

Some new results on function spaces of Lorentz-Sobolev type

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Workshop at UCM
(March, 2026)

Let $(A, \|\cdot\|)$ be a complex quasi-Banach space with constant $c_A \geq 1$ in triangle inequality.

Let $0 < p \leq 1$ be such that $2^{1/p-1} = c_A$.

By the Aoki-Rolewicz theorem, there is another quasi-norm $\|\cdot\|_*$ on A such that $\|\cdot\| \sim \|\cdot\|_*$ and

$$\|x + y\|_*^p \leq \|x\|_*^p + \|y\|_*^p, \quad x, y \in A.$$

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By a **quasi-Banach (respectively, p -Banach) couple** $\bar{A} = (A_0, A_1)$ we mean two quasi-Banach spaces (respectively, two p -Banach spaces) $A_j, j = 0, 1$, such that $A_j \hookrightarrow \mathcal{A}$.

The Peetre's K -functional: For $a \in A_0 + A_1$

$$K(t, a) = K(t, a; A_0, A_1) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j\}, t > 0.$$

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For $0 < \theta < 1$ and $0 < q \leq \infty$, the **real interpolation space** $(A_0, A_1)_{\theta, q}$ consists of all $a \in A_0 + A_1$ which have a finite quasi-norm

$$\|a\|_{\theta, q} = \left(\int_0^\infty (t^{-\theta} K(t, a))^q \frac{dt}{t} \right)^{\frac{1}{q}} \quad \text{if } 0 < q < \infty,$$

$$\|a\|_{\theta, \infty} = \sup_{t > 0} \{t^{-\theta} K(t, a)\}.$$

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Interpolation property: Let $\bar{B} = (B_0, B_1)$ be another quasi-Banach couple. If $T : A_0 + A_1 \rightarrow B_0 + B_1$ is a linear operator whose restrictions $T : A_j \rightarrow B_j$ are bounded for $j = 0, 1$, then

$$T : (A_0, A_1)_{\theta, q} \rightarrow (B_0, B_1)_{\theta, q} \text{ is bounded}$$

with

$$\|T\|_{(A_0, A_1)_{\theta, q}, (B_0, B_1)_{\theta, q}} \leq \|T\|_{A_0, B_0}^{1-\theta} \|T\|_{A_1, B_1}^{\theta}.$$

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▷ L.M. Fernández-Cabrera and A. Martínez, J. Fourier Anal. Appl. 24 (2018) 1181-1203.

Bilinear interpolation theorem.- Let (A_0, A_1) , (B_0, B_1) be quasi-Banach couples and let (E_0, E_1) be an r -Banach couple, $0 < r \leq 1$. Let $0 < \theta < 1$ and $0 < q_0, q_1, q \leq \infty$ with

$$\frac{1}{q} = \begin{cases} 1/q_0 + 1/q_1 - 1/r & \text{if } q_0, q_1 \geq r, \\ 1/\max(q_0, q_1) & \text{if } q_0 < r \text{ or } q_1 < r. \end{cases}$$

If $T : (A_0 \cap A_1) \times (B_0 \cap B_1) \longrightarrow (E_0 \cap E_1)$ is a bilinear operator such that

$$\|T(a, b)\|_{E_j} \leq M_j \|a\|_{A_j} \|b\|_{B_j}, \quad a \in A_0 \cap A_1, b \in B_0 \cap B_1, j = 0, 1$$

then

$$\|T(a, b)\|_{(E_0, E_1)_{\theta, q}} \leq CM_0^{1-\theta} M_1^{\theta} \|a\|_{(A_0, A_1)_{\theta, q_0}} \|b\|_{(B_0, B_1)_{\theta, q_1}}$$

for any $a \in A_0 \cap A_1, b \in B_0 \cap B_1$. Furthermore, if $q_j < \infty$ for $j = 0, 1$, then T may be uniquely extended to a bounded bilinear operator

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- Conditions on q_0, q_1, q come from Young's inequality.
- If $q_0 = q_1 = r$ then $q = r$ and $T : (A_0, A_1)_{\theta, r} \times (B_0, B_1)_{\theta, r} \longrightarrow (E_0, E_1)_{\theta, r}$ is bounded.

Examples.

Fix $n \in \mathbb{N}$ and consider \mathbb{R}^n endowed with the Lebesgue measure.

Let A be a quasi-Banach space and let $0 < p \leq \infty$.

We put $L_p(A) = L_p(\mathbb{R}^n; A)$ for the usual *vector-valued* L_p -space in the sense of the Bochner integral.

