Open Problems

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- ▷ J. Peetre, Duke Univ. Math. Series, Durham, 1976, p. 104 and p.110.
- ▷ H. Triebel, Seminar on Function Spaces, Jena, 2019.

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- We denote by $\ell_q^s(A)$ the space of all sequences $(a_k)_{k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}} \subseteq A$ such that

$$\|(a_k)\|_{\ell_q^s(A)} = \Big(\sum_{k=0}^{\infty}\big(2^{ks}\|a_k\|_A\big)^q\Big)^{1/q} < \infty$$

(with the usual modification if $q=\infty$).

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For $0 < q_0, q_1, r \leq \infty, -\infty < s_0 \neq s_1 < \infty, 0 < \theta < 1$ and $s = (1-\theta)s_0 + \theta s_1$, we have

$$(\ell_{q_0}^{s_0}(A), \ell_{q_1}^{s_1}(A))_{\theta,r} = \ell_r^s(A).$$

No relationship between q_0, q_1 and r.

If we replace A by a quasi-Banach couple (A_0, A_1) then, for $0 < q_0, q_1 \leq \infty, -\infty < s_0 \neq s_1 < \infty, 0 < \theta < 1, s = (1-\theta)s_0 + \theta s_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$, we have

$$(\ell_{q_0}^{s_0}(A_0), \ell_{q_1}^{s_1}(A_1))_{\theta,q} = \ell_q^s((A_0, A_1)_{\theta,q}).$$

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• • • To find a precise description of

$$(\ell_{q_0}^{s_0}(A_0), \ell_{q_1}^{s_1}(A_1))_{\theta,r} \quad \text{if } r \neq q.$$

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provided that $0 < \theta < 1, -\infty < s_0, s_1 < \infty, s = (1-\theta)s_0 + \theta s_1, 0 < p_0 \neq p_1 < \infty, 1/p = (1-\theta)/p_0 + \theta/p_1, 0 < q_0, q_1 < \infty$ and $1/q = (1-\theta)/q_0 + \theta/q_1$.

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The diagonal case $p_0 = q_0$ and $p_1 = q_1$ has been studied in the papers

▷ V.L. Krepkogorskii, Russian Acad. Sci. Sb. Math. 82 (1995) 315-326.

▷ B.F. Besoy, D.D. Haroske, H. Triebel, Math. Nachr. 295 (2022) 1669-1689.

$$B^s_{\min\{q,r\}} L_{p,r} \hookrightarrow (B^{s_0}_{p_0,q_0}, B^{s_1}_{p_1,q_1})_{\theta,r} \hookrightarrow B^s_{\max\{q,r\}} L_{p,r}.$$

and the exponents $\min\{q,r\}$ and $\max\{q,r\}$ are best possible, at least if $0 \le p_0, p_1, q_0, q_1, r \le \infty, 0 < p < \infty, 0 < \theta < 1, -\infty < s_0 \ne s_1 < \infty, 1/p = (1-\theta)/p_0 + \theta/p_1, 1/q = (1-\theta)/q_0 + \theta/q_1, s = (1-\theta)s_0 + \theta s_1$, and

$$\frac{s_0 - s_1}{n/p_0 - n/p_1} < 1.$$

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