Limiting K- and J-spaces in the real interpolation, their relationship and duals

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K-functional and J-functional

Definition

Let (X_0, X_1) be a compatible couple.

(i) The Peetre K-functional is defined for each $f \in X_0 + X_1$ and $t > 0$ by

$$K(f, t; X_0, X_1) := \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1\},$$

where the infimum extends over all representations $f = f_0 + f_1$ of f with $f_0 \in X_0$ and $f_1 \in X_1$.

(ii) The Peetre J-functional is defined for each $f \in X_0 \cap X_1$ and $t > 0$ by

$$J(f, t; X_0, X_1) := \max\{\|f\|_{X_0}, t\,\|f\|_{X_1}\}.$$

K-spaces

Definition

Let (X_0, X_1) be a compatible couple. For $0 \le \theta \le 1$, $1 \le q \le \infty$, and $v \in \mathcal{W}(0,\infty)$, we put

$$(X_0, X_1)_{\theta,q,v;K} := \{f \in X_0 + X_1 : \|f\|_{\theta,q,v;K} < \infty\},$$

where

$$\|f\|_{\theta,q,v;K} \equiv \|f\|_{(X_0,X_1)_{\theta,q,v;K}} := \left\| t^{-\theta-1/q}\, v(t)\, K(f,t;X_0,X_1) \right\|_{q;(0,\infty)}.$$

Remark

If $v = 1$ on $(0,\infty)$ in this definition, then we obtain the classical space $(X_0, X_1)_{\theta,q;K}$.

Modifications:

Replacing $(0,\infty)$ by $(0,1)$ or by $(1,\infty)$, we get the spaces $(X_0, X_1)_{\theta,q,v;K;(0,1)}$ equip with $\left\| t^{-\theta-1/q}\, v(t)\, K(f,t;X_0,X_1) \right\|_{q;(0,1)}$ or $(X_0, X_1)_{\theta,q,v;K;(1,\infty)}$ equip with $\left\| t^{-\theta-1/q}\, v(t)\, K(f,t;X_0,X_1) \right\|_{q;(1,\infty)}$.

J-spaces

Definition

Let (X_0, X_1) be a compatible couple, $0 \le \theta \le 1$, $1 \le q \le \infty$, and let $v \in \mathcal{W}(0,\infty)$. The space

$$(X_0, X_1)_{\theta,q,v;J}$$

consists of all $f \in X_0 + X_1$ for which there is a strongly measurable function

$$u : (0,\infty) \to X_0 \cap X_1$$

such that

$$f = \int_0^\infty u(s)\,\frac{ds}{s} \quad (\mathit{convergence\ in}\ X_0 + X_1) \tag{1}$$

and for which the functional

$$\|f\|_{\theta,q,v;J} \equiv \|f\|_{(X_0,X_1)_{\theta,q,v;J}} := \inf \left\| t^{-\theta-1/q}\, v(t)\, J(u(t), t; X_0, X_1) \right\|_{q;(0,\infty)}$$

is finite (the infimum extends over all representations (1) of f).

Modifications: $(X_0, X_1)_{\theta,q,v;J;(0,1)}$ and $(X_0, X_1)_{\theta,q,v;J;(1,\infty)}$.

Slowly varying functions

Definition (Slowly varying functions)

$b \in SV(0,+\infty) \iff b \in \mathcal{M}^+(0,+\infty)$, b is finite on $(0,+\infty)$, and $\forall \varepsilon > 0 \quad \exists\ h_\varepsilon \in \mathcal{M}^+(0,+\infty;\uparrow), \quad \exists\ h_{-\varepsilon} \in \mathcal{M}^+(0,+\infty;\downarrow)$ such that

$$t^\varepsilon b(t) \approx h_\varepsilon(t) \ \text{ and } \quad t^{-\varepsilon} b(t) \approx h_{-\varepsilon}(t) \quad \forall\ t > 0$$

Examples

$$\ell(t) := 1 + |\log t|, \quad t > 0$$

$$\ell_1 := \ell, \qquad \ell_i := \ell_1 \circ \ell_{i-1}, \quad i > 1,\ i \in \mathbb{N}$$

1. $b(t) = \boldsymbol{\ell}^{\boldsymbol{\alpha}}(t) := \prod_{i=1}^m \ell_i^{\alpha_i}(t), \quad t > 0, \quad m \in \mathbb{N},$ $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$
2. $b(t) = \exp\left(|\log t|^\beta\right), \quad t > 0, \quad \beta \in (0,1)$

Properties of K-spaces

Theorem

Let (X_0, X_1) *and* (Y_0, Y_1) *be compatible couples,* $0 \le \theta \le 1$, $1 \le q \le \infty$, *and let* $v \in \mathcal{W}(0,\infty)$.

