more recent papers by Grafakos and Slavíkowa [19], Seeger and Trebels [32], Hobus and Saal [20], Besoy, Cobos and Triebel [6], Besoy, Haroske and Triebel [7], Besoy and Cobos [5], Sun, Yang and Yuan [35] and Feichtinger, Sun, Yang and Yuan [16].

Besoy, Cobos and Triebel [6, Theorem 5.5] used the characterization by interpolation of Triebel-Lizorkin-Lorentz spaces and the bilinear interpolation theorem for the real method to show that $F^s_q L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra if $s > n/p$, $0 < r \leq \min(1, q)$ and $r < p$. For Besov-Lorentz spaces they derive in [6, Theorem 7.2] that $B^s_q L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra if $s > n/p$, $0 < q \leq r \leq 1$ and $r < p$. They conjecture in [6, Conjectures 5.6 and 7.3] that spaces $F^s_q L_{p,r}(\mathbb{R}^n)$ and $B^s_q L_{p,r}(\mathbb{R}^n)$ are multiplication algebras provided that $s > n/p$ without any condition on r and q.

A step forward in the direction of these conjectures has been taken by Besoy and Cobos [5, Theorems 6.5 to 6.7]. This time they base the arguments on the description of Besov-Lorentz spaces as approximation spaces and on the bilinear interpolation theorem for the complex interpolation method. After their results, it remains open the case $p < r$ for both spaces $B^s_q L_{p,r}(\mathbb{R}^n)$ and $F^s_q L_{p,r}(\mathbb{R}^n)$, and also the case $0 < q \leq 1$ and $q < r$ for spaces $F^s_q L_{p,r}(\mathbb{R}^n)$.

The goal of the present paper is to prove both conjectures of [6]. As for the Triebel-Lizorkin-Lorentz spaces, we follow a direct approach based on the method of paramultiplication. This method has been used by Peetre [27, Chapter 7] and Triebel [37, 2.6]. It is important in microlocal analysis and in the theory of Calderón-Zygmund singular integral operators (see [33, Remark 2.2/2] and [31, pp. 257-258] for detailed references). Our arguments are a combination of interpolation results and ideas developed by Triebel [37, 2.6.2], [39, 2.8.2 and 2.8.3] and Franke [17]. Sharp embeddings between spaces with Lorentz smoothness obtained by Seeger and Trebels [32] are important in our arguments as well. We also show that condition $s > n/p$ is necessary for $F^s_q L_{p,r}(\mathbb{R}^n)$ to be a multiplication algebra at least for a certain range of parameters.

Concerning Besov-Lorentz spaces, we follow a different approach. We first show that the description of $B^s_q L_{p,r}(\mathbb{R}^n)$ as approximation spaces modeled on $F^0_2 L_{p,r}(\mathbb{R}^n)$ (see [5, Theorem 6.1]) still works for $r = \infty$ and $0 < p \leq 1$. For this purpose, we introduce the local Hardy-Lorentz spaces $h_{p,\infty}$ in Section 3 and establish some of their properties. Then we prove the conjecture for Besov-Lorentz spaces. We base the arguments on our achievement in spaces $F^s_q L_{p,r}(\mathbb{R}^n)$ and a new characterization of spaces $B^s_q L_{p,r}(\mathbb{R}^n)$ as approximation spaces. We also show that condition $s > n/p$ is necessary for $B^s_q L_{p,r}(\mathbb{R}^n)$ to be a multiplication algebra at least for a certain range of parameters. At this point our results are based on duality. For this reason, we describe in Section 3 the dual space of $B^s_q L_{p,q}(\mathbb{R}^n)$ for some values of the parameters.

2 Preliminaries

By a *quasi-Banach couple* $\overline{A} = (A_0, A_1)$ it is meant two quasi-Banach spaces A_0, A_1 continuously embedded in some Hausdorff topological vector space.

Let $a \in A_0 + A_1$ and $t > 0$. *Peetre's K-functional* is defined by

$$K(t, a) = K(t, a; A_0, A_1) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j, j = 0, 1\}$$

For $0<\theta<1$ and $0<r\le\infty$, the *real interpolation space* $(A_0,A_1)_{\theta,r}$ consists of all those $a\in A_0+A_1$ having a finite quasi-norm

$$\|a\|_{(A_0,A_1)_{\theta,r}}=\left(\int_0^\infty [t^{-\theta}K(t,a)]^r\frac{dt}{t}\right)^{1/r}$$

(the integral should be replaced by the supremun when $r=\infty$). The space $(A_0,A_1)_{\theta,r}$ is a Banach space provided A_0 and A_1 are Banach spaces and $1\le r\le\infty$. Otherwise $(A_0,A_1)_{\theta,r}$ is a quasi-Banach space.

Assume (B_0,B_1) is another quasi-Banach couple and let T be a linear operator such that $T: A_j\longrightarrow B_j$ is bounded for $j=0,1$, with norm $\|T\|_{A_j,B_j}$. Then $T:(A_0,A_1)_{\theta,r}\longrightarrow(B_0,B_1)_{\theta,r}$ is also bounded with

$$\|T\|_{(A_0,A_1)_{\theta,r},(B_0,B_1)_{\theta,r}}\le\|T\|^{1-\theta}_{A_0,B_0}\|T\|^{\theta}_{A_1,B_1}$$

This interpolation property still holds for sublinear operators if the couple in the target is formed by quasi-Banach spaces of measurable functions with the lattice property (that is, if $|f|$ is smaller than $|g|$ almost everywhere, then the quasi-norm of f is smaller than the quasi-norm of g) (see, for example, [24] or [2]). We refer to the books by Bergh and Löfström [4], Triebel [38], Bennett and Sharpley [3] and Brudnyǐ and Krugljak [8] for further details on the real interpolation method.

If $q<\infty$ then $A_0\cap A_1$ is dense in $(A_0,A_1)_{\theta,q}$. We put A_j° for the closure of $A_0\cap A_1$ in A_j, and $(A_0,A_1)^\circ_{\theta,\infty}$ for the closure of $A_0\cap A_1$ in $(A_0,A_1)_{\theta,\infty}$.

Subsequently, if $1\le q\le\infty$ we put $\dfrac{1}{q}+\dfrac{1}{q'}=1$. If $0<q<1$, we set $q'=\infty$.

Concerning duality, note that the dual space of a quasi-normed space is either $\{0\}$ or a normed space. If (A_0,A_1) is a Banach couple such that $A_0\cap A_1$ is dense in A_0 and A_1, then we have with equivalent norms

$$(A_0,A_1)'_{\theta,q}=(A'_0,A'_1)_{\theta,q'}\quad\text{ if }\quad 0<q<\infty \tag{1}$$

and

$$\left((A_0,A_1)^\circ_{\theta,\infty}\right)'=(A'_0,A'_1)_{\theta,1} \tag{2}$$

(see [4, 28, 38]).

Let $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $s\in\mathbb{R}$, $0<q\le\infty$ and let A be a quasi-Banach space. We write $l^s_q(A)$ for the space of all sequences $(a_k)_{k\in\mathbb{N}_0}\subseteq A$ having a finite quasi-norm

$$\|(a_k)\|_{l^s_q(A)}=\left(\sum_{k=0}^\infty 2^{ksq}\|a_k\|_A^q\right)^{1/q}$$

(with the usual modification if $q=\infty$). If $A=\mathbb{C}$, we simply write l^s_q.

Extending [38, Theorem 1.18.2] and [4, Theorem 5.6.1] to quasi-Banach spaces A, it turns out that with equivalent quasi-norms we have

$$\left(l^{s_0}_{q_0}(A),l^{s_1}_{q_1}(A)\right)_{\theta,r}=l^s_r(A). \tag{3}$$

Here $0<q_0,q_1,r\le\infty$, $-\infty<s_0\ne s_1<\infty$, $0<\theta<1$ and $s=(1-\theta)s_0+\theta s_1$.

Given $n\in\mathbb{N}$, we write $\mathbb{R}^n$ for the Euclidean n-space endowed with the Lebesgue measure and put $|x|$ for the norm in $\mathbb{R}^n$.

If A is a quasi-Banach space and $0<p\le\infty$, we put $L_p(A)=L_p(\mathbb{R}^n,A)$ for the usual *vector-valued L_p-space* in the sense of the Bochner integral. We write $L_p(\mathbb{R}^n)$ if $A=\mathbb{C}$.

Let $0<p<\infty$ and $0<r\le\infty$. The *Lorentz space* $L_{p,r}(A)=L_{p,r}(\mathbb{R}^n,A)$ is formed by all (equivalence classes of) strongly measurable functions from $\mathbb{R}^n$ into A which have a finite quasi-norm

$$\|f\|_{L_{p,r}(A)}=\left(\int_0^\infty [t^{1/p}f^*(t)]^r\frac{dt}{t}\right)^{1/r}$$

(the integral should be replaced by the supremum if $r = \infty$) . Here f^* is the *non-increasing rearrangement* of f defined by

$$f^*(t) = \inf\{s > 0 : |\{x \in \mathbb{R}^n : \|f(x)\|_A > s\}| \le t\}$$

where $|W|$ stands for the Lebesgue measure of the set W (see [3, 4, 13]). If $p = r$, then $L_{p,p}(A) = L_p(A)$. We write $L_{p,r}(\mathbb{R}^n)$ when $A = \mathbb{C}$.

Applying the real method to a couple of Lebesgue spaces one obtains Lorentz spaces: If $0 < p_0 < p < p_1 < \infty$ and $\theta = \dfrac{p_1(p-p_0)}{p(p_1-p_0)}$, then with equivalent quasi-norm we have

$$(L_{p_0}(A), L_{p_1}(A))_{\theta,r} = L_{p,r}(A) \tag{4}$$

(see [4, Theorem 5.2.1] or [38, p. 134 and Remark 5 on p. 135]). We refer to [3, 4, 13, 18, 21, 38] for other properties of Lorentz spaces.

Note that for $0 < p < \infty, 0 < r \le \infty$ and $\lambda > 0$ we have

$$\|f(\lambda\,\cdot)\|_{L_{p,r}(\mathbb{R}^n)} = \lambda^{-n/p}\|f\|_{L_{p,r}(\mathbb{R}^n)}. \tag{5}$$

The dual spaces of Lorentz spaces can be found, for example, in [18, Theorem 1.4.17]. In particular, we have

$$\big(L_{p,q}(\mathbb{R}^n)\big)' = \begin{cases} L_{p',q'}(\mathbb{R}^n) & \text{if } 1 < p, q < \infty, \\ L_{p',\infty}(\mathbb{R}^n) & \text{if } 1 < p < \infty \text{ and } 0 < q \le 1. \end{cases}$$

3 Function spaces of Lorentz-Sobolev type

We write $\mathcal{S}(\mathbb{R}^n)$ for the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ for its topological dual, the space of all tempered distributions. For $f \in \mathcal{S}'(\mathbb{R}^n)$, we put $\mathcal{F}f = \hat{f}$ for its Fourier transform and $\mathcal{F}^{-1}f = f^\vee$ for the inverse Fourier transform.

Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\varphi_0(x) = 1 \ \text{ if } |x| \le 1 \text{ and } \varphi_0(x) = 0 \ \text{ if } |x| \ge 3/2.$$

For $k \in \mathbb{N}$, we put

$$\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x) = \varphi_1(2^{-k+1}x).$$

The sequence $(\varphi_k)_{k\in\mathbb{N}_0}$ is a smooth dyadic resolution of unity, $\sum_{k=0}^{\infty}\varphi_k(x) = 1$ for all $x \in \mathbb{R}^n$.

