On the dual of Besov-Lorentz spaces

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Abstract

We work with Besov spaces with Lorentz smoothness $B^s_q L_{p,r}(\mathbb{R}^n)$. Here $-\infty < s < \infty$ and $0 < p, q, r < \infty$. We determine the dual of $B^s_q L_{p,r}(\mathbb{R}^n)$ with the help of its characterization in terms of wavelets. In particular, when $p = 1$ and $1 < r < \infty$, the dual spaces are new Besov spaces defined by using the limiting Lorentz sequence spaces $\ell_{\infty,r}$. We apply the results to determine the dual of certain Triebel-Lizorkin-Lorentz spaces $F^s_q L_{p,r}(\mathbb{R}^n)$.

Keywords: Besov-Lorentz spaces, Triebel-Lizorkin-Lorentz spaces, dual spaces of Besov-Lorentz spaces, characterizations in terms of wavelets.

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Dedicated to the memory of Professor Des W. Evans

1 Introduction

Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ and Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ are very important in the theory of function spaces because of their many useful properties (see, for example, [27, 28]). But these scales of spaces are not closed by real interpolation (see [21, 26]). This is the reason for introducing Besov spaces with Lorentz smoothness $B^s_q L_{p,r}(\mathbb{R}^n)$ and the corresponding Triebel-Lizorkin-Lorentz spaces $F^s_q L_{p,r}(\mathbb{R}^n)$. They are defined by replacing the Lebesgue space $L_p(\mathbb{R}^n)$ by the more general Lorentz space $L_{p,r}(\mathbb{R}^n)$ in the Fourier-analytical definition of Besov and Triebel-Lizorkin spaces. It turns out that

$$\left(F^s_{p_0,q}(\mathbb{R}^n), F^s_{p_1,q}(\mathbb{R}^n)\right)_{\theta,r} = F^s_q L_{p,r}(\mathbb{R}^n)$$

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where $-\infty < s < \infty, 0 < q, r \leq \infty, 0 < \theta < 1, 0 < p_0 \neq p_1 < \infty, 1/p = (1-\theta)/p_0 + \theta/p_1$. For Besov spaces, we have that

$$\left(B^{s_0}_{p_0,q_0}(\mathbb{R}^n), B^{s_1}_{p_1,q_1}(\mathbb{R}^n)\right)_{\theta,q} = B^s_q L_{p,q}(\mathbb{R}^n)$$

provided that $0 < \theta < 1, -\infty < s_0, s_1 < \infty, s = (1-\theta)s_0 + \theta s_1, 0 < p_0 \neq p_1 < \infty, 1/p = (1-\theta)/p_0 + \theta/p_1, 0 < q_0, q_1 < \infty$ and $1/q = (1-\theta)/q_0 + \theta/q_1$ (see [21, 26, 7]).

Spaces $B^s_q L_{p,r}(\mathbb{R}^n)$ and $F^s_q L_{p,r}(\mathbb{R}^n)$ have been used since the early 1960s in different contexts. For example, Besov-Lorentz spaces appear in the books by Peetre [21], Triebel [26] and Edmunds and Triebel [16], Hardy-Lorentz spaces have been used in the papers by Fefferman, Riviere and Sagher [17] and Almeida and Caetano [2, 3]. Spaces $F^s_2 L_{p,r}(\mathbb{R}^n)$ appear in the papers by Stein [25], Caetano [9] and Cianchi and Pick [10]. Lorentz smoothness spaces in connection with wavelet theory have been studied by Yang, Cheng and Peng [32], Almeida [1], Besoy, Cobos and Triebel [7] and Besoy and Cobos [6]. In the context of partial differential equations, those spaces have been considered by Xiang and Yan [30, 31] and Hobus and Saal [19]. Embeddings and other properties of Lorentz smoothness spaces have been studied by Seeger and Trebels [24], Besoy, Haroske and Triebel [8] and Cobos, Fernández-Cabrera and Kühn [12].

The dual space of $B^s_q L_{p,r}(\mathbb{R}^n)$ has been determined by the present authors in [12] for $-\infty < s < \infty, 0 < q < \infty$ and $1 < p < \infty, 1 \leq r < \infty$, or $0 < p < 1$ and $0 < r < \infty$, or $p = 1$ and $0 < r \leq 1$. Our aim in this paper is to determine the dual of $B^s_q L_{p,r}(\mathbb{R}^n)$ for the remaining values of parameters. In contrast to [12], where the results are based on the interpolation properties of Besov-Lorentz spaces, our approach here is based on the characterization of $B^s_q L_{p,r}(\mathbb{R}^n)$ in terms of wavelets (see [1, 6]). This method allows us to cover the whole range of parameters: $-\infty < s < \infty, 0 < p, q, r < \infty$.

We show that the dual of $B^s_q L_{p,r}(\mathbb{R}^n)$ can be identified with another Besov-Lorentz space except when $p = 1$ and $1 < r < \infty$, then new Besov spaces arise, which are defined by using the limiting Lorentz sequence spaces $\ell_{\infty,r}$.

As an application of the results, we determine dual spaces of Triebel-Lizorkin-Lorentz spaces in a case which was not considered in [12].

The plan of the paper is simple. We start by reviewing Lorentz sequences spaces $\ell_{p,r}$ in Section 2. We also show there that the dual of $\ell_{1,q}$ is the limiting Lorentz space $\ell_{\infty,q'}$ for $1 < q < \infty$ and $1/q + 1/q' = 1$. A duality result for vector-valued sequence spaces is also established. In Section 3 we review definitions of spaces $B^s_q L_{p,r}(\mathbb{R}^n)$ and $F^s_q L_{p,r}(\mathbb{R}^n)$, and the characterization of Besov-Lorentz spaces in terms of wavelets. Finally, in Section 4, we introduce the new Besov spaces $B^s_q L_{\infty,u}(\mathbb{R}^n)$ and we establish the duality results for Besov-Lorentz spaces and for Triebel-Lizorkin-Lorentz spaces.

