Zeitschrift für Analysis und ihre Anwendungen
(submitted)

Convexity estimates for the outer measure of bilinear operators

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Abstract. Let $\mathcal{I}$ be a surjective closed operator ideal with associated outer measure $\gamma_{\mathcal{I}}$, and let $\gamma_{\mathfrak{I}}$ be the extension of $\gamma_{\mathcal{I}}$ to bilinear operators. We establish a convexity inequality for the measure $\gamma_{\mathfrak{I}}$ of a bilinear operator interpolated by the real method. Our arguments do not require that $\mathcal{I}$ be symmetric.

Dedicated to the memory of Professor Mikio Kato

1. Introduction

Interpolation theory provides useful methods for establishing inequalities in very general contexts. In fact, techniques of interpolation theory are handy tools in harmonic analysis, operator theory, partial differential equations, functional analysis and some other areas of mathematics (see the monographs [1, 5, 6, 10, 19, 29, 42, 43]).

Among the most well-known interpolation results, we can mention the Riesz-Thorin theorem: Let $(U, \mu), (V, \nu)$ be σ-finite measure spaces, let $1 \leq p_0, p_1 \leq \infty$, $1 \leq q_0, q_1 \leq \infty$ and assume that R is a linear operator such that $R : L_{p_i}(U) \longrightarrow L_{q_i}(V)$ boundedly for $i = 0, 1$. Then $R : L_p(U) \longrightarrow L_q(V)$ is bounded with

$$\|R\|_{L_p(U),L_q(V)} \leq C\|R\|^{1-\theta}_{L_{p_0}(U),L_{q_0}(V)}\|R\|^{\theta}_{L_{p_1}(U),L_{q_1}(V)} \tag{1.1}$$

provided that $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$. Here $1 \leq C \leq \sqrt{2}$, with $C = 1$ for the case of complex spaces.

2020 Mathematics Subject Classification. Primary 46B70, 47L20; Secondary 46G25, 47L22.
Keywords. Measures associated to bilinear operators; real interpolation; convexity inequalities; projective tensor products.

The convexity inequality (1.1) can be extended to compatible couples of abstract Banach spaces $(A_0, A_1), (B_0, B_1)$ and bounded linear operators R between the couples if we replace L_p, L_q by the interpolation spaces generated by the real (or the complex) method: The norm of the interpolated operator satisfies that

$$\|R\| \leq \|R_0\|^{1-\theta}\|R_1\|^{\theta}$$

where $R_i, i = 0, 1$, are the restrictions of R to the spaces of the couples.

A similar inequality has been established for the measure of non-compactness $\omega(R)$ of a linear operator R interpolated by the real method [16, 41]. Namely,

$$\omega(R) \leq C\omega(R_0)^{1-\theta}\omega(R_1)^{\theta}$$

where $C > 0$ is a constant independent of R. In a more general way, if we replace the measure of non-compactness by the outer measure associated to an operator ideal (see [2]) then an estimate of convexity type also holds (see [17, 18, 25]).

Recently, a lot of work has been done for establishing similar inequalities for bilinear operators and measures associated to ideals of bilinear operators. See, for example, the papers by Mastyło and Silva [33], Besoy and Cobos [7] and Cobos, Fernández-Cabrera and Martínez [12] for the case of the measure of non-compactness, those by Manzano, Rueda and Sánchez-Pérez [31, 32] for the case of general measures but with only a Banach couple in the domain or in the target of the operator, and the paper by the present authors [15] for general couples and outer measures.

In [15] we have extended the outer measure $\gamma_{\mathcal{I}}$ associated to an ideal $\mathcal{I}$ of linear operators to a measure $\gamma_{\mathfrak{I}}$ for bilinear operators. Assuming that $\mathcal{I}$ is an injective, surjective and symmetric ideal satisfying a Σ_r-type condition, we have established a convexity inequality for the measure $\gamma_{\mathfrak{I}}(\tau T)$. Here T is a bilinear operator interpolated by the real method and τ is a certain isometric embedding (see [15, Theorem 3.4]). In particular, the result applies to $\mathcal{I} = \mathcal{W}$, the ideal of weakly compact operators, giving a convexity estimate for the measure of weak non-compactness. Techniques of [15] are based on duality for bilinear operators (see [38]). Symmetry of $\mathcal{I}$ is essential for the arguments.

Our goal in this paper is to develop a different strategy to establish the convexity inequality for the outer measure of a bilinear operator interpolated by the real method. The new techniques that we expound do not use duality. They rely on tensor products and on an interpolation formula for projective tensor products. As a result we get rid of the assumption of symmetry on the ideal $\mathcal{I}$. Besides weakly compact operators, the new convexity inequality applies, for example, to the classes of bilinear operators that correspond to the ideals of Banach-Saks operators, Rosenthal operators and Asplund operators.

We also show that assumptions for the convexity inequality can be significantly simplified if the couple in the target of the bilinear operator reduces to a single Banach space. In this special case, the result applies to a larger class of ideals of bilinear operators including, for example, strictly cosingular operators.

2. Projective tensor products and real interpolation

Let A, B, E be normed spaces. We put $\mathfrak{B}(A \times B, E)$ for the space of all bounded bilinear operators T from $A \times B$ into E with the norm

$$\|T\| = \|T\|_{A\times B,E} = \sup\left\{\|T(a,b)\|_E : a \in U_A\,,\ b \in U_B\right\}.$$

Here U_A is the closed unit ball of A. We write

$$U_A \times U_B = \left\{(a,b) \,:\, a \in U_A, b \in U_B\right\}.$$

Hence

$$\|T\|_{A\times B,E} = \sup\left\{\|T(a,b)\|_E : (a,b) \in U_A \times U_B\right\}.$$

Let $A \otimes B$ be the tensor product of A and B. For $u \in A \otimes B$, we put

$$\pi(u) = \inf\left\{\sum_{k=1}^{n}\|a_k\|_A\|b_k\|_B \,:\, u = \sum_{k=1}^{n} a_k \otimes b_k\right\}.$$

The *projective tensor product* of A and B is the normed space $A \otimes_\pi B = (A \otimes B, \pi)$. We write $A\widehat{\otimes}_\pi B$ for the completion of $A \otimes_\pi B$ (see [21, 23, 39]).

Let $\chi : A \times B \longrightarrow A \otimes_\pi B$ be the bounded bilinear operator defined by $\chi(a,b) = a \otimes b$. Given any $T \in \mathfrak{B}(A \times B, E)$, the *linearization* $\widetilde{T}$ of T is the bounded linear operator $\widetilde{T} \in \mathcal{L}(A\widehat{\otimes}_\pi B, E)$ defined by

$$\widetilde{T}\Big(\sum_{k=1}^{n} a_k \otimes b_k\Big) = \sum_{k=1}^{n} T(a_k, b_k)\,, \qquad \sum_{k=1}^{n} a_k \otimes b_k \in A \otimes B.$$

Clearly, the restriction of $\widetilde{T}$ to $A \otimes_\pi B$ is also bounded $\widetilde{T} \in \mathcal{L}(A \otimes_\pi B, E)$.

If $J \subseteq E$, we write $co(J)$ for the convex hull of J.