The *Triebel-Lizorkin-Lorentz space* $F^s_q L_{p,r}(\mathbb{R}^n)$ is formed by all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ having a finite quasi-norm

$$\|f\|_{F^s_q L_{p,r}(\mathbb{R}^n)} = \left\| \left(\sum_{k=0}^{\infty} 2^{ksq} |(\varphi_k \hat{f})^\vee|^q \right)^{1/q} \right\|_{L_{p,r}(\mathbb{R}^n)}$$

The *Besov-Lorentz space* $B^s_q L_{p,r}(\mathbb{R}^n)$ is defined as the collection of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that the quasi-norm

$$\|f\|_{B^s_q L_{p,r}(\mathbb{R}^n)} = \left(\sum_{k=0}^{\infty} 2^{ksq} \|(\varphi_k \hat{f})^\vee\|^q_{L_{p,r}(\mathbb{R}^n)} \right)^{1/q}$$

is finite.

Note that if $p = r$ then $F^s_q L_{p,p}(\mathbb{R}^n)$ coincides with the classical Triebel-Lizorkin space $F^s_{p,q}(\mathbb{R}^n)$, and $B^s_q L_{p,p}(\mathbb{R}^n)$ is the classical Besov space $B^s_{p,q}(\mathbb{R}^n)$.

As was shown in [5, Proposition 2.3], we have

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow F^s_q L_{p,r}(\mathbb{R}^n),\ B^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$$

where $\hookrightarrow$ means continuous embedding.

The spaces with Lorentz smoothness arise by real interpolation between classical spaces. Namely for $s \in \mathbb{R}, 0 < q, r \leq \infty$, $0 < p_0 \neq p_1 < \infty$, $0 < \theta < 1$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, with equivalent quasi-norms we have the following

$$\left(F^s_{p_0,q}(\mathbb{R}^n), F^s_{p_1,q}(\mathbb{R}^n)\right)_{\theta,r} = F^s_q L_{p,r}(\mathbb{R}^n) \tag{6}$$

(see [6, Theorem 3.6] and [38, Theorem 2.4.2/(c)].

Let $-\infty < s_0, s_1 < \infty$, $0 < \theta < 1$, $0 < q_0, q_1 < \infty$, $0 < p_0 \neq p_1 < \infty$, $s = (1-\theta)s_0 + \theta s_1$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, then we have the following with equivalent quasi-norms

$$\left(B^{s_0}_{p_0,q_0}(\mathbb{R}^n), B^{s_1}_{p_1,q_1}(\mathbb{R}^n)\right)_{\theta,q} = B^s_q L_{p,q}(\mathbb{R}^n) \tag{7}$$

(see [6, Theorem 6.5] and [38, Theorem 2.4.1/(c)]; see also [2, Theorem 4.10]).

Furthermore, the following interpolation formula holds.

Theorem 3.1. *Let $-\infty < s_0 \neq s_1 < \infty$, $0 < \theta < 1$, $s = (1-\theta)s_0 + \theta s_1$, $0 < p < \infty$ and $0 < q_0, q_1, q, r \leq \infty$. Then with equivalent quasi-norms we have*

$$\left(F^{s_0}_{q_0} L_{p,r}(\mathbb{R}^n), F^{s_1}_{q_1} L_{p,r}(\mathbb{R}^n)\right)_{\theta,q} = B^s_q L_{p,r}(\mathbb{R}^n) \tag{8}$$

Proof The result is known for $r < \infty$ (see [6, Theorem 6.7]). Assume $r = \infty$. Pick $0 < \tau_j < \min(p, q_j)$, $j = 0, 1$. By [5, Theorem 5.3] we have

$$B^s_q L_{p,\infty}(\mathbb{R}^n) = \left(B^{s_0}_{\tau_0} L_{p,\infty}(\mathbb{R}^n), B^{s_1}_{\tau_1} L_{p,\infty}(\mathbb{R}^n)\right)_{\theta,q} = \left(B^{s_0}_{\infty} L_{p,\infty}(\mathbb{R}^n), B^{s_1}_{\infty} L_{p,\infty}(\mathbb{R}^n)\right)_{\theta,q}.$$

Now the result follows having in mind the embeddings

$$B^{s_j}_{\tau_j} L_{p,\infty}(\mathbb{R}^n) \hookrightarrow F^{s_j}_{q_j} L_{p,\infty}(\mathbb{R}^n) \hookrightarrow B^{s_j}_{\infty} L_{p,\infty}(\mathbb{R}^n),\ j = 0, 1,$$

(see [32, Theorems 1.1 and 1.2]).

□

By means of these interpolation formulae, it was shown in [6] how to transfer a number of properties of the usual Triebel-Lizorkin and Besov spaces to Lorentz-Sobolev spaces. Next we continue this research.

Lemma 3.2. *Let $-\infty < s < \infty$ and $0 < p, q, r < \infty$. Then $\mathcal{S}(\mathbb{R}^n)$ is dense in $F^s_q L_{p,r}(\mathbb{R}^n)$ and in $B^s_q L_{p,r}(\mathbb{R}^n)$.*

Proof Take $0 < p_0 < p < p_1 < \infty$ and $0 < \theta < 1$ with $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. By (6) we have that

$$\left(F^s_{p_0,q}(\mathbb{R}^n), F^s_{p_1,q}(\mathbb{R}^n)\right)_{\theta,r} = F^s_q L_{p,r}(\mathbb{R}^n).$$

Since $r < \infty$, $F^s_{p_0,q}(\mathbb{R}^n) \cap F^s_{p_1,q}(\mathbb{R}^n)$ is dense in $F^s_q L_{p,r}(\mathbb{R}^n)$. Moreover, by [39, Theorem 2.3.3], one can approximate any element of $F^s_{p_j,q}(\mathbb{R}^n)$ by functions of $\mathcal{S}(\mathbb{R}^n)$ because $q < \infty$. Consequently, $\mathcal{S}(\mathbb{R}^n)$ is dense in $F^s_q L_{p,r}(\mathbb{R}^n)$.

The proof for $B^s_q L_{p,r}(\mathbb{R}^n)$ can be carried out similarly but using now (8). □

Concerning duality, according to [39, Section 2.11], we know that

$$\left(B^s_{p,q}(\mathbb{R}^n)\right)' = B^{-s}_{p',q'}(\mathbb{R}^n)$$

provided that $-\infty < s < \infty$, $1 \le p < \infty$ and $0 < q < \infty$. As before, we put $\frac{1}{q} + \frac{1}{q'} = 1$ if $1 \le q < \infty$, and $q' = \infty$ if $0 < q \le 1$. For Triebel-Lizorkin spaces we have

$$\big(F^s_{p,q}(\mathbb{R}^n)\big)' = F^{-s}_{p',q'}(\mathbb{R}^n)$$

provided that $-\infty < s < \infty$, $1 \le p < \infty$ and $1 < q < \infty$. Next we determine dual spaces of Lorentz-Sobolev spaces. We put $f^s_q L_{p,\infty}(\mathbb{R}^n)$ (respectively, $b^s_\infty L_{p,r}(\mathbb{R}^n)$) for the closure of $\mathcal{S}(\mathbb{R}^n)$ in $F^s_q L_{p,\infty}(\mathbb{R}^n)$ (respectively, $B^s_\infty L_{p,r}(\mathbb{R}^n)$).

Theorem 3.3. *Let $-\infty < s < \infty$, $1 < p, q < \infty$ and $0 < r < \infty$. Then we have with equivalence of norms*

$$\big(F^s_q L_{p,r}(\mathbb{R}^n)\big)' = F^{-s}_{q'} L_{p',r'}(\mathbb{R}^n)$$

and

$$\big(f^s_q L_{p,\infty}(\mathbb{R}^n)\big)' = F^{-s}_{q'} L_{p',1}(\mathbb{R}^n).$$

Proof Pick $1 < p_0 < p < p_1 < \infty$ and $0 < \theta < 1$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. By Lemma 3.2, $F^s_{p_0,q}(\mathbb{R}^n) \cap F^s_{p_1,q}(\mathbb{R}^n)$ is dense in $F^s_{p_j,q}(\mathbb{R}^n)$ for $j = 0, 1$. Since $\frac{1}{p'} = \frac{1-\theta}{p_0'} + \frac{\theta}{p_1'}$, according to (6) and (1), we obtain

$$\begin{aligned}\big(F^s_q L_{p,r}(\mathbb{R}^n)\big)' &= \big((F^s_{p_0,q}(\mathbb{R}^n), F^s_{p_1,q}(\mathbb{R}^n))_{\theta,r}\big)' = (F^{-s}_{p_0',q'}(\mathbb{R}^n), F^{-s}_{p_1',q'}(\mathbb{R}^n))_{\theta,r'}\\ &= F^{-s}_{q'} L_{p',r'}(\mathbb{R}^n).\end{aligned}$$

To establish the second formula of the statement, since we have $F^s_q L_{p,\infty}(\mathbb{R}^n) = (F^s_{p_0,q}(\mathbb{R}^n), F^s_{p_1,q}(\mathbb{R}^n))_{\theta,\infty}$ and $f^s_q L_{p,\infty}(\mathbb{R}^n) = \big(F^s_q L_{p,\infty}(\mathbb{R}^n)\big)^\circ$, using (2), we obtain

$$\begin{aligned}\big(f^s_q L_{p,\infty}(\mathbb{R}^n)\big)' &= \big((F^s_{p_0,q}(\mathbb{R}^n), F^s_{p_1,q}(\mathbb{R}^n))^\circ_{\theta,\infty}\big)' = (F^{-s}_{p_0',q'}(\mathbb{R}^n), F^{-s}_{p_1',q'}(\mathbb{R}^n))_{\theta,1}\\ &= F^{-s}_{q'} L_{p',1}(\mathbb{R}^n).\end{aligned}$$

□

Theorem 3.4. *Let $-\infty < s < \infty$, $1 < p < \infty$, $1 \le r < \infty$ and $0 < q < \infty$. Then we have with equivalence of norms*

$$\big(B^s_q L_{p,r}(\mathbb{R}^n)\big)' = B^{-s}_{q'} L_{p',r'}(\mathbb{R}^n)$$

and

$$(b^s_\infty L_{p,r}(\mathbb{R}^n))' = B^{-s}_1 L_{p',r'}(\mathbb{R}^n).$$

The first formula also holds if $-\infty < s < \infty$, $1 < p < \infty$, $0 < r \le 1$ and $0 < q \le 1$.

