

# Mapping properties of Fourier transforms in function spaces, some recent results

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joint work with Leszek Skrzypczak (Poznań) and Hans Triebel (Jena)

# Fourier transform

The setting

$$(\mathcal{F}\varphi)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} \varphi(x) dx, \quad \varphi \in \mathcal{S}(\mathbb{R}^n), \xi \in \mathbb{R}^n$$

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- ▶  $\mathcal{F}\mathcal{S}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n)$  bijection,  $\mathcal{F}^{-1}\varphi(x) = \mathcal{F}\varphi(-x)$
- ▶ extension to  $\mathcal{S}'(\mathbb{R}^n)$ :  $\mathcal{F}f(\varphi) = f(\mathcal{F}\varphi)$ ,  $f \in \mathcal{S}'(\mathbb{R}^n)$ ,  $\varphi \in \mathcal{S}(\mathbb{R}^n)$   
 $\curvearrowright \mathcal{F}\mathcal{S}'(\mathbb{R}^n) = \mathcal{S}'(\mathbb{R}^n)$  bijection
- ▶  $\mathcal{F}L_2(\mathbb{R}^n) = L_2(\mathbb{R}^n)$  unitary,  $\|\mathcal{F}f\|_{L_2(\mathbb{R}^n)} = \|f\|_{L_2(\mathbb{R}^n)}$
- ▶ interplay with  $D^\alpha$ ,  $\alpha \in \mathbb{N}_0^n$ , that is,

$$\mathcal{F}(D^\alpha f) = i^{|\alpha|} x^\alpha \mathcal{F}f, \quad f \in \mathcal{S}'(\mathbb{R}^n)$$

# The Fourier transform

The classical Hausdorff-Young inequality

- ▶  $\mathcal{F}L_2(\mathbb{R}^n) = L_2(\mathbb{R}^n)$  unitary,  $\|\mathcal{F}f|_{L_2(\mathbb{R}^n)}\| = \|f|_{L_2(\mathbb{R}^n)}\|$
- ▶  $\mathcal{F} : L_1(\mathbb{R}^n) \rightarrow L_\infty(\mathbb{R}^n)$ ,  $\mathcal{F}f$  uniformly continuous,

$$\|\mathcal{F}f|_{L_\infty(\mathbb{R}^n)}\| \leq (2\pi)^{-\frac{n}{2}} \|f|_{L_1(\mathbb{R}^n)}\|$$

- ▶  $f \in L_p(\mathbb{R}^n)$ ,  $1 \leq p \leq 2 \iff \mathcal{F}f$  regular distribution

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- ▶ via interpolation:  $\mathcal{F} : L_p(\mathbb{R}^n) \rightarrow L_{p'}(\mathbb{R}^n)$ ,  $1 \leq p \leq 2$ , and  $\frac{1}{p} + \frac{1}{p'} = 1$ ,

$$\|\mathcal{F} : L_p(\mathbb{R}^n) \rightarrow L_{p'}(\mathbb{R}^n)\| \leq (2\pi)^{-n(\frac{1}{p} - \frac{1}{2})}$$

Hausdorff-Young inequality

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Hausdorff-Young inequality

**Question:** What happens if  $p > 2$ ? Is  $\mathcal{F}f$ ,  $f \in L_p(\mathbb{R}^n)$ , still a regular distribution? What else can be said?

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- ▶  $\mathcal{F}$  nuclear ✓

here:  $X_1, X_2 \dots$  Besov or Sobolev spaces

# The talk is based on our joint papers:



D.D. Haroske and L. Skrzypczak.

Entropy and approximation numbers of embeddings of function spaces with Muckenhoupt weights, I.

*Rev. Mat. Complut.*, 21(1):135-177, 2008.



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Nuclear embeddings in weighted function spaces.

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D.D. Haroske, L. Skrzypczak, and H. Triebel.

Nuclear Fourier transforms.

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D.D. Haroske, L. Skrzypczak, and H. Triebel.

Mapping properties of Fourier transforms, revisited.

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## Introduction

Fourier transform

Function spaces of Besov and Sobolev type

## Continuous embeddings

Tools

Weighted setting

## Compact embeddings

General criterion

Digression

Entropy and approximation numbers

## Nuclear maps

The concept and recent results

Nuclearity of  $\mathcal{F}$

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# Classical Smoothness Spaces

Sobolev spaces on  $\mathbb{R}^n$

- ▶ classical:  $1 \leq p < \infty$ ,  $k \in \mathbb{N}_0$

Sobolev space  $W_p^k = \{f \in L_p : D^\alpha f \in L_p \text{ for } \alpha \in \mathbb{N}_0^n, |\alpha| \leq k\}$

$$\|f|W_p^k\| = \sum_{|\alpha| \leq k} \|D^\alpha f|L_p\|$$

- ▶ via Fourier analysis:  $1 < p < \infty$ ,  $s \in \mathbb{R}$

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From Sobolev to Besov

starting point:

$$\|f\|_{H_2^s} \sim \left( \sum_{j=0}^{\infty} 2^{2js} \|\varphi_j \mathcal{F}f\|_{L_2}^2 \right)^{\frac{1}{2}} \sim \left\| (2^{js} \|\mathcal{F}^{-1}(\varphi_j \mathcal{F}f)\|_{L_2})_j \right\|_{\ell_2}$$

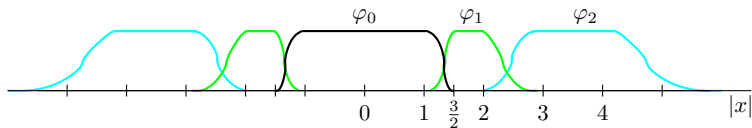
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$s \in \mathbb{R}$ ,  $(\varphi_j)_j$  smooth dyadic partition of unity



$$\varphi_j \in C_0^\infty, \quad \text{supp } \varphi_j \subset \{x \in \mathbb{R}^n : 2^{j-1} < |x| < 2^{j+1}\}, \quad \sum_{j=0}^{\infty} \varphi_j(x) = 1$$

