Degrees of non-compactness for some operators in Analysis *(mainly Fourier Transform)*

(based on Edmunds–L.; Edmunds–Gurka–L.; L., L.–Mihula; Chuah–L.–Yao)

Function Spaces, Interpolation Theory and Related Topics 2026, (ChatGPT product)

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Talk plan:

1. s-numbers for compactness: approximation, Kolmogorov, entropy
2. Measures of non-compactness, essential norm, maximal non-compactness
3. Maximally non-compact embeddings on ℓ- and Lorentz scales
4. Strict singularity (s.s.), finite strict singularity (f.s.s.), Bernstein numbers
5. Embeddings revisited via Bernstein numbers
6. Fourier: $L^p \to L^{p'}$ (maximally non-compact but f.s.s.)
7. Fourier: $L^p \to L^{p',p}$ (failure of strict singularity)
8. Triebel diagram and qualitative compactness criterion
9. Fourier into Besov scales ($L^p \to B^s_p$ and $B^s_p \to L^p$): s.s./f.s.s. transitions
10. Embeddings between Function spaces

Main theme

For a bounded linear map $T : X \to Y$ (Banach or quasi-Banach):

- Compactness can be quantified by *rates* $a_n(T), d_n(T), e_n(T) \downarrow 0$.
- For *non-compact* maps, these sequences typically stabilize at a positive level, and become too coarse to distinguish different "degrees" of non-compactness.

We will use them mainly as motivation for strict singularity and Bernstein numbers.

Definitions: approximation/Kolmogorov/entropy numbers

Let $T \in \mathcal{B}(X,Y)$.

Definition

$$a_n(T) := \inf\{\|T - F\| : F \in \mathcal{B}(X,Y),\ \operatorname{rank}(F) < n\},$$
$$d_n(T) := \inf_{Y_n \subset Y, \dim Y_n = n} \ \sup_{z \in T(B_X)} \ \inf_{y \in Y_n} \|y - z\|_Y ,$$
$$e_n(T) := \inf\Big\{\varepsilon > 0 :\ T(B_X) \subset \bigcup_{j=1}^{2^{n-1}} (b_j + \varepsilon B_Y)\Big\}.$$

T is compact $\iff d_n(T) \to 0 \iff e_n(T) \to 0$

If X has the approximation property, then: T is compact $\iff a_n(T) \to 0$

Hausdorff measure of non-compactness and essential norm

Definition

The *Hausdorff (ball) measure of non-compactness* of $T : X \to Y$ is

$$\beta(T) := \inf\Big\{\rho > 0 : \; T(B_X) \subset \bigcup_{j=1}^{m} (y_j + \rho B_Y) \text{ for some } y_1, \dots, y_m \in Y\Big\}.$$

Definition

The *essential norm* of T is

$$\|T\|_e := \inf\{\|T - K\| : K : X \to Y \text{ compact}\}.$$

Always $0 \leq \|T\|_e \leq \beta(T) \leq \|T\|$. In many concrete operator ideals, one has comparability $\beta(T) \asymp \|T\|_e$.

Maximal non-compactness

Definition

T is *maximally non-compact* if $\|T\|_e = \|T\|$.

- This is a sharp non-compactness statement: *no* compact perturbation improves the operator norm.
- Intuitively, in harmonic analysis, maximal non-compactness often follows from translation invariance plus weak convergence.

Lorentz sequence norms (reminder)

For a sequence $x = (x_k)$, let (x_k^*) be its non-increasing rearrangement. For $0 < p < \infty$, $0 < q \leq \infty$:

$$\|x\|_{\ell^{p,q}} = \begin{cases} \left(\sum_{k \geq 1} \left(k^{1/p - 1/q} x_k^*\right)^q \right)^{1/q}, & q < \infty, \\ \sup_{k \geq 1} \, k^{1/p} x_k^*, & q = \infty. \end{cases}$$

The embedding $\ell^{p,q_0} \hookrightarrow \ell^{p,q_1}$ holds for $q_0 \leq q_1$.