A. *If*

$$\left\| t^{-\theta-1/q}\, v(t) \min\{1, t\} \right\|_{q,(0,\infty)} < \infty, \tag{2}$$

then:

(i) *The space* $(X_0, X_1)_{\theta,q,v;K}$ *is an intermediate space between* X_0 *and* X_1, *that is,*

$$X_0 \cap X_1 \hookrightarrow (X_0, X_1)_{\theta,q,v;K} \hookrightarrow X_0 + X_1.$$

(ii) *The space* $(X_0, X_1)_{\theta,q,v;K}$ *is a Banach space.*

B. *If condition* (2) *is not satisfied, then* $(X_0, X_1)_{\theta,q,v;K} = \{0\}$.

C. *Moreover, if* $v \in SV(0,\infty)$ *satisfies* (2), *then* $(X_0, X_1)_{\theta,q,v;K}$ *and* $(Y_0, Y_1)_{\theta,q,v;K}$ *are interpolation spaces with respect to couples* (X_0, X_1) *and* (Y_0, Y_1).

Properties of J-spaces

Theorem

Let (X_0, X_1) *and* (Y_0, Y_1) *are compatible couples,* $0 \le \theta \le 1$, $1 \le q \le \infty$, *and let* $v \in \mathcal{W}(0,\infty)$.

A. *If*

$$\left\| t^{\theta - 1/q'} \frac{1}{v(t)} \min\left\{1, \frac{1}{t}\right\} \right\|_{q',(0,\infty)} < \infty, \quad \left(\frac{1}{q} + \frac{1}{q'} = 1\right) \tag{3}$$

then:

(i) *The space* $(X_0, X_1)_{\theta,q,v;J}$ *is an intermediate space between* X_0 *and* X_1, *that is,*

$$X_0 \cap X_1 \hookrightarrow (X_0, X_1)_{\theta,q,v;J} \hookrightarrow X_0 + X_1.$$

(ii) *The space* $(X_0, X_1)_{\theta,q,v;J}$ *is a Banach space.*

B. **(iii)** *If condition (3) is not satisfied, then the functional* $\|.\|_{\theta,q,v;J}$ *vanishes on* $X_0 \cap X_1$ *and thus it is not a norm provided that* $X_0 \cap X_1 \neq \{0\}$.

C. *Moreover, if* $v \in SV(0,\infty)$ *satisfies* (3), *then* $(X_0, X_1)_{\theta,q,v;J}$ *and* $(Y_0, Y_1)_{\theta,q,v;J}$ *are interpolation spaces with respect to couples* (X_0, X_1) *and* (Y_0, Y_1).

Main results for K-spaces

Theorem

Let (X_0, X_1) *be a compatible couple and* $1 \leq q < \infty$*. If* $b \in SV(0,\infty)$ *satisfies*

$$\int_x^\infty t^{-1} b^q(t)\, dt < \infty \quad \textit{for all} \;\; x > 0, \qquad \int_0^\infty t^{-1} b^q(t)\, dt = \infty, \tag{4}$$

and $a \in SV(0,\infty)$ *is defined by*

$$a(x) := b^{-q/q'}(x) \int_x^\infty t^{-1} b^q(t)\, dt \quad \textit{for all} \;\; x > 0, \tag{5}$$

then

$$(X_0, X_1)_{0,q,b;K} = (X_0, X_1)_{0,q,a;J},$$

the space $X_0 \cap X_1$ *is dense in* $(X_0, X_1)_{0,q,b;K}$.