Proof Pick $1 < q_0, q_1 < \infty$, $-\infty < s_0 < s < s_1 < \infty$ and $0 < \theta < 1$ with $s = (1-\theta)s_0 + \theta s_1$. By Theorem 3.1 we have

$$\big(F^{s_0}_{q_0} L_{p,r}(\mathbb{R}^n), F^{s_1}_{q_1} L_{p,r}(\mathbb{R}^n)\big)_{\theta,q} = B^s_q L_{p,r}(\mathbb{R}^n)$$

and Lemma 3.2 yields that $F^{s_0}_{q_0} L_{p,r}(\mathbb{R}^n) \cap F^{s_1}_{q_1} L_{p,r}(\mathbb{R}^n)$ is dense in $F^{s_j}_{q_j} L_{p,r}(\mathbb{R}^n)$ for $j = 0, 1$. Therefore, if $1 \le r < \infty$, we can apply (1), Theorem 3.3, and again Theorem 3.1 to derive that

$$\big(B^s_q L_{p,r}(\mathbb{R}^n)\big)' = \Big(F^{-s_0}_{q_0'} L_{p',r'}(\mathbb{R}^n), F^{-s_1}_{q_1'} L_{p',r'}(\mathbb{R}^n)\Big)_{\theta,q'} = B^{-s}_{q'} L_{p',r'}(\mathbb{R}^n).$$

For the second part of the statement, since $B^s_\infty L_{p,r}(\mathbb{R}^n) = \big(F^{s_0}_{q_0} L_{p,r}(\mathbb{R}^n), F^{s_1}_{q_1} L_{p,r}(\mathbb{R}^n)\big)_{\theta,\infty}$ and $b^s_\infty L_{p,r}(\mathbb{R}^n) = \big(B^s_\infty L_{p,r}(\mathbb{R}^n)\big)^\circ$, we have

$$\begin{aligned}\big(b^s_\infty L_{p,r}(\mathbb{R}^n)\big)' &= \big((F^{s_0}_{q_0} L_{p,r}(\mathbb{R}^n), F^{s_1}_{q_1} L_{p,r}(\mathbb{R}^n))^\circ_{\theta,\infty}\big)'\\ &= (F^{-s_0}_{q_0'} L_{p',r'}(\mathbb{R}^n), F^{-s_1}_{q_1'} L_{p',r'}(\mathbb{R}^n))_{\theta,1} = B^{-s}_1 L_{p',r'}(\mathbb{R}^n).\end{aligned}$$

Finally, if $0 < r < 1$, then spaces $F^s_q L_{p,r}(\mathbb{R}^n)$ and $B^s_q L_{p,r}(\mathbb{R}^n)$ are quasi-Banach spaces. For this reason we cannot determine the dual by using the interpolation results, because they require Banach couples. But if $0<r\le 1$ we have that $L_{p,r}(\mathbb{R}^n)\hookrightarrow L_{p,1}(\mathbb{R}^n)$, therefore $B^s_qL_{p,r}(\mathbb{R}^n)\hookrightarrow B^s_qL_{p,1}(\mathbb{R}^n)$. According to our previous achievement, if $0<q\le 1$ we get that

$$B^{-s}_\infty L_{p',\infty}(\mathbb{R}^n)=\big(B^s_qL_{p,1}(\mathbb{R}^n)\big)'\hookrightarrow\big(B^s_qL_{p,r}(\mathbb{R}^n)\big)'$$

In order to check the converse embedding, consider a smooth dyadic resolution of unity $(\varphi_k)_{k\in\mathbb{N}_0}$, take any $g\in\big(B^s_qL_{p,r}(\mathbb{R}^n)\big)'$ and any $\varphi\in\mathcal{S}(\mathbb{R}^n)$. We have

$$\begin{aligned}|(\mathcal{F}^{-1}\varphi_k\mathcal{F}g)(\varphi)|&=|g(\mathcal{F}\varphi_k\mathcal{F}^{-1}\varphi)|\\&\le\|\mathcal{F}\varphi_k\mathcal{F}^{-1}\varphi\|_{B^s_qL_{p,r}(\mathbb{R}^n)}\,\|g\|_{\left(B^s_qL_{p,r}(\mathbb{R}^n)\right)'}\\&\le c\,2^{sk}\,\|\varphi\|_{L_{p,r}(\mathbb{R}^n)}\,\|g\|_{\left(B^s_qL_{p,r}(\mathbb{R}^n)\right)'}.\end{aligned}$$

Since $\big(L_{p,r}(\mathbb{R}^n)\big)'=L_{p',\infty}(\mathbb{R}^n)$, we obtain that

$$\sup_{k\in\mathbb{N}_0}2^{-sk}\|\mathcal{F}^{-1}\varphi_k\mathcal{F}g\|_{Lp',\infty(\mathbb{R}^n)}\le c\|g\|_{\left(B^s_qL_{p,r}(\mathbb{R}^n)\right)'}$$

and therefore $g\in B^{-s}_\infty L_{p',\infty}(\mathbb{R}^n)$. □

Next we show that if $0<p<1$ then the dual of $B^s_qL_{p,r}(\mathbb{R}^n)$ coincides with the dual of the Besov space $B^s_{p,q}(\mathbb{R}^n)$. Therefore (see [39, 2.11.3]), smoothness of the dual space is better than the expected value $-s$.

Theorem 3.5. *Let $-\infty<s<\infty$, $0<q<\infty$. Assume that $0<p<1$ and $0<r<\infty$, or $p=1$ and $0<r\le 1$. Then we have with equivalence of norms*

$$\big(B^s_qL_{p,r}(\mathbb{R}^n)\big)'=B^{-s+n(1/p-1)}_{\infty,q'}(\mathbb{R}^n).$$

Proof By [32, Theorem 1.5], we have $B^s_qL_{p,r}(\mathbb{R}^n)\hookrightarrow B^{s-n/p+n}_{1,q}(\mathbb{R}^n)$. Besides, the embedding is dense. Hence, using [39, Theorem 2.11.2/(i)], we have that

$$B^{-s+n/p-n}_{\infty,q'}(\mathbb{R}^n)\hookrightarrow\big(B^s_qL_{p,r}(\mathbb{R}^n)\big)'.$$

Conversely, take a smooth dyadic resolution of unity $(\varphi_k)_{k\in\mathbb{N}_0}$ and let $g\in\big(B^s_qL_{p,r}(\mathbb{R}^n)\big)'$. We have

$$|(\mathcal{F}^{-1}\varphi_k\mathcal{F}g)(x)|=|g(\mathcal{F}^{-1}\varphi_k)(x-\cdot)|\le\|(\mathcal{F}^{-1}\varphi_k)(x-\cdot)\|_{B^s_qL_{p,r}(\mathbb{R}^n)}\,\|g\|_{\left(B^s_qL_{p,r}(\mathbb{R}^n)\right)'}.\tag{9}$$

Let $(\psi_m)_{m\in\mathbb{N}_0}$ be another smooth dyadic resolution of unity and put $\psi_{-1}\equiv 0$. We know that $\varphi_k(y)=\varphi_1(2^{-k+1}y)$ and $\psi_k(y)=\psi_1(2^{-k+1}y)$. Hence

$$\begin{aligned}\|(\mathcal{F}^{-1}\varphi_k)(x-\cdot)\|_{B^s_qL_{p,r}(\mathbb{R}^n)}&=\Big(\sum_{m=0}^{\infty}2^{smq}\|\big(\mathcal{F}^{-1}\psi_m\mathcal{F}\big)\big((\mathcal{F}^{-1}\varphi_k)(x-\cdot)\big)\|^q_{L_{p,r}(\mathbb{R}^n)}\Big)^{1/q}\\&=\Big(\sum_{j=-1}^{1}2^{s(k+j)q}\|\big(\mathcal{F}^{-1}\psi_{k+j}\varphi_k\big)(x+\cdot)\|^q_{L_{p,r}(\mathbb{R}^n)}\Big)^{1/q}\\&=\Big(\sum_{j=-1}^{1}2^{s(k+j)q}\|\mathcal{F}^{-1}\psi_1(2^{-k-j+1}\cdot)\varphi_1(2^{-k+1}\cdot)\|^q_{L_{p,r}(\mathbb{R}^n)}\Big)^{1/q}\\&\le c_1\,2^{sk+nk(1-1/p)}\,\|\mathcal{F}^{-1}\psi_1\varphi_1\|_{L_{p,r}(\mathbb{R}^n)}\\&\le c_2\,2^{sk+nk(1-1/p)}\end{aligned}$$

where we have also used (5), that $\mathcal{F}^{-1}\varphi_1\in\mathcal{S}(\mathbb{R}^n)$ and that

$$\|\mathcal{F}^{-1}\psi_1\varphi_1\|_{L_{p,r}(\mathbb{R}^n)}=\|\mathcal{F}^{-1}\psi_1\mathcal{F}\mathcal{F}^{-1}\varphi_1\|_{L_{p,r}(\mathbb{R}^n)}\le\|\mathcal{F}^{-1}\varphi_1\|_{F^0_2L_{p,r}(\mathbb{R}^n)}<\infty.$$

This yields

$$\|(\mathcal{F}^{-1}\varphi_k)(x-\cdot)\|_{B^s_qL_{p,r}(\mathbb{R}^n)}\le c_2\,2^{sk+nk(1-1/p)}.$$

Therefore

$$2^{-sk+nk(1/p-1)}\|\mathcal{F}^{-1}\varphi_k\mathcal{F}g\|_{L_\infty(\mathbb{R}^n)}\le c_2\,\|g\|_{\left(B^s_qL_{p,r}(\mathbb{R}^n)\right)'}.\tag{10}$$

If $0 < q \le 1$, we conclude the wanted equality

$$\big(B^s_q L_{p,r}(\mathbb{R}^n)\big)' = B^{-s+n(1/p-1)}_{\infty,\infty}(\mathbb{R}^n).$$

For $1 < q < \infty$, having in mind (10), we can pick $x_k \in \mathbb{R}^n$ such that

$$\|\mathcal{F}^{-1}\varphi_k \mathcal{F} g\|_{L_\infty(\mathbb{R}^n)} \le 2|(\mathcal{F}^{-1}\varphi_k \mathcal{F} g)(x_k)|.$$

Let $N \in \mathbb{N}$ and for $k = 0, \dots, N$ take any complex numbers a_k. Put

$$\varphi(x) = \sum_{k=0}^{N} 2^{-sk+nk(1/p-1)} a_k (\mathcal{F}^{-1}\varphi_k)(x_k - x).$$

Then $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with

$$\|\varphi\|_{B^s_q L_{p,r}(\mathbb{R}^n)} \le c_3 \,\Big(\sum_{k=0}^{N} |a_k|^q\Big)^{1/q}$$

where c_3 is independent of N and of $\{a_0, \dots, a_N\}$. Using (9), we have

$$\begin{aligned} \Big|\sum_{k=0}^{N} 2^{-sk+nk(1/p-1)} a_k (\mathcal{F}^{-1}\varphi_k \mathcal{F} g)(x_k)\Big| &= |g(\varphi)| \\ &\le \|\varphi\|_{B^s_q L_{p,r}(\mathbb{R}^n)} \, \|g\|_{\left(B^s_q L_{p,r}(\mathbb{R}^n)\right)'} \\ &\le c_3 \Big(\sum_{k=0}^{N} |a_k|^q\Big)^{1/q} \|g\|_{\left(B^s_q L_{p,r}(\mathbb{R}^n)\right)'}. \end{aligned}$$

Consequently,

$$\|(2^{-sk+nk(1/p-1)}\mathcal{F}^{-1}\varphi_k \mathcal{F} g)\|_{\ell_{q'}(L_\infty(\mathbb{R}^n))} \le 2\, c_3 \, \|g\|_{\left(B^s_q L_{p,r}(\mathbb{R}^n)\right)'}$$

This completes the proof. □

As for Triebel-Lizorkin-Lorentz spaces with $0 < p < 1$ we have the following result.