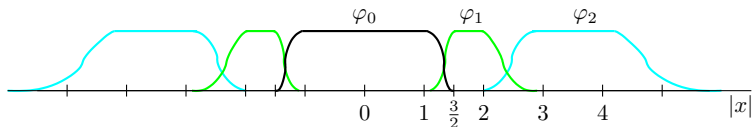
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$$\text{Besov space } B_{p,q}^s \subset \mathcal{S}' : \quad \|f\|_{B_{p,q}^s} = \left\| \left( 2^{js} \|\mathcal{F}^{-1}(\varphi_j \mathcal{F}f)\|_{L_p} \right)_j \right\|_{\ell_q}$$

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Besov and Triebel-Lizorkin spaces on  $\mathbb{R}^n$

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Rem.

- ▶ (quasi-)Banach spaces,  $\mathcal{S} \hookrightarrow A_{p,q}^s \hookrightarrow \mathcal{S}'$
- ▶ (classical) Besov spaces for  $p, q \geq 1$  and  $s \geq 0$  via differences
- ▶  $p = q = \infty$ :  $B_{\infty,\infty}^s = C^s$ ,  $s > 0$  Hölder-Zygmund spaces
- ▶  $1 < p < \infty$ ,  $s \in \mathbb{R}$ :  $F_{p,2}^s = H_p^s$ , i.e.,  $F_{p,2}^k = W_p^k$ ,  $k \in \mathbb{N}_0$

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Convention:  $A_{p,q}^s$  with  $A \in \{B, F\}$ ,  $p < \infty$  when  $A = F$

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## The setting, revisited

idea: generalise  $\mathcal{F} : L_2(\mathbb{R}^n) \rightarrow L_2(\mathbb{R}^n)$  to

$$\mathcal{F} : H_p^{s_1}(\mathbb{R}^n) \rightarrow H_p^{s_2}(\mathbb{R}^n), \quad 1 < p < \infty,$$

and further to

$$\mathcal{F} : B_{p,p}^{s_1}(\mathbb{R}^n) \rightarrow B_{p,p}^{s_2}(\mathbb{R}^n), \quad 1 \leq p \leq \infty,$$

↪ integrability  $p$  fixed, smoothness parameters  $s_1, s_2$  vary

--> Does this admit continuity of  $\mathcal{F}$ , or even compactness, let alone nuclearity?

# Besov and Sobolev spaces

## Notational adaptation

Let  $n \in \mathbb{N}$ ,  $1 \leq p \leq \infty$ ,  $s \in \mathbb{R}$ .

- ▶  $B_p^s(\mathbb{R}^n) = B_{p,p}^s(\mathbb{R}^n)$
- ▶  $d_p^n = 2n\left(\frac{1}{p} - \frac{1}{2}\right)$ ,  $x_p^n = \max(0, d_p^n)$ ,  $y_p^n = \min(0, d_p^n)$
- ▶ source and target spaces:

$$X_p^s B(\mathbb{R}^n) = B_p^{x_p^n + s}(\mathbb{R}^n), \quad Y_p^s B(\mathbb{R}^n) = B_p^{y_p^n - s}(\mathbb{R}^n), \quad 1 \leq p \leq \infty$$

$$X_p^s H(\mathbb{R}^n) = H_p^{x_p^n + s}(\mathbb{R}^n), \quad Y_p^s H(\mathbb{R}^n) = H_p^{y_p^n - s}(\mathbb{R}^n), \quad 1 < p < \infty$$

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**Rem.**  $p = 2 \curvearrowright d_p^n = x_p^n = y_p^n = 0$ :

$$X_2^s B(\mathbb{R}^n) = B_2^s(\mathbb{R}^n) = H_2^s(\mathbb{R}^n) = X_2^s H(\mathbb{R}^n),$$

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--> study now

$$\mathcal{F} : X_p^{s_1} A(\mathbb{R}^n) \hookrightarrow Y_p^{s_2} A(\mathbb{R}^n), \quad A \in \{B, H\}$$

**Rem.**  $p = 2 \curvearrowright d_p^n = x_p^n = y_p^n = 0$ :

$$X_2^s B(\mathbb{R}^n) = B_2^s(\mathbb{R}^n) = H_2^s(\mathbb{R}^n) = X_2^s H(\mathbb{R}^n),$$

$$Y_2^s B(\mathbb{R}^n) = B_2^{-s}(\mathbb{R}^n) = H_2^{-s}(\mathbb{R}^n) = Y_2^s H(\mathbb{R}^n)$$

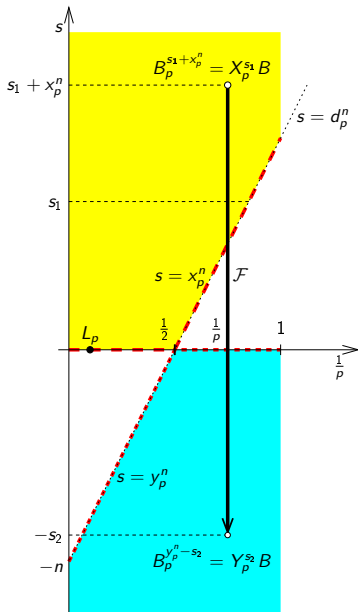
$$d_p^n = 2n \left( \frac{1}{p} - \frac{1}{2} \right)$$

$$x_p^n = \max(0, d_p^n)$$

$$y_p^n = \min(0, d_p^n)$$

$$\mathcal{F} : X_p^{s_1} B(\mathbb{R}^n) \hookrightarrow Y_p^{s_2} B(\mathbb{R}^n)$$

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## Introduction

Fourier transform

## Function spaces of Besov and Sobolev type

## Continuous embeddings

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## Compact embeddings

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# Fourier transform

## Continuity

### Theorem 1

(i) Let  $1 \leq p \leq \infty$  and  $s_1, s_2 \in \mathbb{R}$ . Then

$$\mathcal{F} : X_p^{s_1} B(\mathbb{R}^n) \hookrightarrow Y_p^{s_2} B(\mathbb{R}^n)$$

is **continuous** if, and only if, both  $s_1 \geq 0$  and  $s_2 \geq 0$ .