Model embeddings (and why they are maximally non-compact)

Consider:

- $I_1 : \ell^{p_0}(\mathbb{N}) \hookrightarrow \ell^{p_1}(\mathbb{N})$ for $1 \leq p_0 \leq p_1 \leq \infty$,
- $I_2 : \ell^{p,q_0}(\mathbb{N}) \hookrightarrow \ell^{p,q_1}(\mathbb{N})$ for $0 < q_0 \leq q_1 \leq \infty$.

Maximal non-compactness (sketch). Let (e_k) be the unit vector basis. For any compact K, one has (after passing to a subsequence) $Ke_{k_j} \to 0$ in the target. Since $\left\|Ie_{k_j}\right\| = 1$ for all j,

$$\left\|(I-K)e_{k_j}\right\| \to 1, \qquad \Rightarrow \qquad \|I-K\| \geq 1 = \|I\|.$$

Hence $\|I\|_e = \|I\| = 1$.

Intermission - Compact Operators

$T : X \to Y$ is compact

$\forall \|x_j\|_X = 1$, $(Tx_j)_{j=1}^{\infty} \subset Y$ has conv subseq.

Compactness is POWERFUL! (PDEs, geometric analysis, etc.)

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Fredholm index is stable

$S : X \to Y$ Fredholm and $T : X \to Y$ compact $\Rightarrow \operatorname{ind}(S+T) = \operatorname{ind} S$.

Spectral theorem

If $T : X \to X$ compact, then $\sigma(T)$ are eigenvalues except 0.

Intermission - Compact Operators (cont.)

$T : X \to Y$ is cpt if $\forall\ \|x_j\|_X = 1$, $(Tx_j)_{j=1}^{\infty} \subset Y$ has conv subseq.

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- If $T : X \to X$ compact, then $\sigma(T)$ are eigenvalues except 0.

But compactness may be TOO strong!

$$\operatorname{ind} S = \dim \operatorname{Ker} S - \dim(Y/S).$$
$\lambda \in \sigma(T)$ is eigenvalue if $\{x \in X : Tx = \lambda x\} \neq \{0\}$

These are stated via **subspaces**, not via sequences.

Strict singularity vs finite strict singularity

Note: Fredholmness and spectral thm holds for weaker notion of cpt.

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Definition

$T : X \to Y$ is *strictly singular* (s.s.) if there is *no* infinite-dimensional subspace $Z \subset X$ such that $T|_Z : Z \to T(Z)$ is an isomorphism.

Definition

T is *finitely strictly singular* (f.s.s.) if for every $\varepsilon > 0$ there exists N_ε such that whenever $\dim E > N_\varepsilon$, there exists $x \in E$ with $\|x\|_X = 1$ and $\|Tx\|_Y \leq \varepsilon$.

Hierarchy: compact $\Rightarrow$ f.s.s. $\Rightarrow$ s.s. (reverse implications fail).

Note: If X, Y are Hilbert spaces then: s.s. $\iff$ f.s.s $\iff$ s.s.

Bernstein numbers

Definition

The n-th *Bernstein number* of T is

$$b_n(T) := \sup\Big\{ \inf_{\substack{x \in X_n \\ \|x\|_X = 1}} \|Tx\|_Y \ : \ X_n \subset X,\ \dim X_n = n \Big\}.$$

- $\|T\| = b_1(T) \geq b_2(T) \geq \cdots$.
- Always $b_n(T) \leq d_n(T) \leq a_n(T)$.
- T is f.s.s. $\iff$ $b_n(T) \to 0$.

Cosingularity and duality (useful for future)

Definition

$T : X \to Y$ is *strictly cosingular* if there is no closed $N \subset Y$ with $codim(N) = \infty$ such that $Q_N T : X \to Y/N$ is surjective.

Definition (Mityagin numbers)

$$m_n(T) := \sup_{\rho \geq 0} \sup_{\substack{N \subset Y \\ N \geq n}} \{\rho : Q_N T(B_X) \supset \rho B_{Y/N}\}.$$

- $b_n(T) = m_n(T^*)$.
- T is (F)SS $\Leftrightarrow$ T^* is (finitely) strictly cosingular.