Moreover, if $1 < q < \infty$ *and* $X_0 \cap X_1$ *is dense in* X_0 *and in* X_1*, then*

$$(X_0, X_1)'_{0,q,b;K} = (X'_0, X'_1)_{0,q',\tilde{b};J} = (X'_0, X'_1)_{0,q',\tilde{a};K},$$

where $\tilde{b}(x) := \frac{1}{b(1/x)}$ *and* $\tilde{a}(x) := \frac{1}{a(1/x)}$ *for all* $x > 0$.

Theorem

Let (X_0, X_1) *be a compatible couple and* $1 \le q < \infty$. *If* $b \in SV(0,\infty)$ *satisfies*

$$\int_0^\infty t^{-1} b^q(t)\, dt < \infty, \tag{6}$$

the function $B \in SV(0,\infty)$ *is given by*

$$B(x) := b(x) \ \ \text{if } x \in [1,\infty), \qquad B(x) \approx \beta(x) \ \ \text{if } x \in (0,1), \tag{7}$$

where $\beta \in SV(0,\infty)$ *is such that*

$$\int_x^\infty t^{-1}\beta^q(t)\, dt < \infty \quad \text{for all } x > 0, \qquad \int_0^\infty t^{-1}\beta^q(t)\, dt = \infty,$$

and

$$A(x) := B^{-q/q'}(x) \int_x^\infty t^{-1} B^q(t)\, dt \quad \text{for all } x > 0, \tag{8}$$

then

$$\begin{aligned}(X_0, X_1)_{0,q,b;K} &= (X_0, X_0 + X_1)_{0,q,B;K} = (X_0, X_0 + X_1)_{0,q,A;J} \\ &= X_0 + (X_0, X_1)_{0,q,B;K} = X_0 + (X_0, X_1)_{0,q,A;J},\end{aligned}$$

the space X_0 *is dense in* $(X_0, X_1)_{0,q,b;K}$,

$$(X_0, X_1)_{0,q,b;K} = (X_0, X_1)_{0,q,b;K;(1,\infty)},$$

$$(X_0, X_0 + X_1)_{0,q,B;K} = (X_0, X_0 + X_1)_{0,q,B;K;(1,\infty)},$$

$$(X_0, X_0 + X_1)_{0,q,A;J} = (X_0, X_0 + X_1)_{0,q,A;J;(1,\infty)}.$$

Theorem

If the assumptions of the previous theorem are satisfied, $1 < q < \infty$*, and* $X_0 \cap X_1$ *is dense in* X_0 *and in* X_1*, then*

$$\begin{aligned}(X_0, X_1)'_{0,q,b;K} &= (X'_0, X'_0 \cap X'_1)_{0,q',\tilde{B};J} = (X'_0, X'_0 \cap X'_1)_{0,q',\tilde{A};K}\\ &= X'_0 \cap (X'_0, X'_1)_{0,q',\tilde{B};J} = X'_0 \cap (X'_0, X'_1)_{0,q',\tilde{A};K}\\ &= (X'_0, X'_1)_{0,q',\tilde{b};J},\end{aligned}$$

where $\tilde{b}(x) := \frac{1}{b(1/x)}$, $\tilde{B}(x) := \frac{1}{B(1/x)}$ *and* $\tilde{A}(x) := \frac{1}{A(1/x)}$ *for all* $x > 0$*.*
Moreover,

$$(X_0, X_1)_{0,q,b;K} = (X_0, X_1)_{0,q,b;K;(1,\infty)},$$

$$(X'_0, X'_0 \cap X'_1)_{0,q',\tilde{B};J} = (X'_0, X'_0 \cap X'_1)_{0,q',\tilde{B};J;(0,1)},$$

$$(X'_0, X'_0 \cap X'_1)_{0,q',\tilde{A};K} = (X'_0, X'_0 \cap X'_1)_{0,q',\tilde{A};K;(0,1)},$$

$$(X'_0, X'_1)_{0,q',\tilde{b};J} = (X'_0, X'_1)_{0,q',\tilde{b};J;(0,1)}.$$

The case $q = \infty$

Theorem

Let (X_0, X_1) *be a compatible couple. If* $b \in SV(0,\infty) \cap AC(0,\infty)$ *satisfies*