(ii) Let  $1 < p < \infty$  and  $s_1, s_2 \in \mathbb{R}$ . Then

$$\mathcal{F} : X_p^{s_1} H(\mathbb{R}^n) \hookrightarrow Y_p^{s_2} H(\mathbb{R}^n)$$

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# Fourier transform

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### Rem.

- ▶ classical:  $p = 2, s_1 = s_2 = 0 \curvearrowright \mathcal{F} : L_2(\mathbb{R}^n) \rightarrow L_2(\mathbb{R}^n)$
- ▶ Triebel (2022), H./Skrzypczak/Triebel (2025)

# Tools

## Wavelet decomposition

$Q_{\nu,m}$  cubes, centre/corner at  $2^{-\nu}m$ , side length  $2^{-\nu}$ ,  $\nu \in \mathbb{N}_0$ ,  $m \in \mathbb{Z}^n$   
 $\phi, \psi_i \in C^{N_1}(\mathbb{R}^n)$  with  $\text{supp } \phi, \text{supp } \psi_i \subset [-N_2, N_2]^n$ ,  $i = 1, \dots, 2^n - 1$ ,

$$\phi_{\nu,m}(x) = 2^{\nu n/2} \phi(2^\nu x - m), \quad \psi_{i,\nu,m}(x) = 2^{\nu n/2} \psi_i(2^\nu x - m)$$

# Tools

## Wavelet decomposition

$Q_{\nu,m}$  cubes, centre/corner at  $2^{-\nu}m$ , side length  $2^{-\nu}$ ,  $\nu \in \mathbb{N}_0$ ,  $m \in \mathbb{Z}^n$   
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## Proposition 2

Let  $0 < p < \infty$ ,  $0 < q \leq \infty$ ,  $s \in \mathbb{R}$  with  $N_1 > |s|$ ,  $\sigma = s + \frac{n}{2} - \frac{n}{p}$ . Then  $f \in \mathcal{S}'(\mathbb{R}^n)$  belongs to  $B_{p,q}^s(\mathbb{R}^n)$ , if, and only if,

$$\|f|B_{p,q}^s(\mathbb{R}^n)\|^* = \left\| \left\{ \langle f, \phi_{0,m} \rangle \right\}_m | \ell_p \right\| + \sum_{i=1}^{2^n-1} \left\| \left\{ \langle f, \psi_{i,\nu,m} \rangle \right\}_{\nu,m} | b_{p,q}^\sigma \right\|$$

is finite. Furthermore,  $\|f|B_{p,q}^s(\mathbb{R}^n)\|^* \sim \|f|B_{p,q}^s(\mathbb{R}^n)\|$ .

**Rem.**  $\|\gamma| \ell_p\| \sim \left( \sum_{m \in \mathbb{Z}^n} |\gamma_m|^p \right)^{1/p}$

$$\|\lambda| b_{pq}^s\| \sim \left\| \left\{ 2^{\nu s} \left( \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu,m}|^p \right)^{1/p} \right\}_{\nu \in \mathbb{N}_0} | \ell_q \right\|$$

## Strategy of the proofs - some tools

- ▶ use **wavelet decomposition**  $\curvearrowright$  reduction of some arguments for function spaces to sequence spaces
- ▶ benefit from elementary embeddings:

$$\begin{array}{ccccc}
 & B_{\rho, \min(\rho, q)}^s(\mathbb{R}^n) & \hookrightarrow & F_{\rho, q}^s(\mathbb{R}^n) & \hookrightarrow & B_{\rho, \max(\rho, q)}^s(\mathbb{R}^n) \\
 \xRightarrow{q=2} & B_{\rho, \min(\rho, 2)}^s(\mathbb{R}^n) & \hookrightarrow & F_{\rho, 2}^s(\mathbb{R}^n) & \hookrightarrow & B_{\rho, \max(\rho, 2)}^s(\mathbb{R}^n) \\
 \xRightarrow{1 < \rho < \infty} & B_{\rho, \min(\rho, 2)}^s(\mathbb{R}^n) & \hookrightarrow & H_{\rho}^s(\mathbb{R}^n) & \hookrightarrow & B_{\rho, \max(\rho, 2)}^s(\mathbb{R}^n)
 \end{array}$$

$$\curvearrowright \begin{cases} B_{\rho}^s(\mathbb{R}^n) \hookrightarrow H_{\rho}^s(\mathbb{R}^n), & p \leq 2, \\ H_{\rho}^s(\mathbb{R}^n) \hookrightarrow B_{\rho}^s(\mathbb{R}^n), & p \geq 2, \end{cases} \quad s \in \mathbb{R}$$

- ▶ apply duality arguments when possible,

$$[B_{\rho}^s(\mathbb{R}^n)]' = B_{\rho'}^{-s}(\mathbb{R}^n), \quad 1 \leq \rho < \infty, \quad s \in \mathbb{R}, \quad \frac{1}{\rho} + \frac{1}{\rho'} = 1,$$

$$[H_{\rho}^s(\mathbb{R}^n)]' = H_{\rho'}^{-s}(\mathbb{R}^n), \quad 1 < \rho < \infty, \quad s \in \mathbb{R}$$

and  $\mathcal{F}' = \mathcal{F}$  in the dual pairing  $(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$

