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Interpolation of closed ideals of bilinear operators

Fernando COBOS
Departamento de Análisis Matemático y Matemática Aplicada, Facultad de Matemáticas, Universidad Complutense de Madrid, Plaza de Ciencias 3, 28040 Madrid, Spain
E-mail: *cobos@mat.ucm.es*

Luz M. FERNÁNDEZ-CABRERA
Sección Departamental del Departamento de Análisis Matemático y Matemática Aplicada, Facultad de Estudios Estadísticos, Universidad Complutense de Madrid, 28040 Madrid, Spain
E-mail: $luz\_fernandez-c$@*mat.ucm.es*

Antón MARTíNEZ
Departamento de Matemática Aplicada I, Escuela de Ingeniería Industrial, Universidad de Vigo, 36200 Vigo, Spain
E-mail: *antonmar@uvigo.es*

Dedicated to Professor Baldomero Rubio on the occasion of his 85th birthday

Abstract We extend the (outer) measure $\gamma_{\mathcal{I}}$ associated to an operator ideal $\mathcal{I}$ to a measure $\gamma_{\mathfrak{I}}$ for bounded bilinear operators. If $\mathcal{I}$ is surjective and closed, and $\mathfrak{I}$ is the class of those bilinear operators such that $\gamma_{\mathfrak{I}}(T) = 0$, we prove that $\mathfrak{I}$ coincides with the composition bideal $\mathcal{I} \circ \mathfrak{B}$.
If $\mathcal{I}$ satisfies the Σ_r-condition, we establish a simple necessary and sufficient condition for an interpolated operator by the real method to belong to $\mathfrak{I}$. Furthermore, if in addition $\mathcal{I}$ is symmetric, we prove a formula for the measure $\gamma_{\mathfrak{I}}$ of an operator interpolated by the real method. In particular, results apply to weakly compact operators.

Keywords Measures associated to bideals of operators, real interpolation, measure of weak non-compactness of a bilinear operator, adjoint operator of a bilinear operator.

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Secondary 46G25, 47L20, 47H60

1 Introduction

Let E, F be Banach spaces and let $\mathcal{L}(E, F)$ be the space of all bounded linear operators from E into F, endowed with the usual operator norm.

An *operator ideal* $\mathcal{I}$ is a method of ascribing to each pair (E, F) of Banach spaces, a linear subspace $\mathcal{I}(E, F)$ of $\mathcal{L}(E, F)$ such that

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(i) $\mathcal{I}(E,F)$ contains the finite rank operators from E into F,
and

(ii) for all Banach spaces E, F, X, Y, whenever $R \in \mathcal{L}(X,E)$, $T \in \mathcal{I}(E,F)$, $S \in \mathcal{L}(F,Y)$, then the composed operator STR belongs to $\mathcal{I}(X,Y)$.

See the monographs by Pietsch [50, 51] and Diestel, Jarchow and Tonge [31].

The operator ideal $\mathcal{I}$ is said to be *closed* if $\mathcal{I}(E,F)$ is a closed subspace of $\mathcal{L}(E,F)$ for all Banach spaces E and F. Important examples of closed operator ideals are *compact linear operators* $\mathcal{K}$, *weakly compact operators* $\mathcal{W}$, *Banach-Saks operators* $\mathcal{BS}$, *Rosenthal operators* $\mathcal{R}$ and *Asplund* (=*decomposing*) *operators* $\mathcal{A}$. Besides [31, 50, 51], we refer to the papers by Beauzamy [5], Heinrich [43], Edgar [34] and Stegal [54] for properties of these ideals. Each one of these ideals is also injective and surjective. Let us recall that an operator ideal $\mathcal{I}$ is said to be *injective* if for every isometric embedding $\iota \in \mathcal{L}(F,Y)$ and every operator $T \in \mathcal{L}(E,F)$, if follows from $\iota T \in \mathcal{I}(E,Y)$ that $T \in \mathcal{I}(E,F)$. Injectivity means that it does not depend on the size of the target space F whether or not $T \in \mathcal{L}(E,F)$ belongs to $\mathcal{I}$.

