Spaces of integrable functions associated to vector measures and limiting real interpolation

Antonio Manzano,
Universidad de Burgos, Spain

Joint work with Antonio Fernández (Universidad de Sevilla)

Mini Workshop "Function Spaces, Interpolation Theory and Related Topics 2026"
Madrid, 10-12 March, 2026

Among all interpolation methods, the *real method* $(\cdot,\cdot)_{\theta,q}$, $0<\theta<1$ and $1\le q\le\infty$, has been the most studied. Probably it is due to its flexible construction. Thus, there exist different equivalent definitions of this method, being specially prominent those ones using the K- and J-functionals of Peetre.

Among all interpolation methods, the *real method* $(\cdot,\cdot)_{\theta,q}$, $0<\theta<1$ and $1\le q\le\infty$, has been the most studied. Probably it is due to its flexible construction. Thus, there exist different equivalent definitions of this method, being specially prominent those ones using the K- and J-functionals of Peetre. Let us recall the construction of the space $(A_0,A_1)_{\theta,q}$.

• Let $\bar{A}=(A_0,A_1)$ be a Banach couple; i.e., A_0 and A_1 are Banach spaces continuously embedded in some Hausdorff topological vector space. Then, the sum and intersection spaces

$$\Sigma(\bar{A}) := A_0 + A_1 \quad \text{and} \quad \Delta(\bar{A}) := A_0 \cap A_1,$$

are Banach spaces when endowed with the norms $K(1,\cdot)$ and $J(1,\cdot)$, respectively. For $t>0$, the K-functional and the J-functional are given by

$$K(t,a;A_0,A_1) := \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} \;:\; a=a_0+a_1,\; a_i\in A_i\},\; a\in\Sigma(\bar{A}).$$

$$J(t,a;A_0,A_1) := \max\{\|a\|_{A_0},\, t\|a\|_{A_1}\},\; a\in\Delta(\bar{A}).$$

Among all interpolation methods, the *real method* $(\cdot,\cdot)_{\theta,q}$, $0<\theta<1$ and $1\le q\le\infty$, has been the most studied. Probably it is due to its flexible construction. Thus, there exist different equivalent definitions of this method, being specially prominent those ones using the K- and J-functionals of Peetre. Let us recall the construction of the space $(A_0,A_1)_{\theta,q}$.

• Let $\bar{A}=(A_0,A_1)$ be a Banach couple; i.e., A_0 and A_1 are Banach spaces continuously embedded in some Hausdorff topological vector space. Then, the sum and intersection spaces

$$\Sigma(\bar{A}) := A_0 + A_1 \quad \text{and} \quad \Delta(\bar{A}) := A_0 \cap A_1,$$

are Banach spaces when endowed with the norms $K(1,\cdot)$ and $J(1,\cdot)$, respectively. For $t>0$, the K-functional and the J-functional are given by

$$K(t,a;A_0,A_1) := \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} \ : \ a=a_0+a_1,\ a_i\in A_i\},\ a\in\Sigma(\bar{A}).$$

$$J(t,a;A_0,A_1) := \max\{\|a\|_{A_0},\ t\|a\|_{A_1}\},\ a\in\Delta(\bar{A}).$$

• Let $0<\theta<1$, $1\le q\le\infty$, the real interpolation space $\bar{A}_{\theta,q}=(A_0,A_1)_{\theta,q}$ is formed by all elements $a\in A_0+A_1$ with a finite norm

$$\|a\|_{\theta,q;K} := \left(\int_0^\infty \left(t^{-\theta} K(t,a;A_0,A_1)\right)^q \frac{dt}{t}\right)^{1/q}, \quad \text{when } 1\le q<\infty,$$

$$\|a\|_{\theta,q;K} := \sup_{t>0} t^{-\theta} K(t,a;A_0,A_1), \quad \text{when } q=\infty.$$

• The space $(A_0, A_1)_{\theta,q}$ can be also described using the J-functional. Namely, an element $a \in (A_0, A_1)_{\theta,q}$ if and only if

$$a = \int_0^\infty u(t)\frac{dt}{t} \quad \text{(convergence in } \Sigma(\bar{A})\text{)},$$

where $u(t)$ is a strongly measurable function with values in $\Delta(\bar{A})$ and

$$\left(\int_0^\infty \left(t^{-\theta} J(t, u(t); A_0, A_1)\right)^q \frac{dt}{t}\right)^{1/q} < \infty, \quad \text{if } 1 \le q < \infty \qquad (1)$$

(integral is replaced by supremum when $q = \infty$)
The infimum of (1) taken over all possible representations $u(t)$ gives an equivalent norm $\|\cdot\|_{\theta,\,q;J}$ to $\|\cdot\|_{\theta,\,q;K}$.

Lorentz spaces $L^{p,q}$ can be obtained applying $(\cdot,\cdot)_{\theta,q}$ to a couple of Lebesgue spaces.

• Let (Ω, Σ, μ) be a σ-finite measure space. For any $0 < \theta = \dfrac{1}{p} < 1$ and $1 \le q \le \infty$, then

$$(L^\infty(\mu), L^1(\mu))_{\theta,q} = L^{p,q}(\mu).$$

In particular,

$$(L^\infty(\mu), L^1(\mu))_{\frac{1}{p},p} = L^p(\mu), \quad 1 < p < \infty.$$

The definition of the real method $(\cdot,\cdot)_{\theta,q}$ has been generalised in several directions. For instance, an extension of this method consists in replacing in its construction the function t^θ by a more general function $f(t)$.

- $\bar{A}_{\theta,q} = (A_0, A_1)_{\theta,q}$: $a \in A_0 + A_1$ with a finite norm

$$\|a\|_{\theta,q} := \left(\int_0^\infty \left(\frac{K(t, a; A_0, A_1)}{t^\theta} \right)^q \frac{dt}{t} \right)^{1/q}, \quad \text{when} \;\; 1 \leq q < \infty,$$

$$\|a\|_{\theta,q} := \sup_{t>0} \frac{K(t, a; A_0, A_1)}{t^\theta}, \quad \text{when} \;\; q = \infty.$$

J. Gustavsson, *A function parameter in connection with interpolation of Banach spaces*, Math. Scand. **42** (1978), 289–305.

S. Janson, *Minimal and Maximal Methods of Interpolation*, J. Funct. Anal. **44** (1981), 50–73.

L.E. Persson, *Interpolation with a parameter function*, Math. Scand. **59** (1986), 199–222.

The definition of the real method $(\cdot,\cdot)_{\theta,q}$ has been generalised in several directions. For instance, an extension of this method consists in replacing in its construction the function t^θ by a more general function $f(t)$.

- $\bar{A}_{f,q} = (A_0, A_1)_{f,q}$: $a \in A_0 + A_1$ with a finite norm

$$\|a\|_{f,q} := \left(\int_0^\infty \left(\frac{K(t,a;A_0,A_1)}{f(t)} \right)^q \frac{dt}{t} \right)^{1/q}, \quad \text{when } 1 \le q < \infty,$$

$$\|a\|_{f,q} := \sup_{t>0} \frac{K(t,a;A_0,A_1)}{f(t)}, \quad \text{when } q = \infty.$$

where f is a positive function defined on $(0,\infty)$ that satisfies certain suitable conditions for the main theorems of interpolation theory to be still valid.

J. Gustavsson, *A function parameter in connection with interpolation of Banach spaces*, Math. Scand. **42** (1978), 289–305.

S. Janson, *Minimal and Maximal Methods of Interpolation*, J. Funct. Anal. **44** (1981), 50–73.

L.E. Persson, *Interpolation with a parameter function*, Math. Scand. **59** (1986), 199–222.

An advantage of the *real method with a function parameter* $(\cdot,\cdot)_{f,q}$ is that working with a couple of Lebesgue spaces one obtains a larger class of spaces than Lebesgue and Lorentz spaces provided by the classical real method $(\cdot,\cdot)_{\theta,q}$. In fact,

- Let (Ω,Σ,μ) be a σ-finite measure space. Then

$$\big(L^{\infty}(\mu),L^{1}(\mu)\big)_{f,q}=L^{p,q}(\log L)^{\alpha}(\mu),$$

when $f(t)=t^{\frac{1}{p}}(1+|\log t|)^{-\alpha}$, $1<p<\infty, 1\leq q\leq\infty$ and $\alpha\in\mathbb{R}$.

An advantage of the *real method with a function parameter* $(\cdot,\cdot)_{f,q}$ is that working with a couple of Lebesgue spaces one obtains a larger class of spaces than Lebesgue and Lorentz spaces provided by the classical real method $(\cdot,\cdot)_{\theta,q}$. In fact,

- Let (Ω,Σ,μ) be a σ-finite measure space. Then

$$\big(L^\infty(\mu),L^1(\mu)\big)_{f,q}=L^{p,q}(\log L)^\alpha(\mu),$$

when $f(t)=t^{\frac{1}{p}}(1+|\log t|)^{-\alpha}$, $1<p<\infty, 1\le q\le\infty$ and $\alpha\in\mathbb{R}$.

Some authors have investigated the *logarithmic interpolation spaces* $(A_0,A_1)_{\theta,q,\mathbb{A}}$.

- These spaces correspond to the case where $f(t)=t^\theta\ell^{\mathbb{A}}(t)$, being $\ell^{\mathbb{A}}(t)$ a broken logarithmic function, i.e., $\mathbb{A}=(\alpha_0,\alpha_\infty)\in\mathbb{R}^2$, $\ell(t)=1+|\log t|$ and

$$\ell^{\mathbb{A}}(t)=\begin{cases}\ell^{\alpha_0}(t), & \text{if } 0<t\le 1,\\ \ell^{\alpha_\infty}(t), & \text{if } 1<t<\infty.\end{cases}$$

J. Gustavsson, *A function parameter in connection with interpolation of Banach spaces*, Math. Scand. **42** (1978), 289–305.

W. D. Evans and B. Opic, *Real interpolation with logarithmic functors and reiteration*, Canad. J. Math. **52** (2000), 920–960.

W. D. Evans, B. Opic and L. Pick, *Real interpolation with logarithmic functors*, J. Inequal. Appl. **7** (2002), 187–269.

In the construction of $(A_0, A_1)_{\theta,q,\mathbb{A}}$ it is possible to consider $1 \leq q \leq \infty$ *and not only* $0 < \theta < 1$, *but* θ *even taking the values* 0 *and* 1.