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## Connection to weighted spaces, shift operators

'admissible' weight  $w_\alpha(x) = (1 + |x|^2)^{\alpha/2}$ ,  $x \in \mathbb{R}^n$ ,  $\alpha \in \mathbb{R}$  (special Muckenhoupt weight)

$$f \in B_{p,q}^s(\mathbb{R}^n, w_\alpha) \iff w_\alpha f \in B_{p,q}^s(\mathbb{R}^n), \quad \mathcal{S}'(\mathbb{R}^n) = \bigcup_{\alpha, s \in \mathbb{R}} B_{p,q}^s(\mathbb{R}^n, w_\alpha)$$

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special shift operators:

► *weight shift*  $W_\beta$ ,  $\beta \in \mathbb{R}$

$$W_\beta : f \mapsto w_\beta f, \quad f \in S'(\mathbb{R}^n)$$

$$\curvearrowright W_\beta A_{p,q}^s(\mathbb{R}^n, w_\alpha) = A_{p,q}^s(\mathbb{R}^n, w_{\alpha-\beta}), \quad \alpha \in \mathbb{R}$$

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$\curvearrowright$  appropriate decomposition:  $\mathcal{F} = W_{-\gamma} \circ I_{-\beta} \circ \mathcal{F} \circ W_\beta \circ I_\gamma$

# Hausdorff-Young inequality, revisited

Weighted and lifted setting

## Corollary 3

Let  $s, \sigma \in \mathbb{R}$ .

- ▶ If  $1 \leq p \leq 2$ , then

$$\mathcal{F} : B_p^s(\mathbb{R}^n, w_\sigma) \hookrightarrow B_{p'}^\sigma(\mathbb{R}^n, w_s)$$

is continuous.

- ▶ If  $1 < p \leq 2$ , then

$$\mathcal{F} : H_p^s(\mathbb{R}^n, w_\sigma) \hookrightarrow H_{p'}^\sigma(\mathbb{R}^n, w_s)$$

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Weighted and lifted setting

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### Rem.

- ▶  $s = \sigma = 0 \iff \mathcal{F} : L_p(\mathbb{R}^n) \hookrightarrow L_{p'}(\mathbb{R}^n)$ ,  $p \leq 2$ , classical Hausdorff-Young inequality
- ▶ proof by appropriate decomposition via shift operators

# Hausdorff-Young inequality, revisited

Idea of proof

consider  $\mathcal{F} : B_p^s(\mathbb{R}^n, w_\sigma) \hookrightarrow B_{p'}^\sigma(\mathbb{R}^n, w_s), \quad p \leq 2$

# Hausdorff-Young inequality, revisited

Idea of proof

consider  $\mathcal{F} : B_p^s(\mathbb{R}^n, w_\sigma) \hookrightarrow B_{p'}^\sigma(\mathbb{R}^n, w_s), \quad p \leq 2$

**Step 1:** reduction via appropriate lifts  $I_\gamma, W_\beta$  (isomorphisms) to

$$\mathcal{F} : B_p^0(\mathbb{R}^n) \hookrightarrow B_{p'}^0(\mathbb{R}^n),$$

since

$$B_p^s(\mathbb{R}^n, w_\sigma) \xrightarrow{I_s} B_p^0(\mathbb{R}^n, w_\sigma) \xrightarrow{W_\sigma} B_p^0(\mathbb{R}^n) \xrightarrow{\mathcal{F}} B_{p'}^0(\mathbb{R}^n) \\ \xrightarrow{I_{-\sigma}} B_{p'}^\sigma(\mathbb{R}^n) \xrightarrow{W_{-s}} B_{p'}^\sigma(\mathbb{R}^n, w_s)$$

# Hausdorff-Young inequality, revisited

Idea of proof

consider  $\mathcal{F} : B_p^s(\mathbb{R}^n, w_\sigma) \hookrightarrow B_{p'}^\sigma(\mathbb{R}^n, w_s)$ ,  $p \leq 2$

**Step 1:** reduction via appropriate lifts  $I_\gamma$ ,  $W_\beta$  (isomorphisms) to

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**Step 2:** apply elementary embeddings,

$$B_p^0(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n), \quad p \leq 2, \quad L_{p'}(\mathbb{R}^n) \hookrightarrow B_{p'}^0(\mathbb{R}^n), \quad p' \geq 2,$$

and the classical Hausdorff-Young inequality

$$\mathcal{F} : B_p^0(\mathbb{R}^n) \xrightarrow{\text{id}} L_p(\mathbb{R}^n) \xrightarrow{\mathcal{F}} L_{p'}(\mathbb{R}^n) \xrightarrow{\text{id}} B_{p'}^0(\mathbb{R}^n)$$

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## Compactness

### Theorem 4

(i) Let  $1 \leq p \leq \infty$  and  $s_1, s_2 \in \mathbb{R}$ . Then

$$\mathcal{F} : X_p^{s_1} B(\mathbb{R}^n) \hookrightarrow Y_p^{s_2} B(\mathbb{R}^n)$$

is compact if, and only if, both  $s_1 > 0$  and  $s_2 > 0$ .

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### Rem.