Embeddings and Bernstein numbers: two different behaviors

Ball measures / essential norms do not distinguish I_1 and I_2 (both maximally non-compact). Bernstein numbers and S.S. do:

- For $I_1 : \ell^{p_0} \hookrightarrow \ell^{p_1}$ with $p_0 < p_1$,

$$b_n(I_1) = n^{\frac{1}{p_1} - \frac{1}{p_0}} \longrightarrow 0,$$

  hence I_1 is f.s.s.

  Idea: take $E_n = \operatorname{span}\{e_1, \dots, e_n\}$; on E_n, one has $\|x\|_{p_1} \geq n^{1/p_1 - 1/p_0} \|x\|_{p_0}$ with equality for $x = (1, \dots, 1, 0, \dots)$.

- For $I_2 : \ell^{p,q_0} \hookrightarrow \ell^{p,q_1}$ with $q_0 < q_1$, one has $b_n(I_2) \gtrsim 1$ (uniformly in n), hence I_2 is *not* f.s.s.; in particular $b_n(I_2)$ does not detect compactness but it detects failure of f.s.s.

  Idea: use a carefully chosen block basis so that ℓ^{p,q_0} and ℓ^{p,q_1} norms are uniformly equivalent on the span.

Take-away

Observation

Even among *maximally non-compact* maps, strict singularity/f.s.s. and Bernstein numbers detect genuine structural transitions ("optimal" vs "non-optimal" target scales).

Fourier transform on $\mathbb{R}^n$

We use the normalization (on Schwartz functions)

$$(\mathcal{F}f)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx.$$

Classical endpoints:

$$\mathcal{F} : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n), \qquad \mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n).$$

Interpolation gives boundedness $\mathcal{F} : L^p \to L^{p'}$ for $1 \le p \le 2$, and real interpolation yields the Lorentz improvement $\mathcal{F} : L^p \to L^{p',p}$.

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$\mathcal{F} : L^2 \to L^2$ is non-S.S. b.c. it's invertible.
$\mathcal{F} : L^1 \to L^\infty$, is S.S. but not F.S.S.;
$\mathcal{F} : L^p \to L^{p'}$, when $1 < p < 2$, is ...
$\mathcal{F} : L^p \to L^{p',p}$, when $1 < p < 2$, is ...

Maximal non-compactness of $\mathcal{F} : L^p \to L^{p'}$ (sketch)

Claim

For $1 < p < 2$, $\mathcal{F} : L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$ is maximally non-compact, i.e. $\|\mathcal{F}\|_e = \|\mathcal{F}\|$.

- Fix $f \in \mathcal{S}(\mathbb{R}^n)$ with $\|f\|_{L^p} = 1$ and $\|\mathcal{F}f\|_{L^{p'}}$ close to $\|\mathcal{F}\|$.
- Let $\tau_a f(x) = f(x-a)$; then $\tau_{a_k} f \rightharpoonup 0$ in L^p as $|a_k| \to \infty$.
- For any compact $K : L^p \to L^{p'}$, compactness + weak convergence give $\|K(\tau_{a_k} f)\|_{L^{p'}} \to 0$.
- Yet $\mathcal{F}(\tau_{a_k} f)(\xi) = e^{-ia_k \cdot \xi}(\mathcal{F}f)(\xi)$, so $\|\mathcal{F}(\tau_{a_k} f)\|_{L^{p'}} = \|\mathcal{F}f\|_{L^{p'}}$.

Letting $\|\mathcal{F}f\| \uparrow \|\mathcal{F}\|$ yields $\|\mathcal{F} - K\| \geq \|\mathcal{F}\|$ for all compact K.

Interpolation theorem (Lefèvre–Rodríguez-Piazza) and Bernstein decay

Theorem ([LRP14, Thm. 5.1])

Let (Ω, μ) *and* (Δ, ν) *be measure spaces and let*

$$T : L^1(\mu) + L^2(\mu) \longrightarrow L^2(\nu) + L^\infty(\nu)$$

be linear, inducing bounded operators

$$T_1 : L^1(\mu) \to L^\infty(\nu), \qquad T_2 : L^2(\mu) \to L^2(\nu).$$

Then for every $1 < p < 2$*, the interpolated operator*

$$T_p : L^p(\mu) \to L^{p'}(\nu), \qquad \frac{1}{p} + \frac{1}{p'} = 1,$$

is finitely strictly singular. Moreover, after scaling so that $\|T_1\| \le 1$ *and* $\|T_2\| \le 1$*, there exists* $K_p > 0$ *such that*

$$b_n(T_p) \le K_p\, n^{-1/r} \qquad (n \in \mathbb{N}), \qquad \frac{1}{r} = \frac{1}{p} - \frac{1}{2}.$$

Fourier $L^p \to L^{p'}$: maximally non-compact but f.s.s.