In the construction of $(A_0, A_1)_{\theta,q,\mathbb{A}}$ it is possible to consider $1 \leq q \leq \infty$ *and not only* $0 < \theta < 1$, *but* θ *even taking the values* 0 *and* 1. *For these limiting values* $\theta = 0$ *and* $\theta = 1$, *the extra function* $\ell^{\mathbb{A}}(t)$ *is essential* to get a meaningful definition of $(A_0, A_1)_{\theta,q,\mathbb{A}}$.

In the construction of $(A_0, A_1)_{\theta,q,\mathbb{A}}$ it is possible to consider $1 \leq q \leq \infty$ *and not only* $0 < \theta < 1$, *but* θ *even taking the values* 0 *and* 1. *For these limiting values* $\theta = 0$ *and* $\theta = 1$, *the extra function* $\ell^{\mathbb{A}}(t)$ *is essential* to get a meaningful definition of $(A_0, A_1)_{\theta,q,\mathbb{A}}$.

A different approach consists in considering ordered Banach couples (A_0, A_1), say $A_0 \subseteq A_1$ (where "$\subseteq$" means continuous inclusion). Then just making a natural modification in the definition of the real method $(A_0, A_1)_{\theta,q}$, two limiting classes of real interpolation spaces, $(A_0, A_1)_{1,q;K}$ and $(A_0, A_1)_{0,q;J}$, can be defined *without involving any auxiliary function.*

In the construction of $(A_0, A_1)_{\theta,q,\mathbb{A}}$ it is possible to consider $1 \le q \le \infty$ *and not only* $0 < \theta < 1$, *but* θ *even taking the values* 0 *and* 1. *For these limiting values* $\theta = 0$ *and* $\theta = 1$, *the extra function* $\ell^{\mathbb{A}}(t)$ *is essential* to get a meaningful definition of $(A_0, A_1)_{\theta,q,\mathbb{A}}$.

A different approach consists in considering ordered Banach couples (A_0, A_1), say $A_0 \subseteq A_1$ (where "$\subseteq$" means continuous inclusion). Then just making a natural modification in the definition of the real method $(A_0, A_1)_{\theta,q}$, two limiting classes of real interpolation spaces, $(A_0, A_1)_{1,q;K}$ and $(A_0, A_1)_{0,q;J}$, can be defined *without involving any auxiliary function.*

- If $A_0 \subseteq A_1$ and $1 \le q \le \infty$, the limiting K-space $\bar{A}_{1,q;K} = (A_0, A_1)_{1,q;K}$, consists of those $a \in A_1$ which have a finite norm

$$\|a\|_{1,q;K} := \left(\int_1^\infty \left[\frac{K(t,a)}{t}\right]^q \frac{dt}{t}\right)^{\frac{1}{q}}, 1 \le q < \infty \left(\sup_{t>1} \frac{K(t,a)}{t}, q = \infty\right).$$

In the construction of $(A_0, A_1)_{\theta,q,\mathbb{A}}$ it is possible to consider $1 \le q \le \infty$ *and not only* $0 < \theta < 1$, *but* θ *even taking the values* 0 *and* 1. *For these limiting values* $\theta = 0$ *and* $\theta = 1$, *the extra function* $\ell^{\mathbb{A}}(t)$ *is essential* to get a meaningful definition of $(A_0, A_1)_{\theta,q,\mathbb{A}}$.

A different approach consists in considering ordered Banach couples (A_0, A_1), say $A_0 \subseteq A_1$ (where "$\subseteq$" means continuous inclusion). Then just making a natural modification in the definition of the real method $(A_0, A_1)_{\theta,q}$, two limiting classes of real interpolation spaces, $(A_0, A_1)_{1,q;K}$ and $(A_0, A_1)_{0,q;J}$, can be defined *without involving any auxiliary function.*

• If $A_0 \subseteq A_1$ and $1 \le q \le \infty$, the limiting K-space $\bar{A}_{1,q;K} = (A_0, A_1)_{1,q;K}$, consists of those $a \in A_1$ which have a finite norm

$$\|a\|_{1,q;K} := \left(\int_1^\infty \left[\frac{K(t,a)}{t} \right]^q \frac{dt}{t} \right)^{\frac{1}{q}}, \ 1 \le q < \infty \ \left(\sup_{t>1} \frac{K(t,a)}{t}, q = \infty \right).$$

On the other hand, the limiting J-space $\bar{A}_{0,q;J} = (A_0, A_1)_{0,q;J}$, is formed by all elements $a \in A_1$ for which there is a strongly measurable function $u(t)$ with values in A_0 such that $a = \int_1^\infty u(t) \frac{dt}{t}$ (convergence in A_1) and

$$\left(\int_1^\infty J(t, u(t))^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty, \ 1 \le q < \infty \ \left(\sup_{t>1} J(t, u(t)) < \infty, \ q = \infty \right).$$

$\|a\|_{\bar{A}_{0,q;J}}$:=infimum of the last quantity over all representations $u(t)$ of a.

The spaces $(A_0, A_1)_{1,q;K}$ were studied for the first time by Gomez and Milman in 1986.

M. E. Gomez and M. Milman, *Extrapolation spaces and almost-everywhere convergence of singular integrals*, J. London Math. Soc. **34** (1986), 305–316.

The spaces $(A_0, A_1)_{1,q;K}$ were studied for the first time by Gomez and Milman in 1986.

M. E. Gomez and M. Milman, *Extrapolation spaces and almost-everywhere convergence of singular integrals*, J. London Math. Soc. **34** (1986), 305–316.

More recently, the extreme interpolation methods $(A_0, A_1)_{1,q;K}$ and $(A_0, A_1)_{0,q;J}$ have been investigated by Cobos, Fernández-Cabrera, Kühn and Ullrich in 2009

F. Cobos, L. M. Fernández-Cabrera, T. Kühn and T. Ullrich, *On a extreme class of real interpolation spaces*, J. Funct. Anal. **256** (2009), 2321–2366.

and subsequently in different papers by these authors and by others.

The spaces $(A_0, A_1)_{1,q;K}$ were studied for the first time by Gomez and Milman in 1986.

M. E. Gomez and M. Milman, *Extrapolation spaces and almost-everywhere convergence of singular integrals*, J. London Math. Soc. **34** (1986), 305–316.

More recently, the extreme interpolation methods $(A_0, A_1)_{1,q;K}$ and $(A_0, A_1)_{0,q;J}$ have been investigated by Cobos, Fernández-Cabrera, Kühn and Ullrich in 2009

F. Cobos, L. M. Fernández-Cabrera, T. Kühn and T. Ullrich, *On a extreme class of real interpolation spaces*, J. Funct. Anal. **256** (2009), 2321–2366.

and subsequently in different papers by these authors and by others.

- $(A_0, A_1)_{1,q;K}$ and $(A_0, A_1)_{0,q;J}$ are intermediate Banach spaces with respect to (A_0, A_1) for every $1 \le q \le \infty$, i.e.

$$A_0 \subseteq (A_0, A_1)_{1,q;K} \subseteq A_1, \qquad A_0 \subseteq (A_0, A_1)_{0,q;J} \subseteq A_1,$$

and

$$(A_0, A_1)_{1,\infty,K} = A_1, \qquad (A_0, A_1)_{0,1;J} = A_0.$$

Moreover, limiting K- and J-spaces are increasing with q in the sense that

$$\text{if } p < q,\ (A_0, A_1)_{1,p;K} \subseteq (A_0, A_1)_{1,q;K} \text{ and } (A_0, A_1)_{0,p;J} \subseteq (A_0, A_1)_{0,q;J}.$$

As an intrinsic problem related to interpolation theory, it has been constantly investigated the description of the spaces obtained by applying an interpolation method to concrete compatible couples of spaces.

As an intrinsic problem related to interpolation theory, it has been constantly investigated the description of the spaces obtained by applying an interpolation method to concrete compatible couples of spaces. In recent years, this question has awakened a lot of attention when the couple is formed by spaces of scalar integrable functions associated to a vector measure. This issue immediately shows differences with the case of a scalar measure.

As an intrinsic problem related to interpolation theory, it has been constantly investigated the description of the spaces obtained by applying an interpolation method to concrete compatible couples of spaces. In recent years, this question has awakened a lot of attention when the couple is formed by spaces of scalar integrable functions associated to a vector measure. This issue immediately shows differences with the case of a scalar measure.

- If m is a vector measure,

$$(L^\infty(m), L^1(m))_{\frac{1}{p},p} \text{ is different from } L^p(m), \quad 1 < p < \infty.$$

As an intrinsic problem related to interpolation theory, it has been constantly investigated the description of the spaces obtained by applying an interpolation method to concrete compatible couples of spaces. In recent years, this question has awakened a lot of attention when the couple is formed by spaces of scalar integrable functions associated to a vector measure. This issue immediately shows differences with the case of a scalar measure.

- If m is a vector measure,

$$(L^\infty(m), L^1(m))_{\frac{1}{p},p} \text{ is different from } L^p(m), \quad 1 < p < \infty.$$

In fact, the inclusion mapping $L^\infty(m) \subseteq L^1(m)$ is weakly compact and thus, by Beauzamy's result, $(L^\infty(m), L^1(m))_{\frac{1}{p},p}$ is reflexive for $1 < p < \infty$.

Theorem (Beauzamy, Lecture Notes in Math. (1978))

Let $0 < \theta < 1$ and $1 < q < \infty$.

$$(A_0, A_1)_{\theta,q} \text{ is reflexive } \Leftrightarrow I : A_0 \cap A_1 \to A_0 + A_1 \text{ is weakly compact.}$$

As an intrinsic problem related to interpolation theory, it has been constantly investigated the description of the spaces obtained by applying an interpolation method to concrete compatible couples of spaces. In recent years, this question has awakened a lot of attention when the couple is formed by spaces of scalar integrable functions associated to a vector measure. This issue immediately shows differences with the case of a scalar measure.

- If m is a vector measure,

$$(L^\infty(m), L^1(m))_{\frac{1}{p},p} \text{ is different from } L^p(m), \quad 1 < p < \infty.$$

In fact, the inclusion mapping $L^\infty(m) \subseteq L^1(m)$ is weakly compact and thus, by Beauzamy's result, $(L^\infty(m), L^1(m))_{\frac{1}{p},p}$ is reflexive for $1 < p < \infty$.

Theorem (Beauzamy, Lecture Notes in Math. (1978))

Let $0 < \theta < 1$ and $1 < q < \infty$.

$$(A_0, A_1)_{\theta,q} \text{ is reflexive } \Leftrightarrow I : A_0 \cap A_1 \to A_0 + A_1 \text{ is weakly compact.}$$

However, $L^p(m)$, $p > 1$, is not reflexive whenever $L^1(m) \neq L^1_w(m)$.