$$b \text{ is strictly decreasing}, \quad b(0) = \infty, \quad b(\infty) = 0, \tag{9}$$

and if

$$a(x) := \frac{b^2(x)}{x(-b'(x))} \quad \text{for a.a. } x > 0, \tag{10}$$

then

$$(X_0, X_1)_{0,\infty,b;K} = (X_0, X_1)_{0,\infty,a;J}.$$

Theorem

Let (X_0, X_1) *be a compatible couple and let* $b \in SV(0,\infty) \cap AC(0,\infty)$ *satisfy*

$$b \text{ is strictly decreasing}, \quad b(0) < \infty, \quad b(\infty) = 0. \tag{11}$$

Assume that the function $B \in SV(0,\infty)$ *is given by*

$$B(x) := b(x) \ \text{ if } x \in [1,\infty), \qquad B(x) := c\,\beta(x) \ \text{ if } x \in (0,1), \tag{12}$$

where $\beta \in SV(0,\infty) \cap AC(0,\infty)$ *is such that*

$$\beta \text{ is strictly decreasing}, \quad \beta(0) = \infty, \quad \beta(\infty) = 0, \tag{13}$$

and c *is a positive constant chosen in such a way that* $B \in AC(0,\infty)$*. If*

$$A(x) := \frac{B^2(x)}{x(-B'(x))} \quad \text{for a.a. } x > 0, \tag{14}$$

then

$$\begin{aligned}(X_0, X_1)_{0,\infty,b;K} &= (X_0, X_0 + X_1)_{0,\infty,B;K} = (X_0, X_0 + X_1)_{0,\infty,A;J} \\ &= X_0 + (X_0, X_1)_{0,\infty,B;K} = X_0 + (X_0, X_1)_{0,\infty,A;J}\end{aligned}$$

Remark

Note that, under the assumptions of the previous theorem

$$(X_0, X_1)_{0,\infty,b;K} = (X_0, X_1)_{0,\infty,b;K;(1,\infty)},$$

$$(X_0, X_0 + X_1)_{0,\infty,B;K} = (X_0, X_0 + X_1)_{0,\infty,B;K;(1,\infty)}.$$

Moreover, if $\|A(t)\|_{\infty,(1,e)} < \infty$, then also

$$(X_0, X_0 + X_1)_{0,\infty,A;J} = (X_0, X_0 + X_1)_{0,\infty,A;J;(1,\infty)}.$$

In a particular case, when b is of logarithmic type, some mentioned results on a description of limiting K- spaces by means of limiting J- spaces have been obtained by F. Cobos and T. Kühn (if (X_0, X_1) is an ordered couple) and by F. Cobos and A. Segurado (for a general couple (X_0, X_1)). Note that in the mentioned particular case these authors have also described duals of limiting K-spaces.

Main results for J-spaces

Theorem

Let (X_0, X_1) *be a compatible couple and* $1 < q \le \infty$*. If* $a \in SV(0,\infty)$ *satisfies*

$$\int_0^x t^{-1} a^{-q'}(t)\,dt < \infty \quad \text{for all} \;\; x > 0, \qquad \int_0^\infty t^{-1} a^{-q'}(t)\,dt = \infty, \tag{15}$$

and $b \in SV(0,\infty)$ *is defined by*

$$b(x) := a^{-q'/q}(x) \Big(\int_0^x t^{-1} a^{-q'}(t)\,dt \Big)^{-1} \quad \text{for all} \;\; x > 0, \tag{16}$$

then

$$(X_0, X_1)_{0,q,a;J} = (X_0, X_1)_{0,q,b;K}.$$

If $1 < q < \infty$*, then the space* $X_0 \cap X_1$ *is dense in* $(X_0, X_1)_{0,q,a;J}$*.*

Moreover, if $1 < q < \infty$ *and* $X_0 \cap X_1$ *is dense in* X_0 *and in* X_1*, then*

$$(X_0, X_1)'_{0,q,a;J} = (X_0', X_1')_{0,q',\tilde{a};K} = (X_0', X_1')_{0,q',\tilde{b};J},$$

where $\tilde{a}(x) := \frac{1}{a(1/x)}$ *and* $\tilde{b}(x) := \frac{1}{b(1/x)}$ *for all* $x > 0$*.*