From the above:

For $1 < p < 2$, the Fourier transform

$$\mathcal{F} : L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$$

is non-compact and *finitely strictly singular*; moreover,

$$b_n\big(\mathcal{F} : L^p \to L^{p'}\big) \;\lesssim\; n^{-1/r}, \qquad \frac{1}{r} = \frac{1}{p} - \frac{1}{2}, \quad r \geq 2.$$

Where is $\mathcal{F}$ invertible? (an infinite-dimensional subspace)

Question

What about $1 < p < 2$, $\mathcal{F} : L^p(\mathbb{R}^n) \to L^{p',p}(\mathbb{R}^n)$?

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One can construct an infinite-dimensional closed subspace.

$$X = \overline{\operatorname{span}}\{\varphi_j : \ j \in \mathbb{N}\} \subset L^p(\mathbb{R}^n)$$

with building blocks φ_j chosen to be *almost disjoint* in space and frequency so that

$$\|f\|_{L^p} = 1, \ f \in X \quad \Longrightarrow \quad \|\mathcal{F}f\|_{L^{p',p}} \geq c > 0.$$

Hence $\mathcal{F}|_X$ is bounded below and therefore an isomorphism onto its image; thus $\mathcal{F} : L^p \to L^{p',p}$ is *not* strictly singular.

Interpolation to Lorentz targets (Bernstein-number estimate)

Question

What about $1 < p < 2$, $\mathcal{F} : L^p(\mathbb{R}^n) \to L^{p',r}(\mathbb{R}^n)$, with $r > p$ (i.e. when $L^{p',p} \hookrightarrow L^{p',r}$)?

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Lemma (Interpolation lemma)

Let $1 < p < 2$ and suppose $T : L^p(\mathbb{R}^n) \to L^{p',p}(\mathbb{R}^n)$ bounded, and $T : L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$ is f.s.s.
Then for every r with $p < r < p'$, $T : L^p(\mathbb{R}^n) \to L^{p',r}(\mathbb{R}^n)$ is FSS and

$$b_n\big(T : L^p \to L^{p',r}\big) \;\lesssim\; b_n\big(T : L^p \to L^{p'}\big)^{\theta}, \quad \theta = \frac{\frac{1}{p} - \frac{1}{r}}{\frac{1}{p} - \frac{1}{p'}} \in (0,1).$$

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Lemma

If $1 < p < 2$ and $r > p$, then $\mathcal{F} : L^p(\mathbb{R}^n) \to L^{p',r}(\mathbb{R}^n)$, is f.s.s.

Trichotomy for $\mathcal{F} : L^p \to L^{p',r}$, $1 < p < 2$

Theorem (Trichotomy for Lorentz targets)

Let $1 < p < 2$ *and* $p < r \le \infty$. *Then*

$$\mathcal{F} : L^p(\mathbb{R}^n) \to L^{p',r}(\mathbb{R}^n)$$

- *is* **unbounded** *if* $r < p$;
- *is* **non-compact and not strictly singular** *if* $r = p$;
- *is* **non-compact and FSS** *if* $r > p$.

Moreover, for $p < r < p'$,

$$b_n\big(\mathcal{F} : L^p \to L^{p',r}\big) \;\lesssim\; n^{\frac{1}{2r} - \frac{1}{2p}}.$$

($L^{p',p}$ is the *optimal* Lorentz target among $L^{p',r}$ and also among all r.i. space.)