- If m is a vector measure,

$$(L^\infty(m), L^1(m))_{\frac{1}{p},p} \text{ is different from } L^p(m), \quad 1 < p < \infty.$$

A. Fernández, F. Mayoral and F. Naranjo, *Real interpolation method on spaces of scalar integrable functions with respect to vector measures*, J. Math. Anal. Appl. **376** (2011), 203–211.

Theorem (Fernández, Mayoral and Naranjo, J. Math. Anal. Appl. (2011))

If $0 < \theta = \dfrac{1}{p} < 1$ and $1 \leq q \leq \infty$, it holds

$$\big(L^\infty(m), L^1(m)\big)_{\theta,q} = \big(L^\infty(m), L^1_w(m)\big)_{\theta,q} = L^{p,q}(\|m\|).$$

- If m is a vector measure,

$$(L^\infty(m), L^1(m))_{\frac{1}{p},p} \text{ is different from } L^p(m), \quad 1 < p < \infty.$$

A. Fernández, F. Mayoral and F. Naranjo, *Real interpolation method on spaces of scalar integrable functions with respect to vector measures*, J. Math. Anal. Appl. **376** (2011), 203–211.

Theorem (Fernández, Mayoral and Naranjo, J. Math. Anal. Appl. (2011))

If $0 < \theta = \dfrac{1}{p} < 1$ and $1 \leq q \leq \infty$, it holds

$$\big(L^\infty(m), L^1(m)\big)_{\theta,q} = \big(L^\infty(m), L^1_w(m)\big)_{\theta,q} = L^{p,q}(\|m\|).$$

Some preliminaries about spaces of integrable functions with respect to a vector measure:

- Let Σ be a σ-algebra of a nonempty set Ω and let E be a Banach space. A mapping $m : \Sigma \to E$ such that $m(\emptyset) = 0$ and is countably additive (i.e., $\sum m(A_n)$ converges to $m(\cup A_n)$ in E, for any sequence $(A_n) \subset \Sigma$ of disjoint sets) is said to be a vector measure. When $E = \mathbb{R}$ and m does not take negative values, as usual, we name *positive scalar measure*.

Given a vector measure $m : \Sigma \to E$, $L^0(m)$ denotes the space of measurable functions $f : \Omega \to \mathbb{R}$. Two functions $f, g \in L^0(m)$ will be identified if are equal m-a.e., that is, if $\left\{w \in \Omega : f(w) \neq g(w)\right\}$ is m-null. It means that $\|m\|\left(\left\{w \in \Omega : f(w) \neq g(w)\right\}\right) = 0$, where $\|m\|$ is the semivariation of m:

$$\|m\|(A) := \sup\left\{|\langle m, x^*\rangle|(A) : x^* \in B(E^*)\right\}, \ A \in \Sigma.$$

Given a vector measure $m : \Sigma \to E$, $L^0(m)$ denotes the space of measurable functions $f : \Omega \to \mathbb{R}$. Two functions $f, g \in L^0(m)$ will be identified if are equal m-a.e., that is, if $\{w \in \Omega : f(w) \neq g(w)\}$ is m-null. It means that $\|m\|(\{w \in \Omega : f(w) \neq g(w)\}) = 0$, where $\|m\|$ is the semivariation of m:

$$\|m\|(A) := \sup \{|\langle m, x^* \rangle| (A) : x^* \in B(E^*)\}, \ A \in \Sigma.$$

- $f \in L^0(m)$ is called integrable (w.r.t.m) if
  - i) $f \in L^1(|\langle m, x^* \rangle|)$, for all $x^* \in E^*$; i.e. f is weakly integrable (w.r.t.m)
  - ii) given any $A \in \Sigma$, there exists an element $\int_A f dm \in E$ such that $\langle \int_A f dm, x^* \rangle = \int_A f \, d\langle m, x^* \rangle$, for all $x^* \in E^*$.

Let

$L^1_w(m) := \{f : \ f \text{ is weakly integrable}\}$,
$L^1(m) := \{f : \ f \text{ is integrable}\}$,

endowed with the norm

$$\|f\|_1 := \sup \left\{ \int_\Omega |f| \, d \, |\langle m, x^* \rangle| : x^* \in B(E^*) \right\}.$$

Given a vector measure $m : \Sigma \to E$, $L^0(m)$ denotes the space of measurable functions $f : \Omega \to \mathbb{R}$. Two functions $f, g \in L^0(m)$ will be identified if are equal m-a.e., that is, if $\{w \in \Omega : f(w) \neq g(w)\}$ is m-null. It means that $\|m\|(\{w \in \Omega : f(w) \neq g(w)\}) = 0$, where $\|m\|$ is the semivariation of m:

$$\|m\|(A) := \sup \{|\langle m, x^* \rangle| (A) : x^* \in B(E^*)\}, \ A \in \Sigma.$$

- $f \in L^0(m)$ is called integrable (w.r.t.m) if
  - i) $f \in L^1 (|\langle m, x^* \rangle|)$, for all $x^* \in E^*$; i.e. f is weakly integrable (w.r.t.m)
  - ii) given any $A \in \Sigma$, there exists an element $\int_A f dm \in E$ such that $\left\langle \int_A f dm, x^* \right\rangle = \int_A f \, d\langle m, x^* \rangle$, for all $x^* \in E^*$.

Let

$L^1_w(m) := \{f : \ f \text{ is weakly integrable}\}$,
$L^1(m) := \{f : \ f \text{ is integrable}\}$,

endowed with the norm

$$\|f\|_1 := \sup \left\{ \int_\Omega |f| \, d \, |\langle m, x^* \rangle| : x^* \in B(E^*) \right\}.$$

- $L^\infty(m) := \{f : \ f \text{ is essentially bounded}\}$ equipped with the essential supremum norm. It holds that $L^\infty(m) \subseteq L^1(m)$ with

$$\|f\|_{L^1(m)} \leq \|f\|_{L^\infty(m)} \|m\| (\Omega), \ f \in L^\infty(m).$$

- A function $f \in L^0(m)$ is said to be
  i) weakly p-integrable (w.r.t.m), $1 < p < \infty$, when $|f|^p \in L^1_w(m)$,
  ii) p-integrable (w.r.t.m), $1 < p < \infty$, when $|f|^p \in L^1(m)$.

Let

$$L^p_w(m) := \{f : f \text{ is weakly } p\text{-integrable}\},$$
$$L^p(m) := \{f : f \text{ is } p\text{-integrable}\},$$

with the norm

$$\|f\|_p := \sup\left\{\left(\int_\Omega |f|^p \, d\,|\langle m, x^*\rangle|\right)^{1/p} : x^* \in B(E^*)\right\}.$$

- A function $f \in L^0(m)$ is said to be
  i) weakly p-integrable (w.r.t.m), $1 < p < \infty$, when $|f|^p \in L^1_w(m)$,
  ii) p-integrable (w.r.t.m), $1 < p < \infty$, when $|f|^p \in L^1(m)$.

Let

$L^p_w(m) := \{f : f \text{ is weakly } p\text{-integrable}\}$,

$L^p(m) := \{f : f \text{ is } p\text{-integrable}\}$,

with the norm

$$\|f\|_p := \sup\left\{\left(\int_\Omega |f|^p \, d\,|\langle m, x^*\rangle|\right)^{1/p} : x^* \in B(E^*)\right\}.$$

- Some properties:
  - For every real number $p \geq 1$, $L^p(m)$ and $L^p_w(m)$ are *Banach lattices with, respectively, order continuous norm and the Fatou property.*
  - $L^p(m)$ and $L^p_w(m)$ *may not be reflexive for* $p > 1$.
  - If $1 < p_1 < p_2 < \infty$, then $L^\infty(m) \subseteq L^{p_2}(m) \subseteq L^{p_2}_w(m) \subseteq L^{p_1}(m) \subseteq L^{p_1}_w(m) \subseteq L^1(m) \subseteq L^1_w(m)$.
  - If m is a finite positive scalar measure, $L^p(m)$ and $L^p_w(m)$ coincide with the classical Lebesgue space L^p.

A. Fernández, F. Mayoral, F. Naranjo, C. Sáez and E.A. Sánchez–Pérez, *Spaces of p-integrable functions with respect to a vector measure*, Positivity **10** (2006), 1–16.

This kind of spaces finds applications in some important problems. For instance, when studying the representation of abstract Banach lattices as spaces of integrable functions.

This kind of spaces finds applications in some important problems. For instance, when studying the representation of abstract Banach lattices as spaces of integrable functions.

Thus, it is worthy to mention that *every order continuous Banach lattice* E *with a weak order unit can be represented (order isometrically) as an* $L^1(m)$*-space of a vector measure* m *defined on a* σ*-algebra.*

G. P. Curbera, *Operators into L^1 of a vector measure and applications to Banach lattices*, Math. Ann. **293** (1992), 317-330.

A similar representation can be obtained using an $L^p(m)$*-space, in the case when* E *is in addition* p*-convex.*

E. A. Sánchez-Pérez, *Compactness arguments for spaces of p-integrable functions with respect to a vector measure and factorization of operators through Lebesgue-Bochner spaces*, Illinois J. Math. **45** (2001), 907–923.

A. Fernández, F. Mayoral, F. Naranjo, C. Sáez and E.A. Sánchez–Pérez, *Spaces of p-integrable functions with respect to a vector measure*, Positivity **10** (2006), 1–16.

If order continuity fails but E *has a Fatou type property and a weak order unit belonging to its order continuous part, then* E *can be identified (order isometrically) with a space* $L^1_w(m)$ *associated to a vector measure* m *on a* σ*-algebra.*

G. P. Curbera and W. J. Ricker, *Banach lattices with the Fatou property and optimal domains of kernel operators*, Indag. Math. (N.S.) **17** (2006), 187–204.

For a Banach lattice E *that is in addition* p*-convex, an analogous representation as an* $L^p_w(m)$*-space holds.*

G. P. Curbera and W. J. Ricker, *The Fatou property in p-convex Banach lattices*, J. Math. Anal. Appl. **328** (2007), 287–294.

• *When m is a finite positive scalar measure, $\|m\|$ and m coincide*. But in general, for an arbitrary vector measure m, it holds that

$$L^p(m) \neq L^p(\|m\|) := L^{p,p}(\|m\|), \ 1 \leq p < \infty.$$

For $1 \leq p < q < \infty$, we have the following continuous inclusions:

$$L^\infty(m) \subseteq L^{q,1}(\|m\|) \subseteq L^q(\|m\|) \subseteq L^p(\|m\|) \subseteq L^p(m) \subseteq L^p_w(m) \subseteq L^{p,\infty}(\|m\|).$$

A. Fernández, F. Mayoral and F. Naranjo, *Real interpolation method on spaces of scalar integrable functions with respect to vector measures*, J. Math. Anal. Appl. **376** (2011), 203–211.