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For $0 < r \leq \infty, 0 < p_0 \neq p_1 \leq \infty, 0 < \theta < 1$ and $1/p = (1 - \theta)/p_0 + \theta/p_1$, we have with equivalent quasi-norms

$$(L_{p_0}(A), L_{p_1}(A))_{\theta, r} = L_{p, r}(A).$$

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For $0 < p < \infty$ and $0 < r \leq \infty$, the *Lorentz space* $L_{p, r}(A) = L_{p, r}(\mathbb{R}^n; A)$ consists of all (equivalence classes of) strongly measurable functions with values in A which have a finite quasi-norm

$$\|f\|_{L_{p, r}(A)} = \left(\int_0^\infty (t^{1/p} f^*(t))^r \frac{dt}{t} \right)^{1/r}$$

Here f^* is the *non-increasing rearrangement* of f

$$f^*(t) = \inf\{s > 0 : |\{x \in \mathbb{R}^n : \|f(x)\|_A > s\}| \leq t\}$$

- For $p = r$ we have $L_{p, p}(A) = L_p(A)$.

Lions and Peetre proved that if A is replaced by a Banach couple (A_0, A_1) then

$$(L_{p_0}(A_0), L_{p_1}(A_1))_{\theta, p} = L_p((A_0, A_1)_{\theta, p})$$

provided that $1 \leq p_0, p_1 < \infty, 0 < \theta < 1$ and $1/p = (1 - \theta)/p_0 + \theta/p_1$.

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• We denote by $\ell_q^s(A)$ the space of all sequences $(a_k)_{k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}} \subseteq A$ such that

$$\|(a_k)\|_{\ell_q^s(A)} = \left(\sum_{k=0}^{\infty} (2^{ks} \|a_k\|_A)^q \right)^{1/q} < \infty$$

(with the usual modification if $q = \infty$). We put ℓ_q^s if $A = \mathbb{C}$.

For $0 < q_0, q_1, r \leq \infty$, $-\infty < s_0 \neq s_1 < \infty$, $0 < \theta < 1$ and $s = (1 - \theta)s_0 + \theta s_1$, we have

$$(\ell_{q_0}^{s_0}(A), \ell_{q_1}^{s_1}(A))_{\theta, r} = \ell_r^s(A).$$

If we replace A by a quasi-Banach couple (A_0, A_1) then, for $0 < q_0, q_1 \leq \infty$, $-\infty < s_0 \neq s_1 < \infty$, $0 < \theta < 1$, $s = (1 - \theta)s_0 + \theta s_1$ and $1/q = (1 - \theta)/q_0 + \theta/q_1$, we have

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- **Function spaces**

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• Function spaces

Take $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\varphi_0(x) = 0 \quad \text{if } |x| \geq 3/2 \quad \text{and} \quad \varphi_0(x) = 1 \quad \text{if } |x| \leq 1.$$

. For $j \in \mathbb{N}$ we put

$$\varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x), \quad x \in \mathbb{R}^n.$$

Then $(\varphi_j)_{j=0}^{\infty}$ forms a dyadic resolution of unity: $\sum_{j=0}^{\infty} \varphi_j(x) = 1$, $x \in \mathbb{R}^n$.

For $-\infty < s < \infty, 0 < p, q \leq \infty$

$$B_{p,q}^s = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{B_{p,q}^s} = \left(\sum_{j=0}^{\infty} (2^{js} \|\mathcal{F}^{-1}(\varphi_j \mathcal{F}f)\|_{L_p})^q \right)^{1/q} < \infty \right\}.$$

Here $\mathcal{F}f$ denotes de Fourier transform of f and \mathcal{F}^{-1} the inverse Fourier transform.

$$(\mathcal{F}^{-1}(\varphi_j \mathcal{F}f)) \in \ell_q^s(L_p)$$

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For $s \in \mathbb{R}, 0 < p < \infty$ and $0 < q \leq \infty$, we have

$$B_{p, \min\{p,q\}}^s \hookrightarrow F_{p,q}^s \hookrightarrow B_{p, \max\{p,q\}}^s, \quad \text{so} \quad F_{p,p}^s = B_{p,p}^s.$$

They are quasi-Banach spaces (Banach spaces if $p \geq 1$ and $q \geq 1$).

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- Triebel-Lizorkin-Lorentz spaces ($L_p \rightsquigarrow L_{p,r}$)

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However, if we allow that p changes then the resulting spaces are outside the scale of Besov spaces.

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- These spaces with Lorentz smoothness have been used during the last 50 years by a number of authors in different contexts.

- **Besov-Lorentz spaces**

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- ▷ J. Peetre, Duke Univ. Math. Series, Durham, 1976.