The ideal $\mathcal{I}$ is called *surjective* if for every metric surjection $Q \in \mathcal{L}(X,E)$ and every operator $T \in \mathcal{L}(E,F)$ if follows from $TQ \in \mathcal{I}(X,F)$ that $T \in \mathcal{I}(E,F)$. This property means that it does not depend on the size of the source space E whether or not $T \in \mathcal{L}(E,F)$ belongs to $\mathcal{I}$.

Interpolation properties of the ideals $\mathcal{K}$, $\mathcal{W}$, $\mathcal{BS}$, $\mathcal{R}$ and $\mathcal{A}$ have been extensively studied. Results are of two kinds, depending on whether the ideal $\mathcal{I}$ satisfies the Σ_r-condition (see Section 3 for the definition). Ideals $\mathcal{W}$, $\mathcal{BS}$, $\mathcal{R}$ and $\mathcal{A}$ satisfies the Σ_r-condition. For these ideals $\mathcal{I}$, Heinrich [43] has shown a necessary and sufficient condition for an interpolated operator by the real method to belong to $\mathcal{I}$. As a consequence, he has established that if $\mathcal{I} = \mathcal{W}$, $\mathcal{BS}$, $\mathcal{R}$ or $\mathcal{A}$ then $\mathcal{I}$ has the factorization property. That is to say, if $T \in \mathcal{I}(E,F)$ then there exists a Banach space G whose identity operator I_G belongs to $\mathcal{I}$ and there are bounded linear operators $T_1 \in \mathcal{L}(E,G)$ and $T_2 \in \mathcal{L}(G,F)$ such that $T = T_2T_1$. For the case $\mathcal{I} = \mathcal{W}$, a previous direct proof of the factorization property was given by Davies, Figiel, Johnson and Pelczyński [28], and for the case $\mathcal{I} = \mathcal{BS}$ and $\mathcal{R}$, by Beauzamy [3, 5].

The ideal $\mathcal{K}$ of compact operators does not satisfies the Σ_r-condition. Interpolation results for operators of $\mathcal{K}$ require other techniques. See, for example, the papers by Cwikel [27], Cobos, Kühn and Schonbek [24] and Cobos, Fernández-Martínez and Martínez [23]. The last mentioned paper shows a formula for the measure of non-compactness of an interpolated operator by the real method.

If $\mathcal{I} = \mathcal{K}$, $\mathcal{W}$, $\mathcal{BS}$, $\mathcal{R}$, $\mathcal{A}$ we can also associate to $\mathcal{I}$ a measure $\gamma_{\mathcal{I}}(T)$ which shows the deviation of an operator T from the ideal $\mathcal{I}$: If $T \in \mathcal{L}(E,F)$,

$$\gamma_{\mathcal{I}}(T) = \inf\{\sigma > 0 : T(U_E) \subseteq \sigma U_F + S(U_W),\ S \in \mathcal{I}(W,F),\ W \text{ some Banach space}\}.$$

Here U_E stands for the closed unit ball of E. In fact, operators T of $\mathcal{I}$ are those having $\gamma_{\mathcal{I}}(T) = 0$. The measure $\gamma_{\mathcal{I}}$ was introduced by Astala [2]. When $\mathcal{I} = \mathcal{K}$ then $\gamma_{\mathcal{K}}$ is equal to the measure of non-compactness. If $\mathcal{I} = \mathcal{W}$ then $\gamma_{\mathcal{W}}$ coincides with the measure of weak non-compactness introduced by De Blasi [29]. In addition to [23], we refer to the papers by Cobos, Manzano and Martínez [25], Cobos and Martínez [26] and Fernández-Cabrera and Martínez [36] for the interpolation properties of $\gamma_{\mathcal{I}}$. In this paper we are interested in the corresponding ideals of bilinear operators and their interpolation properties.