- ▶ classical:  $p = 2, s_1 = s_2 = 0 \curvearrowright \mathcal{F} : L_2(\mathbb{R}^n) \rightarrow L_2(\mathbb{R}^n)$  never compact
- ▶ Triebel (2022), H./Skrzypczak/Triebel (2025)

# Connection to weighted embeddings

## Sketch of proof

- ▶ consider first  $1 \leq p \leq 2$ , apply weight and smoothness shifts (as before) to conclude that the compactness of

$$\mathcal{F} : X_p^{s_1} B(\mathbb{R}^n) \hookrightarrow Y_p^{s_2} B(\mathbb{R}^n)$$

is a consequence of the compactness of

$$\text{id}_w : B_p^0(\mathbb{R}^n, w_{s_1}) \hookrightarrow B_p^{-s_2}(\mathbb{R}^n)$$

- ▶ conversely, if  $2 \leq p \leq \infty$  and

$$\mathcal{F} : X_p^{s_1} B(\mathbb{R}^n) \hookrightarrow Y_p^{s_2} B(\mathbb{R}^n)$$

is compact, then this implies the compactness of

$$\text{id}'_w : B_{p'}^0(\mathbb{R}^n, w_{s_1}) \xrightarrow{\mathcal{F}^{-1}} B_p^{s_1}(\mathbb{R}^n) = X_p^{s_1} B(\mathbb{R}^n) \xrightarrow{\mathcal{F}} Y_p^{s_2} B(\mathbb{R}^n) = B_p^{-s_2+d_p^n}(\mathbb{R}^n)$$

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is compact, then this implies the compactness of

$$\text{id}'_w : B_{p'}^0(\mathbb{R}^n, w_{s_1}) \xrightarrow{\mathcal{F}^{-1}} B_p^{s_1}(\mathbb{R}^n) = X_p^{s_1} B(\mathbb{R}^n) \xrightarrow{\mathcal{F}} Y_p^{s_2} B(\mathbb{R}^n) = B_p^{-s_2+d_p^n}(\mathbb{R}^n)$$

- ▶ finally, use duality arguments

--> reduction to (already known) results for **weighted embeddings**

## Digression to special weighted embeddings

$$\text{id}_w : A_{p_1, q_1}^{s_1}(\mathbb{R}^n, w) \longrightarrow A_{p_2, q_2}^{s_2}(\mathbb{R}^n), \quad w \in \mathcal{A}_\infty$$

$$s_1 \geq s_2, \quad 0 < p_1, p_2 \leq \infty \quad (p_i < \infty \text{ when } A = F), \quad 0 < q_1, q_2 \leq \infty$$

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## Notation

- ▶  $\delta = \left(s_1 - \frac{n}{p_1}\right) - \left(s_2 - \frac{n}{p_2}\right)$  differential dimension
- ▶  $\frac{1}{p^*} = \left(\frac{1}{p_2} - \frac{1}{p_1}\right)_+$ ,  $\frac{1}{q^*} = \left(\frac{1}{q_2} - \frac{1}{q_1}\right)_+$

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now:  $w_\beta(x) = (1 + |x|^2)^{\beta/2} =: \langle x \rangle^\beta, \quad \beta \geq 0,$

$p_i = q_i$  (B-case), and  $q_i = 2$  (H-case),  $i = 1, 2$

-->  $\text{id}_\beta : A_{p_1}^{s_1}(\mathbb{R}^n, w_\beta) \rightarrow A_{p_2}^{s_2}(\mathbb{R}^n), \quad A \in \{B, H\}$

# Digression to special weighted embeddings

## Compactness

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Let  $s_i \in \mathbb{R}$ ,  $1 \leq p_i \leq \infty$  ( $B$ -case),  $1 < p_i < \infty$  ( $H$ -case). Then  $\text{id}_\beta$  is **compact** if, and only if,

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### Rem.

- ▶ recall  $\delta = s_1 - s_2 - \frac{n}{p_1} + \frac{n}{p_2}$ ,  $\frac{1}{p^*} = \left(\frac{1}{p_2} - \frac{1}{p_1}\right)_+$
- ▶ recall:  $\text{id} : B_p^0(\mathbb{R}^n, w_{s_1}) \hookrightarrow B_p^{-s_2}(\mathbb{R}^n) \hookrightarrow \frac{1}{p^*} = 0$ ,  $\beta = s_1$ ,  $\delta = s_2$
- ▶ H./Triebel (1994), general: Kühn/Leopold/Sickel/Skrzypczak (2006)

# Reminder: Entropy and approximation numbers

The concept – ‘Measuring’ compactness

## Definition 6

$\mathbb{X}, \mathbb{Y}$  (quasi-) Banach spaces,  $T \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$ ,  $k \in \mathbb{N}$

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(i)  $k$ -th *approximation number* of  $T$

$$a_k(T) := \inf \{ \|T - S\| : S \in \mathcal{L}(\mathbb{X}, \mathbb{Y}), \text{rank } S < k \}$$

(ii)  $k$ -th (*dyadic*) *entropy number* of  $T$

$$e_k(T) = \inf \left\{ \varepsilon > 0 : \exists y_i \in \mathbb{Y} : T(\overline{B}_{\mathbb{X}}) \subseteq \bigcup_{i=1}^{2^{k-1}} \{y_i + \varepsilon \overline{B}_{\mathbb{Y}}\} \right\}$$

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**Rem.**  $T \in \mathcal{L}(\mathbb{X})$  compact, eigenvalue sequence  $\{\mu_k\}_k$

- ▶  $\mathbb{X} = \mathbb{H}$  Hilbert space,  $T$  self-adjoint:  $|\mu_k| = a_k(T)$ ,  $k \in \mathbb{N}$
- ▶ Carl-Triebel inequality:  $|\mu_k| \leq \sqrt{2} e_k(T)$ ,  $k \in \mathbb{N}$

# Digression to special weighted embeddings

## Entropy numbers

$$\text{id}_\beta : A_{p_1}^{s_1}(\mathbb{R}^n, w_\beta) \rightarrow A_{p_2}^{s_2}(\mathbb{R}^n), \quad \beta \geq 0, \quad w_\beta(x) = \langle x \rangle^\beta, \quad A \in \{B, H\}$$

### Proposition 7

Assume that  $\text{id}_\beta$  is compact, i.e.,  $\beta > \frac{n}{p^*}$  and  $\delta > \frac{n}{p^*}$ .