2 Sequence spaces

Fix $n \in \mathbb{N}$. Consider the space $\mathbb{Z}^n$ with the counting measure and let $\ell_q = \ell_q(\mathbb{Z}^n)$ be the usual space of q-summable sequences of complex numbers with indices on $\mathbb{Z}^n$, $0 < q \le \infty$. For a sequence with indices in $\mathbb{Z}^n$, we write sometimes $(u_m)_{m\in\mathbb{Z}^n}$ to emphasize the set of indices. Later we also work with sequences having indices in $\mathbb{N}$. Sometimes we write $(a_k)_{k\in\mathbb{N}}$ to emphasize it.

If $u = (u_m)_{m\in\mathbb{Z}^n}$ is a bounded sequence, we write $\operatorname{card}(u)$ for the cardinality of $\{m \in \mathbb{Z}^n : u_m \neq 0\}$. Given the sequence $x = (x_m)_{m\in\mathbb{Z}^n} \in \ell_\infty(\mathbb{Z}^n)$, the non-increasing rearrangement of x is defined as

$$x_k^* = \inf\{\|x - u\|_{\ell_\infty(\mathbb{Z}^n)} : u \in \ell_\infty(\mathbb{Z}^n), \operatorname{card}(u) < k\},\ \ k \in \mathbb{N}.$$

For $0 < p < \infty$ and $0 < r \le \infty$, the *Lorentz sequence space* $\ell_{p,r} = \ell_{p,r}(\mathbb{Z}^n)$ consists of all sequences $(x_m)_{m\in\mathbb{Z}^n}$ having a finite quasi-norm

$$\|(x_m)\|_{\ell_{p,r}} = \Big(\sum_{k=1}^{\infty}(k^{1/p}x_k^*)^r k^{-1}\Big)^{1/r}.$$

These spaces are ordered lexicographically, meaning that

$$\ell_{p_0,r_0} \hookrightarrow \ell_{p_1,r_1} \text{ for } p_0 < p_1, 0 < r_0, r_1 \le \infty, \text{ and}$$

$$\ell_{p_0,r_0} \hookrightarrow \ell_{p_1,r_1} \text{ for } p_0 = p_1 \text{ and } r_0 < r_1.$$

Here $\hookrightarrow$ stands for continuous embedding. See, for example, [20, 23]. Note that $\ell_{p,p} = \ell_p$.

Subsequently, if $1 \le p \le \infty$ we put $1/p + 1/p' = 1$, and we set $p' = \infty$ if $0 < p < 1$. Using the interpolation properties of Lorentz spaces and the duality theorems for the real method (see [5, 26, 22]), it is not difficult to check the following duality formula

$$(\ell_{p,r})' = \ell_{p',r'} \text{ if } 1 < p < \infty \text{ and } 0 < r < \infty. \tag{2.1}$$

On the other hand, we know that $(\ell_p)' = \ell_\infty$ for $0 < p \le 1$. If $0 < r \le 1$ and $0 < q < 1$, using the embeddings $\ell_q \hookrightarrow \ell_{1,r} \hookrightarrow \ell_1$ it follows that

$$(\ell_{1,r})' = \ell_\infty \text{ if } 0 < r \le 1. \tag{2.2}$$

For $0 < p < 1$, if we pick $\varepsilon > 0$ such that $0 < \varepsilon < p$, then $\ell_{p-\varepsilon} \hookrightarrow \ell_{p,r} \hookrightarrow \ell_1$. This yields that

$$(\ell_{p,r})' = \ell_\infty \text{ if } 0 < p < 1 \text{ and } 0 < r < \infty. \tag{2.3}$$

In order to characterize the dual of $\ell_{1,r}$ for $1 < r < \infty$, we need the limiting Lorentz space

$$\ell_{\infty,r} = \Big\{(x_m)_{m\in\mathbb{Z}^n} : \|(x_m)\|_{\ell_{\infty,r}} = \Big(\sum_{k=1}^{\infty} x_k^{*r} k^{-1}\Big)^{1/r} < \infty\Big\}.$$

Spaces $\ell_{\infty,r}$ have been considered in connection with limiting operator ideals [14] and limiting approximation spaces [13, 11].

Proposition 2.1. *For $1 < r < \infty$, we have $(\ell_{1,r})' = \ell_{\infty,r'}$.*

Proof. Take any $\mu = (\mu_m) \in \ell_{\infty,r'}$. Then μ defines a bounded linear functional f on $\ell_{1,r}$ assigning to any $\xi = (\xi_m) \in \ell_{1,r}$ the value $f(\xi) = \Sigma_{m\in\mathbb{Z}^n}\xi_m\mu_m$. Indeed, using a property of rearrangements [18, Theorem 363] and the Hölder inequality, we obtain

$$\begin{aligned}|f(\xi)| &\le \sum_{m\in\mathbb{Z}^n} |\xi_m||\mu_m| \le \sum_{k=1}^{\infty} \xi_k^*\mu_k^* \\ &= \sum_{k=1}^{\infty} \big(\xi_k^* k^{1-1/r}\big)\big(\mu_k^* k^{-1/r'}\big) \\ &\le \Big(\sum_{k=1}^{\infty} \big(\xi_k^* k^{1-1/r}\big)^r\Big)^{1/r}\Big(\sum_{k=1}^{\infty} \big(\mu_k^* k^{-1/r'}\big)^{r'}\Big)^{1/r'} \\ &= \Big(\sum_{k=1}^{\infty} \big(\xi_k^* k\big)^r k^{-1}\Big)^{1/r}\Big(\sum_{k=1}^{\infty} \mu_k^{*r'} k^{-1}\Big)^{1/r'} \\ &= \|\xi\|_{\ell_{1,r}}\|\mu\|_{\ell_{\infty,r'}}.\end{aligned}$$

Hence, $f \in (\ell_{1,r})'$ and $\|f\|_{(\ell_{1,r})'} \le \|\mu\|_{\ell_{\infty,r'}}$.