- **Hardy-Lorentz spaces**

- ▷ C. Fefferman, N.M. Riviere, Y. Sagher, Trans. Am. Math. Soc. 191 (1974) 75-81.

- ▷ A. Almeida, A.M. Caetano, Acta Math. Sin. Engl. Ser. 26 (2010) 1673-1692.

- **Bessel-Lorentz potential spaces ($F_2^s L_{p,r}$)**

- ▷ E.M. Stein, Ann. Math. 113 (1981) 383-385.

- ▷ A. Cianchi, L. Pick, Ann. Inst. Fourier 60 (2010) 939-951.

- ▷ L. Grafacos, L. Slavíková, Int. Math. Res. Not. IMRN 15 (2019) 4764-4783.

- **Triebel-Lizorkin-Lorentz spaces in connection with wavelet theory**

- ▷ Q. Yand, Z. Cheng, L. Peng, Acta Math. Sci. Ser. A (Chin. Ed.) 25 (2005) 130-144.

- **Triebel-Lizorkin-Lorentz spaces in the context of partial differential equations**

- ▷ Z. Xiang, W. Yan, J. Evol. Equ. 11 (2011) 241-263.

- ▷ P. Hobus, J. Saal, Z. Anal. Anwendungen 38 (2019) 41-72.

- **Embeddings between spaces of Lorentz smoothness**

- ▷ A. Seeger, W. Trebels, Math. Ann. 373 (2019) 1017-1056.

- For $1 < p_0 \neq p_1 < \infty$, $-\infty < s < \infty$, $1 \leq q, r \leq \infty$, $0 < \theta < 1$ and $1/p = (1 - \theta)/p_0 + \theta/p_1$

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Consider the bounded and linear operator

$$J : F_{p, q}^s \longrightarrow L_p(\ell_q^s), \quad J(f) = (\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)),$$

Put $\widetilde{\varphi}_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}$ with $\varphi_{-1} \equiv 0$, define

$$R : L_p(\ell_q^s) \longrightarrow F_{p, q}^s, \quad R(f_j) = \sum_{j=0}^{\infty} \mathcal{F}^{-1}(\widetilde{\varphi}_j \mathcal{F} f_j) \quad (\text{convergence in } \mathcal{S}'(\mathbb{R}^n)).$$

It turns out that $R(Jf) = f$. Hence J is an isomorphic mapping from

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- But the operator R is meaningless for $0 < p_j < 1$.

A [new approach](#) has been given by

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We have been able to transfer a number of important properties of the usual $F_{p,q}^s$ spaces to spaces with Lorentz smoothness.

▷ B.F. Besoy, D.H. Haroske, H. Triebel, Math. Nachr. 295 (2022) 1669-1689.

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For $0 < p < \infty, s > n/p$ and $0 < q \leq \infty$, we have $F_{p,q}^s \hookrightarrow L_\infty$.

Hence, it follows from the characterization by interpolation that

$$F_q^s L_{p,r} \hookrightarrow L_\infty \quad \text{provided that } 0 < p < \infty, s > n/p, 0 < q, r \leq \infty.$$

So, if $f, g \in F_q^s L_{p,r}$ then fg makes sense.

Theorem.- Let $0 < p < \infty, s > n/p, 0 < q \leq \infty, 0 < r \leq \min\{1, q\}$ and $r < p$.
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Theorem.- Let $0 < p < \infty, s > n/p, 0 < q \leq \infty, 0 < r \leq \min\{1, q\}$ and $r < p$. Then $F_q^s L_{p,r}$ is a multiplication algebra.

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- It is conjectured that the theorem should hold without conditions on r .

- $F_q^s L_{p,1}$ is a Banach space and a multiplication algebra provided that $1 < p < \infty$, $s > n/p$ and $1 \leq q \leq \infty$.

This allows to use also the other main interpolation method, the complex method.

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Theorem.- Let $1 < r < p < \infty, s > n/p$ and $1 \leq q \leq \infty$. Then $F_q^s L_{p,r}$ is a multiplication algebra.

- It remains open the case $p < r$, and also the case $0 < q \leq 1$ and $q < r$.

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We follow a direct approach based on the method of paramultiplication. Our arguments are a combination of interpolation results and ideas developed by Peetre, Triebel and Franke.

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We follow a direct approach based on the method of paramultiplication. Our arguments are a combination of interpolation results and ideas developed by Peetre, Triebel and Franke.

Condition $s > n/p$ is necessary for $F_q^s L_{p,r}$ to be a multiplication algebra at least for a certain range of parameters.