Theorem 3.6. *Let $-\infty < s < \infty$, $0 < p < 1$, $p \le r \le 1$ and $0 < q < \infty$, or $p = 1$, $0 < q \le 1$ and $0 < r \le 1$. Then we have with equivalence of norms*

$$\big(F^s_q L_{p,r}(\mathbb{R}^n)\big)' = B^{-s+n(1/p-1)}_{\infty,\infty}(\mathbb{R}^n).$$

Proof According to [32, Theorems 1.1 and 1.2], in the first case for parameters we have

$$B^s_{p,\min(p,q,r)}(\mathbb{R}^n) \hookrightarrow F^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow B^{s-n(1/p-1)}_{1,1}(\mathbb{R}^n).$$

Therefore, by [39, Theorems 2.11.2/(i) and 2.11.3/(i)], we obtain that

$$B^{-s+n(1/p-1)}_{\infty,\infty}(\mathbb{R}^n) \hookrightarrow \big(F^s_q L_{p,r}(\mathbb{R}^n)\big)' \hookrightarrow B^{-s+n(1/p-1)}_{\infty,\infty}(\mathbb{R}^n).$$

If $p = 1, 0 < q \le 1$ and $0 < r \le 1$, we have

$$B^s_{\min(1,q,r)} L_{1,r}(\mathbb{R}^n) \hookrightarrow F^s_q L_{1,r}(\mathbb{R}^n) \hookrightarrow B^s_{1,1}(\mathbb{R}^n).$$

Hence, using Theorem 3.5 we conclude that $\big(F^s_q L_{1,r}(\mathbb{R}^n)\big)' = B^{-s}_{\infty,\infty}(\mathbb{R}^n)$.

□

As it is pointed out in [6, Theorem 6.4], in the definition of $B^s_q L_{p,r}(\mathbb{R}^n)$ we can replace the role of the Lorentz space $L_{p,r}(\mathbb{R}^n)$ by the *local Hardy-Lorentz space* $h_{p,r}$. Recall that $h_{p,r}$ is formed by all those $f \in \mathcal{S}'(\mathbb{R}^n)$ having a finite quasi-norm

$$\|f\|_{h_{p,r}} = \|\sup_{0<t<1} |(\psi(t\,\cdot)\hat{f})^\vee|\|_{L_{p,r}(\mathbb{R}^n)}.$$

Here $0 < p, r < \infty$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ is compactly supported and with $\psi(x) = 1$ if $|x| \le 1$.

Local Hardy-Lorentz spaces have been studied by Almeida and Caetano [2], who proved that if $0 < p_0 \ne p_1 < \infty$, $0 < r < \infty$, $0 < \theta < 1$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, then we have with equivalence of quasi-norms

$$(h_{p_0}, h_{p_1})_{\theta,r} = h_{p,r} \tag{11}$$

(see [2, Theorems 4.3 and 4.7]).

As far as we know, the case $r = \infty$ has not been considered in the literature. Since we will need some results involving $h_{p,\infty}$ in Section 5, next we introduce local Hardy-Lorentz spaces with $r = \infty$ by using real interpolation.

Definition 3.7. *For $0 < p < \infty$, pick $0 < p_0, p_1 < \infty$ and $0 < \theta < 1$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. We put*

$$h_{p,\infty} = (h_{p_0}, h_{p_1})_{\theta,\infty}$$

Lemma 3.8. *The space $h_{p,\infty}$ does not depend of the particular choice of $0 < p_0, p_1 < \infty$ and $0 < \theta < 1$ provided that $1/p = (1-\theta)/p_0 + \theta/p_1$.*

Proof Assume that $0 < q_0, q_1 < \infty$ and $0 < \eta < 1$ satisfy that $1/p = (1-\eta)/q_0 + \eta/q_1$. We should show that, with equivalence of quasi-norms

$$(h_{p_0}, h_{p_1})_{\theta,\infty} = (h_{q_0}, h_{q_1})_{\eta,\infty}.$$

Pick $0 < u, v < \infty$ such that $u < p_0, p_1, q_0, q_1 < v$. We can find $0 < \alpha, \beta, \tau, \rho < 1$ such that

$$\frac{1}{p_0} = \frac{1-\alpha}{u} + \frac{\alpha}{v}\ ,\ \frac{1}{p_1} = \frac{1-\beta}{u} + \frac{\beta}{v}\ ,\ \frac{1}{q_0} = \frac{1-\tau}{u} + \frac{\tau}{v}\ ,\ \frac{1}{q_1} = \frac{1-\rho}{u} + \frac{\rho}{v}\ .$$

Then we have

$$\begin{aligned}\frac{1}{p} &= (1-\theta)\left(\frac{1-\alpha}{u} + \frac{\alpha}{v}\right) + \theta\left(\frac{1-\beta}{u} + \frac{\beta}{v}\right)\\ &= \frac{(1-\theta)(1-\alpha)}{u} + \frac{(1-\theta)\alpha}{v} + \frac{\theta(1-\beta)}{u} + \frac{\theta\beta}{v}\\ &= \frac{1-\alpha+\theta\alpha-\theta\beta}{u} + \frac{\alpha-\theta\alpha+\theta\beta}{v}\\ &= \frac{1-\tau+\eta\tau-\eta\rho}{u} + \frac{\tau-\eta\tau+\eta\rho}{v}.\end{aligned}$$

Since the function $f(t) = (1-t)/u + t/v$ is injective, we get that

$$\alpha - \theta\alpha + \theta\beta = \tau - \eta\tau + \eta\rho.$$

Consequently, using (11) and the reiteration theorem [4, Theorem 3.11.5], we derive

$$\begin{aligned}(h_{p_0}, h_{p_1})_{\theta,\infty} &= \big((h_u, h_v)_{\alpha,p_0}, (h_u, h_v)_{\beta,p_1}\big)_{\theta,\infty} = (h_u, h_v)_{\alpha-\theta\alpha+\theta\beta,\infty}\\ &= (h_u, h_v)_{\tau-\eta\tau+\eta\rho,\infty} = ((h_u, h_v)_{\tau,q_0}, (h_u, h_v)_{\rho,q_1})_{\eta,\infty}\\ &= (h_{q_0}, h_{q_1})_{\eta,\infty}.\end{aligned}$$

□

As it was pointed out in [6, (6.3)]

$$h_{p,r} = F_2^0 L_{p,r}(\mathbb{R}^n) \text{ provided that } r < \infty. \tag{12}$$

Next we show that the equality also holds when $r = \infty$.

Proposition 3.9. *Let $0 < p < \infty$. We have, with equivalence of quasi-norms,*

$$h_{p,\infty} = F_2^0 L_{p,\infty}(\mathbb{R}^n).$$

Proof Pick $0 < p_0 < p < p_1 < \infty$ and $0 < \theta < 1$ with $1/p = (1-\theta)/p_0 + \theta/p_1$. Using (12) and (6), we obtain

$$\begin{aligned}h_{p,\infty} &= (h_{p_0}, h_{p_1})_{\theta,\infty} = \left(F^0_{p_0,2}(\mathbb{R}^n), F^0_{p_1,2}(\mathbb{R}^n)\right)_{\theta,\infty}\\ &= F_2^0 L_{p,\infty}(\mathbb{R}^n).\end{aligned}$$

□

Since $F_2^0 L_{p,r}(\mathbb{R}^n) = L_{p,r}(\mathbb{R}^n)$ provided $1 < p < \infty$ and $0 < r \le \infty$ (see [48, Theorem 5] and [6, (3.1)], it follows from the previous result that

$$h_{p,\infty} = L_{p,\infty}(\mathbb{R}^n) \text{ provided that } 1 < p < \infty.$$

We are going to need also the space

$$\bar{h}_{p,\infty} = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\bar{h}_{p,\infty}} = \| \sup_{0<t<1} |(\psi(t\,\cdot)\hat{f})^\vee| \|_{L_{p,\infty}(\mathbb{R}^n)} < \infty \right\}$$

where $0 < p < \infty$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ is compactly supported and with $\psi(x) = 1$ if $|x| \le 1$.

Lemma 3.10. *Let $0 < p < \infty$. Then $h_{p,\infty} \hookrightarrow \bar{h}_{p,\infty}$.*

Proof Pick $0 < p_0 < p < p_1 < \infty$ and $0 < \theta < 1$ with $1/p = (1-\theta)/p_0 + \theta/p_1$. The map $T : h_{p_j} \longrightarrow L_{p_j}(\mathbb{R}^n)$ defined by $Tf = \sup_{0<t<1} |(\psi(t\,\cdot)\hat{f})^\vee|$ is sublinear and bounded. Therefore, by the interpolation property, there is a constant $C > 0$ such that

$$\|f\|_{\bar{h}_{p,\infty}} = \| \sup_{0<t<1} |(\psi(t\,\cdot)\hat{f})^\vee| \|_{L_{p,\infty}(\mathbb{R}^n)} \le C \|f\|_{h_{p,\infty}}\,,\ f \in h_{p,\infty}.$$

□

As we have mentioned before, if $r < \infty$ then $B_q^s L_{p,r}(\mathbb{R}^n)$ coincides with $B_q^s h_{p,r}(\mathbb{R}^n)$, the space obtained replacing the role of $L_{p,r}(\mathbb{R}^n)$ by $h_{p,r}$ in the definition of Besov-Lorentz spaces. This equality is important to describe the interpolation properties of Besov-Lorentz spaces (see [6]).

Let $B_q^s h_{p,\infty}(\mathbb{R}^n)$ be the Besov-Lorentz space defined as $B_q^s L_{p,\infty}(\mathbb{R}^n)$ but replacing the role of $L_{p,\infty}(\mathbb{R}^n)$ by the space $h_{p,\infty}$.

The following result complements [6, Theorem 6.4].

Theorem 3.11. *Let $-\infty < s < \infty$, $0 < q \le \infty$ and $0 < p < \infty$. Then we have with equivalence of quasi-norms*

$$B_q^s L_{p,\infty}(\mathbb{R}^n) = B_q^s h_{p,\infty}(\mathbb{R}^n).$$

Proof It is enough to show that there are constants $c_0, c_1 > 0$ such that for any $k \in \mathbb{N}_0$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ we have

$$c_0 \|(\varphi_k \hat{f})^\vee\|_{L_{p,\infty}(\mathbb{R}^n)} \le \|(\varphi_k \hat{f})^\vee\|_{h_{p,\infty}} \le c_1 \|(\varphi_k \hat{f})^\vee\|_{L_{p,\infty}(\mathbb{R}^n)}$$

where $(\varphi_k)_{k=0}^\infty$ is a smooth dyadic resolution of unity. Since $\psi(2^{-(k+1)}\,\cdot)\varphi_k = \varphi_k$, using Lemma 3.10, we obtain

$$\begin{aligned}
\|(\varphi_k \hat{f})^\vee\|_{L_{p,\infty}(\mathbb{R}^n)} &= \|(\psi(2^{-(k+1)}\,\cdot)\varphi_k \hat{f})^\vee\|_{L_{p,\infty}(\mathbb{R}^n)} \\
&\le \| \sup_{0<t<1} |(\psi(t\,\cdot)\varphi_k \hat{f})^\vee| \|_{L_{p,\infty}(\mathbb{R}^n)} = \|(\varphi_k \hat{f})^\vee\|_{\bar{h}_{p,\infty}} \\
&\le c \|(\varphi_k \hat{f})^\vee\|_{h_{p,\infty}}.
\end{aligned}$$

Conversely, by Proposition 3.9 we know that $\|f\|_{h_{p,\infty}} \approx \|f\|_{F_2^0 L_{p,\infty}(\mathbb{R}^n)}$. Since [6, Theorem 6.3] works for $r = \infty$ and we also have (5), then we can proceed as [6, Theorem 6.4, p. 832] to derive

$$\|(\varphi_k \hat{f})^\vee\|_{h_{p,\infty}} \approx \|(\varphi_k \hat{f})^\vee\|_{F_2^0 L_{p,\infty}(\mathbb{R}^n)} \le c_1 \|(\varphi_k \hat{f})^\vee\|_{L_{p,\infty}(\mathbb{R}^n)}.$$

□

Next we show some other results that will be useful in the next section.

Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and let $(\varphi_k)_{k \in \mathbb{N}_0}$ be a smooth dyadic resolution of unity. We put

$$f_k = \mathcal{F}^{-1}\varphi_k \mathcal{F} f \text{ and } f^k = \sum_{j=0}^{k} f_j = \mathcal{F}^{-1}\varphi_0(2^{-k}\,\cdot)\mathcal{F} f\,,\ k \in \mathbb{N}_0. \tag{13}$$

By [39, (5) p. 49], if $f \in F^s_{p,q}(\mathbb{R}^n)$ and $q < \infty$, then (f^k) converges to f in $F^s_{p,q}(\mathbb{R}^n)$. Since $\mathcal{F}f^k$ has compact support, it follows that

$$W = \{f \in F^s_{p,q}(\mathbb{R}^n) : \text{supp } \mathcal{F}f \text{ is compact}\} \text{ is dense in } F^s_{p,q}(\mathbb{R}^n) \text{ if } q < \infty. \quad (14)$$

Note that, according to the Paley-Wiener-Schwartz theorem, if $f \in W$ then f is an analytic function.

For $k \in \mathbb{N}$, consider the linear operator $R_k(f) = f^k$. There is a constant $c > 0$ independent of k such that

$$\|R_k f\|_{F^s_{p,q}(\mathbb{R}^n)} \leq c\|f\|_{F^s_{p,q}(\mathbb{R}^n)} \ , \ f \in F^s_{p,q}(\mathbb{R}^n)$$

(see [40, (15) p. 112]). Interpolating the operator R_k and using (6) we obtain that there is a constant $C > 0$ independent of k such that

$$\|f^k\|_{F^s_q L_{p,r}(\mathbb{R}^n)} \leq C\|f\|_{F^s_q L_{p,r}(\mathbb{R}^n)} \ , \ f \in F^s_q L_{p,r}(\mathbb{R}^n). \quad (15)$$

From density of $A_0 \cap A_1$ in $(A_0, A_1)_{\theta,r}$ for $r < \infty$ and (14) we derive that if $-\infty < s < \infty$ and $0 < p, q, r < \infty$ then

$$\text{The set } \{f \in F^s_q L_{p,r}(\mathbb{R}^n) : \text{supp } \mathcal{F}f \text{ is compact } \} \text{ is dense in } F^s_q L_{p,r}(\mathbb{R}^n). \quad (16)$$

Let Γ be a dense subspace of $F^s_q L_{p,r}(\mathbb{R}^n)$ and let E be a quasi-Banach space. If

$$T : (\Gamma, \|\cdot\|_{F^s_q L_{p,r}(\mathbb{R}^n)}) \times (\Gamma, \|\cdot\|_{F^s_q L_{p,r}(\mathbb{R}^n)}) \longrightarrow E$$

is a bilinear operator such that

$$\|T(f,g)\|_E \leq M\|f\|_{F^s_q L_{p,r}(\mathbb{R}^n)}\|g\|_{F^s_q L_{p,r}(\mathbb{R}^n)}, \ f, g \in \Gamma,$$

then we can extend T to a bounded bilinear operator in the product $F^s_q L_{p,r}(\mathbb{R}^n) \times F^s_q L_{p,r}(\mathbb{R}^n)$. Indeed, given any $(f,g) \in F^s_q L_{p,r}(\mathbb{R}^n) \times F^s_q L_{p,r}(\mathbb{R}^n)$ it suffices to choose sequences $(u_m), (v_m) \subseteq \Gamma$ such that $u_m \to f$ and $v_m \to g$ in $F^s_q L_{p,r}(\mathbb{R}^n)$ and define $T(f,g) = \lim_{m\to\infty} T(u_m, v_m)$. In the next section we shall use this remark with Γ being the set in (16).

Let A be a quasi-Banach space with $\mathcal{S}(\mathbb{R}^n) \hookrightarrow A \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$. Following [17, 2.6], we say that A has the *Fatou property* if there is a constant $C > 0$ such that from

$$\liminf_{m\to\infty} \|g_m\|_A \leq D \text{ and } g_m \to g \text{ in } \mathcal{S}'(\mathbb{R}^n) \text{ for the weak topology}$$

it follows that

$$g \in A \text{ and } \|g\|_A \leq CD.$$

Lemma 3.12. *Let $-\infty < s < \infty, 0 < p < \infty$ and $0 < q, r \leq \infty$. Then $F^s_q L_{p,r}(\mathbb{R}^n))$ has the Fatou property.*

Proof We start by showing that the Lorentz space $L_{p,r}(\mathbb{R}^n)$ satisfies that if (f_m) is a sequence of functions which converges almost everywhere to some function f and $\liminf_{m\to\infty} \|f_m\|_{L_{p,r}(\mathbb{R}^n)} < \infty$ then

$$f \in L_{p,r}(\mathbb{R}^n) \text{ and } \|f\|_{L_{p,r}(\mathbb{R}^n)} \leq \liminf_{m\to\infty} \|f_m\|_{L_{p,r}(\mathbb{R}^n)}. \quad (17)$$

Indeed, let $h_k(x) = \inf_{m\geq k} |f_m(x)|$. We have that $0 \leq h_k \uparrow |f|$ a.e. Passing to the non-increasing rearrangements we obtain $h^*_k \uparrow f^*$. Hence

$$\|f\|_{L_{p,r}(\mathbb{R}^n)} = \left(\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t}\right)^{1/q} = \lim_{k\to\infty} \left(\int_0^\infty (t^{1/p} h^*_k(t))^q \frac{dt}{t}\right)^{1/q}$$

$$\leq \lim_{k\to\infty}\left(\inf_{m\geq k}\|f_m\|_{L_{p,r}(\mathbb{R}^n)}\right) = \liminf_{k\to\infty}\|f_k\|_{L_{p,r}(\mathbb{R}^n)}.$$

Now, having (17), we can complete the proof by proceeding as [17, Theorem 2.6.1] for the case of Triebel-Lizorkin spaces. Assume that $(g^{(m)}) \subseteq F^s_q L_{p,r}(\mathbb{R}^n)$ satisfies that

$$\liminf_{m\to\infty}\|g^{(m)}\|_{F^s_q L_{p,r}(\mathbb{R}^n)} \leq D \text{ and } g^{(m)} \to g \text{ in } \mathcal{S}'(\mathbb{R}^n) \text{ for the weak topology.}$$

For $f \in \mathcal{S}'(\mathbb{R}^n)$ we put, as in (13), $f_k = \mathcal{F}^{-1}\varphi_k\mathcal{F}f$. So f_k is an analytic function and $f_k(x) = f\left((\mathcal{F}^{-1}\varphi_k)(x-\cdot)\right)$. Hence, if k and x are fixed,

$$g_k^{(m)}(x) \to g_k(x) \text{ as } m \to \infty.$$

Let $h^{(m)}(x) = \|(2^{ks}g_k^{(m)}(x))\|_{\ell_q}$. We have

$$\begin{aligned}\liminf_{m\to\infty}\|h^{(m)}\|_{L_{p,r}(\mathbb{R}^n)} &= \liminf_{m\to\infty}\|\|(2^{ks}g_k^{(m)}(x))\|_{\ell_q}\|_{L_{p,r}(\mathbb{R}^n)}\\ &= \liminf_{m\to\infty}\|g^{(m)}\|_{F^s_q L_{p,r}(\mathbb{R}^n)} \leq D.\end{aligned}$$

Using (17), we obtain that $\liminf_{m\to\infty} h^{(m)}(x)$ exists almost everywhere and

$$\|\liminf_{m\to\infty} h^{(m)}(x)\|_{L_{p,r}(\mathbb{R}^n)} \leq \liminf_{m\to\infty}\|h^{(m)}\|_{L_{p,r}(\mathbb{R}^n)} < \infty.$$

For any $x \in \mathbb{R}^n$ such that $\liminf_{m\to\infty} h^{(m)}(x)$ exists and for any $N \in \mathbb{N}$, we have

$$\|(2^{ks}g_k(x))_{k=0}^N\|_{\ell_q} \leq \liminf_{m\to\infty}\|(2^{ks}g_k^{(m)}(x))_{k=0}^N\|_{\ell_q} \leq \liminf_{m\to\infty} h^{(m)}(x). \tag{18}$$

Therefore, $\|(2^{ks}g_k(\cdot))_{k=0}^\infty\|_{\ell_q}$ exists almost everywhere. Moreover, as $\|(2^{ks}g_k(\cdot))_{k=0}^N\|_{\ell_q}$ is Lebesgue measurable for each N, we have that $\|(2^{ks}g_k(\cdot))_{k=0}^\infty\|_{\ell_q}$ is also measurable. By (18), we obtain

$$\|(2^{ks}g_k(x))_{k=0}^\infty\|_{\ell_q} \leq \liminf_{m\to\infty} h^{(m)}(x).$$

Whence, using (17), we derive

$$\begin{aligned}\|g\|_{F^s_q L_{p,r}(\mathbb{R}^n)} &= \|\|(2^{ks}g_k(x))\|_{\ell_q}\|_{L_{p,r}(\mathbb{R}^n)} \leq \|\liminf_{m\to\infty} h^{(m)}(x)\|_{L_{p,r}(\mathbb{R}^n)}\\ &\leq \liminf_{m\to\infty}\|h^{(m)}(x)\|_{L_{p,r}(\mathbb{R}^n)} = \liminf_{m\to\infty}\|g^{(m)}\|_{F^s_q L_{p,r}(\mathbb{R}^n)} \leq D.\end{aligned}$$

□

In our later computations we shall also need the Peetre's maximal function: Let $a > 0$ and $(h_k)_{k=0}^\infty$ be a sequence of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{C}$. The *Peetre's maximal function* is defined by

$$(h_k)^*_a(x) = \sup_{y\in\mathbb{R}^n}\frac{|h_k(x-y)|}{(1+2^k|y|)^a}.$$

Let $-\infty < s < \infty, 0 < p < \infty$ and $0 < q, r \leq \infty$. Given $f \in F^s_{p,q}(\mathbb{R}^n)$, let again $f_k = (\varphi_k\hat{f})^\vee, k \in \mathbb{N}_0$. If $a > n/\min(p,q)$, then the operator $Tf = (f_k)^*_a$ is sublinear and bounded from $F^s_{p,q}(\mathbb{R}^n)$ into $L_p(\ell^s_q)$ (see [40, Theorem 2.3.2]). By the interpolation property and formulae (6) and (4) with $A = \ell^s_q$, we derive that the following maximal inequality holds

$$\|(f_k)^*_a\|_{L_{p,r}(\ell^s_q)} \leq C\|f\|_{F^s_q L_{p,r}(\mathbb{R}^n)}. \tag{19}$$

4 Multiplication algebras and Triebel-Lizorkin-Lorentz spaces

Let $A = F^s_q L_{p,r}(\mathbb{R}^n)$ or $B^s_q L_{p,r}(\mathbb{R}^n)$. We say that A is a *multiplication algebra* if $fg \in A$ whenever $f, g \in A$, and there is a constant $C > 0$ such that $\|fg\|_A \leq C\|f\|_A\|g\|_A$ for any $f, g \in A$.