Lemma 2.1. *Given any norm spaces A, B, E and any $T \in \mathfrak{B}(A \times B, E)$, we have*

$$T(U_A \times U_B) \subseteq \widetilde{T}\big(U_{A\otimes_\pi B}\big) \subseteq \overline{co\left(T\big(U_A \times U_B\big)\right)}$$

Proof. Since $U_{A\otimes_\pi B} \subseteq U_{A\widehat{\otimes}_\pi B}$, the result is a direct consequence of [13, Lemma 2.2] or [39, Proposition 2.2]. ∎

By a *normed couple* $\bar{A} = (A_0, A_1)$ we mean two normed spaces A_0, A_1 which are continuously embedded in the same Hausdorff topological vector space. We write $A_0 + A_1$ for their sum and $A_0 \cap A_1$ for their intersection. These spaces are norm spaces with the norms

$$\|a\|_{A_0+A_1} = \inf\left\{\|a_0\|_{A_0} + \|a_1\|_{A_1} : a = a_0 + a_1\,,\ a_i \in A_i\right\}$$

and

$$\|a\|_{A_0\cap A_1} = \max\left\{\|a\|_{A_0}\,,\ \|a\|_{A_1}\right\}.$$

If A_0 and A_1 are complete we refer to $\bar{A} = (A_0, A_1)$ as a Banach couple.

The *Peetre K-functional* is given by

$$K(t,a) = K(t,a;A_0,A_1) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1,\ a_i \in A_i\}.$$

For $0 < \theta < 1$ and $1 \le q \le \infty$, the *real interpolation space* realized as a K-space in discrete form $\bar{A}_{\theta,q} = (A_0, A_1)_{\theta,q}$ consists of all $a \in A_0 + A_1$ having a finite norm

$$\|a\|_{\theta,q} = \left(\sum_{m=-\infty}^{\infty}\left(2^{-\theta m}K(2^m,a)\right)^q\right)^{\frac{1}{q}}$$

(the sum should be replaced by the supremum if $q = \infty$). If $\bar{A}$ is a Banach couple then $\bar{A}_{\theta,q}$ is a Banach space. See [5, 6, 10, 42].

The real interpolation space can be also characterized in terms of the J-functional

$$J(t,a) = J(t,a;A_0,A_1) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\},\ a \in A_0 \cap A_1.$$

It turns out that $\bar{A}_{\theta,q}$ is formed by all those $a \in A_0 + A_1$ which can be represented by $a = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $A_0 + A_1$) where $(u_m) \subseteq A_0 \cap A_1$ and $(2^{-\theta m}J(2^m, u_m)) \in \ell_q$. Moreover

$$\|a\|_{\theta,q;J} = \inf\left\{\left\|\left(2^{-\theta m}J(2^m,u_m)\right)\right\|_{\ell_q} : a = \sum_{m=-\infty}^{\infty} u_m\right\}$$

defines an equivalent norm to $\|\cdot\|_{\theta,q}$.

Let $\bar{B} = (B_0, B_1)$ be another normed couple, and let $R : A_0 + A_1 \longrightarrow B_0 + B_1$ be a bounded linear operator such that the restrictions $R : A_i \longrightarrow B_i$ are bounded for $i = 0, 1$. The following convexity inequality holds

$$\|R\|_{\bar{A}_{\theta,q},\bar{B}_{\theta,q}} \le 2^\theta\ \|R\|_{A_0,B_0}^{1-\theta}\|R\|_{A_1,B_1}^{\theta}. \tag{2.1}$$

The constant 2^θ is due to the fact that we have introduced the real interpolation spaces through series. This discrete form of $\bar{A}_{\theta,q}$ will be useful in our future considerations. It is not difficult to check that the constant 2^θ does not appear in (2.1) if $\|R\|_{A_1,B_1}/\|R\|_{A_0,B_0} = 2^k$ for some $k \in \mathbb{Z}$.

The real interpolation spaces can be defined equivalently by changing the series to integrals (see [6, 42]). Working with that continuous representation, the constant 2^θ does not appear in the estimate of the norm of the interpolated operator.

Take another normed couple $\bar{E} = (E_0, E_1)$. We denote by $\mathfrak{B}(\bar{A}\times\bar{B}, \bar{E})$ the collection of all bounded bilinear operators $T : (A_0 + A_1)\times(B_0 + B_1) \longrightarrow E_0 + E_1$ such that the restrictions $T : A_i \times B_i \longrightarrow E_i$ are bounded for $i = 0, 1$. If $E_0 = E_1 = E$, then we write $\mathfrak{B}(\bar{A}\times\bar{B}, E)$. The argument in the proof of [26, Theorem 3.1] shows that if $0 < \theta < 1$, $1 \le p, q, r \le \infty$ and $1/p + 1/q = 1 + 1/r$ then

$$T : (A_0, A_1)_{\theta,p} \times (B_0, B_1)_{\theta,q} \longrightarrow (E_0, E_1)_{\theta,r}$$

is bounded with

$$\|T\|_{\bar{A}_{\theta,p}\times\bar{B}_{\theta,q},\bar{E}_{\theta,r}} \le C\, \|T\|^{1-\theta}_{A_0\times B_0,E_0}\|T\|^{\theta}_{A_1\times B_1,E_1} \tag{2.2}$$

where $C > 0$ is a constant independent of T.

Using the normed couples $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, we may consider the projective tensor products

$$A_0 \otimes_\pi B_0\,, \quad A_1 \otimes_\pi B_1\,, \quad (A_0 + A_1)\otimes_\pi (B_0 + B_1).$$

The continuous embeddings

$$A_i \hookrightarrow A_0 + A_1\,, \quad B_i \hookrightarrow B_0 + B_1\,, \quad i = 0, 1,$$

and [30, Section 41.1.8, p. 175 and Section 41.3.1, p. 179] (or [39, p. 7 and 17]) yield that the following continuous embedding holds

$$A_i \otimes_\pi B_i \hookrightarrow (A_0 + A_1)\otimes_\pi (B_0 + B_1)\,, \quad i = 0, 1.$$

Hence, $(A_0 \otimes_\pi B_0, A_1 \otimes_\pi B_1)$ is a normed couple.

Next we establish an interpolation formula for projective tensor products. We put $\{A_0 \cap A_1\}_{\theta,p}$ for $A_0 \cap A_1$ normed by $\|\cdot\|_{\theta,p}$ and $\{A_0 \cap A_1\}_i$ for $A_0 \cap A_1$ normed by $\|\cdot\|_{A_i}$, $i = 0, 1$.

Lemma 2.2. *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be normed couples. Let $0 < \theta < 1$ and $1 \leq p, q, r \leq \infty$ with $1/p + 1/q = 1 + 1/r$. Then the following continuous embedding holds*

$$\{A_0 \cap A_1\}_{\theta,p} \otimes_\pi \bar{B}_{\theta,q} \hookrightarrow (A_0 \otimes_\pi B_0, A_1 \otimes_\pi B_1)_{\theta,r}.$$

Proof. Consider the bounded bilinear operator

$$\chi : (A_0 \cap A_1) \times (B_0 + B_1) \longrightarrow (A_0 + A_1) \otimes_\pi (B_0 + B_1)$$

defined by $\chi(a,b) = a \otimes b$. It is clear that restrictions

$$\chi : \{A_0 \cap A_1\}_i \times B_i \longrightarrow A_i \otimes_\pi B_i\,, \quad i = 0, 1,$$

are bounded with norm $\|\chi\|_i \leq 1$. Now we show that

$$\chi : (A_0 \cap A_1) \times (B_0 + B_1) \longrightarrow (A_0 \otimes_\pi B_0) + (A_1 \otimes_\pi B_1)$$

is bounded. Let $a \in A_0 \cap A_1$ and $b \in B_0 + B_1$. Take any decomposition $b = b_0 + b_1$ with $b_i \in B_i$. We have

$$\begin{aligned} \|\chi(a,b)\|_{(A_0 \otimes_\pi B_0)+(A_1 \otimes_\pi B_1)} &\leq \|a \otimes b_0\|_{A_0 \otimes_\pi B_0} + \|a \otimes b_1\|_{A_1 \otimes_\pi B_1} \\ &\leq \|a\|_{A_0 \cap A_1}(\|b_0\|_{B_0} + \|b_1\|_{B_1}). \end{aligned}$$