Recently it has been shown that compact bilinear operators occur rather naturally in harmonic analysis. See, for example, the papers by Bényi and Torres [8], Bényi and Oh [7], Hu [44], Torres, Xue and Yan [57], Tao, Xue, Yang and Yuan [55] and Tao, Yang, Yang [56]. In fact, commutators of bilinear Calderón-Zygmund operators and multiplication by functions in the subspace CMO of BMO are compact bilinear operators from $L_p(\mathbb{R}^n)\times L_q(\mathbb{R}^n) \longrightarrow L_r(\mathbb{R}^n)$ for $1 < p, q < \infty$, $1/2 < r < \infty$ and $1/r = 1/p+1/q$. This has been a motivation for the study of the interpolation properties of compact bilinear operators. This research was iniciated by Calderón [14] already in his foundational paper on the complex method. Results for the real method have been established in the last few years, in the papers by Fernandez and Silva [35], Fernández-Cabrera and Martínez [37, 38], Mastyło and Silva [49] and Cobos, Fernández-Cabrera and Martínez [18, 19, 21]. These papers show sufficient conditions for an interpolated bilinear operator to be compact. Quantitative estimates in terms of the measure of non-compactness have been also established in the papers by Mastyło and Silva [48], Besoy and Cobos [10] and Cobos, Fernández-Cabrera and Martínez [20].

But for weakly compact bilinear operators and the corresponding classes of bilinear operators to $\mathcal{BS}$, $\mathcal{R}$ and $\mathcal{A}$ the research on their interpolation properties is just starting. Manzano, Rueda and Sánchez-Pérez [46, 47] have established interpolation results in certain degenerate situations, where the couple in the target reduces to a single Banach space or both couples in the source reduce to single Banach spaces. Very recently, the present authors [22] (see also [16]) have established interpolation results for weakly compact bilinear operators acting among arbitrary Banach couples. In the present paper, we continue those investigations.

Given an operator ideal $\mathcal{I}$, we first extend the measure $\gamma_{\mathcal{I}}$ to a measure $\gamma_{\mathfrak{I}}$ for the class of bounded bilinear operators. Among other results, we show in Section 2 that if $\mathcal{I}$ is surjective and closed, then the class $\mathfrak{I}$ of all bounded bilinear operators T such that $\gamma_{\mathfrak{I}}(T) = 0$ coincides with the composition bideal $\mathcal{I} \circ \mathfrak{B}$ (see [11, 39]). In other words, a bounded bilinear operator $T \in \mathfrak{B}(A \times B, E)$ belongs to $\mathfrak{I}$ if and only if there are a Banach space X, a bounded bilinear operator $S : A \times B \longrightarrow X$ and a linear operator $R : X \longrightarrow E$ belonging to $\mathcal{I}$ such that $T = RS$. This condition is also equivalent to the fact that $\widetilde{T} \in \mathcal{I}(A\widehat{\otimes}_\pi B, E)$. Here $A\widehat{\otimes}_\pi B$ is the projective tensor product of the Banach spaces A and B, and $\widetilde{T}$ is the linearization of T. Furthermore, we show that $\gamma_{\mathfrak{I}}(T)$ coincides with $\gamma_{\mathcal{I}}(\widetilde{T})$ and we establish relationships between $\gamma_{\mathfrak{I}}(T)$ and the (inner) measure $\beta_{\mathcal{I}}(T^\times)$ of the adjoint operator $T^\times$ of T.

In Section 3 we study the interpolation properties of the bilinear operators of $\mathfrak{I}$. If $\mathcal{I}$ satisfies the Σ_r-condition, we establish a simple necessary and sufficient condition for an interpolated operator by the real method to belong to $\mathfrak{I}$. For the special case $\mathcal{I} = \mathcal{W}$, we recover a recent result of Cobos and Fernández-Cabrera [16]. Furthermore, we prove a formula for the measure $\gamma_{\mathfrak{I}}$ of an operator interpolated by the real method. In the particular case $\mathcal{I} = \mathcal{W}$, the formula gives a quantitative estimate for the qualitative result establish by the authors in [22, Theorem 3.1].

2 Bideals and measures

In what follows A, B, E stand for Banach spaces on the field $\mathbb{K}$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We put U_A for the closed unit ball of A.