Let $0 < p < \infty$, $s > n/p$ and $0 < q, r \leq \infty$. Then $F^s_{p,q}(\mathbb{R}^n)$ and $B^s_{p,q}(\mathbb{R}^n)$ are continuously embedded in $L_\infty(\mathbb{R}^n)$ (see [17, 33, 39]). Using the embeddings between spaces with Lorentz smoothness and spaces $F^s_{p,q}(\mathbb{R}^n)$, $B^s_{p,q}(\mathbb{R}^n)$ (see [32]), it is not hard to check that

$$F^s_q L_{p,r}(\mathbb{R}^n),\ B^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n) \tag{20}$$

provided that $s > n/p$. Therefore, for $f, g \in F^s_q L_{p,r}(\mathbb{R}^n)$ the product fg makes sense pointwise almost everywhere.

As we pointed out in the Introduction, $F^s_{p,q}(\mathbb{R}^n)$ is a multiplication algebra if $s > n/p$. Next we prove Conjecture 5.6 of [6] on Triebel-Lizorkin-Lorentz spaces. We combine ideas developed by Triebel [37, 2.6.2], [39, 2.8.2 and 2.8.3] and Franke [17] with interpolation results.

Theorem 4.1. *Let $0 < p < \infty$, $s > n/p$, and $0 < q, r \le \infty$. Then $F^s_q L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra.*

Proof Let $f, g \in F^s_q L_{p,r}(\mathbb{R}^n)$. Using the smooth dyadic resolution of unity $(\varphi_k)_{k\in\mathbb{N}_0}$, we put for $l, j = 0, 1, 2, \ldots$

$$f_l = \mathcal{F}^{-1}\varphi_l \mathcal{F} f, \quad f^l = \sum_{k=0}^{l} f_k, \quad g_j = \mathcal{F}^{-1}\varphi_j \mathcal{F} g \text{ and } g^j = \sum_{k=0}^{j} g_k$$

Note that

$$f^l = \mathcal{F}^{-1}(\varphi_0(2^{-l}\cdot)\mathcal{F}f) \text{ and } g^j = \mathcal{F}^{-1}(\varphi_0(2^{-j}\cdot)\mathcal{F}g).$$

We assume for the moment that $\mathcal{F}f$ and $\mathcal{F}g$ have a compact support. In $\mathcal{S}'(\mathbb{R}^n)$ we have

$$fg = \lim_{l\to\infty} f^l g^l = \sum_{j=2}^{\infty} f_j g^{j-2} + \sum_{l=2}^{\infty} f^{l-2} g_l + \sum_{|l-j|\le 1} f_l g_j.$$

Note that

$$\operatorname{supp} \mathcal{F} f_j g^{j-2} \subseteq \{x : 2^{j-3} \le |x| \le 15 \cdot 2^{j-3}\}, \ j-2 \in \mathbb{N}_0$$

and

$$\operatorname{supp} \mathcal{F} f_l g_j \subseteq \{x : |x| \le 9 \cdot 2^{l-1}\}, \ |l-j| \le 1.$$

Since

$$\operatorname{supp} \varphi_k \subseteq \{x : 2^{k-1} \le |x| \le 3 \cdot 2^{k-1}\}, \ k \in \mathbb{N},$$

the support conditions yield that

$$\begin{aligned}\mathcal{F}^{-1}\varphi_k \mathcal{F} fg &= \mathcal{F}^{-1}\varphi_k \mathcal{F} \sum_{j=2}^{\infty} f_j g^{j-2} + \mathcal{F}^{-1}\varphi_k \mathcal{F} \sum_{l=2}^{\infty} f^{l-2} g_l + \mathcal{F}^{-1}\varphi_k \mathcal{F} \sum_{|l-j|\le 1} f_l\, g_j \\ &= \mathcal{F}^{-1}\varphi_k \sum_{j=-1}^{3} \mathcal{F} f_{k+j} g^{k+j-2} + \mathcal{F}^{-1}\varphi_k \sum_{l=-1}^{3} \mathcal{F} f^{k+l-2} g_{k+l} + \sum_{l=k-3}^{\infty} \mathcal{F}^{-1}\varphi_k \sum_{j=-1}^{1} \mathcal{F} f_l\, g_{l+j}.\end{aligned}$$

In order to check that fg belongs to $F^s_q L_{p,r}(\mathbb{R}^n)$, that is $(\mathcal{F}^{-1}\varphi_k \mathcal{F} fg)$ belongs to $L_{p,r}(l^s_q)$, it suffices to estimate the following three model cases:

$\Sigma'_k(f,g) = \mathcal{F}^{-1}\varphi_k \mathcal{F} f_k\, g$
$\Sigma''_k(f,g) = \Sigma'_k(g,f)$,
$\Sigma'''_k(f,g) = \sum_{l=k}^{\infty} (\mathcal{F}^{-1}\varphi_k \mathcal{F} f_l\, g_l)$.

Using the Peetre's maximal function, we have

$$\begin{aligned}|\Sigma'_k(f,g)(x)| &= \left| \int_{\mathbb{R}^n} (\mathcal{F}^{-1}\varphi_k)(y) f_k(x-y) g(x-y)\, dy \right| \\ &\le c_1 (f_k)^*_a(x) \|g\|_{L_\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\mathcal{F}^{-1}\varphi_k(y)|(1+2^k|y|)^a dy \\ &= c_1 (f_k)^*_a(x) \|g\|_{L_\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\mathcal{F}^{-1}\varphi_1(2^{-k+1}y)|(1+2^k|y|)^a dy \\ &= c_1 (f_k)^*_a(x) \|g\|_{L_\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} 2^{-n(-k+1)} |(\mathcal{F}^{-1}\varphi_1)(2^{k-1}y)|(1+2^k|y|)^a dy \\ &= c_1 (f_k)^*_a(x) \|g\|_{L_\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\mathcal{F}^{-1}\varphi_1(\xi)|(1+2|\xi|)^a d\xi \\ &\le c_2 (f_k)^*_a(x) \|g\|_{L_\infty(\mathbb{R}^n)}.\end{aligned}$$

Pick $a > \dfrac{n}{\min(p,q)}$. According to (19), we derive

$$\|(\Sigma'_k(f,g))\|_{L_{p,r}(l^s_q)} \le c_3 \|f\|_{F^s_q L_{p,r}(\mathbb{R}^n)} \|g\|_{L_\infty} \le c_4 \|f\|_{F^s_q L_{p,r}(\mathbb{R}^n)} \|g\|_{F^s_q L_{p,r}(\mathbb{R}^n)}$$

where we have used (20) in the last inequality.

Changing the roles of f and g, we get

$$\|(\Sigma''_k(f,g))\|_{L_{p,r}(l^s_q)} = \|(\Sigma'_k(g,f))\|_{L_{p,r}(l^s_q)} \le c_4 \|f\|_{F^s_q L_{p,r}(\mathbb{R}^n)} \|g\|_{F^s_q L_{p,r}(\mathbb{R}^n)}.$$

In order to estimate the remaining term, pick $0 < \tau < \infty$ and note that $s > n\left(\frac{1}{\min(p,1)} - 1\right)$. By [17, Step 3, p. 48] there is a constant $c_5 > 0$ such that for any $u \in F^s_{p,q}(\mathbb{R}^n)$ and $v \in B^{n/\tau}_{\tau,\infty}(\mathbb{R}^n)$ we have

$$\|(2^{ks}\Sigma'''_k(u,v))\|_{L_p(l_q)} \le c_5 \|u\|_{F^s_{p,q}(\mathbb{R}^n)} \|v\|_{B^{n/\tau}_{\tau,\infty}(\mathbb{R}^n)}. \tag{21}$$

Let $0 < p_1 < p < p_2 < \infty$ and $0 < \theta < 1$ such that $1/p = (1-\theta)/p_0 + \theta/p_1$ and $n/p_1 < s$. Then

$$s > n\left(\frac{1}{\min(p_1,1)} - 1\right) \ge n\left(\frac{1}{\min(p,1)} - 1\right) \ge n\left(\frac{1}{\min(p_2,1)} - 1\right).$$

Fix $v \in B^{n/\tau}_{\tau,\infty}(\mathbb{R}^n)$ and consider the linear operator $u \longrightarrow (2^{ks}\Sigma'''_k(u,v))$. By (21), this operator is bounded from $F^s_{p_i,q}(\mathbb{R}^n)$ into $L_{p_i}(l_q)$. Using (4) and (6), we get that there is a constant $c_6 > 0$ such that for any $u \in F^s_q L_{p,r}(\mathbb{R}^n)$ we have

$$\|(2^{ks}\Sigma'''_k(u,v))\|_{L_{p,r}(l_q)} \le c_6 \|u\|_{F^s_q L_{p,r}(\mathbb{R}^n)} \|v\|_{B^{n/\tau}_{\tau,\infty}(\mathbb{R}^n)}.$$

Pick now $\tau = 2p$. By [32, Theorem 1.2] the embedding $F^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow B^{n/2p}_{2p,\infty}(\mathbb{R}^n)$ holds. Therefore

$$\|(2^{ks}\Sigma'''_k(f,g))\|_{L_{p,r}(l_q)} \le c_7 \|f\|_{F^s_q L_{p,r}(\mathbb{R}^n)} \|g\|_{F^s_q L_{p,r}(\mathbb{R}^n)}.$$

Consequently,

$$(\Gamma, \|\cdot\|_{F^s_q L_{p,r}(\mathbb{R}^n)}) \cdot (\Gamma, \|\cdot\|_{F^s_q L_{p,r}(\mathbb{R}^n)}) \hookrightarrow F^s_q L_{p,r}(\mathbb{R}^n) \tag{22}$$

where

$$\Gamma = \{f \in F^s_q L_{p,r}(\mathbb{R}^n) : \operatorname{supp} \mathcal{F}f \text{ is compact}\}.$$

If $\max\{r,q\} < \infty$ then, as we pointed out in (16), Γ is dense in $F^s_q L_{p,r}(\mathbb{R}^n)$. Therefore, the product can be extended to $F^s_q L_{p,r}(\mathbb{R}^n) \times F^s_q L_{p,r}(\mathbb{R}^n)$ with the effect that $F^s_q L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra.

Finally, if $\max\{r,q\} = \infty$, given any $f, g \in F^s_q L_{p,r}(\mathbb{R}^n)$ we consider

$$f^k = \sum_{l=0}^{k} \mathcal{F}\varphi_l \mathcal{F} f \quad \text{and} \quad g^k = \sum_{l=0}^{k} \mathcal{F}\varphi_l \mathcal{F} g, \quad k \in \mathbb{N}_0.$$

By (15), there is a constant $d_1 > 0$ independent of f and k such that

$$\|f^k\|_{F^s_q L_{p,r}(\mathbb{R}^n)} \le d_1 \|f\|_{F^s_q L_{p,r}(\mathbb{R}^n)}.$$

Let

$$s_1 = \frac{n}{2p+2} + \frac{1}{2}\left(s - \frac{n}{p}\right).$$

According to [32, Theorem 1.6], we have

$$F^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow F^{s_1}_{2p+2,2}(\mathbb{R}^n),$$

and the space $F^{s_1}_{2p+2,2}(\mathbb{R}^n)$ is a multiplication algebra because $s_1 > n/(2p+2)$. Since

$$f^k \longrightarrow f \quad \text{and} \quad g^k \longrightarrow g \quad \text{in } F^{s_1}_{2p+2,2}(\mathbb{R}^n)$$

(see [39, p. 49]), we have that

$$f^k g^k \longrightarrow fg \quad \text{in } F^{s_1}_{2p+2,2}(\mathbb{R}^n)$$

and in particular in $\mathcal{S}'(\mathbb{R}^n)$. Furthermore, by (22), we have

$$\begin{aligned} \|f^k g^k\|_{F^s_q L_{p,r}(\mathbb{R}^n)} &\le c \|f^k\|_{F^s_q L_{p,r}(\mathbb{R}^n)} \|g^k\|_{F^s_q L_{p,r}(\mathbb{R}^n)} \\ &\le c\, d_1^2 \|f\|_{F^s_q L_{p,r}(\mathbb{R}^n)} \|g\|_{F^s_q L_{p,r}(\mathbb{R}^n)}. \end{aligned}$$

Consequently, using the Fatou property of $F^s_q L_{p,r}(\mathbb{R}^n)$ (Lemma 3.12), we conclude that $fg \in F^s_q L_{p,r}(\mathbb{R}^n)$ with

$$\|fg\|_{F^s_q L_{p,r}(\mathbb{R}^n)} \le C \|f\|_{F^s_q L_{p,r}(\mathbb{R}^n)} \|g\|_{F^s_q L_{p,r}(\mathbb{R}^n)}.$$

The proof is complete.