Theorem

Let (X_0, X_1) *be a compatible couple and* $1 < q \le \infty$. *If* $a \in SV(0,\infty)$ *satisfies*

$$\int_0^\infty t^{-1} a^{-q'}(t)\,dt < \infty, \tag{17}$$

the function $A \in SV(0,\infty)$ *is given by*

$$A(x) := a(x) \quad \text{if} \;\; x \in (0,1], \qquad A(x) \approx \alpha(x) \quad \text{if} \;\; x \in (1,\infty), \tag{18}$$

where the function $\alpha \in SV(0,\infty)$ *is such that*

$$\int_0^x t^{-1}\alpha^{-q'}(t)\,dt < \infty \quad \text{for all} \;\; x > 0, \qquad \int_0^\infty t^{-1}\alpha^{-q'}(t)\,dt = \infty,$$

and

$$B(x) := A^{-q'/q}(x) \left(\int_0^x t^{-1} A^{-q'}(t)\,dt \right)^{-1} \quad \text{for all} \;\; x > 0, \tag{19}$$

then

$$\begin{aligned}(X_0, X_1)_{0,q,a;J} &= (X_0, X_0 \cap X_1)_{0,q,A;J} = (X_0, X_0 \cap X_1)_{0,q,B;K} \\ &= X_0 \cap (X_0, X_1)_{0,q,A;J} = X_0 \cap (X_0, X_1)_{0,q,B;K},\end{aligned}$$

$$(X_0, X_1)_{0,q,a;J} = (X_0, X_1)_{0,q,a;J;(0,1)},$$

$$(X_0, X_0 \cap X_1)_{0,q,A;J} = (X_0, X_0 \cap X_1)_{0,q,A;J;(0,1)},$$

$$(X_0, X_0 \cap X_1)_{0,q,B;K} = (X_0, X_0 \cap X_1)_{0,q,B;K;(0,1)},$$

If $1 < q < \infty$, *then* $X_0 \cap X_1$ *is dense in* $(X_0, X_1)_{0,q,a;J}$.

Theorem

If the assumptions of the previous theorem are satisfied, $1 < q < \infty$*, and* $X_0 \cap X_1$ *is dense in* X_0 *and in* X_1*, then*

$$\begin{aligned}(X_0, X_1)'_{0,q,a;J} =& (X_0', X_0' + X_1')_{0,q';\tilde{A};K} = (X_0', X_0' + X_1')_{0,q';\tilde{B};J}\\ =& X_0' + (X_0', X_1')_{0,q';\tilde{A};K} = X_0' + (X_0', X_1')_{0,q';\tilde{B};J}\\ =& (X_0', X_1')_{0,q';\tilde{a};K},\end{aligned}$$

where $\tilde{a}(x) := \frac{1}{a(1/x)}, \tilde{A}(x) := \frac{1}{A(1/x)}$ *and* $\tilde{B}(x) := \frac{1}{B(1/x)}$ *for all* $x > 0$*.*
Moreover,

$$(X_0, X_1)_{0,q,a;J} = (X_0, X_1)_{0,q,a;J;(0,1)},$$

$$(X_0', X_0' + X_1')_{0,q';\tilde{A};K} = (X_0', X_0' + X_1')_{0,q';\tilde{A};K;(1,\infty)},$$

$$(X_0', X_0' + X_1')_{0,q';\tilde{B};J} = (X_0', X_0 + X_1')_{0,q';\tilde{B};J;(1,\infty)},$$

$$(X_0', X_1')_{0,q';\tilde{a};K} = (X_0', X_1')_{0,q';\tilde{a};K;(1,\infty)}.$$

The case $q = 1$

Theorem

Let (X_0, X_1) *be a compatible couple. If* $a \in SV(0,\infty) \cap AC(0,\infty)$ *satisfies*

$$a \text{ is strictly decreasing}, \quad a(0) = \infty, \quad a(\infty) = 0, \tag{20}$$

and if

$$b(x) := -x\, a'(x) \quad \text{for a.a. } x > 0, \tag{21}$$

then

$$(X_0, X_1)_{0,1,a;J} = (X_0, X_1)_{0,1,b;K},$$

the space $X_0 \cap X_1$ *is dense in* $(X_0, X_1)_{0,1,a;J}$.