Let $T : A \times B \longrightarrow E$ be a bilinear operator. We say that T is *bounded* if

$$\|T\| = \|T\|_{A\times B,E} = \sup\left\{\|T(a, b)\|_E : a \in U_A\,,\, b \in U_B\right\} < \infty.$$

We write $\mathfrak{B}(A \times B, E)$ for the Banach space of all bounded bilinear operators from $A \times B$ into E with the norm $\|\cdot\|_{A\times B,E}$. If $E = \mathbb{K}$, then we simply put $\mathfrak{B}(A \times B)$ instead of $\mathfrak{B}(A \times B, \mathbb{K})$.

Write

$$\mathfrak{B} = \bigcup_{A,B,E} \mathfrak{B}(A \times B, E).$$

Following [52], we say that a subset $\mathfrak{J}$ of $\mathfrak{B}$ is a *bideal* if the following conditions are satisfied:

(a) $\mathfrak{J}(A \times B, E) := \mathfrak{J} \cap \mathfrak{B}(A \times B, E)$ is a $\mathbb{K}$-vector space.

(b) For each $\phi \in \mathfrak{B}(A \times B)$ and $z \in E$, we have that $\phi \otimes z$ belongs to $\mathfrak{J}(A \times B, E)$, where

$$(\phi \otimes z)(a, b) = \phi(a, b)z.$$

(c) For all Banach spaces A, B, E, X, Y, Z, whenever $S_1 \in \mathcal{L}(X, A)$, $S_2 \in \mathcal{L}(Y, B)$, $T \in \mathfrak{J}(A \times B, E)$, $R \in \mathcal{L}(E, Z)$, then the composed operator $RT(S_1, S_2)$ belongs to $\mathfrak{J}(X \times Y, Z)$. Here $T(S_1, S_2)(x, y) = T(S_1 x, S_2 y)$.

We say that the bideal $\mathfrak{J}$ is *closed* if $\mathfrak{J}(A \times B, E)$ is a closed subspace of $\mathfrak{B}(A \times B, E)$ for all Banach spaces A, B, E.

Injectivity and surjectivity are defined as in the linear case: The bideal $\mathfrak{J}$ is called *injective* if whenever $T \in \mathfrak{B}(A \times B, E)$, $\iota \in \mathcal{L}(E, Y)$ is an isometric embedding and $\iota T \in \mathfrak{J}(A \times B, Y)$, then $T \in \mathfrak{J}(A \times B, E)$.

The bideal $\mathfrak{J}$ is said to be *surjective* if whenever $T \in \mathfrak{B}(A \times B, E)$ and $Q_1 \in \mathcal{L}(X, A)$, $Q_2 \in \mathcal{L}(Y, B)$ are metric surjections with $T(Q_1, Q_2) \in \mathfrak{J}(X \times Y, E)$, then $T \in \mathfrak{J}(A \times B, E)$.

We recall that $Q \in \mathcal{L}(F, W)$ is a metric surjection if the image by Q of the open unit ball of F coincides with the open unit ball of W.

Next we study a version of the measure $\gamma_{\mathcal{I}}$ for bilinear operators. We put

$$U_A \times U_B = \{(x, y) \,:\, x \in U_A, y \in U_B\}.$$

Definition 2.1 Let $\mathcal{I}$ be an operator ideal of linear operators. Given any bilinear operator $T \in \mathfrak{B}(A \times B, E)$ we put

$$\begin{aligned}\gamma_{\mathfrak{J}}(T) &= \gamma_{\mathfrak{J}}(T : A \times B \longrightarrow E)\\ &= \inf\{\sigma > 0 : T(U_A \times U_B) \subseteq R(U_Z) + \sigma U_E, R \in \mathcal{I}(Z, E),\ Z \text{ some Banach space}\}.\end{aligned}$$

We denote by $\mathfrak{J}$ the collection of all $T \in \mathfrak{B}$ such that $\gamma_{\mathfrak{J}}(T) = 0$.