Conversely, take any $f \in (\ell_{1,r})'$ and put $\mu_m = f(e_m)$ where $e_m = (\delta_h^m)_{h\in\mathbb{Z}^n}$ is the sequence with 0 in all coordinates except for the one in m which is 1. We claim that (μ_m) is a null sequence. That is, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for any $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$ with $|m_j| \ge N$ for $j = 1, \dots, n$ we have that $|f(e_m)| \le \varepsilon$. Indeed, if this were not the case, then there would exist $L > 0$ such that $|f(e_{m_j})| > L$ for a certain subsequence $(f(e_{m_j}))$. Let $\sigma_{m_j} \in \mathbb{C}$ with $|\sigma_{m_j}| = 1$ and $\sigma_{m_j} f(e_{m_j}) = f(\sigma_{m_j} e_{m_j}) = |f(e_{m_j})| > L$. For any $K \in \mathbb{N}$, we have

$$\begin{aligned}f\Big(\sum_{j=1}^{K} \frac{\sigma_{m_j}}{j} e_{m_j}\Big) &= \sum_{j=1}^{K} \frac{1}{j}\sigma_{m_j} f(e_{m_j}) \ge L\sum_{j=1}^{K}\frac{1}{j} \ge L\int_1^{K+1}\frac{dx}{x} \\ &= L\log(K+1).\end{aligned}$$

But

$$\begin{aligned}f\Big(\sum_{j=1}^{K} \frac{\sigma_{m_j}}{j} e_{m_j}\Big) &= |f\Big(\sum_{j=1}^{K} \frac{\sigma_{m_j}}{j} e_{m_j}\Big)| \le \|f\|_{(\ell_{1,r})'}\|\sum_{j=1}^{K} \frac{\sigma_{m_j}}{j} e_{m_j}\|_{\ell_{1,r}} \\ &= \|f\|_{(\ell_{1,r})'}\big(\sum_{j=1}^{K}\frac{1}{j}\big)^{1/r} \le \|f\|_{(\ell_{1,r})'}\big(1 + \int_1^K \frac{1}{x}dx\big)^{1/r} \\ &= \|f\|_{(\ell_{1,r})'}(1+\log K)^{1/r}.\end{aligned}$$

This yields $L\log(K+1) \le \|f\|_{(\ell_{1,r})'}(1+\log K)^{1/r}$ for any $K \in \mathbb{N}$ with $r > 1$, which is impossible.

Finally we show that $(f(e_m))$ belongs to $\ell_{\infty,r'}$ with $\|(f(e_m))\|_{\ell_{\infty,r'}} \le \|f\|_{(\ell_{1,r})'}$. Let $(f(e_m)^*) = (\Gamma_k)$ be the decreasing rearrangement of $(f(e_m))$. Note that, since $(f(e_m))$ is a null sequence, for any $k \in \mathbb{N}$ we have that $\Gamma_k = f(e_{\tilde{\mathbf{k}}})$ for some $\tilde{\mathbf{k}} \in \mathbb{Z}^n$. Let $\sigma_k \in \mathbb{C}$ with $|\sigma_k| = 1$ and $\sigma_k \Gamma_k = |\Gamma_k|$. For any $K \in \mathbb{N}$, we have

$$\begin{aligned}\Big(\sum_{k=1}^{K} |\Gamma_k|^{r'} \frac{1}{k}\Big)^{1/r'} &= \Big(\sum_{k=1}^{K} \big(\frac{1}{k^{1/r'}} \sigma_k \Gamma_k\big)^{r'}\Big)^{1/r'} \\ &= \sum_{k=1}^{K} \delta_k \frac{1}{k^{1/r'}} \sigma_k \Gamma_k\end{aligned}$$

where $(\delta_k)_{1\le k\le K}$ is a non-increasing sequence of non-negative numbers with $\sum_{k=1}^{K} \delta_k^r = 1$. Therefore

$$\begin{aligned}\Big(\sum_{k=1}^{K} |\Gamma_k|^{r'} \frac{1}{k}\Big)^{1/r'} &= f\Big(\sum_{k=1}^{K} \delta_k \frac{1}{k^{1/r'}} \sigma_k e_{\tilde{\mathbf{k}}}\Big) \\ &\le \|f\|_{(\ell_{1,r})'} \|\sum_{k=1}^{K} \delta_k \frac{1}{k^{1/r'}} \sigma_k e_{\tilde{\mathbf{k}}}\|_{\ell_{1,r}} \\ &= \|f\|_{(\ell_{1,r})'} \Big(\sum_{k=1}^{K} \big(k\delta_k \frac{1}{k^{1/r'}}\big)^r \frac{1}{k}\Big)^{1/r} \\ &= \|f\|_{(\ell_{1,r})'} \Big(\sum_{k=1}^{K} \delta_k^r\Big)^{1/r} = \|f\|_{(\ell_{1,r})'}.\end{aligned}$$

This yields that $(f(e_m))$ belongs to $\ell_{\infty,r'}$ with

$$\|(f(e_m))\|_{\ell_{\infty,r'}} = \Big(\sum_{k=1}^{\infty} |\Gamma_k|^{r'} \frac{1}{k}\Big)^{1/r'} \le \|f\|_{(\ell_{1,r})'}.$$

□

Let $(A_k)_{k\in\mathbb{N}}$ be a sequence of quasi-Banach spaces. We write c_k for the constant in the quasi-triangle inequality in A_k, and we assume that $\sup_{k\in\mathbb{N}}\{c_k\} < \infty$. Given $0 < q < \infty$, we put $\ell_q(A_k)$ for the collection of all sequences $a = (a_k)_{k\in\mathbb{N}}$ such that $a_k \in A_k$ and the quasi-norm

$$\|a\|_{\ell_q(A_k)} = \big(\sum_{k=1}^{\infty} \|a_k\|_{A_k}^q\big)^{1/q}$$

is finite.

The Banach case of the next result is known, but for completeness we include details.