Triebel parameters and the “limiting” lines

For fixed $1 < p < \infty$ define

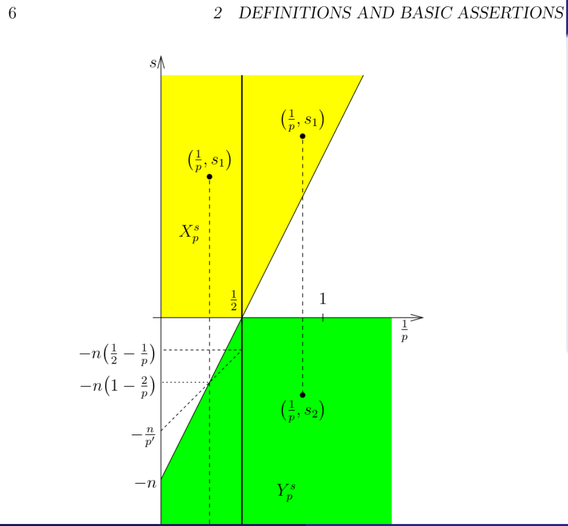
$$d_p^n := 2n\Big(\frac{1}{p} - \frac{1}{2}\Big), \qquad \tau_p^{n+} := \max(0, d_p^n), \qquad \tau_p^{n-} := \min(0, d_p^n).$$

Triebel introduces source/target classes $X_p^{s_1}$ and $Y_p^{s_2}$ (built from L^p and B_p^s) with the natural admissible region

$$s_1 \geq \tau_p^{n+}, \qquad s_2 \leq \tau_p^{n-},$$

and calls the boundary cases $s_1 = \tau_p^{n+}$ and $s_2 = \tau_p^{n-}$ *limiting*.

Triebel's diagram and Theorem 3.2 (qualitative compactness)



Theorem (Triebel, Thm. 3.2)

Let $1 < p < \infty$ *and* $\tau_p^{n+} = \max(0, d_p^n)$, $\tau_p^{n-} = \min(0, d_p^n)$. *For* $s_1 \geq \tau_p^{n+}$ *and* $s_2 \leq \tau_p^{n-}$,

$$\mathcal{F} : X_p^{s_1}(\mathbb{R}^n) \to Y_p^{s_2}(\mathbb{R}^n)$$

is continuous. It is compact iff both inequalities are strict:

$$s_1 > \tau_p^{n+} \quad \text{and} \quad s_2 < \tau_p^{n-}.$$

Interpretation: limiting spaces form *barriers* to compactness; the strict interior gives compactness.

Let $1 < p < \infty$ and $d_p^n = 2n\left(\frac{1}{p} - \frac{1}{2}\right)$, $\tau_p^{n+} = \max(0, d_p^n)$, $\tau_p^{n-} = \min(0, d_p^n)$. Triebel singles out $B_p^s = B_{p,p}^s$ and defines piecewise:

$$X_p^s(\mathbb{R}^n) = \begin{cases} L^p(\mathbb{R}^n), & 2 \le p < \infty,\ s = 0, \\ B_p^s(\mathbb{R}^n), & 2 \le p < \infty,\ s > 0, \\ B_p^s(\mathbb{R}^n), & 1 < p \le 2,\ s \ge d_p^n, \end{cases} \qquad Y_p^s(\mathbb{R}^n) = \begin{cases} B_p^s(\mathbb{R}^n), & 2 \le p < \infty,\ s \le d_p^n, \\ B_p^s(\mathbb{R}^n), & 1 < p \le 2,\ s < 0, \\ L^p(\mathbb{R}^n), & 1 < p \le 2,\ s = 0. \end{cases}$$

The *limiting* spaces are exactly $X_p^{\tau_p^{n+}}$ and $Y_p^{\tau_p^{n-}}$.

Why Besov spaces for $p > 2$?

For $p > 2$, the Fourier transform does not improve integrability the same way. We have a non-compact map:

$$\mathcal{F} : L^p(\mathbb{R}^n) \to B^s_p(\mathbb{R}^n), \qquad s \leq d^n_p < 0.$$

- $s = d^n_p$ is the limiting (critical) smoothness.
- Q: Quality of "non-compactness" when the borderline map is "optimal" $(s = d^n_p)$.
- Q: Quality of "non-compactness" when $s < d^n_p$, i.e. "subcritical" case.