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(ii) If  $\beta = \delta$  and  $A = B$ , then

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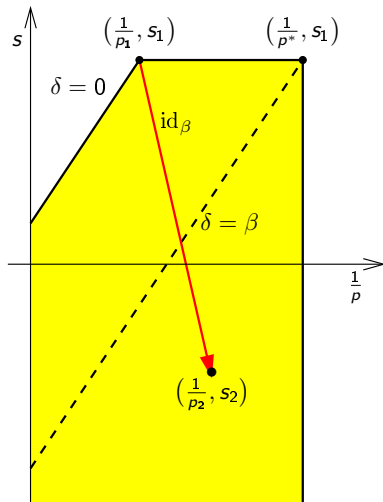
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**Rem.** H./Triebel (1994), H. (1995), H./Triebel (2005),  
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## Digression to special weighted embeddings

## Entropy numbers



# Fourier transform

Entropy numbers – via weighted embeddings

## Theorem 8

Let  $1 < p < \infty$ ,  $s_1 > 0$ ,  $s_2 > 0$ .

- ▶ There exist positive constants  $C, c > 0$  such that for  $k \in \mathbb{N}$ ,  $k \geq 2$ ,

$$e_k(\mathcal{F} : X_p^{s_1} B(\mathbb{R}^n) \hookrightarrow Y_p^{s_2} B(\mathbb{R}^n)) \leq c \begin{cases} k^{-\frac{1}{n} \min(s_1, s_2)}, & s_1 \neq s_2, \\ \left(\frac{k}{\log k}\right)^{-\frac{s_1}{n}}, & s_1 = s_2, \end{cases}$$

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- ▶ The same estimates hold for  $e_k(\mathcal{F} : X_p^{s_1} H(\mathbb{R}^n) \hookrightarrow Y_p^{s_2} H(\mathbb{R}^n))$ .
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$$e_k(\mathcal{F} : H^{s_1}(\mathbb{R}^n) \hookrightarrow H^{-s_2}(\mathbb{R}^n)) \sim \begin{cases} k^{-\frac{1}{n} \min(s_1, s_2)}, & s_1 \neq s_2, \\ \left(\frac{k}{\log k}\right)^{-\frac{s_1}{n}}, & s_1 = s_2. \end{cases}$$

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# Fourier transform

Entropy numbers – recent improvement for  $p \neq 2$

**Rem.** improvement (2025): use further embeddings

$$\mathcal{F} : L_p(\mathbb{R}^n) \hookrightarrow B_{2,p}^{n(\frac{1}{p}-\frac{1}{2})}(\mathbb{R}^n), \quad p \geq 2, \quad \mathcal{F} : B_{2,p}^{n(\frac{1}{p}-\frac{1}{2})}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n), \quad p \leq 2$$

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Entropy numbers – recent improvement for  $p \neq 2$

## Theorem 9

Let  $1 < p < \infty$ ,  $s_1 > 0$ ,  $s_2 > 0$ ,  $\tau(p) = \left|1 - \frac{2}{p}\right| = \frac{|d_p^n|}{n}$ .

► If  $s_1 \neq s_2$ , then for  $k \in \mathbb{N}$ ,

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If  $s_1 = s_2 = s$ , then there are  $c > 0$ ,  $C > 0$ , s.t. that for all  $k \in \mathbb{N}$ ,  $k \geq 2$ ,

$$\begin{aligned} ck^{-\frac{s}{n} - \left|\frac{1}{p} - \frac{1}{2}\right|} (\log k)^{\frac{s}{n}} &\leq e_k(\mathcal{F} : X_p^s B(\mathbb{R}^n) \hookrightarrow Y_p^s B(\mathbb{R}^n)) \\ &\leq Ck^{-\frac{s}{n} - \left|\frac{1}{p} - \frac{1}{2}\right|} (\log k)^{\frac{s}{n} + \tau(p)}. \end{aligned}$$

► The same estimates hold for  $e_k(\mathcal{F} : X_p^{s_1} H(\mathbb{R}^n) \hookrightarrow Y_p^{s_2} H(\mathbb{R}^n))$ .

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# Digression to special weighted embeddings

## Approximation numbers

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### Proposition 10

Assume that  $\min(\beta, \delta) > \frac{n}{p^*}$  and  $\beta \neq \delta$ . Then  $a_k(\text{id}_\beta) \sim k^{-\varkappa}$ ,  $k \in \mathbb{N}$ , with

$$\varkappa = \begin{cases} \frac{\min(\delta, \beta)}{n} & , \quad p_1 \leq p_2 \leq 2 \quad \text{or} \quad 2 \leq p_1 \leq p_2, & \text{A} \\ \frac{\min(\delta, \beta)}{n} - \frac{1}{p_2} + \frac{1}{p_1} & , \quad p^* < p_2 \leq p_1, & \text{B} \\ \frac{s_1 - s_2}{n} - \max\left(\frac{1}{2} - \frac{1}{p_2}, \frac{1}{p_1} - \frac{1}{2}\right) & , \quad p_1 < 2 < p_2, \quad \beta > \delta > \frac{n}{\min(p_1', p_2)}, & \text{C} \\ \frac{\beta}{n} + \min\left(\frac{1}{2} - \frac{1}{p_2}, \frac{1}{p_1} - \frac{1}{2}\right) & , \quad p_1 < 2 < p_2, \quad \delta > \beta > \frac{n}{\min(p_1', p_2)}, & \text{D} \\ \frac{\min(\delta, \beta)}{n} \cdot \frac{\min(p_1', p_2)}{2} & , \quad p_1 < 2 < p_2, \quad \min(\delta, \beta) < \frac{n}{\min(p_1', p_2)}. & \text{E} \end{cases}$$