Taking the infimum over all possible representations of b we obtain that

$$\|\chi\|_{(A_0 \cap A_1)\times(B_0+B_1),(A_0 \otimes_\pi B_0)+(A_1 \otimes_\pi B_1)} \leq 1.$$

Next we show that

$$\chi : \{A_0 \cap A_1\}_{\theta,p} \times \bar{B}_{\theta,q} \longrightarrow (A_0 \otimes_\pi B_0, A_1 \otimes_\pi B_1)_{\theta,r}$$

is bounded. We consider on $\bar{B}_{\theta,q}$ the norm defined by the J-functional. Given any $a \in A_0 \cap A_1$ and $b \in \bar{B}_{\theta,q}$, take any J-representation $b = \sum_{k=-\infty}^{\infty} u_k$ of b. There are decompositions $a = a_{0,n,k} + a_{1,n,k}$ with $a_{i,n,k} \in A_i$ such that for given $\varepsilon > 0$

$$\|a_{0,n,k}\|_{A_0} + 2^{n-k}\|a_{1,n,k}\|_{A_1} \leq (1+\varepsilon)K(2^{n-k}, a; A_0, A_1).$$

Note that in fact $a_{i,n,k} \in A_0 \cap A_1$ because $a \in A_0 \cap A_1$. We have

$$\begin{aligned} K(2^n, \chi(a,b); A_0 \otimes_\pi B_0, A_1 \otimes_\pi B_1) &\leq \sum_{k=-\infty}^{\infty} K(2^n, \chi(a,u_k); A_0 \otimes_\pi B_0, A_1 \otimes_\pi B_1) \\ &\leq \sum_{k=-\infty}^{\infty} \left(\|\chi(a_{0,n,k}, u_k)\|_{A_0 \otimes_\pi B_0} + 2^n \|\chi(a_{1,n,k}, u_k)\|_{A_1 \otimes_\pi B_1}\right) \\ &\leq \sum_{k=-\infty}^{\infty} \left(\|a_{0,n,k}\|_{A_0} \|u_k\|_{B_0} + 2^{n-k} \|a_{1,n,k}\|_{A_1} 2^k \|u_k\|_{B_1}\right) \\ &\leq \sum_{k=-\infty}^{\infty} \left(\|a_{0,n,k}\|_{A_0} + 2^{n-k} \|a_{1,n,k}\|_{A_1}\right) J(2^k, u_k; B_0, B_1) \\ &\leq (1+\varepsilon) \sum_{m=-\infty}^{\infty} K(2^m, a; A_0, A_1) J(2^{n-m}, u_{n-m}; B_0, B_1). \end{aligned}$$

Hence

$$2^{-\theta n} K(2^n, \chi(a,b)) \leq \sum_{m=-\infty}^{\infty} 2^{-\theta m} K(2^m, a; A_0, A_1) 2^{-\theta(n-m)} J(2^{n-m}, u_{n-m}; B_0, B_1).$$

Using Young's inequality for convolution, we obtain that for any $a \in A_0 \cap A_1$ and $b \in \bar{B}_{\theta,q}$, $\chi(a,b)$ belongs to $(A_0 \otimes_\pi B_0, A_1 \otimes_\pi B_1)_{\theta,r}$ with

$$\|\chi(a,b)\|_{(A_0 \otimes_\pi B_0, A_1 \otimes_\pi B_1)_{\theta,r}} \leq \|a\|_{\bar{A}_{\theta,p}} \|b\|_{\bar{B}_{\theta,q}}. \tag{2.3}$$

Consequently, $\{A_0 \cap A_1\}_{\theta,p} \otimes_\pi \bar{B}_{\theta,q} \subseteq (A_0 \otimes_\pi B_0, A_1 \otimes_\pi B_1)_{\theta,r}$. Moreover, given any $z = \sum_{j=1}^n a_j \otimes b_j \in \{A_0 \cap A_1\}_{\theta,p} \otimes_\pi \bar{B}_{\theta,q}$ with $\{a_j\}_{j=1}^n \subseteq A_0 \cap A_1$ and $\{b_j\}_{j=1}^n \subseteq \bar{B}_{\theta,q}$, using (2.3) we obtain

$$\begin{aligned} \|z\|_{(A_0 \otimes_\pi B_0, A_1 \otimes_\pi B_1)_{\theta,r}} &\leq \sum_{j=1}^{n} \|a_j \otimes b_j\|_{(A_0 \otimes_\pi B_0, A_1 \otimes_\pi B_1)_{\theta,r}} \\ &\leq \sum_{j=1}^{n} \|a_j\|_{\bar{A}_{\theta,p}} \|b_j\|_{\bar{B}_{\theta,q}}. \end{aligned}$$

This yields that

$$\|z\|_{(A_0 \otimes_\pi B_0, A_1 \otimes_\pi B_1)_{\theta,r}} \leq \|z\|_{\{A_0 \cap A_1\}_{\theta,p} \otimes_\pi \bar{B}_{\theta,q}}$$

and finishes the proof. ∎

3. Operator ideals, bilinear operators and measures

An *operator ideal* $\mathcal{I}$ is a method of ascribing to each pair (E, F) of Banach spaces, a linear subspace $\mathcal{I}(E, F)$ of $\mathcal{L}(E, F)$ such that

(i) $\mathcal{I}(E,F)$ contains the finite rank operators from E into F;

and

(ii) for all Banach spaces E, F, X, Y, whenever $V \in \mathcal{L}(X,E)$, $R \in \mathcal{I}(E,F)$, $S \in \mathcal{L}(F,Y)$, then the composed operator $SRV \in \mathcal{I}(X,Y)$.

See [22, 36, 37].

We say that $\mathcal{I}$ is *closed* if $\mathcal{I}(E,F)$ is a closed subspace of $\mathcal{L}(E,F)$ for all Banach spaces E and F.

We say that $\mathcal{I}$ is *surjective* if for every metric surjection $Q \in \mathcal{L}(X,E)$ and every operator $R \in \mathcal{L}(E,F)$, if follows from $RQ \in \mathcal{I}(X,F)$ that $R \in \mathcal{I}(E,F)$. We recall that a surjection $Q \in \mathcal{L}(X,E)$ is said to be metric if the image by Q of the open unit ball of X coincides with the open unit ball of E.

Compact linear operators $\mathcal{K}$, weakly compact operators $\mathcal{W}$, Banach-Saks operators $\mathcal{BS}$, Rosenthal operators $\mathcal{R}$, Asplund (= decomposing) operators $\mathcal{A}$ and strictly cosingular operators $\mathcal{C}$ are examples of surjective closed operator ideals (see [4, 24, 28, 36, 40]).

If (E_m) is a sequence of Banach spaces, $1 < q < \infty$ and (λ_m) is a sequence of non-negative numbers, we denote by $\ell_q(\lambda_m E_m)$ the collection of all sequences $x = (x_m)$ such that $x_m \in E_m$ for each $m \in \mathbb{Z}$ and

$$\|x\|_{\ell_q(\lambda_m E_m)} = \left(\sum_{m=-\infty}^{\infty} \left(\lambda_m \|x_m\|_{E_m}\right)^q \right)^{\frac{1}{q}} < \infty.$$

We put $Q_k : \ell_q(\lambda_m E_m) \longrightarrow \lambda_k E_k$ for the projection $Q_k(x_m) = x_k$, and $P_n : \lambda_n E_n \longrightarrow \ell_q(\lambda_m E_m)$ for the embedding $P_n y = (\delta_m^n y)$ where δ_m^n is the Kronecker delta.