□

Necessary and sufficient conditions are known for Triebel-Lizorkin spaces to be multiplication algebras (see [17]). Next we show that condition $s > n/p$ is also necessary for $F^s_q L_{p,r}(\mathbb{R}^n)$ to be a multiplication algebra at least for a certain range of parameters. In the proof we use an idea of Maz'ya and Shaposnikowa [25] (see also [31, Remark 4.3.2/1]).

Theorem 4.2. *Let $s > 0$, $1 < p, r < \infty$ and $0 < q < \infty$. If $F^s_q L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra, then $s > n/p$.*

Proof Since $\max\{q, r\} < \infty$, it follows from Lemma 3.3 that $\mathcal{S}(\mathbb{R}^n)$ is dense in $F^s_q L_{p,r}(\mathbb{R}^n)$. Take any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and consider the linear operator $T_\varphi(f) = \varphi f$. By the assumption, $T_\varphi : F^s_q L_{p,r}(\mathbb{R}^n) \longrightarrow F^s_q L_{p,r}(\mathbb{R}^n)$ is bounded.

Let $p < v < \infty$ such that $s > \frac{n}{p} - \frac{n}{v} > 0$. We know that $L_v(\mathbb{R}^n) = F^0_{v,2}(\mathbb{R}^n)$ and, by [32, Theorem 1.6], we have

$$F^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow L_v(\mathbb{R}^n). \tag{23}$$

Given any cube $Q \subseteq \mathbb{R}^n$, we can find $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\psi = 1$ on Q. Then

$$\|\varphi\|_{L_\infty(Q)} = \lim_{m\to\infty} \left(\int_Q |\varphi(x)|^{vm} |\psi(x)|^v dx \right)^{1/(vm)}.$$

Embedding (23) yields that

$$\begin{aligned} \|\varphi^m \psi\|^{1/m}_{L_v(Q)} &\le c_1 \|\varphi^m \psi\|^{1/m}_{F^s_q L_{p,r}(\mathbb{R}^n)} \\ &\le c_1 \|T^m_\varphi(\psi)\|^{1/m}_{F^s_q L_{p,r}(\mathbb{R}^n)} \\ &\le c_1 \|T_\varphi\|_{F^s_q L_{p,r}(\mathbb{R}^n), F^s_q L_{p,r}(\mathbb{R}^n)} \|\psi\|^{1/m}_{F^s_q L_{p,r}(\mathbb{R}^n)} \end{aligned}$$

Passing to the limit when $m \to \infty$ we derive that

$$\begin{aligned} \|\varphi\|_{L_\infty(\mathbb{R}^n)} &\le c_1 \|T_\varphi\|_{F^s_q L_{p,r}(\mathbb{R}^n), F^s_q L_{p,r}(\mathbb{R}^n)} \\ &\le c_2 \|\varphi\|_{F^s_q L_{p,r}(\mathbb{R}^n)}. \end{aligned}$$

This yields that $F^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$.

Now we finish the proof of the necessity of condition $s > n/p$ proceeding by contradiction. If $s \le n/p$, we pick $0 < u < p$. Then $s - n/p + n/u \le n/u$. By [32, Theorems 1.1 and 1.5], it follows that

$$B^{n/u}_{u,r}(\mathbb{R}^n) \hookrightarrow B^{s-n/p+n/u}_{u,r}(\mathbb{R}^n) \hookrightarrow F^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$$

but this is not possible because, according to [37, p. 134], since $r > 1$ the space $B^{n/u}_{u,r}(\mathbb{R}^n)$ has essentially unbounded functions. □

5 Multiplication algebras and Besov-Lorentz spaces

Next we study Conjecture 7.3 of [6] on Besov-Lorentz spaces. The approach that we follow is based on the characterization of spaces $B^s_q L_{p,r}(\mathbb{R}^n)$ as approximation spaces.

Let $(X, \|\cdot\|_X)$ be a quasi-Banach space and let $(D_k)_{k\in\mathbb{N}_0}$ be a sequence of subsets of X satisfying the three following conditions:

$$D_0 = \{0\} \subseteq D_1 \subseteq \cdots \subseteq D_k \subseteq \cdots \subseteq X,$$

$$\lambda D_k \subseteq D_k \text{ for any } \lambda \in \mathbb{C} \text{ and any } k \in \mathbb{N}_0,$$
$$D_k + D_m \subseteq D_{k+m} \text{ for any } k, m \in \mathbb{N}_0.$$

Given any $f \in X$, let $E_0(f) = \|f\|_X$ and

$$E_k(f) = \inf\{\|f - g\|_X : g \in D_k\},\ k \in \mathbb{N}.$$

Let $\alpha > 0$ and $0 < q \leq \infty$. The *approximation space* $[X]_q^\alpha = (X; D_m)_q^\alpha$ consists of all those $f \in X$ having a finite quasi-norm

$$\|f\|_{X_q^\alpha} = \|(k^{\alpha-1/q}E_{k-1}(f))\|_{\ell_q}$$

(see [11, 12, 29, 30]). An equivalent quasi-norm in $[X]_q^\alpha$ is

$$\|f\|^\diamond_{X_q^\alpha} = \left(\|f\|_X^q + \sum_{k=1}^\infty 2^{k\alpha q}E_{2^k}(f)^q\right)^{1/q} \tag{24}$$

(with the usual modification if $q = \infty$) (see [30, Proposition 2]).

Let $0 < q \leq \infty, s > 0, 0 < p < \infty$ and $0 < r < \infty$, allowing also $r = \infty$ if $1 < p < \infty$. Let $D_0 = \{0\}$ and

$$D_k = \{g \in F_2^0 L_{p,r}(\mathbb{R}^n) : \operatorname{supp} \mathcal{F}g \subseteq \{x : |x| \leq k\}\}, \, k \in \mathbb{N}.$$

According to [5, Theorem 6.1] we have with equivalence of quasi-norms

$$[F_2^0 L_{p,r}(\mathbb{R}^n)]_q^s = B_q^s L_{p,r}(\mathbb{R}^n). \tag{25}$$

Note that $F_2^0 L_{p,r}(\mathbb{R}^n)$ coincides with equivalence of quasi-norms with $L_{p,r}(\mathbb{R}^n)$ provided that $1 < p < \infty$ and $0 < r \leq \infty$ (see [6, (3.1)]). Moreover, $F_2^0 L_{p,r}(\mathbb{R}^n)$ is equal to the local Hardy-Lorentz space $h_{p,r}$ for $0 < p < \infty$ and $0 < r \leq \infty$ (see Proposition 3.9 and [6, (6.3)].

The following result shows that (25) also holds if $r = \infty$ and $0 < p \leq 1$.

Theorem 5.1. *Let $s > 0$ and $0 < p \leq 1$. Then we have with equivalence of quasi-norms*

$$[F_2^0 L_{p,\infty}(\mathbb{R}^n)]_q^s = B_q^s L_{p,\infty}(\mathbb{R}^n). \tag{26}$$

Proof In view of Proposition 3.9 and Theorem 3.11, we can proceed as in [5, Theorem 6.1]. □

Next we show a new approximation formula where $F_2^0 L_{p,r}(\mathbb{R}^n)$ is replaced by other Triebel-Lizorkin-Lorentz space, now with positive smoothness. In what follows, given a Lorentz-Sobolev space A which contains the sets D_k, we put $[A]_q^\tau$ for the corresponding approximation space.

Theorem 5.2. *Let $s, \tau > 0, 0 < p < \infty, 0 < q, q_0 \leq \infty$ with $p \neq q_0$ and $0 < r \leq \infty$. Then we have with equivalence of quasi-norms*

$$[F_{q_0}^s L_{p,r}(\mathbb{R}^n)]_q^\tau = B_q^{s+\tau} L_{p,r}(\mathbb{R}^n).$$

Proof According to [32, Theorem 1.1] we have

$$B_{\min(p,q_0,r)}^s L_{p,r}(\mathbb{R}^n) \hookrightarrow F_{q_0}^s L_{p,r}(\mathbb{R}^n),$$

and by (25) and (26) we know that

$$[F_2^0 L_{p,r}(\mathbb{R}^n)]_{\min(p,q_0,r)}^s = B_{\min(p,q_0,r)}^s L_{p,r}(\mathbb{R}^n).$$

Therefore, using the reiteration theorem for approximation spaces [30, p. 123], we obtain

$$\begin{aligned} B_q^{s+\tau} L_{p,r}(\mathbb{R}^n) &= [F_2^0 L_{p,r}(\mathbb{R}^n)]_q^{s+\tau} = [[F_2^0 L_{p,r}(\mathbb{R}^n)]_{\min(p,q_0,r)}^s]_q^\tau \\ &= [B_{\min(p,q_0,r)}^s L_{p,r}(\mathbb{R}^n)]_q^\tau \hookrightarrow [F_{q_0}^s L_{p,r}(\mathbb{R}^n)]_q^\tau. \end{aligned}$$

Conversely, since

$$F_{q_0}^s L_{p,r}(\mathbb{R}^n) \hookrightarrow B_{\max(p,q_0,r)}^s L_{p,r}(\mathbb{R}^n)$$

(see [32, Theorem 1.2]), using again (25), (26) and the reiteration theorem, we derive

$$\begin{aligned} [F_{q_0}^s L_{p,r}(\mathbb{R}^n)]_q^\tau &\hookrightarrow [B_{\max(p,q_0,r)}^s L_{p,r}(\mathbb{R}^n)]_q^\tau = [[F_2^0 L_{p,r}(\mathbb{R}^n)]_{\max(p,q_0,r)}^s]_q^\tau \\ &= [F_2^0 L_{p,r}(\mathbb{R}^n)]_q^{s+\tau} = B_q^{s+\tau} L_{p,r}(\mathbb{R}^n). \end{aligned}$$

□

Next we prove Conjecture 7.3 of [6].