Besov definition

Fix a dyadic resolution of unity $(\phi_j)_{j\geq 0} \subset \mathcal{S}(\mathbb{R}^n)$. For $s \in \mathbb{R}$, $1 \leq p \leq \infty$ (here $q = p$),

$$\|g\|_{B_p^s} \asymp \left(\sum_{j\geq 0} 2^{jsp} \left\| (\phi_j \widehat{g})^\vee \right\|_{L^p}^p \right)^{1/p} .$$

Heuristic: B_p^s is an L^p-based Littlewood–Paley scale; scaling and frequency localization determine the critical exponents.

Boundedness/unboundedness for $\mathcal{F} : L^p \to B^s_p$ $(p > 2)$: a scaling test

Pick $\psi \in \mathcal{S}(\mathbb{R}^n)$ supported in an annulus and define f_j by $\widehat{f_j}(\xi) = \psi(2^{-j}\xi)$. Then

$$\|f_j\|_{L^p} \asymp 2^{jn(1-1/p)}, \qquad \|\mathcal{F} f_j\|_{B^s_p} \asymp 2^{js} \left\|\psi^\vee\right\|_{L^p}.$$

Comparing growth yields the critical inequality $s \le d^n_p$ for boundedness.

Failure of strict singularity at $s = d_p^n$: isomorphism on a block system

Choose (f_j) with essentially disjoint spatial supports so that for $f = \sum_j \alpha_j f_j$,

$$\|f\|_{L^p}^p \asymp \sum_j |\alpha_j|^p.$$

Arrange dyadic separation in frequency so that

$$\|\mathcal{F} f\|_{B_p^{d_p^n}}^p \asymp \sum_j |\alpha_j|^p.$$

Hence $\mathcal{F}$ is an isomorphism on $\overline{\mathrm{span}}\{f_j\} \cong \ell^p$, so $\mathcal{F}$ is not s.s. at the critical line.

Main trichotomy: $\mathcal{F} : L^p \to B^s_p$

Theorem (Trichotomy for Besov targets)

Let $2 < p < \infty$ *and* $d^n_p < 0$. *Then*

$$\mathcal{F} : L^p(\mathbb{R}^n) \to B^s_p(\mathbb{R}^n)$$

- *is* **unbounded** *if* $s > d^n_p$;
- *is* **non-compact and not strictly singular** *if* $s = d^n_p$;
- *is* **non-compact and FSS** *if* $s < d^n_p$.

Trichotomy: $\mathcal{F} : B^s_p \to L^p$ for $p < 2$

Theorem (Dual trichotomy)

Let $1 < p < 2$ *and* $d^n_p > 0$. *Then*

$$\mathcal{F} : B^s_p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

- *is* **unbounded** *if* $s < d^n_p$;
- *is* **non-compact and not strictly singular** *if* $s = d^n_p$;
- *is* **non-compact and FSS** *if* $s > d^n_p$.

Cosingularity trichotomies

Using $b_n(T) = m_n(T^*)$ and duality of $\mathcal{F}$ (up to reflection), one obtains:

- For $1 < p < 2$, $\mathcal{F} : B^s_p \to L^p$ is
  - unbounded if $s < d^n_p$;
  - non-compact and *not strictly cosingular* if $s = d^n_p$;
  - non-compact and *finitely strictly cosingular* if $s > d^n_p$.
- For $2 < p < \infty$, $\mathcal{F} : L^p \to B^s_p$ is
  - unbounded if $0 > s > d^n_p$;
  - non-compact and *not strictly cosingular* if $s = d^n_p$;
  - non-compact and *finitely strictly cosingular* if $s < d^n_p$.

A second "optimal" Besov scale

There is also an optimal mixed-Besov scale:

Corollary (Optimal $B^{d_p^n/2}_{2,p}$ scale)

- *If* $2 < p < \infty$, *then* $\mathcal{F} : L^p(\mathbb{R}^n) \to B^{d_p^n/2}_{2,p}(\mathbb{R}^n)$ *is not strictly singular.*
- *If* $1 < p < 2$, *then* $\mathcal{F} : B^{d_p^n/2}_{2,p}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ *is not strictly singular.*

Takeaways

- For each mapping scale, the Fourier transform exhibits a sharp *trichotomy*:

  unbounded / non-compact non-SS (optimal) / non-compact FSS.