- *When m is a finite positive scalar measure, $\|m\|$ and m coincide*. But in general, for an arbitrary vector measure m, it holds that

$$L^p(m) \neq L^p(\|m\|) := L^{p,p}(\|m\|), \ 1 \leq p < \infty.$$

For $1 \leq p < q < \infty$, we have the following continuous inclusions:

$$L^\infty(m) \subseteq L^{q,1}(\|m\|) \subseteq L^q(\|m\|) \subseteq L^p(\|m\|) \subseteq L^p(m) \subseteq L^p_w(m) \subseteq L^{p,\infty}(\|m\|).$$

A. Fernández, F. Mayoral and F. Naranjo, *Real interpolation method on spaces of scalar integrable functions with respect to vector measures*, J. Math. Anal. Appl. **376** (2011), 203–211.

- Lorentz-Zygmund space $L^{p,q}(\log L)^\alpha(\|m\|)$, for a vector measure m

For $1 \leq p, q \leq \infty$ and $\alpha \in \mathbb{R}$, the space $L^{p,q}(\log L)^\alpha(\|m\|)$ is formed by all functions $f \in L^0(m)$ for which

$$\|f\|_{L^{p,q}(\log L)^\alpha(\|m\|)} := \left(\int_0^\infty \left[t^{\frac{1}{p}} (1 + |\log t|)^\alpha f_*(t) \right]^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty$$

(with the usual modification if $q = \infty$). Here f_* is the decreasing rearrangement (w.r.t.m) of f given by

$$f_*(t) := \inf\{s > 0 : \|m\|(\{w \in \Omega : |f(w)| > s\}) \leq t\}.$$

As far as we know, there is no description regarding the spaces obtained when applying limiting interpolation methods to Banach couples formed by function spaces associated to vector measures.

As far as we know, there is no description regarding the spaces obtained when applying limiting interpolation methods to Banach couples formed by function spaces associated to vector measures. We have obtained some results about this question when considering $(0, q; J)$-methods.

As far as we know, there is no description regarding the spaces obtained when applying limiting interpolation methods to Banach couples formed by function spaces associated to vector measures. We have obtained some results about this question when considering $(0, q; J)$-methods.

Our main result in this sense is the following:

Theorem 1

For $1 < q \leq \infty$, it holds that

$$\left(L^\infty(m), L^1(m)\right)_{0,q;J} = \left(L^\infty(m), L^1_w(m)\right)_{0,q;J} = L^{\infty,q}(\log L)^{-1}(\|m\|).$$

To establish Theorem 1, it is very helpful to us the next important description of $(0, q; J)$-spaces in terms of the K-functional:

Theorem (Cobos, Fernández-Cabrera, Kühn and Ullrich, J. Funct. Anal. (2009))

If A_0, A_1 are Banach spaces such that $A_0 \subseteq A_1$ and $1 < q \leq \infty$, then

$$(A_0, A_1)_{0,q;J} = (A_0, A_1)_{\log,q;K},$$

where the space $\bar{A}_{\log,q;K} = (A_0, A_1)_{\log,q;K}$, $1 < q \leq \infty$, consists of all elements $a \in A_1$ for which the following norm is finite

$$\|a\|_{\log,q;K} := \begin{cases} \left(\int_1^\infty \left[\frac{K(t,a)}{1+\log t} \right]^q \frac{dt}{t} \right)^{\frac{1}{q}}, & 1 < q < \infty, \\ \sup_{t>1} \left\{ \frac{K(t,a)}{1+\log t} \right\}, & q = \infty. \end{cases}$$

F. Cobos, L. M. Fernández-Cabrera, T. Kühn and T. Ullrich, *On an extreme class of real interpolation spaces*, J. Funct. Anal. **256** (2009), 2321–2366.

These estimates for the K-functional that apply to the Banach couples $(L^\infty(m), L^1_w(m))$ and $(L^\infty(m), L^1(m))$ are very useful to us too:

Theorem (Fernández, Mayoral and Naranjo, J. Math. Anal. Appl. (2011))

If $f \in L^1_w(m)$, then

$$f_*(t) \preccurlyeq K\left(t^{-1}, f; L^\infty(m), L^1_w(m)\right).$$

On the other hand, if $f \in L^1(m)$, then

$$K\left(t, f; L^\infty(m), L^1(m)\right) \preccurlyeq t \int_0^{t^{-1}} f_*(s)ds.$$

A. Fernández, F. Mayoral and F. Naranjo, *Real interpolation method on spaces of scalar integrable functions with respect to vector measures*, J. Math. Anal. Appl. **376** (2011), 203–211.

Theorem 1

For $1 < q \leq \infty$, it holds that

$$\left(L^{\infty}(m), L^{1}(m)\right)_{0,q;J} = \left(L^{\infty}(m), L^{1}_{w}(m)\right)_{0,q;J} = L^{\infty,q}(\log L)^{-1}(\|m\|).$$

Theorem 1

For $1 < q \leq \infty$, it holds that

$$\left(L^\infty(m), L^1(m)\right)_{0,q;J} = \left(L^\infty(m), L^1_w(m)\right)_{0,q;J} = L^{\infty,q}(\log L)^{-1}(\|m\|).$$

Sketch of the proof

We assume that $q < \infty$ (if $q = \infty$ the proof follows with menor changes). Put $m_\Omega := \|m\|(\Omega)$. Note that $f_*(t) = 0$ if $t \geq m_\Omega$. Due to $L^1(m) \subseteq L^1_w(m)$,

$$\left(L^\infty(m), L^1(m)\right)_{0,q;J} \subseteq \left(L^\infty(m), L^1_w(m)\right)_{0,q;J}.$$

In order to see that

$$\left(L^\infty(m), L^1_w(m)\right)_{0,q;J} \subseteq L^{\infty,q}(\log L)^{-1}(\|m\|),$$

take any $f \in \left(L^\infty(m), L^1_w(m)\right)_{0,q;J}$. Since $f_*(t) \preccurlyeq K\left(t^{-1}, f; L^\infty(m), L^1_w(m)\right)$,

$$\|f\|_{L^{\infty,q}(\log L)^{-1}(\|m\|)} = \left(\int_0^{m_\Omega} \left[(1+|\log t|)^{-1} f_*(t)\right]^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

$$\preccurlyeq \left(\int_0^{m_\Omega} \left[(1+|\log t|)^{-1} K(t^{-1}, f; L^\infty(m), L^1_w(m))\right]^q \frac{dt}{t}\right)^{\frac{1}{q}}.$$

The change of variables $u = \dfrac{m_\Omega}{t}$ gives that

$$\|f\|_{L^{\infty,q}(\log L)^{-1}(\|m\|)}$$
$$\preccurlyeq \left(\int_1^\infty \left[\left(1+\left|\log \frac{m_\Omega}{u}\right|\right)^{-1} K\left(\frac{u}{m_\Omega}, f; L^\infty(m), L^1_w(m)\right)\right]^q \frac{du}{u}\right)^{\frac{1}{q}}$$
$$\leq \max\left\{1, \frac{1}{m_\Omega}\right\} \left(\int_1^\infty \left[\left(1+\left|\log \frac{m_\Omega}{u}\right|\right)^{-1} K(u, f; L^\infty(m), L^1_w(m))\right]^q \frac{du}{u}\right)^{\frac{1}{q}}.$$

Using that

$$\left(1+\left|\log\left(\frac{m_\Omega}{u}\right)\right|\right)^{-1} \leq \frac{1+|\log m_\Omega|}{1+|\log u|}$$

and keeping in mind the equivalent description of $(\cdot,\cdot)_{0,q;J}$ as $(\cdot,\cdot)_{\log,q;K}$,

$$\|f\|_{L^{\infty,q}(\log L)^{-1}(\|m\|)}$$
$$\preccurlyeq \max\left\{1, \frac{1}{m_\Omega}\right\}(1+|\log m_\Omega|) \left(\int_1^\infty \left[\frac{K\left(u, f; L^\infty(m), L^1_w(m)\right)}{1+\log u}\right]^q \frac{du}{u}\right)^{\frac{1}{q}}$$
$$= \max\left\{1, \frac{1}{m_\Omega}\right\}(1+|\log m_\Omega|) \, \|f\|_{(L^\infty(m), L^1_w(m))_{\log,q;K}} \simeq \|f\|_{(L^\infty(m), L^1_w(m))_{0,q;J}}.$$

Now we prove that

$$L^{\infty,q}(\log L)^{-1}(\|m\|) \subseteq \left(L^{\infty}(m), L^{1}(m)\right)_{0,q;J}.$$

We will use Hardy inequality (for non-increasing functions) with the weight $W(t) = \dfrac{1}{t(1+|\log t|)^q}$.

Theorem (Ariño and Muckenhoupt, Trans. Amer. Math. Soc. (1990))

If $1 \leq q < \infty$ and W is a non-negative function on $(0,+\infty)$, are equivalent:

1) There exists $C > 0$ such that the Hardy inequality

$$\int_0^\infty \left[\frac{1}{t}\int_0^t g(s)\,ds\right]^q W(t)\,dt \leq C\int_0^\infty g(t)^q W(t)\,dt$$

holds for every non-negative non-increasing function g on $(0,+\infty)$.