### Rem.

- ▶ in case of **D** only estimate from below
- ▶ H. (1995, 1997), Caetano (1998), Skrzypczak (2005),  
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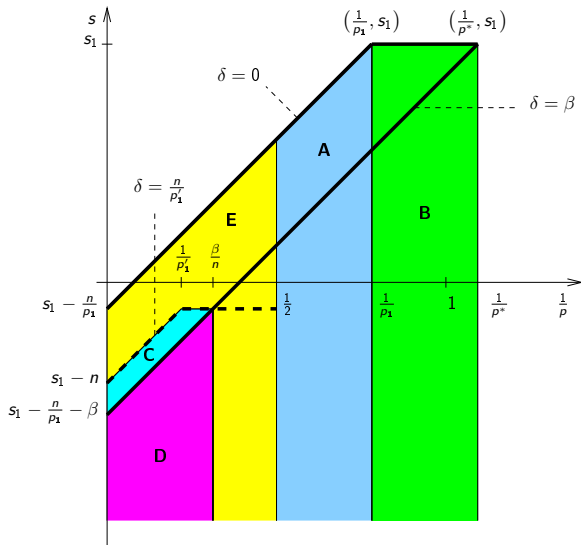
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## Digression to special weighted embeddings

## Approximation numbers



# Fourier transform

## Approximation numbers

### Theorem 11

Let  $1 < p < \infty$ ,  $s_1 > 0$ ,  $s_2 > 0$ .

$$\blacktriangleright p = 2: a_k(\mathcal{F} : H^{s_1}(\mathbb{R}^n) \hookrightarrow H^{-s_2}(\mathbb{R}^n)) \sim \begin{cases} k^{-\frac{1}{n} \min(s_1, s_2)} & \text{if } s_1 \neq s_2 \\ \left(\frac{k}{\log k}\right)^{-\frac{s_1}{n}} & \text{if } s_1 = s_2 \end{cases}$$

$s_1 \neq s_2$ :  $\exists c_2 > c_1 > 0$  such that for  $k \in \mathbb{N}$ ,

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# Fourier transform

## Approximation numbers

### Theorem 11

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## Introduction

Fourier transform

## Function spaces of Besov and Sobolev type

## Continuous embeddings

Tools

Weighted setting

## Compact embeddings

General criterion

Digression

Entropy and approximation numbers

## Nuclear maps

The concept and recent results

Nuclearity of  $\mathcal{F}$

# The concept and recent results

## Nuclear maps

$X, Y$  Banach spaces,  $X'$  dual space of  $X$ ,  $T \in \mathcal{L}(X, Y)$

- ▶  $T$  *nuclear*, i.e.  $T \in \mathcal{N}(X, Y)$ , if there exist elements  $a_j \in X'$ ,  $y_j \in Y$ ,  $j \in \mathbb{N}$ , such that

$$\sum_{j=1}^{\infty} \|a_j\|_{X'} \|y_j\|_Y < \infty \quad \text{and} \quad Tx = \sum_{j=1}^{\infty} a_j(x)y_j, \quad x \in X$$

- ▶ *nuclear norm*  $\nu(T) = \inf \left\{ \sum_{j=1}^{\infty} \|a_j\|_{X'} \|y_j\|_Y : T = \sum_{j=1}^{\infty} a_j(\cdot)y_j \right\}$
- ▶  $\mathcal{N}(X, Y)$  with  $\nu(\cdot)$  Banach space
- ▶  $\text{rank}(T) < \infty \implies T \in \mathcal{N}(X, Y) \implies T$  compact
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$$\nu(RTS) \leq \|R\| \nu(T) \|S\|$$

**Rem.** Grothendieck (1955), Pietsch (1980, 1987)

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# The concept and recent results

Tong's result (1969)

Let  $r_1, r_2 \in [1, \infty]$ , and  $t(r_1, r_2) \in [1, \infty]$  be given by

$$\frac{1}{t(r_1, r_2)} = \begin{cases} 1, & \text{if } 1 \leq r_2 \leq r_1 \leq \infty, \\ 1 - \frac{1}{r_1} + \frac{1}{r_2}, & \text{if } 1 \leq r_1 \leq r_2 \leq \infty. \end{cases}$$

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## Proposition 12

Let  $1 \leq r_1, r_2 \leq \infty$ ,  $\tau = (\tau_j)_j$ ,  $D_\tau : x = (x_j)_j \mapsto (\tau_j x_j)_j$  diagonal operator.

$D_\tau$  is nuclear  $\iff \tau \in \ell_{t(r_1, r_2)}$ , with  $\ell_{t(r_1, r_2)} = c_0$  if  $t(r_1, r_2) = \infty$ .

Moreover,

$$\nu(D_\tau : \ell_{r_1} \rightarrow \ell_{r_2}) = \|\tau\|_{\ell_{t(r_1, r_2)}}.$$

**Rem.** Tong (1969); Cobos, Domínguez, Kühn (2018)

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Observation

- ▶  $\frac{1}{t(r_1, r_2)} = 1 - \left(\frac{1}{r_1} - \frac{1}{r_2}\right)_+ \geq \frac{1}{r^*} = \left(\frac{1}{r_2} - \frac{1}{r_1}\right)_+$
- ▶  $t(r_1, r_2) = r^* \in \{1, \infty\} \iff \{r_1, r_2\} = \{1, \infty\}$

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**Rem.** observation that *compactness*  $\dashrightarrow$  *nuclearity* of embeddings corresponds to replacement  $r^* \dashrightarrow t(r_1, r_2)$

# Digression to special weighted embeddings

## Nuclearity

$$\text{id}_\beta : A_{p_1}^{s_1}(\mathbb{R}^n, w_\beta) \rightarrow A_{p_2}^{s_2}(\mathbb{R}^n), \quad \beta \geq 0, \quad w_\beta(x) = \langle x \rangle^\beta, \quad A \in \{B, H\}$$

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### Theorem 13

Let  $1 \leq p_1 < \infty$ ,  $1 \leq p_2 \leq \infty$  ( $1 < p_i < \infty$  if  $A = H$ ),  $s_i \in \mathbb{R}$ ,  $i = 1, 2$ .  
Then  $\text{id}_\beta$  is **nuclear** if, and only if,

$$\beta > \frac{n}{t(p_1, p_2)} \quad \text{and} \quad \delta > \frac{n}{t(p_1, p_2)}.$$

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### Rem.