Lemma 2.2 *The set $\mathfrak{J}$ is a surjective and closed bideal.*

Proof First we check statements (a), (b), (c) of the definition of bideal. Take any $T_1, T_2 \in \mathfrak{J}(A \times B, E)$ and any $\lambda_1, \lambda_2 \in \mathbb{K}$. For any $\varepsilon > 0$, there are Banach spaces Z_1, Z_2 and operators $R_1 \in \mathcal{I}(Z_1, E)$, $R_2 \in \mathcal{I}(Z_2, E)$ such that

$$T_j(U_A \times U_B) \subseteq R_j(U_{Z_j}) + \frac{\varepsilon}{2|\lambda_j|} U_E, \quad j = 1, 2.$$

Let $(Z_1 \times Z_2)_{\ell_\infty}$ be the product of Z_1 and Z_2, normed by
$\|(x_1, x_2)\| = \max\{\|x_1\|_{Z_1}, \|x_2\|_{Z_2}\}$. The operator $R : (Z_1 \times Z_2)_{\ell_\infty} \longrightarrow E$, defined by $R(x_1, x_2) = \lambda_1 R_1 x_1 + \lambda_2 R_2 x_2$ belongs to $\mathcal{I}$ because it is the sum of the composed operators

$$\begin{array}{ccccccc}(Z_1 \times Z_2)_{\ell_\infty} & \longrightarrow & Z_j & \longrightarrow & Z_j & \longrightarrow & E\\ (x_1, x_2) & \longrightarrow & x_j & \longrightarrow & \lambda_j x_j & \longrightarrow & R_j(\lambda_j x_j).\end{array}$$

Since

$$\begin{aligned}(\lambda_1 T_1 + \lambda_2 T_2)(U_A \times U_B) &\subseteq \lambda_1 T_1(U_A \times U_B) + \lambda_2 T_2(U_A \times U_B)\\ &\subseteq \lambda_1 R_1(U_{Z_1}) + \frac{\varepsilon}{2} U_E + \lambda_2 R_2(U_{Z_2}) + \frac{\varepsilon}{2} U_E \subseteq R(U_{(Z_1 \times Z_2)_{\ell_\infty}}) + \varepsilon U_E\end{aligned}$$

we derive that $\lambda_1 T_1 + \lambda_2 T_2$ belongs to $\mathfrak{J}$.

Take any $\phi \in \mathfrak{B}(A\times B)$ and any $z \in E$. In order to show that $\phi\otimes z$ belongs to $\mathfrak{I}$, observe that the linear operator $R : \mathbb{K} \longrightarrow E$ defined by $R\lambda = \lambda\|\phi\|z$ belongs to $\mathcal{I}$. We have

$$(\phi\otimes z)(U_A \times U_B) = \big\{\|\phi\|\frac{\phi(a,b)}{\|\phi\|}z \,:\, a \in U_A\,,\, b \in U_B\big\} \subseteq R(U_{\mathbb{K}}).$$

So $\phi\otimes z \in \mathfrak{I}$.

Let now $R \in \mathcal{L}(E,Z)$ and $T \in \mathfrak{I}(A\times B,E)$. Given any $\varepsilon > 0$, there exist a Banach space X and a linear operator $V \in \mathcal{I}(X,E)$ such that

$$T(U_A \times U_B) \subseteq V(U_X) + \frac{\varepsilon}{\|R\|}U_E.$$

Therefore, using the operator $RV \in \mathcal{I}(X,Z)$, we have

$$RT(U_A \times U_B) \subseteq (RV)(U_X) + \varepsilon U_Z.$$

Whence $RT \in \mathfrak{I}$.