Moreover, if $X_0 \cap X_1$ *is dense in* X_0 *and in* X_1, *then*

$$(X_0, X_1)'_{0,1,a;J} = (X'_0, X'_1)_{0,\infty,\tilde{a};K} = (X'_0, X'_1)_{0,\infty,\tilde{b};J},$$

where $\tilde{a}(x) := \frac{1}{a(1/x)}$ *and* $\tilde{b}(x) := \frac{1}{b(1/x)}$ *for all* $x > 0$.

Theorem

Let (X_0, X_1) *be a compatible couple and let* $a \in SV(0,\infty) \cap AC(0,\infty)$ *satisfy*

$$a \text{ is strictly decreasing}, \quad a(0) = \infty, \quad a(\infty) > 0. \tag{22}$$

Assume that the function $A \in SV(0,\infty)$ *is given by*

$$A(x) := a(x) \ \text{ if } x \in (0,1], \qquad A(x) := c\,\alpha(x) \ \text{ if } x \in (1,\infty), \tag{23}$$

where $\alpha \in SV(0,\infty) \cap AC(0,\infty)$ *is such that*

$$\alpha \text{ is strictly decreasing}, \quad \alpha(0) = \infty, \quad \alpha(\infty) = 0,$$

and c *is a positive constant chosen in such a way that* $A \in AC(0,\infty)$. *If*

$$B(x) := -x\,A'(x) \quad \text{for a.a. } x > 0, \tag{24}$$

then

$$(X_0, X_1)_{0,1,a;J} = (X_0, X_0 \cap X_1)_{0,1,A;J} = (X_0, X_0 \cap X_1)_{0,1,B;K},$$

$$(X_0, X_1)_{0,1,a;J} = (X_0, X_1)_{0,1,a;J;(0,1)},$$

$$(X_0, X_0 \cap X_1)_{0,1,A;J} = (X_0, X_0 \cap X_1)_{0,1,A;J;(0,1)},$$

$$(X_0, X_0 \cap X_1)_{0,1,B;K} = (X_0, X_0 \cap X_1)_{0,1,B;K;(0,1)},$$

$$X_0 \cap X_1 \text{ is dense in } (X_0, X_1)_{0,1,a;J}.$$

Theorem

If the assumptions of the previous theorem are satisfied and $X_0 \cap X_1$ is dense in X_0 and in X_1, then

$$\begin{aligned}(X_0, X_1)'_{0,1,a;J} &= (X_0', X_0' + X_1')_{0,\infty,\tilde{A};K} = (X_0', X_0' + X_1')_{0,\infty,\tilde{B};J}\\ &= X_0' + (X_0', X_1')_{0,\infty,\tilde{A};K} = X_0' + (X_0', X_1')_{0,\infty,\tilde{B};J},\\ &= (X_0', X_1')_{0,\infty,\tilde{a};K}\end{aligned}$$

where $\tilde{a}(x) := \frac{1}{a(1/x)}, \tilde{A}(x) := \frac{1}{A(1/x)}$, *and* $\tilde{B}(x) := \frac{1}{B(1/x)}$ *for all* $x > 0$. *Moreover,*

$$(X_0, X_1)_{0,1,a;J} = (X_0, X_1)_{0,1,a;J;(0,1)},$$

$$(X_0', X_1')_{0,\infty,\tilde{a};K} = (X_0', X_1')_{0,\infty,\tilde{a};K;(1,\infty)},$$

$$(X_0', X_0' + X_1')_{0,\infty,\tilde{A};K} = (X_0', X_0' + X_1')_{0,\infty,\tilde{A};K;(1,\infty)}.$$

If $\|\tilde{B}(t)\|_{\infty,(1,e)} < \infty$, *then also*

$$(X_0', X_0' + X_1')_{0,\infty,\tilde{B};J} = (X_0', X_0' + X_1')_{0,\infty,\tilde{B};J;(1,\infty)}.$$