2) W belongs to the class of functions B_q, i.e. there exists $C > 0$ for which

$$\int_r^\infty \frac{W(t)}{t^q}\,dt \leq \frac{C}{r^q}\int_0^r W(t)\,dt, \quad \text{for all } r > 0.$$

We note that for the function $W(t) = \dfrac{1}{t(1+|\log t|)^q}$, it holds that

$$\int_r^\infty \frac{W(t)}{t^q}dt \leq \frac{q-1}{r^q}\int_0^r W(t)dt, \quad \text{for all } \ r>0. \qquad (2)$$

In fact, since $t \mapsto t(1+|\log t|)$ is a non-decreasing function on $(0,\infty)$,

$$\begin{aligned}
\int_r^\infty \frac{W(t)}{t^q}dt &= \int_r^\infty \frac{1}{t^{q+1}(1+|\log t|)^q}dt \\
&\leq \frac{1}{r^{q-1}(1+|\log r|)^{q-1}}\int_r^\infty \frac{1}{t^2(1+|\log t|)}dt \\
&\leq \frac{1}{r^{q-1}(1+|\log r|)^{q-1}}\int_r^\infty \frac{1}{t^2}dt = \frac{1}{r^q(1+|\log r|)^{q-1}}.
\end{aligned}$$

In other words,

$$\int_r^\infty \frac{W(t)}{t^q}dt \leq \frac{1}{r^q(1+|\log r|)^{q-1}}, \quad \text{for every } r>0.$$

On the other hand, if $0 < r \leq 1$, it follows that

$$\int_0^r W(t)dt = \int_0^r \frac{1}{t(1+|\log t|)^q}dt = \frac{1}{q-1} \cdot \frac{1}{(1+|\log r|)^{q-1}},$$

and clearly, for $r > 1$,

$$\int_0^r W(t)dt \geq \int_0^1 \frac{1}{t(1+|\log t|)^q}dt = \frac{1}{q-1}.$$

Thus,

$$\int_0^r W(t)dt \geq \frac{1}{q-1} \cdot \frac{1}{(1+|\log r|)^{q-1}}, \quad \text{for each } r > 0.$$

Therefore, we have seen the validity of the following inequalities

$$\int_r^\infty \frac{W(t)}{t^q}dt \leq \frac{1}{r^q(1+|\log r|)^{q-1}}, \; r>0. \tag{3}$$

and

$$\int_0^r W(t)dt \geq \frac{1}{q-1}\cdot\frac{1}{(1+|\log r|)^{q-1}}, \; r>0. \tag{4}$$

As a consequence of (3) and (4), we get the promised inequality (2) because

$$\int_r^\infty \frac{W(t)}{t^q}dt \leq \frac{1}{r^q(1+|\log r|)^{q-1}} \leq \frac{q-1}{r^q}\int_0^r W(t)dt, \quad \text{for all } r>0.$$

Hardy inequality with $W(t)$ implies that, for every $f \in L^{\infty,q}(\log L)^{-1}(\|m\|)$,

$$\left(\int_0^\infty \frac{1}{t(1+|\log t|)^q}\left[\frac{1}{t}\int_0^t f_*(s)ds\right]^q dt\right)^{\frac{1}{q}} \tag{5}$$

$$\preccurlyeq \quad \left(\int_0^\infty \left[\frac{f_*(t)}{1+|\log t|}\right]^q \frac{dt}{t}\right)^{\frac{1}{q}} = \|f\|_{L^{\infty,q}(\log L)^{-1}(\|m\|)}.$$

It follows in particular that the function $\frac{1}{t}\int_0^t f_*(s)ds$ is finite a.e., and hence $f \in L^1(\|m\|) \subseteq L^1(m)$.

Thus, for any $f \in L^{\infty,q}(\log L)^{-1}(\|m\|)$, using the description of $(\cdot,\cdot)_{0,q;J}$ as $(\cdot,\cdot)_{\log,q;K}$ and the inequality $K\left(t,f;L^{\infty}(m),L^{1}(m)\right) \preccurlyeq t\int_0^{t^{-1}} f_*(s)ds$,

$$\|f\|_{(L^{\infty}(m),L^{1}(m))_{0,q;J}} \simeq \|f\|_{(L^{\infty}(m),L^{1}(m))_{\log,q;K}}$$

$$= \left(\int_1^{\infty}\left[\frac{K\left(t,f;L^{\infty}(m),L^{1}(m)\right)}{1+\log t}\right]^q\frac{dt}{t}\right)^{\frac{1}{q}} \preccurlyeq \left(\int_1^{\infty}\left[\frac{t\int_0^{t^{-1}} f_*(s)ds}{1+\log t}\right]^q\frac{dt}{t}\right)^{\frac{1}{q}}$$

$$= \left(\int_0^{1}\frac{1}{u(1-\log u)^q}\left[\frac{1}{u}\int_0^{u} f_*(s)ds\right]^q du\right)^{\frac{1}{q}} \preccurlyeq \|f\|_{L^{\infty,q}(\log L)^{-1}(\|m\|)}.$$

Then $L^{\infty,q}(\log L)^{-1}(\|m\|) \subseteq \left(L^{\infty}(m),L^{1}(m)\right)_{0,q;J}$ is proved. □

REMARK Applying certain results on variants of Hardy inequality with appropriate parameters, such as those established in

C. Bennett and K. Rudnic, *On Lorentz-Zygmund spaces*, Dissertationes Math. (Rozprawy Mat.) **175** (1980), pp. 67

W. D. Evans, B. Opic and L. Pick, *Real interpolation with logarithmic functors*, J. Inequal. Appl. **7** (2002), 187–269.

it is possible to obtain directly the validity of (5), and thus to deduce that $f \in L^1(\|m\|) \subseteq L^1(m)$ if $f \in L^{\infty,q}(\log L)^{-1}(\|m\|)$. Then, reasoning as in the proof of Theorem 1, it follows that $\|f\|_{(L^\infty(m),L^1(m))_{0,q;J}} \preccurlyeq \|f\|_{L^{\infty,q}(\log L)^{-1}(\|m\|)}$. However, we have preferred to show the concrete calculations for the validity of (2) with the function $W(t) = \dfrac{1}{t(1+|\log t|)^q}$, which leads to the validity of (5). □

Combining the following abstract results,

Theorem A (Cobos, Fernández-Cabrera, Kühn and Ullrich, J. Funct. Anal. (2009))

Assume that A_0, A_1 are Banach spaces such that $A_0 \subseteq A_1$. Let $0 < \theta_0 < \theta_1 < 1$, $1 \leq q \leq \infty$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Then

$$((A_0, A_1)_{\theta_0,q}, (A_0, A_1)_{\theta_1,q})_{0,q;J} = \left\{ f \in A_1 : \left(\int_1^\infty \left[\frac{K(t, f; A_0, A_1)}{t^{\theta_0}(1 + \log t)^{1/q'}} \right]^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\}.$$

Theorem B (Cobos, Fernández-Cabrera, Kühn and Ullrich, J. Funct. Anal. (2009))

Assume that A_0, A_1 are Banach spaces such that $A_0 \subseteq A_1$. Let $0 < \theta < 1$, $1 < q \leq \infty$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Then

i) $((A_0, A_1)_{\theta,q}, A_1)_{0,q;J} = \left\{ f \in A_1 : \left(\int_1^\infty \left[\frac{K(t, f; A_0, A_1)}{t^{\theta}(1 + \log t)^{1/q'}} \right]^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\}.$

ii) $(A_0, (A_0, A_1)_{\theta,q})_{0,q;J} = (A_0, A_1)_{0,q;J}$.

and Theorem 1 (or similar arguments to those used in its proof), we can obtain a description of the limiting J-spaces for other different couples.

Corollary 2

Let $1 < r < p < \infty$, $1 < q \leq \infty$ and $\frac{1}{q} + \frac{1}{q'} = 1$. It holds that:

a) $\big(L^{p,q}(\|m\|), L^1(m)\big)_{0,q;J} = \big(L^{p,q}(\|m\|), L^1_w(m)\big)_{0,q;J} = L^{p,q}(\log L)^{-\frac{1}{q'}}(\|m\|)$.

b) $(L^\infty(m), L^{p,q}(\|m\|))_{0,q;J} = L^{\infty,q}(\log L)^{-1}(\|m\|)$.

c) $(L^{p,q}(\|m\|), L^{r,q}(\|m\|))_{0,q;J} = L^{p,q}(\log L)^{-\frac{1}{q'}}(\|m\|)$.

Corollary 2

Let $1 < r < p < \infty$, $1 < q \leq \infty$ and $\dfrac{1}{q} + \dfrac{1}{q'} = 1$. It holds that:

a) $\big(L^{p,q}(\|m\|), L^1(m)\big)_{0,q;J} = \big(L^{p,q}(\|m\|), L^1_w(m)\big)_{0,q;J} = L^{p,q}(\log L)^{-\frac{1}{q'}}(\|m\|)$.

b) $(L^\infty(m), L^{p,q}(\|m\|))_{0,q;J} = L^{\infty,q}(\log L)^{-1}(\|m\|)$.

c) $(L^{p,q}(\|m\|), L^{r,q}(\|m\|))_{0,q;J} = L^{p,q}(\log L)^{-\frac{1}{q'}}(\|m\|)$.

Furthermore, taking into account the inclusion relationship between the spaces $L^{p,q}(\|m\|)$, $L^p(m)$ and $L^p_w(m)$, and Theorem 1 and Corollary 2, it is possible to deduce the next result.

Corollary 3

For any $1 < p, q < \infty$,

$$(L^\infty(m), L^p(m))_{0,q;J} = (L^\infty(m), L^p_w(m))_{0,q;J} = L^{\infty,q}(\log L)^{-1}(\|m\|).$$

Corollary 2

Let $1 < r < p < \infty$, $1 < q \leq \infty$ and $\frac{1}{q} + \frac{1}{q'} = 1$. It holds that:

a) $\left(L^{p,q}(\|m\|), L^1(m)\right)_{0,q;J} = \left(L^{p,q}(\|m\|), L^1_w(m)\right)_{0,q;J} = L^{p,q}(\log L)^{-\frac{1}{q'}}(\|m\|)$.

b) $(L^\infty(m), L^{p,q}(\|m\|))_{0,q;J} = L^{\infty,q}(\log L)^{-1}(\|m\|)$.

c) $(L^{p,q}(\|m\|), L^{r,q}(\|m\|))_{0,q;J} = L^{p,q}(\log L)^{-\frac{1}{q'}}(\|m\|)$.

Furthermore, taking into account the inclusion relationship between the spaces $L^{p,q}(\|m\|)$, $L^p(m)$ and $L^p_w(m)$, and Theorem 1 and Corollary 2, it is possible to deduce the next result.