To complete the proof of (c), take any bounded linear operators $S_1 \in \mathcal{L}(X_1,A)$, $S_2 \in \mathcal{L}(X_2,B)$ and any $T \in \mathfrak{I}(A\times B,E)$. Given any $\varepsilon > 0$, there are Banach spaces Z and a linear operator $R \in \mathcal{I}(Z,E)$ such that

$$T(U_A \times U_B) \subseteq \frac{1}{\|S_1\|\|S_2\|}R(U_Z) + \frac{\varepsilon}{\|S_1\|\|S_2\|}U_E.$$

Whence

$$T(S_1,S_2)(U_{X_1} \times U_{X_2}) \subseteq \|S_1\|\|S_2\|T(U_A \times U_B) \subseteq R(U_Z) + \varepsilon U_E.$$

Consequently, $\mathfrak{I}$ is a bideal.

In order to show that $\mathfrak{I}$ is closed, assume that $(T_n) \subseteq \mathfrak{I}(A\times B,E)$, $T \in \mathfrak{B}(A\times B,E)$ and $\lim_{n\to\infty} T_n = T$ in $\mathfrak{B}(A\times B,E)$. Take any $\varepsilon > 0$. There is $n \in \mathbb{N}$ such that
$\|T - T_n\|_{A\times B,E} < \varepsilon/2$. So

$$T(U_A \times U_B) \subseteq (T - T_n)(U_A \times U_B) + T_n(U_A \times U_B) \subseteq \frac{\varepsilon}{2}\,U_E + T_n(U_A \times U_B).$$

Since $T_n \in \mathfrak{I}(A\times B,E)$, there exist Z and $R \in \mathcal{I}(Z,E)$ such that

$$T_n(U_A \times U_B) \subseteq R(U_Z) + \frac{\varepsilon}{2}\,U_E.$$

Hence

$$T(U_A \times U_B) \subseteq R(U_Z) + \varepsilon\,U_E.$$

This shows that $T \in \mathfrak{I}(A\times B,E)$.

Finally, we show that $\mathfrak{I}$ is surjective. Let $Q_1 \in \mathcal{L}(X_1,A)$, $Q_2 \in \mathcal{L}(X_2,B)$ metric surjections and let $T \in \mathfrak{B}(A\times B,E)$ such that $T(Q_1,Q_2) \in \mathfrak{I}(X_1\times X_2,E)$. Given any $\varepsilon > 0$, there are Z and $R \in \mathcal{I}(Z,E)$ such that

$$T(Q_1,Q_2)(U_{X_1} \times U_{X_2}) \subseteq \frac{1}{(1+\varepsilon)^2}\,R(U_Z) + \frac{\varepsilon}{(1+\varepsilon)^2}\,U_E.$$

For any $x \in U_A$, $y \in U_B$, since $\|\frac{1}{1+\varepsilon}x\|_A < 1$ and $\|\frac{1}{1+\varepsilon}y\|_B < 1$, there exit $u \in X_1$ with $\|u\|_{X_1} < 1$ and $v \in X_2$ with $\|v\|_{X_2} < 1$ such that $Q_1u = \frac{1}{1+\varepsilon}x$, $Q_2v = \frac{1}{1+\varepsilon}y$. Therefore

$$T(U_A \times U_B) \subseteq (1+\varepsilon)^2\Big(\frac{1}{(1+\varepsilon)^2}\,R(U_Z) + \frac{\varepsilon}{(1+\varepsilon)^2}\,U_E\Big) = R(U_Z) + \varepsilon\,U_E.$$

This completes the proof. □

The following result shows some properties of $\gamma_{\mathfrak{I}}$.

Lemma 2.3 *Let $\mathcal{I}$ be an operator ideal.*

(i) If $T \in \mathfrak{B}(A \times B, E)$ then $\gamma_{\mathfrak{I}}(T) \leq \|T\|_{A\times B,E}$.

(ii) If $T_1, T_2 \in \mathfrak{B}(A \times B, E)$ then $\gamma_{\mathfrak{I}}(T_1 + T_2) \leq \gamma_{\mathfrak{I}}(T_1) + \gamma_{\mathfrak{I}}(T_2)$.

(iii) If $T \in \mathfrak{B}(A \times B, E)$ and $S \in \mathcal{L}(E, W)$, then $\gamma_{\mathfrak{I}}(ST) \leq \gamma_{\mathcal{I}}(S)\, \gamma_{\mathfrak{I}}(T)$.