Theorem 5.3. *Let $0 < p < \infty, s > n/p, 0 < q \leq \infty$ and $0 < r \leq \infty$. Then $B_q^s L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra.*

Proof Pick $0 < q_0 < \infty$ and $s_0 > 0$ such that $q_0 \neq p$ and $n/p < s_0 < s$. Put $\tau = s - s_0$. Then, according to Theorem 5.2, we have

$$[F_{q_0}^{s_0} L_{p,r}(\mathbb{R}^n)]_q^\tau = B_q^s L_{p,r}(\mathbb{R}^n). \tag{27}$$

Now consider the bilinear operator $T(f,g) = fg$. Since $s_0 > n/p$, it follows from Theorem 4.1 that

$$T : F_{q_0}^{s_0} L_{p,r}(\mathbb{R}^n) \times F_{q_0}^{s_0} L_{p,r}(\mathbb{R}^n) \longrightarrow F_{q_0}^{s_0} L_{p,r}(\mathbb{R}^n)$$

is bounded. Take any $m, k \in \mathbb{N}$ and $f \in D_{2^m}, g \in D_{2^k}$. Then $T(f,g) = fg$ belongs to $D_{2^m+2^k}$ because

$$fg = \mathcal{F}^{-1}(\mathcal{F}f * \mathcal{F}g) \text{ and so } \operatorname{supp} \mathcal{F}(fg) \subseteq \{x : |x| \leq 2^m + 2^k\}.$$

Using [5, Lemma 6.2], we obtain that there is $c_1 > 0$ such that for any $k \in \mathbb{N}$ and any $f, g \in F_{q_0}^{s_0} L_{p,r}(\mathbb{R}^n)$ we have

$$E_{2^{k+1}}(fg) \leq c_1 \left(E_{2^k}(f) \|g\|_{F_{q_0}^{s_0} L_{p,r}(\mathbb{R}^n)} + \|f\|_{F_{q_0}^{s_0} L_{p,r}(\mathbb{R}^n)} E_{2^k}(g) \right).$$

Therefore, working with the quasi-norm (24), we obtain

$$\|T(fg)\|_{[F_{q_0}^{s_0} L_{p,r}(\mathbb{R}^n)]_q^\tau} \leq c_2 \left(\|fg\|^q_{F_{q_0}^{s_0} L_{p,r}(\mathbb{R}^n)} + \sum_{k=2}^{\infty} 2^{k\tau q} E_{2^k}(fg)^q \right)^{1/q}$$

$$\leq c_3 \left(\|f\|^q_{F_{q_0}^{s_0} L_{p,r}(\mathbb{R}^n)} \|g\|^q_{F_{q_0}^{s_0} L_{p,r}(\mathbb{R}^n)} + \sum_{k=1}^{\infty} 2^{k\tau q} \left(E_{2^k}(f) \|g\|_{F_{q_0}^{s_0} L_{p,r}(\mathbb{R}^n)} + \|f\|_{F_{q_0}^{s_0} L_{p,r}(\mathbb{R}^n)} E_{2^k}(g) \right)^q \right)^{1/q}$$

$$\leq c_3 \left(\|f\|^q_{F_{q_0}^{s_0} L_{p,r}(\mathbb{R}^n)} + \sum_{k=1}^{\infty} 2^{k\tau q} E_{2^k}(f)^q \right)^{1/q} \left(\|g\|^q_{F_{q_0}^{s_0} L_{p,r}(\mathbb{R}^n)} + \sum_{k=1}^{\infty} 2^{k\tau q} E_{2^k}(g)^q \right)^{1/q}$$

$$\leq c_4 \|f\|_{B_q^s L_{p,r}(\mathbb{R}^n)} \|g\|_{B_q^s L_{p,r}(\mathbb{R}^n)}$$

where we have used (27) in the last inequality. Consequently, $fg \in B_q^s L_{p,r}(\mathbb{R}^n)$ and there is a constant $c_4 > 0$ such that for any $f, g \in B_q^s L_{p,r}(\mathbb{R}^n)$ we have

$$\|fg\|_{B_q^s L_{p,r}(\mathbb{R}^n)} \leq c_4 \|f\|_{B_q^s L_{p,r}(\mathbb{R}^n)} \|g\|_{B_q^s L_{p,r}(\mathbb{R}^n)}.$$

□

Triebel [37, Theorem 2.6.2/1] has established necessary and sufficient conditions for a Besov space to be a multiplication algebra. We finish the paper by showing that condition $s > n/p$ is also necessary for $B_q^s L_{p,r}(\mathbb{R}^n)$ to be a multiplication algebra at least for a certain range of parameters.

Theorem 5.4. *Let*

(i) $s > 0, 1 < p, q < \infty$ *and* $0 < r < \infty$, *or*
(ii) $-\infty < s < \infty$ *and* $1 < p, q, r < \infty$, *or*
(iii) $-\infty < s < \infty, 1 < q < \infty, 0 < p < 1$ *and* $0 < r < \infty$, *or*
(iv) $-\infty < s < \infty, 1 < q < \infty, p = 1$ *and* $0 < r \leq 1$.

If $B_q^s L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra, then $s > n/p$.

Proof For (i) we proceed as in Theorem 4.2. Pick $p < v < \infty$ such that $s > n/p - n/v > 0$. Instead of (23), now we have

$$B_q^s L_{p,r}(\mathbb{R}^n) \hookrightarrow F_{v,2}^0(\mathbb{R}^n) = L_v(\mathbb{R}^n)$$

(see [32, Theorem 1.1]). Arguing as in the proof of Theorem 4.2, we obtain that $B_q^s L_{p,r}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$. To complete the proof, we proceed by contradiction. If $s \leq n/p$, the choice of $0 < u < p$ yields that $s - n/p + n/u \leq n/u$. From [32, Theorem 1.5], it follows that

$$B_{u,q}^{n/u}(\mathbb{R}^n) \hookrightarrow B_{u,q}^{s-n/p+n/u}(\mathbb{R}^n) \hookrightarrow B_q^s L_{p,r}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n).$$

But this is not possible because since $q > 1$ the space $B^{n/u}_{u,q}(\mathbb{R}^n)$ has unbounded functions (see [37, p. 134]).

Next we assume that parameters satisfy (ii). Take any $\varphi \in \mathcal{S}(\mathbb{R}^n)$. If $B^s_q L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra, then the linear operator $T_\varphi : B^s_q L_{p,r}(\mathbb{R}^n) \longrightarrow B^s_q L_{p,r}(\mathbb{R}^n)$ defined by $T_\varphi(f) = \varphi f$ is bounded with

$$\|T_\varphi\|_{B^s_q L_{p,r}(\mathbb{R}^n), B^s_q L_{p,r}(\mathbb{R}^n)} \leq c_1 \|\varphi\|_{B^s_q L_{p,r}(\mathbb{R}^n)}$$

where c_1 is a constant independent of φ. The operator T_φ is self-adjoint. Using the duality formulae of Theorem 3.4, we have that $T_\varphi : B^{-s}_{q'} L_{p',r'}(\mathbb{R}^n) \longrightarrow B^{-s}_{q'} L_{p',r'}(\mathbb{R}^n)$ is also bounded with norm less than or equal to $c_2 \|\varphi\|_{B^s_q L_{p,r}(\mathbb{R}^n)}$. We claim that

$$\left(B^s_q L_{p,r}(\mathbb{R}^n), B^{-s}_{q'} L_{p',r'}(\mathbb{R}^n)\right)_{1/2,2} = B^0_{2,2}(\mathbb{R}^n) = L_2(\mathbb{R}^n). \tag{28}$$

Indeed, if $p \neq 2$ then (28) is a direct consequence of [5, Theorem 5.1]. If $p = 2 = p'$, then we still can obtain (28) proceeding as in the proof of [5, Theorem 5.1] because

$$(\ell_{2,r}, \ell_{2,r'})_{1/2,2} = \ell_2 \text{ and } \left(\ell_q(2^{k(s-n/2)}\ell_{2,r}), \ell_{q'}(2^{k(-s-n/2)}\ell_{2,r'})\right)_{1/2,2} = \ell_2(2^{k(0-n/2)}\ell_2)$$

(see [38, Theorem 1.18.1] and [4, Theorem 5.2.4]). From (28) and the interpolation theorem, we derive that $T_\varphi : L_2(\mathbb{R}^n) \longrightarrow L_2(\mathbb{R}^n)$ is bounded with

$$\|T_\varphi\|_{L_2(\mathbb{R}^n), L_2(\mathbb{R}^n)} \leq c_4 \|\varphi\|_{B^s_q L_{p,r}(\mathbb{R}^n)}$$

where c_4 is independent of φ. Consequently,

$$\|\varphi\|_{L_\infty} = \|T_\varphi\|_{L_2(\mathbb{R}^n), L_2(\mathbb{R}^n)} \leq c_4 \|\varphi\|_{B^s_q L_{p,r}(\mathbb{R}^n)}, \; \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Density of $\mathcal{S}(\mathbb{R}^n)$ in $B^s_q L_{p,r}(\mathbb{R}^n)$ yields that

$$B^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n).$$

To complete the proof, we proceed by contradiction. If $s \leq n/p$, picking $0 < v < p$ we have $s - n/p + n/v \leq n/v$. Therefore, by [32, Theorem 1.5], we derive

$$B^{n/v}_{v,q}(\mathbb{R}^n) \hookrightarrow B^{s-n/p+n/v}_{v,q}(\mathbb{R}^n) \hookrightarrow B^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n) \tag{29}$$

but this is not possible because, since $q > 1$, it follows from [37, pp. 134-135] that $B^{n/v}_{v,q}(\mathbb{R}^n)$ has essentially unbounded functions.

Finally, assume that we have (iii) or (iv). We can follow the ideas of Triebel in [37, Theorem 2.6.2/1, Step 1]. Let $\sigma = -s + n(1/p - 1), u = \infty, v = q'$ and consider the Besov space $B^\sigma_{u,v}(\mathbb{R}^n)$. Take any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and let T_φ the operator defined before. Since $T_\varphi : B^s_q L_{p,r}(\mathbb{R}^n) \longrightarrow B^s_q L_{p,r}(\mathbb{R}^n)$ is bounded, by duality and Theorem 3.5 we have that $T_\varphi : B^\sigma_{u,v}(\mathbb{R}^n) \longrightarrow B^\sigma_{u,v}(\mathbb{R}^n)$ is bounded with norm less than or equal to $c_1 \|\varphi\|_{B^s_q L_{p,r}(\mathbb{R}^n)}$. Let $\left(B^\sigma_{u,v}(\mathbb{R}^n)\right)^\circ$ be the completion of $\mathcal{S}(\mathbb{R}^n)$ in $B^\sigma_{u,v}(\mathbb{R}^n)$. We also have that $T_\varphi : \left(B^\sigma_{u,v}(\mathbb{R}^n)\right)^\circ \longrightarrow \left(B^\sigma_{u,v}(\mathbb{R}^n)\right)^\circ$ is bounded. By duality and using [37, Remarks 2.5.1/7 and 2.5.2/3], we obtain that $T_\varphi : B^{-\sigma}_{u',v'}(\mathbb{R}^n) \longrightarrow B^{-\sigma}_{u',v'}(\mathbb{R}^n)$ is bounded with norm less than or equal to $c_2 \|\varphi\|_{B^s_q L_{p,r}(\mathbb{R}^n)}$. Complex interpolation yields that

$$[B^\sigma_{u,v}(\mathbb{R}^n), B^{-\sigma}_{u',v'}(\mathbb{R}^n)]_{1/2} = L_2(\mathbb{R}^n)$$

(see [38, Theorem 2.4.1/(d)]). Therefore, $T_\varphi : L_2(\mathbb{R}^n) \longrightarrow L_2(\mathbb{R}^n)$ is bounded and so

$$\|\varphi\|_{L_\infty(\mathbb{R}^n)} = \|T_\varphi\|_{L_2(\mathbb{R}^n), L_2(\mathbb{R}^n)} \leq c_3 \|\varphi\|_{B^s_q L_{p,r}(\mathbb{R}^n)} \quad , \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Consequently, $B^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$. Since (29) still holds in the cases (iii) and (iv), we can complete the proof as in the case (ii).

□

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