The operator ideal $\mathcal{I}$ is said to satisfy the Σ_q*-condition* if for any sequences of Banach spaces (E_m), (F_m) and any $R \in \mathcal{L}\left(\ell_q(E_m), \ell_q(F_m)\right)$, it follows from $Q_k R P_n \in \mathcal{I}(E_n, F_k)$ for any n, k that $R \in \mathcal{I}\left(\ell_q(E_m), \ell_q(F_m)\right)$.

If $\mathcal{I}$ satisfies the Σ_q-condition, then $\mathcal{I}$ must be closed.

Operator ideals $\mathcal{W}$, $\mathcal{BS}$, $\mathcal{R}$, $\mathcal{A}$ satisfy the Σ_q-condition for $1 < q < \infty$ (see [28] and [11]). Compact operators $\mathcal{K}$ do not satisfy the Σ_q-condition for any $1 < q < \infty$ because $id_{\ell_q} \notin \mathcal{K}(\ell_q, \ell_q)$. Strictly cosingular operators $\mathcal{C}$ fail also the Σ_q-condition.

Let $\mathcal{I}$ be a surjective and closed operator ideal. Astala defined in [2] the *outer measure* of $R \in \mathcal{L}(E,F)$ by

$$\begin{aligned}\gamma_{\mathcal{I}}(R) &= \gamma_{\mathcal{I}}(R : E \longrightarrow F)\\ &= \inf\{\sigma > 0 : R(U_E) \subseteq S(U_X) + \sigma U_F, S \in \mathcal{I}(X,F),\ X \text{ some Banach space}\}.\end{aligned}$$

For such an ideal $\mathcal{I}$ and $R \in \mathcal{L}(E, F)$, it turns out that $\gamma_{\mathcal{I}}(R) = 0$ if and only if $R \in \mathcal{I}(E, F)$. Hence, $\gamma_{\mathcal{I}}(R)$ shows the deviation of R from the ideal $\mathcal{I}$.

The following extension of $\gamma_{\mathcal{I}}$ to bilinear operators has been studied by the present authors in [15]:

If $T \in \mathfrak{B}(A \times B, E)$, we put

$$\begin{aligned}\gamma_{\mathfrak{I}}(T) &= \gamma_{\mathfrak{I}}(T : A \times B \longrightarrow E)\\ &= \inf\{\sigma > 0 : T(U_A \times U_B) \subseteq \sigma U_E + R(U_Z), R \in \mathcal{I}(Z, E),\ Z \text{ some Banach space}\}.\end{aligned}$$

When $\mathcal{I} = \mathcal{K}$ then it turns out that $\gamma_{\mathfrak{K}}(T)$ is equal to the measure of non-compactness $\omega(T)$ of the bilinear operator T (see [15, Proposition 2.8]). If $\mathcal{I} = \mathcal{W}$ then $\gamma_{\mathfrak{W}}(T)$ coincides with $\eta(T)$, the natural extension to bilinear operators of the measure of weak non-compactness (see [15, Proposition 2.9]).

Let

$$\mathfrak{I} = \left\{T \in \mathfrak{B} \ : \ \gamma_{\mathfrak{I}}(T) = 0\right\}.$$

Then $\mathfrak{I}$ is a *bideal* in the sense of [38], which is always surjective and closed even if $\mathcal{I}$ fails any of these properties (see [15, Lemma 2.2]). Furthermore, if $\mathcal{I}$ is surjective and closed then we show in [15, Corollary 2.5] that a bilinear operator $T \in \mathfrak{B}(A \times B, E)$ belongs to $\mathfrak{I}$ if and only if the linearization $\widetilde{T}$ of T belongs to $\mathcal{I}$. So, $\mathfrak{I}$ is the *composition ideal* generated by $\mathcal{I}$ (see [27, Section 7.3] and [9, Section 3]).

If E is a Banach space, A, B are only normed spaces and $T \in \mathfrak{B}(A \times B, E)$, we define $\gamma_{\mathfrak{I}}(T)$ as if T would be a bilinear operator acting between Banach spaces. Let $\widetilde{T} : A\widehat{\otimes}_\pi B \longrightarrow E$ be the linearization of T. We put $\check{T}$ for the restriction of $\widetilde{T}$ to $A \otimes_\pi B$. We define $\gamma_{\mathcal{I}}(\check{T}) = \gamma_{\mathcal{I}}(\check{T} : A \otimes_\pi B \longrightarrow E)$ as if $\check{T}$ would be a linear operator between Banach spaces.

Lemma 3.1. *Let $\mathcal{I}$ be a surjective and closed ideal. Let A, B, E be Banach spaces and let $T \in \mathfrak{B}(A \times B, E)$. Then*

$$\gamma_{\mathfrak{I}}(T) = \gamma_{\mathcal{I}}(\check{T}).$$

Proof. Since $T(U_A \times U_B) \subseteq \check{T}\big(U_{A\otimes_\pi B}\big)$, it is clear that $\gamma_{\mathfrak{I}}(T) \leq \gamma_{\mathcal{I}}(\check{T})$. On the other hand, we have that $\check{T}\big(U_{A\otimes_\pi B}\big) \subseteq \widetilde{T}\big(U_{A\widehat{\otimes}_\pi B}\big)$ and, by [15, Lemma 2.4], $\gamma_{\mathfrak{I}}(T) = \gamma_{\mathcal{I}}(\widetilde{T})$, therefore we obtain that $\gamma_{\mathcal{I}}(\check{T}) \leq \gamma_{\mathcal{I}}(\widetilde{T}) = \gamma_{\mathfrak{I}}(T)$. ■

Now we establish the main results of the paper. Given a Banach couple $\bar{E} = (E_0, E_1)$ and $m \in \mathbb{Z}$, we put $W_m = (E_0 + E_1, K(2^m, \cdot))$ and we define $\iota : \bar{E}_{\theta,q} \longrightarrow \ell_q(2^{-\theta m} W_m)$ by $\iota x = (\cdots, x, x, x, \cdots)$. It is clear that $\|\iota x\|_{\ell_q(2^{-\theta m}W_m)} = \|x\|_{\bar{E}_{\theta,q}}$.

Theorem 3.2. *Let $0 < \theta < 1$, $1 \le p, q < \infty$, $1 < r < \infty$ with $1/p + 1/q = 1 + 1/r$ and let $\mathcal{I}$ be a surjective operator ideal satisfying the Σ_r-condition. Assume that $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, $\bar{E} = (E_0, E_1)$ are Banach couples and that $T \in \mathfrak{B}(\bar{A} \times \bar{B}, \bar{E})$. Then*

$$\gamma_{\mathfrak{I}}\big(\iota T : \bar{A}_{\theta,p} \times \bar{B}_{\theta,q} \longrightarrow \ell_r(2^{-\theta m} W_m)\big) \\ \le C\, \gamma_{\mathfrak{I}}(T : A_0 \times B_0 \longrightarrow E_0)^{1-\theta} \gamma_{\mathfrak{I}}(T : A_1 \times B_1 \longrightarrow E_1)^{\theta},$$

where the constant $C > 0$ is independent of T.