Applications

Definition

Let $n \in \mathbb{N}$, $1 \le p < \infty$, $1 \le q \le \infty$ and let $b \in SV(0,\infty)$ be such that

$$\|t^{-1/q}\, b(t)\|_{q;(0,1)} = \infty \quad \text{and} \quad \|t^{-1/q}\, b(t)\|_{q;(1,\infty)} < \infty. \tag{25}$$

The Besov space $B^{0,b}_{p,q}(\mathbb{R}^n)$ consists of all $f \in L_p(\mathbb{R}^n)$ for which the norm

$$\|f\|_{B^{0,b}_{p,q}} := \|f\|_p + \|t^{-1/q} b(t) \omega_1(f,t)_p\|_{q;(0,\infty)} \tag{26}$$

is finite.

Theorem

If $n \in \mathbb{N}$, $1 \le p < \infty$, $1 \le q < \infty$, and $b \in SV(0,\infty)$ is such that (25) holds, then the set $C_0^\infty(\mathbb{R}^n)$ is dense in the space $B^{0,b}_{p,q}(\mathbb{R}^n)$.

Remark

Note that, under the assumptions of the previous theorem

$$(L_p(\mathbb{R}^n), W^1_p(\mathbb{R}^n))_{0,q,b;K} = B^{0,b}_{p,q}(\mathbb{R}^n).$$

The space $H_p^s := H_p^s(\mathbb{R}^n)$

If $s \in \mathbb{R}$ and $1 < p < \infty$, then the *fractional Sobolev* (or *Bessel potential*) *space* $H_p^s := H_p^s(\mathbb{R}^n)$ is the collections of all tempered distributions f on $\mathbb{R}^n$ satisfying

$$\|f\|_{H_p^s} = \|\mathcal{F}^{-1}((1+|x|^2)^{s/2}\mathcal{F}f)\|_p < \infty,$$

where the symbol $\mathcal{F}$ denotes the Fourier transform.
Let $s \in \mathbb{R}$ and the operator I_s is given by

$$I_s f = \mathcal{F}^{-1}((1+|x|^2)^{s/2}\mathcal{F}f). \tag{27}$$

If $s, \sigma \in \mathbb{R}$ and $1 < p < \infty$, then I_s is a lift which maps H_p^σ isomorphically onto $H_p^{\sigma - s}$. Consequently,

$$I_s(H_p^\sigma) = H_p^{\sigma - s}.$$

Moreover, if $s \in \mathbb{R}$ and $1 < p < \infty$, then

$$(H_p^s)' = H_{p'}^{-s}.$$

The dual of the space $B^{0,b}_{p,q}(\mathbb{R}^n)$

Theorem

Let $n \in \mathbb{N}$, $1 < p < \infty$, $1 \le q < \infty$, *and let* $b \in SV(0,\infty)$ *be such that*

$$\|t^{-1/q}\, b(t)\|_{q;(0,1)} = \infty \quad \text{and} \quad \|t^{-1/q}\, b(t)\|_{q;(1,\infty)} < \infty.$$

If $a \in SV\,(0,\infty)$ *is defined by*

$$a(x) := b^{-q/q'}(x) \int_x^\infty t^{-1} b^q(t)\, dt \quad \text{for all } x > 0,$$

then $(B^{0,b}_{p,q}(\mathbb{R}^n))'$ *is the set of all functions* $f \in H^{-1}_{p'}(\mathbb{R}^n)$ *which satisfy*

$$N(f) := \|I_{-1} f\|_{p'} + \left\| \tau^{-1/q'} \frac{\omega_1(I_{-1} f, \tau)_{p'}}{\tau a(\tau)} \right\|_{q',(0,1)} < \infty,$$

where the operator I_{-1} *is defined by* (27). *Moreover,*

$$\|f\|_{(B^{0,b}_{p,q})'} \approx N(f) \quad \text{for all} \ \ f \in (B^{0,b}_{p,q})'.$$

On taking $b(x) = (1 + |\ln x|)^\beta$ if $x > 0$, where $\beta > -1/q$, in this theorem, one obtains the result proved by F. Cobos and Ó. Domínguez.