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$$\|T_\varphi(f)\|_{F_q^s L_{p,r}(\mathbb{R}^n)} = \|\varphi f\|_{F_q^s L_{p,r}(\mathbb{R}^n)} \leq c \|\varphi\|_{F_q^s L_{p,r}(\mathbb{R}^n)} \|f\|_{F_q^s L_{p,r}(\mathbb{R}^n)}$$

So $T_\varphi \in \mathcal{L}(F_q^s L_{p,r}(\mathbb{R}^n), F_q^s L_{p,r}(\mathbb{R}^n))$ with $\|T_\varphi\| \leq c \|\varphi\|_{F_q^s L_{p,r}(\mathbb{R}^n)}$.

Pick $p < v < \infty$ such that $s > (n/p) - (n/v) > 0$. Then

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Passing to the limit when $m \rightarrow \infty$ we derive that

$$\begin{aligned} \|\varphi\|_{L_\infty(\mathbb{R}^n)} &\leq c_1 \|T_\varphi\|_{F_q^s L_{p,r}(\mathbb{R}^n), F_q^s L_{p,r}(\mathbb{R}^n)} \\ &\leq c_2 \|\varphi\|_{F_q^s L_{p,r}(\mathbb{R}^n)}. \end{aligned}$$

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Next we turn our attention to [Besov-Lorentz spaces](#)

$$B_q^s L_{p,r} = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{B_q^s L_{p,r}} = \left(\sum_{j=0}^{\infty} (2^{js} \|\mathcal{F}^{-1}(\varphi_j \mathcal{F}f)\|_{L_{p,r}})^q \right)^{1/q} < \infty \right\}.$$

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This time we have for $0 < \theta < 1$, $-\infty < s_0, s_1 < \infty$, $s = (1 - \theta)s_0 + \theta s_1$, $0 < p_0 \neq p_1 < \infty$, $1/p = (1 - \theta)/p_0 + \theta/p_1$, $0 < q_0, q_1 < \infty$ and $1/q = (1 - \theta)/q_0 + \theta/q_1$

$$(B_{p_0, q_0}^{s_0}, B_{p_1, q_1}^{s_1})_{\theta, q} = B_q^s L_{p, q}$$

- $(L_{p_0}(A_0), L_{p_1}(A_1))_{\theta, q} = L_q((A_0, A_1)_{\theta, q}), \quad 1/q = (1 - \theta)/p_0 + \theta/p_1.$

We have $(F_{q_0}^{s_0} L_{p,r}, F_{q_1}^{s_1} L_{p,r})_{\theta,q} = B_q^s L_{p,r}$

provided that $-\infty < s_0 \neq s_1 < \infty, 0 < p, r < \infty, 0 < q_0, q_1, q \leq \infty, 0 < \theta < 1$
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Theorem.- Let $0 < p < \infty, s > n/p, 0 < q \leq \infty$ and $0 < r \leq \infty$. Then $B_q^s L_{p,r}$ is a multiplication algebra.

The proof is based on the theorem for spaces $F_q^s L_{p,r}$ and a characterization of spaces $B_q^s L_{p,r}$ as approximation spaces based on spaces $F_q^s L_{p,r}$.

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Theorem.- Let

- (i) $s > 0, 1 < p, q < \infty$ and $0 < r < \infty$, or
- (ii) $-\infty < s < \infty$ and $1 < p, q, r < \infty$, or
- (iii) $-\infty < s < \infty, 1 < q < \infty, 0 < p < 1$ and $0 < r < \infty$, or
- (iv) $-\infty < s < \infty, 1 < q < \infty, p = 1$ and $0 < r \leq 1$.

If $B_q^s L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra, then $s > n/p$.

Proof of (ii) Take any $\varphi \in \mathcal{S}(\mathbb{R}^n)$. If $B_q^s L_{p,r}(\mathbb{R}^n)$ is a multiplication algebra, then $T_\varphi : B_q^s L_{p,r}(\mathbb{R}^n) \longrightarrow B_q^s L_{p,r}(\mathbb{R}^n)$, $T_\varphi(f) = \varphi f$ is bounded with

$$\|T_\varphi\|_{B_q^s L_{p,r}(\mathbb{R}^n), B_q^s L_{p,r}(\mathbb{R}^n)} \leq c_1 \|\varphi\|_{B_q^s L_{p,r}(\mathbb{R}^n)}$$

where c_1 is a constant independent of φ .

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T_φ is self-adjoint. Therefore, $T_\varphi : B_{q'}^{-s} L_{p',r'}(\mathbb{R}^n) \longrightarrow B_{q'}^{-s} L_{p',r'}(\mathbb{R}^n)$ is also bounded with norm less than or equal to $c_2 \|\varphi\|_{B_q^s L_{p,r}(\mathbb{R}^n)}$. Here $1/p + 1/p' = 1$.