Proof. Proceeding as in [14, Theorem 3.1], we get that $T : \{A_0 \cap A_1\}_{\theta,p} \times \bar{B}_{\theta,q} \longrightarrow \bar{E}_{\theta,r}$ is bounded. So, $\iota T : \{A_0 \cap A_1\}_{\theta,p} \times \bar{B}_{\theta,q} \longrightarrow \ell_r(2^{-\theta m} W_m)$ is also bounded. Since $T\big(U_{\{A_0 \cap A_1\}_{\theta,p}} \times U_{\bar{B}_{\theta,q}}\big) \subseteq \check{T}(U_{\{A_0 \cap A_1\}_{\theta,p} \otimes_\pi \bar{B}_{\theta,q}})$, we have

$$\begin{aligned} &\gamma_{\mathfrak{I}}\big(\iota T : \{A_0 \cap A_1\}_{\theta,p} \times \bar{B}_{\theta,q} \longrightarrow \ell_r(2^{-\theta m} W_m)\big) \\ &\quad \le \gamma_{\mathcal{I}}\big((\iota T)^{\vee} : \{A_0 \cap A_1\}_{\theta,p} \otimes_\pi \bar{B}_{\theta,q} \longrightarrow \ell_r(2^{-\theta m} W_m)\big) \\ &\quad = \gamma_{\mathcal{I}}\big(\iota \check{T} : \{A_0 \cap A_1\}_{\theta,p} \otimes_\pi \bar{B}_{\theta,q} \longrightarrow \ell_r(2^{-\theta m} W_m)\big). \end{aligned} \tag{3.1}$$

Besides, since $\check{T} : A_i \otimes_\pi B_i \longrightarrow E_i$ is bounded for $i = 0, 1$, the interpolation property for linear operators yields that

$$\check{T} : (A_0 \otimes_\pi B_0, A_1 \otimes_\pi B_1)_{\theta,r} \longrightarrow \bar{E}_{\theta,r}$$

is bounded. According to Lemma 2.2, the following factorization holds

$$\begin{array}{ccc} \{A_0 \cap A_1\}_{\theta,p} \otimes_\pi \bar{B}_{\theta,q} & \xrightarrow{\quad \iota \check{T} \quad} & \ell_r(2^{-\theta m} W_m). \\ & {\scriptstyle I} \searrow \qquad \nearrow {\scriptstyle \iota \check{T}} & \\ & (A_0 \otimes_\pi B_0, A_1 \otimes_\pi B_1)_{\theta,r} & \end{array}$$

Therefore

$$\begin{aligned} &\gamma_{\mathcal{I}}\big(\iota \check{T} : \{A_0 \cap A_1\}_{\theta,p} \otimes_\pi \bar{B}_{\theta,q} \longrightarrow \ell_r(2^{-\theta m} W_m)\big) \\ &\quad \le \gamma_{\mathcal{I}}\big(\iota \check{T} : (A_0 \otimes_\pi B_0, A_1 \otimes_\pi B_1)_{\theta,r} \longrightarrow \ell_r(2^{-\theta m} W_m)\big). \end{aligned} \tag{3.2}$$

Next we show that

$$\begin{aligned} &\gamma_{\mathcal{I}}\big(\iota \check{T} : (A_0 \otimes_\pi B_0, A_1 \otimes_\pi B_1)_{\theta,r} \longrightarrow \ell_r(2^{-\theta m} W_m)\big) \\ &\quad \le C\, \gamma_{\mathcal{I}}(\check{T} : A_0 \otimes_\pi B_0 \longrightarrow E_0)^{1-\theta} \gamma_{\mathcal{I}}(\check{T} : A_1 \otimes_\pi B_1 \longrightarrow E_1)^{\theta}. \end{aligned} \tag{3.3}$$

To simplify notation, put $G_i = A_i \otimes_\pi B_i$, $i = 0, 1$. Take any $\sigma_i > \gamma_{\mathcal{I}}(\breve{T} : G_i \longrightarrow E_i)$ and let Z_i be a Banach space and $S_i \in \mathcal{I}(Z_i, E_i)$ such that

$$\breve{T}\left(U_{G_i}\right) \subseteq \sigma_i U_{E_i} + S_i\left(U_{Z_i}\right), \quad i = 0, 1. \tag{3.4}$$

Take any $a \in (G_0, G_1)_{\theta,r}$ with $\|a\|_{\theta,r} \le 1$. Given any $\delta > 0$, pick $d_m = d_m(a) > 0$ such that

$$2^{-\theta m} K(2^m, a) < d_m \quad \text{and} \quad \sum_{m=-\infty}^{\infty} d_m^r < (1+\delta)^r.$$

We can choose $k \in \mathbb{Z}$ such that $2^{k-1} \le \sigma_1/\sigma_0 < 2^k$. Then

$$K\Big(2^m \frac{\sigma_1}{\sigma_0}, a\Big) \le K\big(2^{m+k}, a\big) < 2^{(m+k)\theta} d_{m+k} \le 2^{(m+1)\theta}\Big(\frac{\sigma_1}{\sigma_0}\Big)^{\theta} d_{m+k}.$$

Put

$$\rho_m^0 = 2^{(m+1)\theta}\Big(\frac{\sigma_1}{\sigma_0}\Big)^{\theta} d_{m+k}\,, \quad \rho_m^1 = 2^m \frac{\sigma_1}{\sigma_0}.$$

Since $K(\rho_m^1, a) < \rho_m^0$, we can find a decomposition

$$a = a_m^0 + a_m^1 \quad \text{with} \quad \frac{1}{\rho_m^0} a_m^0 \in U_{G_0} \quad \text{and} \quad \frac{\rho_m^1}{\rho_m^0} a_m^1 \in U_{G_1}.$$

By (3.4), there exists $x_m^i \in U_{Z_i}$ such that

$$\left\| \breve{T}\left(\frac{(\rho_m^i)^i}{\rho_m^0} a_m^i\right) - S_i x_m^i \right\|_{E_i} \le \sigma_i\,, \quad i = 0, 1. \tag{3.5}$$

Then

$$\left\| \breve{T} a_m^i - S_i\Big(2^\theta 2^{m(\theta-i)}\Big(\frac{\sigma_1}{\sigma_0}\Big)^{\theta-i} d_{m+k} x_m^i\Big) \right\|_{E_i} \le 2^\theta 2^{m(\theta-i)} \sigma_0^{1-\theta} \sigma_1^{\theta} d_{m+k}. \tag{3.6}$$

For any $m \in \mathbb{Z}$, let $D_m = (Z_0 \times Z_1)_{\ell_\infty}$ be the product of Z_0 and Z_1, normed by $\|(x_0, x_1)\| = \max\{\|x_0\|_{Z_0}, \|x_1\|_{Z_1}\}$. The operator $S : \ell_r(D_m) \longrightarrow \ell_r(2^{-\theta m} W_m)$ defined by

$$S(x_m^0, x_m^1) = \left(\sum_{i=0}^{1} (1+\delta) 2^\theta 2^{m(\theta-i)} \Big(\frac{\sigma_1}{\sigma_0}\Big)^{\theta-i} S_i x_m^i\right)$$

is bounded because

$$\|S(x_m^0, x_m^1)\|_{\ell_r(2^{-\theta m} W_m)} \le \left(\sum_{m=-\infty}^{\infty} \left(2^{-\theta m} \sum_{i=0}^{1} (1+\delta) 2^\theta 2^{m\theta} \Big(\frac{\sigma_1}{\sigma_0}\Big)^{\theta-i} \|S_i\|_{Z_i,E_i} \|x_m^i\|_{Z_i}\right)^r\right)^{\frac{1}{r}}$$

$$\le 2^{\theta+1}(1+\delta) \max\left\{\Big(\frac{\sigma_1}{\sigma_0}\Big)^{\theta}, \Big(\frac{\sigma_1}{\sigma_0}\Big)^{\theta-1}\right\} \max\left\{\|S_0\|_{Z_0,E_0}\,, \|S_1\|_{Z_1,E_1}\right\} \|(x_m^0, x_m^1)\|_{\ell_r(D_m)}.$$

Since each component

$$(Q_k S P_n)(x^0, x^1) = \begin{cases} 0 & \text{if} \quad k \neq n \\ \sum_{i=0}^{1} (1+\delta) 2^{\theta} 2^{n(\theta-i)} \big(\frac{\sigma_1}{\sigma_0}\big)^{\theta-i} S_i x^i & \text{if} \quad k = n\,, \end{cases}$$

belongs to $\mathcal{I}(D_n, 2^{-\theta k} W_k)$, it follows from the Σ_r-condition that $S \in \mathcal{I}\left(\ell_r(D_m), \ell_r(2^{-\theta m} W_m)\right)$.