Lemma 2.2. *Let (A_k) be a sequence of quasi-Banach spaces with $\sup_{k\in\mathbb{N}}\{c_k\} < \infty$, and $0 < q < \infty$. Then*

$$\big(\ell_q(A_k)\big)' = \ell_{q'}(A_k').$$

Proof. Given any $f = (f_k) \in \ell_{q'}(A_k')$, we can define a functional $F : \ell_q(A_k) \longrightarrow \mathbb{C}$ by assigning to any $a = (a_k)$ the number $F(a) = \sum_{k=1}^{\infty} f_k(a_k)$. This series converges because, if $1 \le q < \infty$, using the Hölder inequality we obtain

$$\begin{aligned}\sum_{k=1}^{\infty} |f_k(a_k)| &\le \sum_{k=1}^{\infty} \|f_k\|_{A_k'} \|a_k\|_{A_k} \\ &\le \Big(\sum_{k=1}^{\infty} \|f_k\|_{A_k'}^{q'} \Big)^{1/q'} \Big(\sum_{k=1}^{\infty} \|a_k\|_{A_k}^{q} \Big)^{1/q} \\ &\le \Big(\sum_{k=1}^{\infty} \|f_k\|_{A_k'}^{q'} \Big)^{1/q'} \|a\|_{\ell_q(A_k)}.\end{aligned}$$

Otherwise, if $0 < q < 1$, using that $\ell_q \hookrightarrow \ell_1$ we have

$$\begin{aligned}\sum_{k=1}^{\infty} |f_k(a_k)| &\le \Big(\sum_{k=1}^{\infty} \big(\|f_k\|_{A_k'} \|a_k\|_{A_k} \big)^q \Big)^{1/q} \le \|f\|_{\ell_\infty(A_k')} \big(\sum_{k=1}^{\infty} \|a_k\|_{A_k}^q \big)^{1/q} \\ &= \|f\|_{\ell_\infty(A_k')} \|a\|_{\ell_q(A_k)}.\end{aligned}$$

Therefore, F belongs to $\big(\ell_q(A_k)\big)'$ and $\|F\|_{\left(\ell_q(A_k)\right)'} \le \|f\|_{\ell_{q'}(A_k')}$.

Conversely, given any $F \in \big(\ell_q(A_k)\big)'$, for any $k \in \mathbb{N}$ and $z \in A_k$, let $f_k(z) = F(0, 0, \dots, 0, z, 0, \dots)$ where the vector z stands in the position k in $(0, 0, \dots, 0, z, 0, \dots)$. It is clear that $f_k \in A_k'$. We claim that for any $x = (x_k) \in \ell_q(A_k)$ we have that $F(x) = \sum_{k=1}^{\infty} f_k(x_k)$. Indeed, since $q < \infty$, we have that $x = \lim_{k\to\infty}(x_1, ..., x_k, 0, 0, \dots)$ in $\ell_q(A_k)$. So

$$\begin{aligned}F(x) &= \lim_{k\to\infty} F\big((x_1, \dots, x_k, 0, 0, \dots)\big) = \lim_{k\to\infty} \sum_{j=1}^{k} F(0, 0, \dots, 0, x_j, 0, \dots) \\ &= \lim_{k\to\infty} \sum_{j=1}^{k} f_j(x_j) = \sum_{k=1}^{\infty} f_k(x_k).\end{aligned}$$

Finally, we show that $f = (f_j)$ belongs to $\ell_{q'}(A_k')$ with $\|f\|_{\ell_{q'}(A_k')} \le \|F\|_{\left(\ell_q(A_k)\right)'}$. We start with the case $1 < q < \infty$. Take any $\varepsilon > 0$ and pick $a_k \in A_k$ with $\|a_k\|_{A_k} = 1$ such that $(1 + \varepsilon)|f_k(a_k)| \ge \|f_k\|_{A_k'}$. Replacing a_k by $\sigma_k a_k$ with some $\sigma_k \in \mathbb{C}$ with $|\sigma_k| = 1$, we may assume that $f_k(a_k) = |f_k(a_k)|$. Let now (μ_k) be any sequence of positive numbers with $\sum_{k=1}^{\infty} \mu_k^q = 1$. We have

$$\begin{aligned}\sum_{k=1}^{\infty} \|f_k\|_{A_k'} \mu_k &\le (1 + \varepsilon) \sum_{k=1}^{\infty} f_k(a_k)\mu_k = (1 + \varepsilon) \sum_{k=1}^{\infty} f_k(\mu_k a_k) \\ &\le (1 + \varepsilon) |F\big((\mu_k a_k)\big)| \\ &\le (1 + \varepsilon) \|F\|_{\left(\ell_q(A_k)\right)'} \|(\mu_k a_k)\|_{\ell_q(A_k)} \\ &= (1 + \varepsilon) \|F\|_{\left(\ell_q(A_k)\right)'}.\end{aligned}$$

This yields that

$$\|f\|_{\ell_{q'}(A_k')} = \Big(\sum_{k=1}^{\infty} \|f_k\|_{A_k'}^{q'} \Big)^{1/q'} \le \|F\|_{\left(\ell_q(A_k)\right)'}.$$

Assume now that $0 < q < 1$. Given any $\varepsilon > 0$, proceeding as before we can pick $a_k \in A_k$ such that $(1+\varepsilon)f_k(a_k) \geq \|f_k\|_{A'_k}$ and $\|a_k\|_{A_k} = 1$. Then

$$\begin{aligned}\|f_k\|_{A'_k} &\leq (1+\varepsilon)f_k(a_k) = (1+\varepsilon)F(0,\dots,0,a_k,0,\dots)\\ &\leq (1+\varepsilon)\|F\|_{\left(\ell_q(A_k)\right)'}\|a_k\|_{A_k}\\ &= (1+\varepsilon)\|F\|_{\left(\ell_q(A_k)\right)'}.\end{aligned}$$

Whence,

$$\|f\|_{\ell_\infty(A'_k)} = \sup_{k\in\mathbb{N}} \|f_k\|_{A'_k} \leq \|F\|_{\left(\ell_q(A_k)\right)'}.$$

□

3 Spaces of Lorentz-Sobolev type

We put $\mathcal{S}(\mathbb{R}^n)$ for the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ for its topological dual, the space of all tempered distributions. For $f \in \mathcal{S}'(\mathbb{R}^n)$, we write $\hat{f}$ for the Fourier transform of f, and $f^\vee$ for the inverse Fourier transform.

Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\varphi_0(x) = 1 \text{ if } |x| \leq 1 \text{ and } \varphi_0(x) = 0 \text{ if } |x| \geq 3/2,$$

where $|x|$ stands for the norm in $\mathbb{R}^n$. For $k \in \mathbb{N}$, let $\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x)$. Since $\sum_{k=0}^\infty \varphi_k(x) = 1$ for every $x \in \mathbb{R}^n$, the sequence $(\varphi_k)_{k\in\mathbb{N}_0}$ is a smooth dyadic resolution of unity. Here $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Let $0 < p < \infty$ and $0 < r \leq \infty$. The *Lorentz space* $L_{p,q}(\mathbb{R}^n)$ consists of all (equivalence classes of) measurable functions from $\mathbb{R}^n$ into $\mathbb{C}$ which have a finite quasi-norm

$$\|f\|_{L_{p,r}(\mathbb{R}^n)} = \Big(\int_0^\infty [t^{1/p}f^*(t)]^r \frac{dt}{t}\Big)^{1/r}$$

(the integral should be replaced by the supremum if $r = \infty$). Here f^* is the non-increasing rearrangement of f. See [5, 4, 15] for properties of Lorentz spaces $L_{p,r}(\mathbb{R}^n)$. If $p = r$, then $L_{p,p}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$.

Let $s \in \mathbb{R}, 0 < p < \infty$ and $0 < q, r \leq \infty$. The *Besov-Lorentz space* $B^s_q L_{p,r}(\mathbb{R}^n)$ collects all $f \in \mathcal{S}'(\mathbb{R}^n)$ having a finite quasi-norm

$$\|f\|_{B^s_q L_{p,r}(\mathbb{R}^n)} = \left(\sum_{k=0}^\infty 2^{ksq}\|(\varphi_k\hat{f})^\vee\|^q_{L_{p,r}(\mathbb{R}^n)}\right)^{1/q}$$

(with the usual modification if $q = \infty$).

The *Triebel-Lizorkin-Lorentz space* $F^s_q L_{p,r}(\mathbb{R}^n)$ is formed by all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that the quasi-norm

$$\|f\|_{F^s_q L_{p,r}(\mathbb{R}^n)} = \left\| \left(\sum_{k=0}^{\infty} 2^{ksq} |(\varphi_k \hat{f})^\vee(\cdot)|^q \right)^{1/q} \right\|_{L_{p,r}(\mathbb{R}^n)}$$

is finite.

We refer to [7, 8, 6, 12] and the papers cited there for properties of Lorentz-Sobolev spaces. When $p = r$, then $B^s_q L_{p,p}(\mathbb{R}^n)$ is the classical Besov space $B^s_{p,q}(\mathbb{R}^n)$, and $F^s_q L_{p,p}(\mathbb{R}^n)$ coincides with the classical Triebel-Lizorkin space $F^s_{p,q}(\mathbb{R}^n)$.

By [6, Proposition 2.3], we have

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^s_q L_{p,r}(\mathbb{R}^n),\ F^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

In fact, according to [12, Lemma 3.2], if $0 < p, q, r < \infty$, then $\mathcal{S}(\mathbb{R}^n)$ is dense in $B^s_q L_{p,r}(\mathbb{R}^n)$ and in $F^s_q L_{p,r}(\mathbb{R}^n)$.

It is well-known that classical Besov spaces (and also Triebel-Lizorkin spaces) can be represented via wavelets (see, for example, [29, Theorem 1.20]). For Besov-Lorentz spaces the characterization was established by Almeida [1, Corollary 3.2] when $q = r$ and by Besoy and Cobos [6, Theorem 4.4] for the general case. Next we recall the result.

For $L \in \mathbb{N}$, we designate by $C^L(\mathbb{R})$ the space of complex-valued continuous functions on $\mathbb{R}$ with continuous bounded derivatives up to order L (inclusive),

Let $\psi_F, \psi_M \in C^L(\mathbb{R})$ be real-valued compactly supported functions with

$$\int_{\mathbb{R}} \psi_F(t)^2 dt = 1,\ \int_{\mathbb{R}} \psi_M(t)^2 dt = 1 \text{ and } \int_{\mathbb{R}} \psi_M(t) t^\ell dt = 0 \text{ for all } \ell \in \mathbb{N}_0 \text{ with } \ell < L.$$

To extend these wavelets to $\mathbb{R}^n$, let

$$G = (G_1, ..., G_n) \in G^0 = \{F, M\}^n,$$

which means that each G_ℓ is either F or M. For $j \in \mathbb{N}$, let

$$G = (G_1, ..., G_n) \in G^j = \{F, M\}^{n*}$$

which means that each G_ℓ is either F or M but at least one of the components of G must be M. So, G^0 has 2^n elements and G^j with $j \in \mathbb{N}$ has $2^n - 1$ elements. Put

$$\psi^j_{G,m}(x) = 2^{jn/2} \prod_{\ell=1}^{n} \psi_{G_\ell}(2^j x_\ell - m_\ell), \quad G \in G^j,\ m \in \mathbb{Z}^n,\ j \in \mathbb{N}_0. \tag{3.1}$$

Then $\{\psi^j_{G,m} : G \in G^j, m \in \mathbb{Z}^n, j \in \mathbb{N}_0\}$ is called a *wavelet system.*

For $-\infty < s < \infty, 0 < p < \infty$ and $0 < q, r \leq \infty$, the space $b^s_q \ell_{p,r}$ is formed by all sequences $(\mu^{j,G}_m) \subset \mathbb{C}$ having a finite quasi-norm

$$\|(\mu^{j,G}_m)\|_{b^s_q \ell_{p,r}} = \Big(\sum_{j=0}^{\infty} 2^{j(s-n/p)q} \sum_{G \in G^j} \|\mu^{j,G}\|^q_{\ell_{p,r}(\mathbb{Z}^n)} \Big)^{1/q}$$

where $\mu^{j,G} = (\mu_m^{j,G})_{m\in\mathbb{Z}^n}$.