(iv) If $T \in \mathfrak{B}(A \times B, E)$ and $S \in \mathcal{L}(E, W)$, then $\gamma_{\mathfrak{I}}(ST) \leq \|S\|_{E,W}\, \gamma_{\mathfrak{I}}(T)$.

Proof Clearly $T(U_A \times U_B) \subseteq \|T\|_{A\times B,E}\, U_E$. Since $R = 0$ belongs to $\mathcal{I}$, statement (i) follows.

To establish (ii), take any $\sigma_j > \gamma_{\mathfrak{I}}(T_j)$. There are Banach spaces X_1, X_2 and linear operators $R_1 \in \mathcal{I}(X_1, E)$, $R_2 \in \mathcal{I}(X_2, E)$ such that

$$T_j(U_A \times U_B) \subseteq R_j(U_{X_j}) + \sigma_j U_E\ , \quad j = 1, 2.$$

Put $X = (X_1 \times X_2)_{\ell_\infty}$ and consider the linear operator $R(x_1, x_2) = R_1 x_1 + R_2 x_2$. Proceeding as in the proof of Lemma 2.2, one can check that $R \in \mathcal{I}(X, E)$. Since

$$\begin{aligned}(T_1 + T_2)(U_A \times U_B) &\subseteq R_1(U_{X_1}) + \sigma_1 U_E + R_2(U_{X_2}) + \sigma_2 U_E \\ &= R(U_{(X_1\times X_2)_{\ell_\infty}}) + (\sigma_1 + \sigma_2) U_E,\end{aligned}$$

passing to the infimum on σ_1 and σ_2, we derive (ii).

To prove (iii), take any $\sigma > \gamma_{\mathfrak{I}}(T)$ and $\mu > \gamma_{\mathcal{I}}(S)$. We can find Banach spaces Z, Y and linear operators $R \in \mathcal{I}(Z, E)$, $V \in \mathcal{I}(Y, W)$ such that

$$T(U_A \times U_B) \subseteq R(U_Z) + \sigma U_E \quad \text{and} \quad S(U_E) \subseteq V(U_Y) + \mu U_W.$$

Consider the linear operators

$$P : (Z \times Y)_{\ell_\infty} \longrightarrow W,\ P(z, y) = (SR)z, \quad L : (Z \times Y)_{\ell_\infty} \longrightarrow W,\ L(z, y) = \sigma V y.$$

They belong to $\mathcal{I}$, so $(P + L) \in \mathcal{I}$. We have

$$\begin{aligned}(ST)(U_A \times U_B) &\subseteq S(R(U_Z) + \sigma U_E) \subseteq (SR)(U_Z) + \sigma V(U_Y) + \sigma\mu U_W \\ &= (P + L)(U_{(Z\times Y)_{\ell_\infty}}) + \sigma\mu U_W.\end{aligned}$$

Then statement (iii) is established by taking the infimum on σ and μ.

Finally, (iv) follows from (iii) using that $\gamma_{\mathcal{I}}(S) \leq \|S\|_{E,W}$. □

Next we consider the tensor product $A \otimes B$ of the Banach spaces A, B. For $u \in A \otimes B$, we put

$$\pi(u) = \inf \Big\{ \sum_{k=1}^{n} \|a_k\|_A \|b_k\|_B \ :\ u = \sum_{k=1}^{n} a_k \otimes b_k \Big\}.$$

We denote by $A\widehat{\otimes}_\pi B$ the *projective tensor product* of A and B. That is to say, $A\widehat{\otimes}_\pi B$ is the completion of $(A \otimes B, \pi)$ (see [30, 32]). We still denote by π the norm of $A\widehat{\otimes}_\pi B$.