Next we show that

$$(\iota \check{T})\big(U_{\bar{G}_{\theta,r}}\big) \subseteq 2^{\theta+1}(1+\delta)\sigma_0^{1-\theta}\sigma_1^{\theta} U_{\ell_q(2^{-\theta m} W_m)} + S(U_{\ell_q(D_m)}).$$

Given any $a \in U_{\bar{G}_{\theta,r}}$, we can take $x_m^i \in U_{Z_i}$ as in (3.5). Let

$$x = \big((1+\delta)^{-1} d_{m+k} x_m^0\,,\ (1+\delta)^{-1} d_{m+k} x_m^1\big).$$

Then x belongs to $U_{\ell_r(D_m)}$ and, using (3.6), we obtain

$$\begin{aligned} &\|(\iota\check{T})a - Sx\|^r_{\ell_r(2^{-\theta m} W_m)} \\ &\quad \le \sum_{m=-\infty}^{\infty} \left(2^{-\theta m} \left(\sum_{i=0}^{1} 2^{mi} \left\| \check{T} a_m^i - 2^{\theta} 2^{m(\theta-i)} \left(\frac{\sigma_1}{\sigma_0}\right)^{\theta-i} d_{m+k} S_i x_m^i \right\|_{E_i} \right)\right)^r \\ &\quad \le \sum_{m=-\infty}^{\infty} \left(2^{-\theta m} \left(\sum_{i=0}^{1} 2^{mi} 2^{\theta} 2^{m(\theta-i)} \sigma_0^{1-\theta} \sigma_1^{\theta} d_{m+k} \right)\right)^r \\ &\quad \le \left(2^{\theta+1}\sigma_0^{1-\theta}\sigma_1^{\theta}\right)^r \sum_{m=-\infty}^{\infty} d_{m+k}^r \le \left(2^{\theta+1}\sigma_0^{1-\theta}\sigma_1^{\theta}\right)^r (1+\delta)^r. \end{aligned}$$

Hence,

$$\gamma_{\mathcal{I}}\big(\iota\check{T} : (G_0, G_1)_{\theta,r} \longrightarrow \ell_r(2^{-\theta m} W_m)\big) \le 2^{\theta+1}(1+\delta)\sigma_0^{1-\theta}\sigma_1^{\theta}.$$

Since $\delta > 0$ is arbitrary, we derive that

$$\gamma_{\mathcal{I}}\big(\iota\check{T} : (G_0, G_1)_{\theta,r} \longrightarrow \ell_r(2^{-\theta m} W_m)\big) \le C\, \gamma_{\mathcal{I}}(\check{T} : G_0 \longrightarrow E_0)^{1-\theta} \gamma_{\mathcal{I}}(\check{T} : G_1 \longrightarrow E_1)^{\theta}.$$

Consequently, it follows from (3.1), (3.2), (3.3) and Lemma 3.1 that

$$\begin{aligned} &\gamma_{\mathfrak{I}}\big(\iota T : \{A_0 \cap A_1\}_{\theta,p} \times \bar{B}_{\theta,q} \longrightarrow \ell_r(2^{-\theta m} W_m)\big) \\ &\qquad \le C\, \gamma_{\mathcal{I}}(\check{T} : A_0 \otimes_\pi B_0 \longrightarrow E_0)^{1-\theta} \gamma_{\mathcal{I}}(\check{T} : A_1 \otimes_\pi B_1 \longrightarrow E_1)^{\theta} \\ &\qquad = C\, \gamma_{\mathfrak{I}}(T : A_0 \times B_0 \longrightarrow E_0)^{1-\theta} \gamma_{\mathfrak{I}}(T : A_1 \times B_1 \longrightarrow E_1)^{\theta}. \end{aligned}$$

To complete the proof, we show that

$$\begin{aligned} &\gamma_{\mathfrak{I}}\big(\iota T : \bar{A}_{\theta,p} \times \bar{B}_{\theta,q} \longrightarrow \ell_r(2^{-\theta m} W_m)\big) \\ &\qquad \le \mu := \gamma_{\mathfrak{I}}\big(\iota T : \{A_0 \cap A_1\}_{\theta,p} \times \bar{B}_{\theta,q} \longrightarrow \ell_r(2^{-\theta m} W_m)\big). \end{aligned} \tag{3.7}$$

Take any $\varepsilon > 0$. By the definition of μ, there are a Banach space Z and an operator $R \in \mathcal{I}(Z, \ell_r(2^{-\theta m} W_m))$ such that

$$\iota T(U_{\{A_0 \cap A_1\}_{\theta,p}} \times U_{\bar{B}_{\theta,q}}) \subseteq (\mu + \varepsilon) U_{\ell_r(2^{-\theta m} W_m)} + R(U_Z).$$

We are going to show that

$$\iota T(U_{\bar{A}_{\theta,p}} \times U_{\bar{B}_{\theta,q}}) \subseteq (\mu + 2\varepsilon) U_{\ell_r(2^{-\theta m} W_m)} + R(U_Z).$$

Take any $a \in U_{\bar{A}_{\theta,p}}$ and $b \in U_{\bar{B}_{\theta,q}}$. Since $p < \infty$, it follows from the J-representation of $\bar{A}_{\theta,p}$ that $A_0 \cap A_1$ is dense in $\bar{A}_{\theta,p}$. Hence, there exists $(a_n) \subseteq U_{\{A_0 \cap A_1\}_{\theta,p}}$ such that $a_n \longrightarrow a$ in $\bar{A}_{\theta,p}$. We can find $N \in \mathbb{N}$ such that

$$\|a - a_N\|_{\theta,q} \leq \varepsilon / \|T : \bar{A}_{\theta,p} \times \bar{B}_{\theta,q} \longrightarrow \ell_r(2^{-\theta m} W_m)\|$$

and pick $z \in U_Z$ such that

$$\|\iota T(a_N, b) - Rz\|_{\ell_r(2^{-\theta m} W_m)} \leq \mu + \varepsilon.$$

Therefore,

$$\begin{aligned} \|\iota T(a,b) - Rz\|_{\ell_r(2^{-\theta m} W_m)} &\leq \|\iota T(a,b) - \iota T(a_N, b)\|_{\ell_r(2^{-\theta m} W_m)} + \|\iota T(a_N, b) - Rz\|_{\ell_r(2^{-\theta m} W_m)} \\ &\leq \|T : \bar{A}_{\theta,p} \times \bar{B}_{\theta,q} \longrightarrow \ell_r(2^{-\theta m} W_m)\| \|a - a_N\|_{\bar{A}_{\theta,p}} \|b\|_{\bar{B}_{\theta,q}} + \mu + \varepsilon \\ &\leq \mu + 2\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this yields (3.7) and finishes the proof. ∎

Theorem 3.2 can be applied not only when $\mathcal{I} = \mathcal{W}$ but also when $\mathcal{I} = \mathcal{BS}$, $\mathcal{R}$ or $\mathcal{A}$.