The characterization of $B_q^s L_{p,r}(\mathbb{R}^n)$ in terms of wavelets is as follows:

Theorem 3.1. *Let $-\infty < s < \infty$, $0 < p < \infty$, $0 < q, r \le \infty$, and let $\psi_{G,m}^j$ be the wavelets in (3.1).*

Assume that

$$L > \max\{s, n\max\{\frac{1}{\min\{p,r\}} - 1, 0\} - s, \frac{n}{p} - s\}. \tag{3.2}$$

Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B_q^s L_{p,r}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f(x) = \sum_{j=0}^{\infty} \sum_{G\in G^j} \sum_{m\in\mathbb{Z}^n} \lambda_m^{j,G} 2^{-jn/2} \psi_{G,m}^j(x), \quad (\lambda_m^{j,G}) \in b_q^s \ell_{p,r} \tag{3.3}$$

unconditional convergence being in $\mathcal{S}'(\mathbb{R}^n)$. The representation (3.3) is unique,

$$\lambda_m^{j,G} = \lambda_m^{j,G}(f) = 2^{jn/2}\langle f, \psi_{G,m}^j\rangle$$

and the operator

$$I : f \longrightarrow (2^{jn/2}\langle f, \psi_{G,m}^j\rangle)_{j\in\mathbb{N}_0, G\in G^j, m\in\mathbb{Z}^n} \tag{3.4}$$

is an isomorphism from $B_q^s L_{p,r}(\mathbb{R}^n)$ onto $b_q^s \ell_{p,r}$.

See [6, Theorem 4.4] for the proof.

4 Duality

We continue using notation introduced in the previous sections. It is going to be useful to complement the scale of the sequence spaces $b_q^s \ell_{p,r}$ with spaces $b_q^s \ell_{\infty,u}$ formed by all sequences $(\mu_m^{j,G}) \subset \mathbb{C}$ such that

$$\|(\mu_m^{j,G})\|_{b_q^s\ell_{\infty,u}} = \Big(\sum_{j=0}^{\infty} 2^{jsq} \sum_{G\in G^j} \|\mu^{j,G}\|_{\ell_{\infty,u}(\mathbb{Z}^n)}^q\Big)^{1/q} < \infty.$$

Here $-\infty < s < \infty, 0 < q < \infty$ and $1 < u < \infty$.

Having in mind Theorem 3.1, it is natural to associate $b_q^s \ell_{\infty,u}$ with the Besov space $B_q^s L_{\infty,u}(\mathbb{R}^n)$ formed by all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$f(x) = \sum_{j=0}^{\infty} \sum_{G\in G^j} \sum_{m\in\mathbb{Z}^n} \lambda_m^{j,G} 2^{-jn/2} \psi_{G,m}^j(x), \text{ with } (\lambda_m^{j,G}) \in b_q^s \ell_{\infty,u}.$$

We put

$$\|f\|_{B_q^s L_{\infty,u}(\mathbb{R}^n)} = \|(\lambda_m^{j,G})\|_{b_q^s \ell_{\infty,u}}.$$

Next we describe the dual space of spaces $B_q^s L_{p,r}(\mathbb{R}^n)$. As we pointed out, $\mathcal{S}(\mathbb{R}^n) \hookrightarrow B_q^s L_{p,r}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ is dense in $B_q^s L_{p,r}(\mathbb{R}^n)$ provided that $0 < p, r, q < \infty$

(see [12, Lemma 3.2]). Therefore, any continuous linear functional on $B^s_q L_{p,r}(\mathbb{R}^n)$ can be interpreted as an element of $\mathcal{S}'(\mathbb{R}^n)$. In fact, the necessary and sufficient condition for $g \in \mathcal{S}'(\mathbb{R}^n)$ to belong to $\big(B^s_q L_{p,r}(\mathbb{R}^n)\big)'$ is that there is $M > 0$ such that

$$|g(\varphi)| \le M \|\varphi\|_{B^s_q L_{p,r}(\mathbb{R}^n)} \text{ for all } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

As before, we put $1/p + 1/p' = 1$ if $1 < p < \infty$ and we write $p' = \infty$ if $0 < p \le 1$.

Theorem 4.1. *Let* $-\infty < s < \infty$ *and* $0 < q < \infty$. *Then we have with equivalence of norms*

$$\big(B^s_q L_{p,r}(\mathbb{R}^n)\big)' = \begin{cases} B^{-s}_{q'} L_{p',r'}(\mathbb{R}^n) & \text{if } 1 < p < \infty \text{ and } 0 < r < \infty, \\ B^{-s+n(1/p-1)}_{\infty,q'}(\mathbb{R}^n) & \text{if either } 0 < p < 1 \text{ and } 0 < r < \infty, \\ & \text{or } p = 1 \text{ and } 0 < r \le 1, \\ B^{-s}_{q'} L_{\infty,r'}(\mathbb{R}^n) & \text{if } p = 1 \text{ and } 1 < r < \infty. \end{cases}$$

Proof. Let $L \in \mathbb{N}$ satisfying (3.2), consider a wavelet system $\{\psi^j_{G,m}\}_{G \in G^j, m \in \mathbb{Z}^n, j \in \mathbb{N}_0}$ in (3.1) and let I be the isomorphism (3.4) between $B^s_q L_{p,r}(\mathbb{R}^n)$ and $b^s_q \ell_{p,r}$. It is useful to describe $b^s_q \ell_{p,r}$ in a slightly different way.