Corollary 3

For any $1 < p, q < \infty$,

$$(L^\infty(m), L^p(m))_{0,q;J} = (L^\infty(m), L^p_w(m))_{0,q;J} = L^{\infty,q}(\log L)^{-1}(\|m\|).$$

Proof

For $1 < q \leq p < \infty$, it holds that $L^{p,q}(\|m\|) \subseteq L^p(\|m\|) \subseteq L^p(m) \subseteq L^1(m)$. Then, it follows from Corollary 2b) and Theorem 1 that

$$L^{\infty,q}(\log L)^{-1}(\|m\|) = (L^\infty(m), L^{p,q}(\|m\|))_{0,\, q;\, J} \subseteq (L^\infty(m), L^p(\|m\|))_{0,q;J}$$

$$\subseteq (L^\infty(m), L^p(m))_{0,q;J} \subseteq \left(L^\infty(m), L^1(m)\right)_{0,q;J} = L^{\infty,q}(\log L)^{-1}(\|m\|).$$

Similarly, if $1 < p < q < \infty$, then $L^q(\|m\|) \subseteq L^p(\|m\|) \subseteq L^p(m) \subseteq L^1(m)$. Applying Corollary 2b) and Theorem 1, we have that

$$L^{\infty,q}(\log L)^{-1}(\|m\|) = (L^\infty(m), L^q(\|m\|))_{0,q;J} \subseteq (L^\infty(m), L^p(\|m\|))_{0,q;J}$$
$$\subseteq (L^\infty(m), L^p(m))_{0,q;J} \subseteq \left(L^\infty(m), L^1(m)\right)_{0,q;J} = L^{\infty,q}(\log L)^{-1}(\|m\|).$$

In conclusion

$$(L^\infty(m), L^p(m))_{0,q;J} = L^{\infty,q}(\log L)^{-1}(\|m\|), \text{ for } 1 < p, q < \infty.$$

By analogous arguments, it holds that

$$(L^\infty(m), L^p_w(m))_{0,q;J} = L^{\infty,q}(\log L)^{-1}(\|m\|), \text{ for } 1 < p, q < \infty.$$

and the proof is finished. □

Similarly, if $1 < p < q < \infty$, then $L^q(\|m\|) \subseteq L^p(\|m\|) \subseteq L^p(m) \subseteq L^1(m)$. Applying Corollary 2b) and Theorem 1, we have that

$$L^{\infty,q}(\log L)^{-1}(\|m\|) = (L^\infty(m), L^q(\|m\|))_{0,q;J} \subseteq (L^\infty(m), L^p(\|m\|))_{0,q;J}$$
$$\subseteq (L^\infty(m), L^p(m))_{0,q;J} \subseteq \left(L^\infty(m), L^1(m)\right)_{0,q;J} = L^{\infty,q}(\log L)^{-1}(\|m\|).$$

In conclusion

$$(L^\infty(m), L^p(m))_{0,q;J} = L^{\infty,q}(\log L)^{-1}(\|m\|), \text{ for } 1 < p, q < \infty.$$

By analogous arguments, it holds that

$$(L^\infty(m), L^p_w(m))_{0,q;J} = L^{\infty,q}(\log L)^{-1}(\|m\|), \text{ for } 1 < p, q < \infty.$$

and the proof is finished. □

Theorem 1 and Corollaries 2 and 3 provide versions in the setting of vector measures of some known results established for the case of positive scalar measures. If in particular m is a finite positive scalar measure, these results can be derived from ours.

Another significant application of scalar integrable function spaces with respect to a vector measure refers to the so-called optimal domain of an operator T acting from a Banach function space X into a Banach space E.

Another significant application of scalar integrable function spaces with respect to a vector measure refers to the so-called optimal domain of an operator T acting from a Banach function space X into a Banach space E.

Namely, *if* X *is an order continuous Banach function space on a finite measure space* (Ω, Σ, μ), *under some natural condition on* T, $L^1(m_T)$ *is the optimal domain for* T *within the class of order continuous Banach function spaces (here, the* E*-valued function* $m_T(A) := T(\chi_A)$, $A \in \Sigma$, *is the vector measure associated to* T), that is, $L^1(m_T)$ is the largest space (in that class of spaces) to which T can be extended as a continuous operator, still with values in E.

G. P. Curbera and W. J. Ricker, *Optimal domains for kernel operators via interpolation*, Math. Nachr. **244** (2002), 47–63.

Another significant application of scalar integrable function spaces with respect to a vector measure refers to the so-called optimal domain of an operator T acting from a Banach function space X into a Banach space E.

Namely, *if X is an order continuous Banach function space on a finite measure space (Ω, Σ, μ), under some natural condition on T, $L^1(m_T)$ is the optimal domain for T within the class of order continuous Banach function spaces (here, the E-valued function $m_T(A) := T(\chi_A)$, $A \in \Sigma$, is the vector measure associated to T)*, that is, $L^1(m_T)$ is the largest space (in that class of spaces) to which T can be extended as a continuous operator, still with values in E.

G. P. Curbera and W. J. Ricker, *Optimal domains for kernel operators via interpolation*, Math. Nachr. **244** (2002), 47–63.

The space $L^p(m_T)$ has also an important optimality property if T is a *p-th power factorable operator* from X into E, in the sense that *$L^p(m_T)$ is maximal among all order continuous Banach function spaces Y that continuously contain X and such that T has an extension from Y into E which is itself p-th power factorable.*

S. Okada, W. J. Ricker and E. A. Sánchez-Pérez, "Optimal domain and integral extension of operators acting in function spaces", Operator Theory: Advances and Applications, Vol. **180**, Birkhäuser-Verlag, Basel, 2008.

• By a *Banach function space* X *on a finite measure space* (Ω, Σ, μ), or *on* μ for short, we mean that X is an ideal of the space of (μ-a.e. equivalence classes of) measurable functions $L^0(\mu)$, endowed with a complete norm $\|\cdot\|_X$ that is compatible with the μ-a.e. order and such that $L^\infty(\mu) \subseteq X$.

A Banach function space X is *order continuous* if every order bounded increasing sequence in X is norm convergent.

• By a *Banach function space* X *on a finite measure space* (Ω, Σ, μ), or *on* μ for short, we mean that X is an ideal of the space of (μ-a.e. equivalence classes of) measurable functions $L^0(\mu)$, endowed with a complete norm $\|\cdot\|_X$ that is compatible with the μ-a.e. order and such that $L^\infty(\mu) \subseteq X$.

A Banach function space X is *order continuous* if every order bounded increasing sequence in X is norm convergent.

• Let X be a Banach function space on a finite measure space (Ω, Σ, μ) and let E be a Banach space. Given $1 \leq p < \infty$, an operator $T : X \to E$ is said to be p-th power factorable if there is a constant $C > 0$ such that

$$\|T(f)\|_E \leq C \|f\|_{X_{[p]}} = C \left\| |f|^{\frac{1}{p}} \right\|_X^p, \ f \in X.$$

This class of operators has turned out to be useful for analyzing some factorization properties of operators between function spaces.

S. Okada, W. J. Ricker and E. A. Sánchez-Pérez, "Optimal domain and integral extension of operators acting in function spaces", Operator Theory: Advances and Applications, Vol. **180**, Birkhäuser-Verlag, Basel, 2008.

O. Galdames and E. A. Sánchez-Pérez., *Factorizing kernel operators*, Integral Equ. Oper. Theory **75** (2013), 13–29.

The interpolation properties of p-th power factorable operators when considering the complex method were studied in

R. del Campo, A. Fernández, O. Galdames, F. Mayoral and F. Naranjo, *Complex interpolation of operators and optimal domains*, Integral Equ. Oper. Theory **80** (2014), 229–238.

The interpolation properties of p-th power factorable operators when considering the complex method were studied in

R. del Campo, A. Fernández, O. Galdames, F. Mayoral and F. Naranjo, *Complex interpolation of operators and optimal domains*, Integral Equ. Oper. Theory **80** (2014), 229–238.

whereas their behaviour under the classical real method has been investigated in

R. del Campo, A. Fernández, F. Mayoral, F. Naranjo and E. A. Sánchez–Pérez, *Lorentz spaces of vector measures and real interpolation of operators*, Quaest. Math. **43** (2020), 591–609.

The interpolation properties of p-th power factorable operators when considering the complex method were studied in

R. del Campo, A. Fernández, O. Galdames, F. Mayoral and F. Naranjo, *Complex interpolation of operators and optimal domains*, Integral Equ. Oper. Theory **80** (2014), 229–238.

whereas their behaviour under the classical real method has been investigated in

R. del Campo, A. Fernández, F. Mayoral, F. Naranjo and E. A. Sánchez–Pérez, *Lorentz spaces of vector measures and real interpolation of operators*, Quaest. Math. **43** (2020), 591–609.

In both cases the authors deal with operators that satisfy an additional (but natural) condition, namely to be μ-determined.

The interpolation properties of p-th power factorable operators when considering the complex method were studied in

R. del Campo, A. Fernández, O. Galdames, F. Mayoral and F. Naranjo, *Complex interpolation of operators and optimal domains*, Integral Equ. Oper. Theory **80** (2014), 229–238.

whereas their behaviour under the classical real method has been investigated in

R. del Campo, A. Fernández, F. Mayoral, F. Naranjo and E. A. Sánchez–Pérez, *Lorentz spaces of vector measures and real interpolation of operators*, Quaest. Math. **43** (2020), 591–609.

In both cases the authors deal with operators that satisfy an additional (but natural) condition, namely to be μ-determined.

As far as we know, nothing is known about interpolation of p-th power factorable operators by limiting methods. We have obtained some positive results when interpolating by $(1, q; K)$-methods.

Some additional definitions and notation:

• Given an order continuous Banach function space X on a finite measure space (Ω, Σ, μ) and a Banach space E, $T : X \to E$ is called μ-determined if μ and m_T have exactly the same null sets, where $m_T : \Sigma \to E$ is the vector measure associated to T, given by $m_T(A) := T(\chi_A)$.

Some additional definitions and notation:

• Given an order continuous Banach function space X on a finite measure space (Ω, Σ, μ) and a Banach space E, $T : X \to E$ is called μ-determined if μ and m_T have exactly the same null sets, where $m_T : \Sigma \to E$ is the vector measure associated to T, given by $m_T(A) := T(\chi_A)$. *In that case* $L^p(m_T)$, $1 \leq p < \infty$, *is an order continuous Banach function space on* (Ω, Σ, μ).

S. Okada, W. J. Ricker and E. A. Sánchez-Pérez, "Optimal domain and integral extension of operators acting in function spaces", Operator Theory: Advances and Applications, Vol. **180**, Birkhäuser-Verlag, Basel, 2008.

Some additional definitions and notation:

• Given an order continuous Banach function space X on a finite measure space (Ω, Σ, μ) and a Banach space E, $T : X \to E$ is called μ-determined if μ and m_T have exactly the same null sets, where $m_T : \Sigma \to E$ is the vector measure associated to T, given by $m_T(A) := T(\chi_A)$. *In that case* $L^p(m_T)$, $1 \leq p < \infty$, *is an order continuous Banach function space on* (Ω, Σ, μ).

S. Okada, W. J. Ricker and E. A. Sánchez-Pérez, "Optimal domain and integral extension of operators acting in function spaces", Operator Theory: Advances and Applications, Vol. **180**, Birkhäuser-Verlag, Basel, 2008.