If we remove the operator ι, then Theorem 3.2 does not hold. We show it with a counterexample based on the ideas of [17, Remark 3.4].

Counterexample 3.3. Let $\mathcal{I} = \mathcal{W}$, so $\mathfrak{I} = \mathfrak{W}$ is the bideal of all weakly compact bilinear operators. The functional $\gamma_{\mathcal{I}}$ is the measure of weak non-compactness introduced by De Blasi [20] and $\gamma_{\mathfrak{I}}$ coincides with the natural extension of $\gamma_{\mathcal{I}}$ to bilinear operators (see [15, Proposition 2.9]).

It follows from [3, Theorem 4] that there are a Banach space E and a sequence of linear operators $S_n \subseteq \mathcal{L}(E, c_0)$, so that

$$\|S_n\| \leq 2\,, \quad \gamma_{\mathcal{W}}(S_n^*) = 1 \quad \text{and} \quad \gamma_{\mathcal{W}}(S_n^{**}) \leq \gamma_{\mathcal{W}}(S_n) \leq 1/n\,, \quad n \in \mathbb{N}. \tag{3.8}$$

Let $\ell_1(U_E)$ be the Banach space of all absolutely summable families of scalars (λ_x) indexed by the elements $x \in U_E$. Let $\ell_\infty(U_E)$ be the dual space of $\ell_1(U_E)$ and let $Q_E : \ell_1(U_E) \longrightarrow E$ be the surjection $Q_E(\lambda_x) = \sum_{x\in U_E} \lambda_x x$. Put $R_n = Q_E^* S_n^* \in \mathcal{L}(\ell_1, \ell_\infty(U_E))$ and consider the subspace $F = Q_E^*(E^*)$ of $\ell_\infty(U_E)$.

Choose $\bar{A} = (\ell_1, \ell_1)$, $\bar{B} = (\mathbb{K}, \mathbb{K})$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is the scalar fields, choose $\bar{E} = (F, \ell_\infty(U_E))$, and consider the bilinear operators $T_n : \ell_1 \times \mathbb{K} \longrightarrow F$ defined by $T_n(x, \lambda) = \lambda R_n x$.

If we suppose that the estimate of Theorem 3.2 holds without ι, then there would exist a constant $C > 0$ such that

$$\begin{aligned} &\gamma_{\mathfrak{W}}\big(T_n : \ell_1 \times \mathbb{K} \longrightarrow (F, \ell_\infty(U_E))_{\theta,r}\big) \\ &\qquad \le C\, \gamma_{\mathfrak{W}}(T_n : \ell_1 \times \mathbb{K} \longrightarrow F)^{1-\theta} \gamma_{\mathfrak{W}}(T_n : \ell_1 \times \mathbb{K} \longrightarrow \ell_\infty(U_E))^{\theta}. \end{aligned} \tag{3.9}$$

Since F is a closed subspace of $\ell_\infty(U_E)$, we have that $(F, \ell_\infty(U_E))_{\theta,r} = F$ with equivalent norms. Moreover, using that $Q_E^* : E^* \longrightarrow F$ is an isometry and using (3.8), we get

$$\gamma_{\mathfrak{W}}(T_n : \ell_1 \times \mathbb{K} \longrightarrow F) = \gamma_{\mathcal{W}}(R_n : \ell_1 \longrightarrow F) = \gamma_{\mathcal{W}}(S_n^* : \ell_1 \longrightarrow E^*) = 1.$$

As for the other term in (3.9), applying [2, Corollary 5.3] and (3.8) we obtain

$$\gamma_{\mathfrak{W}}(T_n : \ell_1 \times \mathbb{K} \longrightarrow \ell_\infty(U_E)) = \gamma_{\mathcal{W}}(R_n : \ell_1 \longrightarrow \ell_\infty(U_E)) = \gamma_{\mathcal{W}}(R_n^*) \le \gamma_{\mathcal{W}}(S_n^{**}) \le 1/n.$$

Therefore, we would obtain from (3.9) that

$$1 \le C/n^\theta\,, \quad n = 1, 2, \ldots$$

which is impossible.

However, as we show next, if the couple $\bar{E} = (E_0, E_1)$ reduces to a single space E, then we can get rid of the operator ι. We can also eliminate the assumption that $\mathcal{I}$ satisfies the Σ_r-condition, as well as the connection between p and q. Moreover, parameters p and q are also allowed to take the value ∞.

Theorem 3.4. *Let $0 < \theta < 1$, $1 \le p, q \le \infty$ and let $\mathcal{I}$ be a surjective closed operator ideal. Assume that $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ are Banach couples, let E be a Banach space and $T \in \mathfrak{B}(\bar{A} \times \bar{B}, E)$. Then*

$$\gamma_{\mathfrak{I}}\big(T : \bar{A}_{\theta,p} \times \bar{B}_{\theta,q} \longrightarrow E\big) \le C\, \gamma_{\mathfrak{I}}(T : A_0 \times B_0 \longrightarrow E)^{1-\theta} \gamma_{\mathfrak{I}}(T : A_1 \times B_1 \longrightarrow E)^{\theta}$$

where $C > 0$ is a constant independent of T.

Proof. Since $T : (A_0 + A_1) \times (B_0 + B_1) \longrightarrow E$ boundedly, it is clear that $T : \bar{A}_{\theta,p} \times \bar{B}_{\theta,q} \longrightarrow E$ is bounded for any $1 \le p, q \le \infty$. Besides, $\bar{A}_{\theta,q} \hookrightarrow \bar{A}_{\theta,\infty}$. Hence, it suffices to establish the result for $p = q = \infty$. Note also that $T : A_0 \times B_1 \longrightarrow E$ and $T : A_1 \times B_0 \longrightarrow E$ are bounded. Let $M_1 = \|T\|_{A_0 \times B_1, E}$ and $M_2 = \|T\|_{A_1 \times B_0, E}$.

Given any $\sigma_i > \gamma_{\mathfrak{I}}(T : A_i \times B_i \longrightarrow E)$, $i = 0, 1$, there are Banach spaces Z_i and linear operators $S_i \in \mathcal{I}(Z_i, E)$ such that

$$T(U_{A_i} \times U_{B_i}) \subseteq \sigma_i U_E + S_i(U_{Z_i}), \quad i = 0, 1. \tag{3.10}$$

We can pick $s, m \in \mathbb{Z}$ such that

$$2^{s-1} \le \sigma_1/\sigma_0 < 2^s \quad \text{and} \quad 2^{m-1} \le \max\left\{\frac{\sigma_1}{2^2 M_1}, \frac{2^2 M_2}{\sigma_0}\right\} < 2^m. \tag{3.11}$$

Given any $a \in \bar{A}_{\theta,\infty}, b \in \bar{B}_{\theta,\infty}$ with $\|a\|_{\bar{A}_{\theta,\infty}} \le 1$ and $\|b\|_{\bar{B}_{\theta,\infty}} \le 1$, we can find decompositions $a = a_0 + a_1$ with $a_i \in A_i$ and $b = b_0 + b_1$ with $b_i \in B_i$ such that

$$\|a_0\|_{A_0} + 2^m \|a_1\|_{A_1} \le 2\,2^{\theta m} \quad , \quad \|b_0\|_{B_0} + 2^{-m+s} \|b_1\|_{B_1} \le 2\,2^{\theta(-m+s)}.$$