If X is a quasi-Banach space, we write $\ell^s_q(X) = \ell_q(2^{ks}X)$ for the collection of all sequences $(x_k)_{k \in \mathbb{N}} \subseteq X$ such that

$$\|(x_k)\|_{\ell^s_q(X)} = \|(x_k)\|_{\ell_q(2^{ks}X)} = \Big(\sum_{k=1}^{\infty} 2^{ksq} \|x_k\|^q_X\Big)^{1/q} < \infty.$$

Let $< \ell_{p,r} >^0$ be the direct sum of 2^n copies of $\ell_{p,r}(\mathbb{Z}^n)$ and for $j \in \mathbb{N}$ let $< \ell_{p,r} >^j = < \ell_{p,r} >$ be the direct sum of $2^n - 1$ copies of $\ell_{p,r}(\mathbb{Z}^n)$. Note that the number of copies of $\ell_{p,r}(\mathbb{Z}^n)$ in $< \ell_{p,r} >^k$ corresponds to the number of elements of G^k. Then we have with equivalence of quasi-norms

$$b^s_q \ell_{p,r} = < \ell_{p,r} >^0 \oplus \ell^{s-n/p}_q (< \ell_{p,r} >).$$

Therefore,

$$I : B^s_q L_{p,r}(\mathbb{R}^n) \longrightarrow < \ell_{p,r} >^0 \oplus \ell^{s-n/p}_q (< \ell_{p,r} >)$$

is an isomorphism. Next we determine the dual of the sequence spaces with the help of the results of Section 3.

By Lemma 2.2, we know that

$$\big(\ell^s_q(X)\big)' = \big(\ell_q(2^{ks}X)\big)' = \ell_{q'}(2^{-ks}X') = \ell^{-s}_{q'}(X').$$

Hence

$$\big(b^s_q \ell_{p,r}\big)' = \Big(< \ell_{p,r} >^0 \oplus \ell^{s-n/p}_q (< \ell_{p,r} >)\Big)' = < (\ell_{p,r})' >^0 \oplus \ell^{-s+n/p}_{q'} (< (\ell_{p,r})' >).$$

Using Proposition 2.1 and the other duality formulae for spaces $\ell_{p,q}$, we obtain

$$\left(b_q^s\ell_{p,r}\right)' = \begin{cases} b_{q'}^{-s+n}\ell_{p',r'} & \text{if } 1<p<\infty, 0<r<\infty, \\ b_{q'}^{-s+n/p}\ell_\infty & \text{if either } 0<p<1, 0<r<\infty, \text{ or } p=1, 0<r\le 1, \\ b_{q'}^{-s+n}\ell_{\infty,r'} & \text{if } p=1, 1<r<\infty. \end{cases} \tag{4.1}$$

The dual operator $\left(I^{-1}\right)'$ is an isomorphism from $\left(B_q^sL_{p,r}(\mathbb{R}^n)\right)'$ onto $(b_q^s\ell_{p,r})'$. Let us determine $\left(I^{-1}\right)'$. For any $g \in \left(B_q^sL_{p,r}(\mathbb{R}^n)\right)'$ and $(\lambda_m^{j,G}) \in b_q^s\ell_{p,r}$, we have

$$\begin{aligned} \langle \left(I^{-1}\right)'g, (\lambda_m^{j,G})\rangle &= \langle g, I^{-1}(\lambda_m^{j,G})\rangle \\ &= \langle g, \sum_{j,G,m} 2^{-jn/2}\lambda_m^{j,G}\psi_{G,m}^j\rangle \\ &= \sum_{j,G,m} 2^{-jn/2}\lambda_m^{j,G}\langle g, \psi_{G,m}^j\rangle. \end{aligned}$$

Therefore,

$$\left(I^{-1}\right)'g = \left(2^{-jn/2}\langle g, \psi_{G,m}^j\rangle\right).$$

Besides, $J(\mu_m^{j,G}) = (2^{jn}\mu_m^{j,G})$ is an isomorphism from $b_v^u\ell_{w,z}$ onto $b_v^{u-n}\ell_{w,z}$. Consequently, the operator

$$\left(J\circ\left(I^{-1}\right)'\right)(g) = \left(2^{jn/2}\langle g, \psi_{G,m}^j\rangle\right)$$

is an isomorphism from $\left(B_q^sL_{p,r}(\mathbb{R}^n)\right)'$ onto $b_v^u\ell_{w,z}$ where

$$b_v^u\ell_{w,z} = \begin{cases} b_{q'}^{-s}\ell_{p',r'} & \text{if } 1<p<\infty, 0<r<\infty, \\ b_{q'}^{-s+n(1/p-1)}\ell_\infty & \text{if either } 0<p<1, 0<r<\infty, \text{ or } p=1, 0<r\le 1, \\ b_{q'}^{-s}\ell_{\infty,r'} & \text{if } p=1, 1<r<\infty. \end{cases}$$

But $\left(J\circ\left(I^{-1}\right)'\right)$ coincides with the operator I and, by Theorem 3.1, $I: B_v^uL_{w,z}(\mathbb{R}^n) \longrightarrow b_v^u\ell_{w,z}$ is an isomorphism. This completes the proof.

□

As for Triebel-Lizorkin-Lorentz spaces, it is shown in [12, Theorems 3.3 and 3.6] that for $-\infty<s<\infty$ we have

$$\left(F_q^sL_{p,r}(\mathbb{R}^n)\right)' = \begin{cases} F_{q'}^{-s}L_{p',r'}(\mathbb{R}^n) & \text{if } 1<p,q<\infty \text{ and } 0<r<\infty, \\ B_{\infty,\infty}^{-s+n(1/p-1)}(\mathbb{R}^n) & \text{if } 0<p<1, p\le r\le 1 \text{ and } 0<q<\infty, \\ & \text{or } p=1, 0<q\le 1 \text{ and } 0<r\le 1. \end{cases}$$

We finish the paper by using Theorem 4.1 to cover one more case.