• If $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ are Banach couples such that $A_0 \subseteq A_1$ and $B_0 \subseteq B_1$, it is said to be that T is an admissible operator between the couples $\bar{A}$ and $\bar{B}$ when T is a continuous linear operator from A_1 to B_1, whose restriction to A_0 defines a continuous linear operator from A_0 to B_0. In that case $T : (A_0, A_1)_{1,q;K} \to (B_0, B_1)_{1,q;K}$ is a continuous operator, with

$$\|T\|_{(A_0,A_1)_{1,q;K},(B_0,B_1)_{1,q;K}} \preccurlyeq \|T\|_{A_1,B_1} \left[1 + \max\left\{0, \log \frac{\|T\|_{A_0,B_0}}{\|T\|_{A_1,B_1}}\right\}\right].$$

F. Cobos, L. M. Fernández-Cabrera, T. Kühn and T. Ullrich, *On an extreme class of real interpolation spaces*, J. Funct. Anal. **256** (2009), 2321–2366.

- For an admissible operator T between $\bar{X} = (X_0, X_1)$ and $\bar{E} = (E_0, E_1)$, where $X_0 \subseteq X_1$ and $E_0 \subseteq E_1$, we denote by T_i the restriction $T : X_i \to E_i$, $i = 0, 1$, and by $T_{1,q;K}$ the operator $T : (X_0, X_1)_{1,q;K} \to (E_0, E_1)_{1,q;K}$. We also set $m_i := m_{T_i}$ $(i = 0, 1,)$ and $m_{1,q;K} := m_{T_{1,q;K}}$.

• For an admissible operator T between $\bar{X} = (X_0, X_1)$ and $\bar{E} = (E_0, E_1)$, where $X_0 \subseteq X_1$ and $E_0 \subseteq E_1$, we denote by T_i the restriction $T : X_i \to E_i$, $i = 0, 1$, and by $T_{1,q;K}$ the operator $T : (X_0, X_1)_{1,q;K} \to (E_0, E_1)_{1,q;K}$. We also set $m_i := m_{T_i}$ $(i = 0, 1,)$ and $m_{1,q;K} := m_{T_{1,q;K}}$.

Our precise interpolation result on p-th power factorable operators is the following:

Theorem 6

Let $\bar{X} = (X_0, X_1)$ be a Banach couple of order continuous Banach function spaces on the same finite measure space such that $X_0 \subseteq X_1$, and let $\bar{E} = (E_0, E_1)$ be a couple of Banach spaces with $E_0 \subseteq E_1$. Assume that T is an admissible operator between the couples $\bar{X}$ and $\bar{E}$ and moreover that T_1 is μ-determined. When $T_0 : X_0 \to E_0$ and $T_1 : X_1 \to E_1$ are p-th power factorable operators for some $1 < p < \infty$, then

$$T_{1,q;K} : (X_0, X_1)_{1,q;K} \to (E_0, E_1)_{1,q;K}$$

is also p-th power factorable for every $1 \leq q < \infty$.

• For an admissible operator T between $\bar{X} = (X_0, X_1)$ and $\bar{E} = (E_0, E_1)$, where $X_0 \subseteq X_1$ and $E_0 \subseteq E_1$, we denote by T_i the restriction $T : X_i \to E_i$, $i = 0, 1$, and by $T_{1,q;K}$ the operator $T : (X_0, X_1)_{1,q;K} \to (E_0, E_1)_{1,q;K}$. We also set $m_i := m_{T_i}$ $(i = 0, 1,)$ and $m_{1,q;K} := m_{T_{1,q;K}}$.

Our precise interpolation result on p-th power factorable operators is the following:

Theorem 6

Let $\bar{X} = (X_0, X_1)$ be a Banach couple of order continuous Banach function spaces on the same finite measure space such that $X_0 \subseteq X_1$, and let $\bar{E} = (E_0, E_1)$ be a couple of Banach spaces with $E_0 \subseteq E_1$. Assume that T is an admissible operator between the couples $\bar{X}$ and $\bar{E}$ and moreover that T_1 is μ-determined. When $T_0 : X_0 \to E_0$ and $T_1 : X_1 \to E_1$ are p-th power factorable operators for some $1 < p < \infty$, then

$$T_{1,q;K} : (X_0, X_1)_{1,q;K} \to (E_0, E_1)_{1,q;K}$$

is also p-th power factorable for every $1 \leq q < \infty$.

It follows from this characterization for a μ-determined p-th power factorable operator:

• *If $T : X \to E$ is a μ-determined operator, then*

T is p-th power factorable if and only if $X \subseteq L^p(m_T)$

and from our Theorems 4 and 5:

Theorem 4

Let $\bar{X} = (X_0, X_1)$ be a Banach couple of order continuous Banach function spaces on the same finite measure space such that $X_0 \subseteq X_1$, and let $\bar{E} = (E_0, E_1)$ be a couple of Banach spaces with $E_0 \subseteq E_1$. Assume that T is an admissible operator between the couples $\bar{X}$ and $\bar{E}$ and moreover that T_1 is μ-determined. For every $1 \leq q < \infty$, it holds that

$$\left(L^1(m_0), L^1(m_1)\right)_{1,q;K} \subseteq L^1(m_{1,q;K}).$$

and from our Theorems 4 and 5:

Theorem 4

Let $\bar{X} = (X_0, X_1)$ be a Banach couple of order continuous Banach function spaces on the same finite measure space such that $X_0 \subseteq X_1$, and let $\bar{E} = (E_0, E_1)$ be a couple of Banach spaces with $E_0 \subseteq E_1$. Assume that T is an admissible operator between the couples $\bar{X}$ and $\bar{E}$ and moreover that T_1 is μ-determined. For every $1 \leq q < \infty$, it holds that

$$\left(L^1(m_0), L^1(m_1)\right)_{1,q;K} \subseteq L^1(m_{1,q;K}).$$

Theorem 4 relates the limiting K-space $\left(L^1(m_0), L^1(m_1)\right)_{1,q;K}$ of the ordered Banach couple formed by the optimal domains of T_0 and T_1 with the optimal domain $L^1(m_{1,q;K})$ of $T_{1,q;K}$. To establish it, we use the crucial fact that $L^1(m_T)$ is the (order continuous) optimal domain for the operator T.

and from our Theorems 4 and 5:

Theorem 4

Let $\bar{X} = (X_0, X_1)$ be a Banach couple of order continuous Banach function spaces on the same finite measure space such that $X_0 \subseteq X_1$, and let $\bar{E} = (E_0, E_1)$ be a couple of Banach spaces with $E_0 \subseteq E_1$. Assume that T is an admissible operator between the couples $\bar{X}$ and $\bar{E}$ and moreover that T_1 is μ-determined. For every $1 \leq q < \infty$, it holds that

$$\left(L^1(m_0), L^1(m_1)\right)_{1,q;K} \subseteq L^1(m_{1,q;K}).$$

Theorem 4 relates the limiting K-space $\left(L^1(m_0), L^1(m_1)\right)_{1,q;K}$ of the ordered Banach couple formed by the optimal domains of T_0 and T_1 with the optimal domain $L^1(m_{1,q;K})$ of $T_{1,q;K}$. To establish it, we use the crucial fact that $L^1(m_T)$ is the (order continuous) optimal domain for the operator T.

Theorem 5

Under the same hypotheses as Theorem 4, for $1 < p < \infty$ and $1 \leq q < \infty$, it holds that

$$(L^p(m_0), L^p(m_1))_{1,q;K} \subseteq L^p\left(m_{1,q;K}\right).$$

Theorem 5

Let $\bar{X} = (X_0, X_1)$ be a Banach couple of order continuous Banach function spaces on the same finite measure space such that $X_0 \subseteq X_1$, and let $\bar{E} = (E_0, E_1)$ be a couple of Banach spaces with $E_0 \subseteq E_1$. Assume that T is an admissible operator between the couples $\bar{X}$ and $\bar{E}$ and moreover that T_1 is μ-determined. For $1 < p < \infty$ and $1 \leq q < \infty$, it holds that

$$(L^p(m_0), L^p(m_1))_{1,q;K} \subseteq L^p\left(m_{1,q;K}\right).$$

Sketch of the proof of Theorem 5

Using that

$$(L^p(m_0), L^p(m_1))_{1,q;K} \subseteq (L^p(m_0), L^p(m_1))_{1,p\,q;K}\,,$$

it is sufficient to check that

$$(L^p(m_0), L^p(m_1))_{1,p\,q;K} \subseteq L^p(m_{1,q;K}).$$

Keeping in mind Theorem 4, we will prove that

$$|f|^p \in \left(L^1(m_0), L^1(m_1)\right)_{1,q;K} \subseteq L^1(m_{1,q;K}), \text{ for } f \in (L^p(m_0), L^p(m_1))_{1,p\,q;K}\,.$$

Applying a known estimate of the K-functional for p-convexifications given in

L. Maligranda, *The K-functional for p-convexifications*, Positivity **17** (2013), 707–710.

we obtain that

$$\begin{aligned} \||f|^p\|^q_{(L^1(m_0),L^1(m_1))_{1,q;K}} &= \int_1^\infty \left[\frac{K\left(t,|f|^p;L^1(m_0),L^1(m_1)\right)}{t}\right]^q \frac{dt}{t} \\ &= p\int_1^\infty \left[\frac{K\left(s^p,|f|^p;L^1(m_0),L^1(m_1)\right)}{s^p}\right]^q \frac{ds}{s} \\ &\simeq \int_1^\infty \left[\frac{K\left(s,|f|;L^p(m_0),L^p(m_1)\right)^p}{s^p}\right]^q \frac{ds}{s} \\ &= \|f\|^{p\,q}_{(L^p(m_0),L^p(m_1))_{1,p\,q;K}}. \end{aligned}$$

In fact, it holds that

$$\|f\|_{L^p(m_{1,q;K})} = \||f|^p\|\,\|^{1/p}_{L^1(m_{1,q;K})} \preccurlyeq \||f|^p\|^{1/p}_{(L^1(m_0),L^1(m_1))_{1,q;K}} \simeq \|f\|_{(L^p(m_0),L^p(m_1))_{1,p\,q;K}},$$

and consequently the space $(L^p(m_0),L^p(m_1))_{1,q;K}$ is continuously embedded into $L^p\left(m_{1,q;K}\right)$. □

Finally, we show some interpolation results for other classes of operators related to p-th power factorable operators, such as bidual (p, q)-power-concave operators or q-concave operators.