- Bernstein numbers quantify the “degree” of non-compactness beyond $\alpha(T)$.
- Borderline non-SS is typically proved by constructing an explicit infinite-dimensional subspace with dyadic separation (almost-diagonal behavior).

Besov embeddings on $\Omega \subset \mathbb{R}^n$

Theorem (Chuah, L, Yao, 2025)

Let $p_0, p_1, q_0, q_1 \in (0, \infty]$, $s_0, s_1 \in \mathbb{R}$, $\Omega \subset \mathbb{R}^n$. $I : B^{s_0}_{p_0 q_0}(\Omega) \hookrightarrow B^{s_1}_{p_1 q_1}(\Omega)$.

1. $B^{s_0}_{p_0 q_0}(\Omega) \not\subset B^{s_1}_{p_1 q_1}(\Omega)$, *i.e. there is no embedding, if and only if:*
   - $s_0 - s_1 < \max(0, \frac{n}{p_0} - \frac{n}{p_1})$; *or* $s_0 - s_1 = \max(0, \frac{n}{p_0} - \frac{n}{p_1})$ *and* $q_0 > q_1$.
2. *The embedding is compact if and only if* $s_0 - s_1 > \max(0, \frac{n}{p_0} - \frac{n}{p_1})$.
3. *The embedding is non-compact but finitely strictly singular if and only if:*
   - $s_0 - s_1 = \frac{n}{p_0} - \frac{n}{p_1} > 0$ *and* $q_0 < q_1$; *or* $s_0 - s_1 = 0$, $p_1 < p_0 = \infty$ *and* $q_0 < q_1$.
4. *The embedding is not finitely strictly singular but strictly singular if and only if:*
   - $s_0 - s_1 = 0$, $p_1 = p_0 = \infty$ *and* $q_0 < q_1$; *or*
   - $s_0 - s_1 = 0$, $p_1 \leq p_0 < \infty$ *and* $q_0 < q_1$.
5. *The embedding is not strictly singular if and only if* $s_0 - s_1 = \max(0, \frac{n}{p_0} - \frac{n}{p_1})$ *and* $q_0 = q_1$.

Besov embeddings on $\mathbb{R}^n$

Theorem (Chuah,L, Yao, 2025)

For $I: \; B^{s_0}_{p_0,q_0}(\mathbb{R}^n) \hookrightarrow B^{s_1}_{p_1,q_1}(\mathbb{R}^n)$ *one has:*

- *no embedding if* $p_0 > p_1$, *or* $s_0 - s_1 < \frac{n}{p_0} - \frac{n}{p_1}$, *or equality with* $q_0 > q_1$;
- *non-compact but FSS if* $s_0 - s_1 > \frac{n}{p_0} - \frac{n}{p_1} > 0$, *or equality with* $q_0 < q_1$;
- *not SS on the "diagonal" regime (examples include* $p_0 = p_1$ *with* $s_0 > s_1$, *and* $s_0 = s_1$ *with* $q_0 \le q_1$*).*

Sobolev embeddings on $\Omega \subset \mathbb{R}^n$

Theorem ((L., Mihula, 2023),(Chuah,L,, Yao, 2025,2026))

$k \in Z_+$, $1 \leq p < \frac{n}{k}$, $\frac{1}{p^*} = \frac{1}{p} - \frac{k}{n}$, Ω *bdd*

- $W^{k,p}(\Omega) \hookrightarrow L^{p*,q}(\Omega)$ *is FSS, for* $q > p$.
- $W^{k,p}(\Omega) \hookrightarrow L^{p*,p}(\Omega)$ *is not SS.*

Open problem

Problem

Identify Besov source/target spaces for which the Fourier transform is

strictly singular but not finitely strictly singular.

- Compare with $\ell^{p,q_0} \hookrightarrow \ell^{p,q_1}$: SS but not FSS.
- Suggests looking for Lorentz/Besov parameter gaps mimicking "q-gap" behavior.

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