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$$\left(B_q^s L_{p,r}(\mathbb{R}^n), B_{q'}^{-s} L_{p',r'}(\mathbb{R}^n) \right)_{1/2,2} = B_{2,2}^0(\mathbb{R}^n) = L_2(\mathbb{R}^n).$$

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Then $T_\varphi : L_2(\mathbb{R}^n) \longrightarrow L_2(\mathbb{R}^n)$ is bounded with

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Consequently,

$$\|\varphi\|_{L_\infty} = \|T_\varphi\|_{L_2(\mathbb{R}^n), L_2(\mathbb{R}^n)} \leq c_3 \|\varphi\|_{B_q^s L_{p,r}(\mathbb{R}^n)}, \varphi \in \mathcal{S}(\mathbb{R}^n).$$

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Density of $\mathcal{S}(\mathbb{R}^n)$ in $B_q^s L_{p,r}(\mathbb{R}^n)$ yields that

$$B_q^s L_{p,r}(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n). \square$$

▷ F. Cobos, L.M. Fernández-Cabrera, T. Kühn, preprint, 2025.

• Dual spaces of all Besov-Lorentz spaces for q and r finite.

We use the characterization of $B_q^s L_{p,r}(\mathbb{R}^n)$ in terms of wavelets

▷ A. Almeida, pp. 601-610, World Scientific, London, 2009. (Case $r = q$).

▷ B.F. Besoy, F. Cobos, J. Funct. Anal. 282 (2022) 109452. (General case).

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Let $\ell_{p,r} = L_{p,r}(\mathbb{Z}^n) = \{(x_m) : \|(x_m)\|_{\ell_{p,r}} = (\sum_{k=1}^{\infty} (k^{1/p} x_k^*)^r k^{-1})^{1/r} < \infty\}$.

Here x^* is the non-increasing rearrangement of x

$x_k^* = \inf\{\|x - u\|_{\ell_{\infty}(\mathbb{Z}^n)} : u \in \ell_{\infty}(\mathbb{Z}^n), \text{card}(u) < k\}, k \in \mathbb{N}$.

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Consider also the limiting Lorentz spaces

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▷ F. Cobos, I. Resina, J. London Math. Soc. 39 (1989) 324-334.

▷ F. Cobos, M. Milman, Numer. Funct. Anal. Optim. 11 (1990) 11-31.

▷ F. Cobos, O. Domínguez, T. Kühn, Constr. Approx. 47 (2018) 453-486.

For $1 \leq p \leq \infty$, we put $1/p + 1/p' = 1$, and we write $p' = \infty$ if $0 < p < 1$. We have

$$(\ell_{p,r})' = \begin{cases} \ell_{p',r'} & \text{if } 1 < p < \infty, 0 < r < \infty, \\ \ell_{\infty} & \text{if either } 0 < p < 1, 0 < r < \infty, \text{ or } p = 1, 0 < r \leq 1, \\ \ell_{\infty,r'} & \text{if } p = 1, 1 < r < \infty. \end{cases}$$

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Let $\langle \ell_{p,r} \rangle^0$ be the direct sum of 2^n copies of $\ell_{p,r}$
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$B_q^s L_{p,r}(\mathbb{R}^n)$ is isomorphic to $\langle \ell_{p,r} \rangle^0 \oplus \ell_q^{s-n/p}(\langle \ell_{p,r} \rangle)$.

Theorem.- Let $-\infty < s < \infty$ and $0 < q < \infty$. Then we have with equivalence of norms

$$(B_q^s L_{p,r}(\mathbb{R}^n))' = \begin{cases} B_{q'}^{-s} L_{p',r'}(\mathbb{R}^n) & \text{if } 1 < p < \infty \text{ and } 0 < r < \infty, \\ B_{\infty,q'}^{-s+n(1/p-1)}(\mathbb{R}^n) & \text{if either } 0 < p < 1 \text{ and } 0 < r < \infty, \\ & \text{or } p = 1 \text{ and } 0 < r \leq 1, \\ B_{q'}^{-s} L_{\infty,r'}(\mathbb{R}^n) & \text{if } p = 1 \text{ and } 1 < r < \infty. \end{cases}$$

Here $B_q^s L_{\infty,r}(\mathbb{R}^n)$ is formed by all distributions such that their coefficients in the wavelet system belong to $\langle \ell_{\infty,r} \rangle^0 \oplus \ell_q^s(\langle \ell_{\infty,r} \rangle)$.