Corollary 4.2. *Let $-\infty<s<\infty, 0<p<1$ and $0<r,q\le 1$. Then we have with equivalence of norms*

$$\left(F_q^sL_{p,r}(\mathbb{R}^n)\right)' = B_{\infty,\infty}^{-s+n(1/p-1)}(\mathbb{R}^n)$$

Proof. By [24, Theorem 1.1], we have that

$$B^s_{\min\{p,q,r\}} L_{p,r}(\mathbb{R}^n) \hookrightarrow F^s_q L_{p,r}(\mathbb{R}^n)$$

except if $p = q < r$, but in this case

$$B^s_{p,p}(\mathbb{R}^n) = F^s_{p,p}(\mathbb{R}^n) \hookrightarrow F^s_p L_{p,r}(\mathbb{R}^n).$$

On the other hand, by [24, Theorem 1.2], we get

$$F^s_q L_{p,r}(\mathbb{R}^n) \hookrightarrow B^s_1 L_{p,1}(\mathbb{R}^n).$$

According to Theorem 4.1, we know that

$$\big(B^s_{\min\{p,q,r\}} L_{p,r}(\mathbb{R}^n)\big)' = \big(B^s_1 L_{p,1}(\mathbb{R}^n)\big)' = B^{-s+n(1/p-1)}_{\infty,\infty}(\mathbb{R}^n).$$

Moreover, see [27, Theorem 2.11.3], $\big(B^s_{p,p}(\mathbb{R}^n)\big)' = B^{-s+n(1/p-1)}_{\infty,\infty}(\mathbb{R}^n)$. Therefore $\big(F^s_q L_{p,r}(\mathbb{R}^n)\big)' = B^{-s+n(1/p-1)}_{\infty,\infty}(\mathbb{R}^n)$.

□

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References

[1] A. Almeida, On interpolation properties of generalized Besov spaces. In: *Further Progress in Analysis*, pp. 601–610. Word Scientific, London (2009).

[2] A. Almeida, A.M. Caetano, Generalized Hardy spaces, Acta Math. Sin. (Engl. Ser.) **26** (2010) 1673–1692.

[3] A. Almeida, A.M. Caetano, Real interpolation of generalized Besov-Hardy spaces and applications, J. Fourier Anal. Appl. **17** (2011) 691–719.

[4] C. Bennett, R. Sharpley, *Interpolation of Operators*, Academic Press, Boston, 1988.

[5] J. Bergh, J. Löfström, *Interpolation Spaces. An introduction*, Springer, Berlin, 1976.

[6] B.F. Besoy, F. Cobos, Function spaces of Lorentz-Sobolev type: Atomic decompositions, characterizations in terms of wavelets, interpolation and multiplications, J. Funct. Anal. **282** (2022) 109452.

[7] B.F. Besoy, F. Cobos, H. Triebel, On function spaces of Lorentz-Sobolev type, Math. Ann. **381** (2021) 807–839.

[8] B.F. Besoy, D.D. Haroske, H. Triebel, Traces of some weighted function spaces and related non-standard real interpolation of Besov spaces, Math. Nachr. **295** (2022) 1669–1689.

[9] A.M. Caetano, Subatomic representation of Bessel potential spaces modelled on Lorentz spaces. In: Textos de Matemática, Série B, 34, “The J.A. Sampaio Martins Anniversary Volume”, Coimbra, 2004, 37–47.

[10] A. Cianchi, L. Pick, An optimal endpoint trace embedding, Ann. Inst. Fourier **60** (2010) 939–951.

[11] F. Cobos, O. Domínguez, T. Kühn, Approximation and entropy numbers of embeddings between approximation spaces, Constr. Approx. **47** (2018) 453–486.

[12] F. Cobos, L.M. Fernández-Cabrera, T. Kühn, Lorentz-Sobolev spaces that are multiplication algebras, J. Fourier Anal. Appl. **32** (2026) 17.

[13] F. Cobos, M. Milman, On a limit class of approximation spaces, Numer. Funct. Anal. Optim. **11** (1990) 11–31.

[14] F. Cobos, I. Resina, Representation theorems for some operator ideals, J. London Math. Soc. **39** (1989) 324–334.

[15] D.E. Edmunds, W.D. Evans, *Hardy operators, function spaces and embeddings*, Springer, Berlin, 2004.

[16] D.E. Edmunds, H. Triebel, Function Spaces, Entropy Numbers and Differential Operators, Cambridge Univ. Press, Cambridge, 1996.

[17] C. Fefferman, N.M. Riviere, Y. Sagher, Interpolation between H^p spaces: The real method, Trans. Amer. Math. Soc. bf 191 (1974), 75–81.

[18] G.H. Hardy, J.E. Littlewood, G. Pólya, *Inequalities*, Cambridge Univ. Press, Cambridge, 1978.

[19] P. Hobus, J. Saal, Triebel-Lizorkin-Lorentz spaces and the Navier-Stokes equations, Z. Anal. Anwendungen **38** (2019) 41–72.

[20] H. König,, *Eigenvalue Distribution of Compact Operators*, Birkhäuser, Basel, 1986.

[21] J. Peetre, *New Thoughts on Besov Spaces*, Duke University of Mathematical Series, Duke University, Durham, 1976.

[22] J. Peetre, Remark on the dual of an interpolation spaces, Math. Scand. **34** (1974) 124–128.

[23] A. Pietsch, *Eigenvalues and s-Numbers*, Cambridge Univ. Press, Cambridge, 1987.

[24] A. Seeger, W. Trebels, Embeddings for spaces of Lorentz-Sobolev type, Math. Ann. **373** (2019) 1017–1056.

[25] E.M. Stein, Editors note: the differentiability of functions in $\mathbb{R}^n$, Ann. of Math. **113** (1981) 383–385.

[26] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam, 1978.

[27] H. Triebel, *Theory of Function Spaces*, Birkhäuser, Basel, 1983.

[28] H. Triebel, *Theory of Function Spaces II*, Birkhäuser, Basel, 1992.

[29] H. Triebel, *Function Spaces and Wavelets on Domains*, European Mathematical Society, Freiburg, 2008.

[30] Z. Xiang, W. Yan, On the well-posedness of the quasi-geostrophic equation in the Triebel-Lizorkin-Lorentz spaces, J. Evol. Equ. **11** (2011) 241–263.

[31] Z. Xiang, W. Yan, On the well-posedness of Boussinesq equation in the Triebel-Lizorkin-Lorentz spaces, Abstract and Applied Analysis (2012), article ID 573084, 17p.

[32] Q. Yang, Z. Cheng, L. Peng, Uniform characterization of function spaces by wavelets, Acta Math. Sci. Ser. A (Chin. Ed.) **25** (2005) 130–144.