• Let X be a Banach function space on a finite measure space (Ω, Σ, μ) and let E be a Banach space. For $1 \leq p, q < \infty$, an operator $T : X \to E$ is said to be bidual (p, q)-power-concave if there exists a constant $C > 0$ such that

$$\sum_{j=1}^{n} \| T(f_j)\|_E^{q/p} \leq C \sup\Big\{ \Big\langle \sum_{j=1}^{n} |f_j|^{q/p}, \xi \Big\rangle : \xi \in B({X_{[q]}}^*) \Big\},$$

for all $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in X$.

Finally, we show some interpolation results for other classes of operators related to p-th power factorable operators, such as bidual (p,q)-power-concave operators or q-concave operators.

• Let X be a Banach function space on a finite measure space (Ω, Σ, μ) and let E be a Banach space. For $1 \leq p, q < \infty$, an operator $T : X \to E$ is said to be bidual (p,q)-power-concave if there exists a constant $C > 0$ such that

$$\sum_{j=1}^{n} \|T(f_j)\|_E^{q/p} \leq C \sup\Big\{ \Big\langle \sum_{j=1}^{n} |f_j|^{q/p}, \xi \Big\rangle : \xi \in B({X_{[q]}}^*) \Big\},$$

for all $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in X$.

Bidual $(1,q)$-power-concave operators are also known as bidual q-concave operators. *A bidual q-concave operator is, in particular, q-concave.*

Finally, we show some interpolation results for other classes of operators related to p-th power factorable operators, such as bidual (p,q)-power-concave operators or q-concave operators.

• Let X be a Banach function space on a finite measure space (Ω,Σ,μ) and let E be a Banach space. For $1 \le p,q < \infty$, an operator $T : X \to E$ is said to be bidual (p,q)-power-concave if there exists a constant $C > 0$ such that

$$\sum_{j=1}^{n} \|T(f_j)\|_E^{q/p} \le C \sup\Big\{ \Big\langle \sum_{j=1}^{n} |f_j|^{q/p}, \xi \Big\rangle : \xi \in B({X_{[q]}}^*) \Big\},$$

for all $n \in \mathbb{N}$ and $f_1,\ldots,f_n \in X$.

Bidual $(1,q)$-power-concave operators are also known as bidual q-concave operators. *A bidual q-concave operator is, in particular, q-concave.*

• *If $T : X \to E$ is μ-determined (being X order continuous), then T is bidual (p,q)-power-concave if and only if there is a function $0 < w \in L^1(\mu)$ such that $X \subseteq L^q(w\,d\mu) \subseteq L^p(m_T)$*, where

$$L^q(w\,d\mu) := \Big\{ f \in L^0(\mu) : \int_\Omega |f|^q w d\mu < \infty \Big\}$$

endowed with the norm $\|f\|_{L^q(w\,d\mu)} := \|\,|f|^q w\,\|_{L^1(\mu)}^{1/q}$.

Theorem 7

Let $\bar{X} = (X_0, X_1)$ be a Banach couple of order continuous Banach function spaces on the same finite measure space such that $X_0 \subseteq X_1$, and let $\bar{E} = (E_0, E_1)$ be a couple of Banach spaces with $E_0 \subseteq E_1$. Assume that T is an admissible operator between the couples $\bar{X}$ and $\bar{E}$ and moreover that T_1 is μ-determined. Whenever $T_0 : X_0 \to E_0$ and $T_1 : X_1 \to E_1$ are bidual (p, q)-power-concave operators for some $1 \leq p < \infty$ and $1 \leq q < \infty$,

$$T_{1,q;K} : (X_0, X_1)_{1,q;K} \to (E_0, E_1)_{1,q;K}$$

is also bidual (p, q)-power-concave.

Theorem 7

Let $\bar{X} = (X_0, X_1)$ be a Banach couple of order continuous Banach function spaces on the same finite measure space such that $X_0 \subseteq X_1$, and let $\bar{E} = (E_0, E_1)$ be a couple of Banach spaces with $E_0 \subseteq E_1$. Assume that T is an admissible operator between the couples $\bar{X}$ and $\bar{E}$ and moreover that T_1 is μ-determined. Whenever $T_0 : X_0 \to E_0$ and $T_1 : X_1 \to E_1$ are bidual (p, q)-power-concave operators for some $1 \leq p < \infty$ and $1 \leq q < \infty$,

$$T_{1,q;K} : (X_0, X_1)_{1,q;K} \to (E_0, E_1)_{1,q;K}$$

is also bidual (p, q)-power-concave.

Its proof uses Theorem 5 and the next fact proved by Cobos, Fernández-Cabrera, Kühn and Ullrich

- For $1 \leq q < \infty$ and any weights $w_0 \geq w_1 > 0$ μ-a.e., it holds that

$$\left(L^q(w_0\, d\mu), L^q(w_1\, d\mu)\right)_{1,q;K} = L^q(w\, d\mu), \text{ where } w := w_1\left(1 + \frac{1}{q}\log\frac{w_0}{w_1}\right).$$

Theorem 7

Let $\bar{X} = (X_0, X_1)$ be a Banach couple of order continuous Banach function spaces on the same finite measure space such that $X_0 \subseteq X_1$, and let $\bar{E} = (E_0, E_1)$ be a couple of Banach spaces with $E_0 \subseteq E_1$. Assume that T is an admissible operator between the couples $\bar{X}$ and $\bar{E}$ and moreover that T_1 is μ-determined. Whenever $T_0 : X_0 \to E_0$ and $T_1 : X_1 \to E_1$ are bidual (p, q)-power-concave operators for some $1 \leq p < \infty$ and $1 \leq q < \infty$,

$$T_{1,q;K} : (X_0, X_1)_{1,q;K} \to (E_0, E_1)_{1,q;K}$$

is also bidual (p, q)-power-concave.

Its proof uses Theorem 5 and the next fact proved by Cobos, Fernández-Cabrera, Kühn and Ullrich

- For $1 \leq q < \infty$ and any weights $w_0 \geq w_1 > 0$ μ-a.e., it holds that

$$(L^q(w_0\, d\mu), L^q(w_1\, d\mu))_{1,q;K} = L^q(w\, d\mu), \text{ where } w := w_1\left(1 + \frac{1}{q}\log\frac{w_0}{w_1}\right).$$

Observe that $w \in L^1(\mu)$ whenever $w_i \in L^1(\mu)$ for $i = 0, 1$, since it is not difficult to check that $0 < w \leq \frac{1}{q} w_0 + \left(1 - \frac{1}{q}\right) w_1$ μ-a.e.

Corollary 8

Let $\bar{X} = (X_0, X_1)$ be a Banach couple of order continuous Banach function spaces on the same finite measure space such that $X_0 \subseteq X_1$, and let $\bar{E} = (E_0, E_1)$ be a couple of Banach spaces with $E_0 \subseteq E_1$. Assume that T is an admissible operator between the couples $\bar{X}$ and $\bar{E}$ and moreover that T_1 is μ-determined. In addition, suppose that X_0 and X_1 are q-convex for $1 \leq q < \infty$. If the operators $T_0 : X_0 \to E_0$ and $T_1 : X_1 \to E_1$ are q-concave,

$$T_{1,q;K} : (X_0, X_1)_{1,q;K} \to (E_0, E_1)_{1,q;K}$$

is q-concave.

Corollary 8

Let $\bar{X} = (X_0, X_1)$ be a Banach couple of order continuous Banach function spaces on the same finite measure space such that $X_0 \subseteq X_1$, and let $\bar{E} = (E_0, E_1)$ be a couple of Banach spaces with $E_0 \subseteq E_1$. Assume that T is an admissible operator between the couples $\bar{X}$ and $\bar{E}$ and moreover that T_1 is μ-determined. In addition, suppose that X_0 and X_1 are q-convex for $1 \leq q < \infty$. If the operators $T_0 : X_0 \to E_0$ and $T_1 : X_1 \to E_1$ are q-concave,

$$T_{1,q;K} : (X_0, X_1)_{1,q;K} \to (E_0, E_1)_{1,q;K}$$

is q-concave.

Proof

Due to X_i is a q-convex order continuous Banach function space,

$$T_i : X_i \to E_i \text{ is a } q\text{-concave operator}$$

is equivalent to

$$T_i : X_i \to E_i \text{ is a bidual } (1, q)\text{-power-concave operator.}$$

Applying Theorem 7 with $p = 1$, we deduce that

$$T_{1,q;K} : (X_0, X_1)_{1,q;K} \to (E_0, E_1)_{1,q;K} \text{ is also bidual } (1, q)\text{-power-concave.}$$

In particular, as mentioned, $T_{1,q;K}$ is q-concave. □

Some references

- M. A. Ariño and B. Muckenhoupt, *Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions*, Trans.Amer.Math.Soc. **320** (1990), 727–735.
- P. L. Butzer and H. Berens, "Semi-groups of operators and approximation", Die Grundlehren der mathematischen Wissenschaften **145**, Springer-Verlag New York, Inc., New York, 1967.
- F. Cobos, L. M. Fernández-Cabrera, T. Kühn and T. Ullrich, *On a extreme class of real interpolation spaces*, J. Funct. Anal. **256** (2009), 2321–2366.
- G. P. Curbera, *Operators into L^1 of a vector measure and applications to Banach lattices*, Math. Ann. **293** (1992), 317–330.
- G. P. Curbera and W. J. Ricker, *Optimal domains for kernel operators via interpolation*, Math. Nachr. **244** (2002), 47–63.
- R. del Campo, A. Fernández, F. Mayoral, F. Naranjo and E. A. Sánchez-Pérez, *Lorentz spaces of vector measures and real interpolation of operators*, Quaest. Math. **43** (2020), 591–609.
- W. D. Evans, B. Opic and L. Pick, *Real interpolation with logarithmic functors*, J. Inequal. Appl. **7** (2002), 187–269.
- A. Fernández and A. Manzano, *On spaces of integrable functions associated to vector measures and limiting real interpolation*, J. Math. Anal. Appl. **540** (2024), Article 128573.
- A. Fernández, F. Mayoral and F. Naranjo, *Real interpolation method on spaces of scalar integrable functions with respect to vector measures*, J. Math. Anal. Appl. **376** (2011), 203–211.
- M. E. Gomez and M. Milman, *Extrapolation spaces and almost-everywhere convergence of singular integrals*, J. London Math. Soc. **34** (1986), 305–316.
- J. Gustavsson, *A function parameter in connection with interpolation of Banach spaces*, Math. Scand. **42** (1978), 289–305.
- L. Maligranda, *The K-functional for p-convexifications*, Positivity **17** (2013), 707–710.
- S. Okada, W. J. Ricker and E. A. Sánchez-Pérez, "Optimal domain and integral extension of operators acting in function spaces". Operator Theory: Advances and Applications, Vol. **180**, Birkhäuser-Verlag, Basel, 2008.