- ▶ recall:  $\text{id}_\beta$  compact  $\iff \beta > \frac{n}{p^*}$  and  $\delta > \frac{n}{p^*}$
- ▶ as before: compactness vs. nuclearity: replace  $p^*$  by  $t(p_1, p_2)$
- ▶ H./Skrzypczak (2020), proof by wavelet decomposition & Tong

# Fourier transform

## Nuclearity

### Theorem 14

Let  $1 < p < \infty$ ,  $s_1, s_2 \in \mathbb{R}$ .

(i) Then

$$\mathcal{F} : X_p^{s_1} B(\mathbb{R}^n) \hookrightarrow Y_p^{s_2} B(\mathbb{R}^n)$$

is **nuclear** if, and only if,  $s_i > \frac{2n}{\max(p, p')}$   $i = 1, 2$ .

(ii) The same is true for

$$\mathcal{F} : X_p^{s_1} H(\mathbb{R}^n) \hookrightarrow Y_p^{s_2} H(\mathbb{R}^n).$$

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### Rem.

- ▶  $\mathcal{F}$  compact  $\iff s_1 > 0$  and  $s_2 > 0$
- ▶ note:  $\frac{2n}{\max(p, p')} = n + \min(d_p^n, d_{p'}^n) \geq 0$ ,  $1 \leq p \leq \infty$
- ▶ Triebel (2022), H./Skrzypczak/Triebel (2023)
- ▶ further results in limiting cases and for spaces of type  $B_{p,q}^s$  and  $F_{p,q}^s$

# Fourier transform

## Compactness and nuclearity

$\mathcal{F}$  is **compact**, if, and only if,

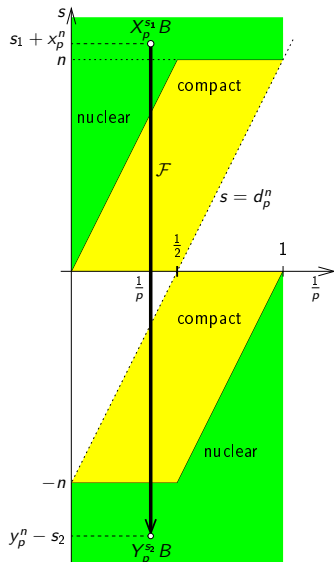
$$s_i > 0, \quad i = 1, 2$$

$\mathcal{F}$  is **nuclear**, if, and only if,

$$s_i > \frac{2n}{\max(p, p')} = n + \min(d_p^n, d_{p'}^n), \quad i = 1, 2$$

**Rem.**

- ▶ H./Skrzypczak/Triebel (2023)
- ▶ duality 'visible' since  $\mathcal{F}' = \mathcal{F}$



# The end

'... but orderly to end where I begun ...' (Hamlet, Shakespeare)

**Goal:** mapping properties of the Fourier transform

$$\mathcal{F} : A_p^{s_1}(\mathbb{R}^n) \rightarrow A_p^{s_2}(\mathbb{R}^n), \quad A \in \{B, H\},$$

$s_i \in \mathbb{R}$ ,  $1 \leq p_i \leq \infty$  ( $A = B$ ),  $1 < p_i < \infty$  ( $A = H$ ),  $i = 1, 2$

- ▶ complete characterisation for **continuity** and **compactness** in case of  $B_p^s(\mathbb{R}^n)$ ,  $1 \leq p \leq \infty$ , and  $H_p^s(\mathbb{R}^n)$ ,  $1 < p < \infty$ ,  $s \in \mathbb{R}$
- ▶ first, but partly two-sided estimates for **entropy** and **approximation numbers**, only sharp if  $p = 2$
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**always:** influence of  $|d_p^n|$ , that is,  $|\frac{1}{p} - \frac{1}{2}|$

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**Goal:** mapping properties of the Fourier transform

$$\mathcal{F} : A_p^{s_1}(\mathbb{R}^n) \rightarrow A_p^{s_2}(\mathbb{R}^n), \quad A \in \{B, H\},$$

$s_i \in \mathbb{R}$ ,  $1 \leq p_i \leq \infty$  ( $A = B$ ),  $1 < p_i < \infty$  ( $A = H$ ),  $i = 1, 2$

- ▶ complete characterisation for **continuity** and **compactness** in case of  $B_p^s(\mathbb{R}^n)$ ,  $1 \leq p \leq \infty$ , and  $H_p^s(\mathbb{R}^n)$ ,  $1 < p < \infty$ ,  $s \in \mathbb{R}$
- ▶ first, but partly two-sided estimates for **entropy and approximation numbers**, only sharp if  $p = 2$
- ▶ complete characterisation for **nuclearity** in case of  $B_p^s(\mathbb{R}^n)$ ,  $1 \leq p < \infty$ , and  $H_p^s(\mathbb{R}^n)$ ,  $1 < p < \infty$ ,  $s \in \mathbb{R}$

always: influence of  $|d_p^n|$ , that is,  $|\frac{1}{p} - \frac{1}{2}|$

Thank you very much for your attention!