Therefore, using (3.10), we obtain

$$\begin{aligned}
T(a, b) &= T(a_0, b_0) + T(a_1, b_1) + T(a_0, b_1) + T(a_1, b_0) \\
&\in \sigma_0\, 2\,2^{\theta m}\, 2\,2^{\theta(-m+s)} U_E + 2\,2^{\theta m}\, 2\,2^{\theta(-m+s)} S_0(U_{Z_0}) \\
&+ \sigma_1\, 2\,2^{(\theta-1)m}\, 2\,2^{(\theta-1)(-m+s)} U_E + 2\,2^{(\theta-1)m}\, 2\,2^{(\theta-1)(-m+s)} S_1(U_{Z_1}) \\
&+ M_1\, 2\,2^{\theta m}\, 2\,2^{(\theta-1)(-m+s)} U_E + M_2\, 2\,2^{(\theta-1)m}\, 2\,2^{\theta(-m+s)} U_E \\
&\subseteq \left[\sigma_0\, 2^2\, 2^{\theta s} + \sigma_1\, 2^2\, 2^{(\theta-1)s} + M_1\, 2^2\, 2^m\, 2^{(\theta-1)s} + M_2\, 2^2\, 2^{-m}\, 2^{\theta s}\right] U_E \\
&+ 2^2\, 2^{\theta s} S_0(U_{Z_0}) + 2^2\, 2^{(\theta-1)s} S_1(U_{Z_1}).
\end{aligned}$$

By (3.11), we have

$$\begin{aligned}
\sigma_0 2^2 2^{\theta s} &\le \sigma_0 2^2 2^{\theta} (\sigma_1/\sigma_0)^{\theta} = 2^{2+\theta} \sigma_0^{1-\theta} \sigma_1^{\theta}, \\
\sigma_1 2^2 2^{(\theta-1)s} &\le \sigma_1 2^2 (\sigma_1/\sigma_0)^{(\theta-1)} = 2^2 \sigma_0^{1-\theta} \sigma_1^{\theta}, \\
M_1 2^2 2^m 2^{(\theta-1)s} &\le 2\sigma_1 (\sigma_1/\sigma_0)^{(\theta-1)} = 2\sigma_0^{1-\theta} \sigma_1^{\theta}, \\
M_2 2^2 2^{-m} 2^{\theta s} &\le 2^{\theta} \sigma_0 (\sigma_1/\sigma_0)^{\theta} = 2^{\theta} \sigma_0^{1-\theta} \sigma_1^{\theta}.
\end{aligned}$$

Therefore

$$T(a, b) \in (2^{2+\theta} + 2^2 + 2 + 2^{\theta}) \sigma_0^{1-\theta} \sigma_1^{\theta} U_E + 2^2 2^{\theta s} S_0(U_{Z_0}) + 2^2 2^{(\theta-1)s} S_1(U_{Z_1}).$$

Let $Z = (Z_0 \times Z_1)_{\ell_\infty}$ be the product of Z_0 and Z_1 with the norm of the maximum, and let $S : Z \longrightarrow E$ be the linear operator defined by

$$S(u, v) = 2^2 2^{\theta s} S_0 u + 2^2 2^{(\theta-1)s} S_1 v.$$

The operator S is the sum of the composed operators

$$\begin{array}{ccccc} Z & \longrightarrow & Z_0 & \longrightarrow & E \\ (u, v) & \longrightarrow & u & \longrightarrow & 2^2 2^{\theta s} S_0 u, \end{array}$$

$$\begin{array}{ccccc} Z & \longrightarrow & Z_1 & \longrightarrow & E \\ (u, v) & \longrightarrow & v & \longrightarrow & 2^2 2^{(\theta-1)s} S_1 v. \end{array}$$

Therefore, S belongs to $\mathcal{I}(Z, E)$ and

$$T\left(U_{\bar{A}_{\theta,\infty}} \times U_{\bar{B}_{\theta,\infty}}\right) \subseteq C\sigma_0^{1-\theta}\sigma_1^{\theta} U_E + S(U_Z)$$

with $C = 2^{2+\theta} + 2^2 + 2 + 2^\theta$. This implies that

$$\gamma_{\mathfrak{I}}\left(T : \bar{A}_{\theta,\infty} \times \bar{B}_{\theta,\infty} \longrightarrow E\right) \le C\gamma_{\mathfrak{I}}(T : A_0 \times B_0 \longrightarrow E)^{1-\theta}\gamma_{\mathfrak{I}}(T : A_1 \times B_1 \longrightarrow E)^{\theta}$$

and completes the proof. ∎

Theorem 3.4 can be applied to operator ideals which do not satisfy the $\sum_r$-condition and so Theorem 3.2 cannot be used with them. This is the case of the ideal C of strictly cosingular operators. Recall (see [34–36]) that an operator $R \in \mathcal{L}(X, E)$ between Banach spaces X, E is said to be *strictly cosingular* (or *Pełczyński operator*) if there is no infinite-codimensional (closed) subspace $F \subseteq E$, such that $\Phi_F T$ is surjective. Here Φ_F stands for the quotient mapping from E onto E/F. The ideal C is closed and surjective. Moreover, $\mathcal{K} \subseteq C$.

One may wonder if a convexity estimate of the type of Theorem 3.4 may hold in the other special case, where $A_0 = A_1$, $B_0 = B_1$ and (E_0, E_1) is an arbitrary Banach couple. The answer is negative as we show next with the help of C and ideas of [8, Counterexample 2.5].

Counterexample 3.5. According to [8, Counterexample 2.5], there are a Banach space X, a Banach couple (E_0, E_1) and an operator $R \in \mathcal{L}(X, E_0 \cap E_1)$ such that $R : X \longrightarrow E_0$ is strictly cosingular but $R : X \longrightarrow (E_0, E_1)_{\theta,r}$ is not strictly cosingular. Here $0 < \theta < 1$ and $1 < r < \infty$. Using [2, Theorem 3.1], we have

$$\gamma_C(R : X \longrightarrow E_0) = 0 \quad \text{but} \quad \gamma_C(R : X \longrightarrow (E_0, E_1)_{\theta,r}) \neq 0.$$

Let $A_0 = A_1 = \mathbb{K}$, the scalar field, and put $T(\lambda, x) = \lambda Rx$. Then $T : \mathbb{K} \times X \longrightarrow E_0 \cap E_1$ is bounded. Since $T(U_{\mathbb{K}} \times U_X) = R(U_X)$, we obtain that

$$\gamma_{\mathfrak{C}}(T : \mathbb{K} \times X \longrightarrow E_0) = 0 \quad \text{and} \quad \gamma_{\mathfrak{C}}(T : \mathbb{K} \times X \longrightarrow (E_0, E_1)_{\theta,r}) \neq 0.$$

Moreover, we have with equivalent norms that $(\mathbb{K}, \mathbb{K})_{\theta,p} = \mathbb{K}$ and $(X, X)_{\theta,q} = X$. Consequently, convexity inequality

$$\gamma_{\mathfrak{C}}(T : (\mathbb{K}, \mathbb{K})_{\theta,p} \times (X, X)_{\theta,q} \longrightarrow (E_0, E_1)_{\theta,r}) \leq C \gamma_{\mathfrak{C}}(T : \mathbb{K} \times X \longrightarrow E_0)^{1-\theta} \gamma_{\mathfrak{C}}(T : \mathbb{K} \times X \longrightarrow E_1)^{\theta}$$

does not hold.

We finish the paper with a remark on interpolation properties of the bideal $\mathfrak{I}$: From Theorem 3.2 we can recover [15, Corollary 3.2]. Furthermore, if $\bar{E} = (E_0, E_1)$ reduces to a single space, then using Theorem 3.4 we can get a result of Manzano, Rueda and Sánches-Pérez [32, Corollary 4.6].

Acknowledgements. The authors would like to thank the referees for their comments.

Funding. F. Cobos and L.M. Fernández-Cabrera have been supported in part by MICIU/AEI/PCI 2021-155073-2.

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