Clearly $\chi : A \times B \longrightarrow A\widehat{\otimes}_\pi B,\ \chi(a, b) = a \otimes b$, is a bounded bilinear operator with $\|\chi\|_{A\times B, A\widehat{\otimes}_\pi B} \leq 1$. To any $T \in \mathfrak{B}(A \times B, E)$ we can associate the bounded linear operator $\widetilde{T} \in \mathcal{L}(A\widehat{\otimes}_\pi B, E)$ defined by

$$\widetilde{T}\big(\sum_{k=1}^{n} a_k \otimes b_k\big) = \sum_{k=1}^{n} T(a_k, b_k)\ , \quad \sum_{k=1}^{n} a_k \otimes b_k \in A \otimes B.$$

Note that $T = \widetilde{T}\chi$. The linear operator $\widetilde{T}$ is called the *linearization* of T. The correspondence $T \longrightarrow \widetilde{T}$ is a isometric isomorphism between $\mathfrak{B}(A \times B, E)$ and $\mathcal{L}(A\widehat{\otimes}_\pi B, E)$ (see [32, Theorem 1, p. 230]).

The following inclusions hold

(2.1) $T(U_A \times U_B) \subseteq \widetilde{T}(U_{A\widehat{\otimes}_\pi B})$,

(2.2) $\widetilde{T}(U_{A\widehat{\otimes}_\pi B}) \subseteq \overline{co\,(T(U_A \times U_B))}$,

where $co\,(T(U_A \times U_B))$ stands for the convex hull of $T(U_A \times U_B)$. See [53, Proposition 2.2] or [22, Lemma 2.2].

Lemma 2.4 *Let $\mathcal{I}$ be an operator ideal. For any $T \in \mathfrak{B}(A \times B, E)$, we have*

$$\gamma_{\mathfrak{I}}(T) = \gamma_{\mathcal{I}}(\widetilde{T}).$$

Proof Inclusion (2.1) implies that $\gamma_{\mathfrak{I}}(T) \leq \gamma_{\mathcal{I}}(\widetilde{T})$. To establish the converse inequality, assume that

$$T(U_A \times U_B) \subseteq R(U_Z) + \sigma U_E$$

for some $\sigma > 0$, some Banach space Z and $R \in \mathcal{I}(Z, E)$. Since the set $R(U_Z) + \sigma U_E$ is convex, we get

$$co\,(T(U_A \times U_B)) \subseteq R(U_Z) + \sigma U_E.$$

Take any $w \in U_{A\widehat{\otimes}_\pi B}$. Then, by (2.2), we have

$$\widetilde{T}w \in \overline{co\,(T(U_A \times U_B))} \subseteq \overline{R(U_Z) + \sigma U_E}.$$

Whence, given any $\varepsilon > 0$, we can find $x \in U_Z$ and $u \in U_E$ such that $\|\widetilde{T}w - Rx - \sigma u\|_E \leq \varepsilon$. So,

$$\|\widetilde{T}w - Rx\|_E \leq \|\widetilde{T}w - Rx - \sigma u\|_E + \|\sigma u\|_E \leq \varepsilon + \sigma.$$

This yields that

$$\widetilde{T}(U_{A\widehat{\otimes}_\pi B}) \subseteq R(U_Z) + (\varepsilon + \sigma)U_E.$$

Therefore, $\gamma_{\mathcal{I}}(\widetilde{T}) \leq \gamma_{\mathfrak{I}}(T)$. □

The following result is a direct consequence of Lemma 2.4, Definition 2.1 and [2, Theorem 3.11].

Corollary 2.5 *Let $\mathcal{I}$ be an operator ideal and let T be a bilinear operator.*

(a) If the linearization $\widetilde{T}$ of T belongs to $\mathcal{I}$, then T belongs to the bideal $\mathfrak{I}$.

(b) If $\mathcal{I}$ is a surjective closed ideal and T belongs to $\mathfrak{I}$, then $\widetilde{T}$ belongs to $\mathcal{I}$.

Corollary 2.6 *Let $\mathcal{I}$ be an injective surjective closed operator ideal. Then the bideal $\mathfrak{I}$ is also injective.*

Proof Let $\iota \in \mathcal{L}(E, Y)$ be an isometric embedding and let $T \in \mathfrak{B}(A \times B, E)$ such that $\iota T \in \mathfrak{I}(A \times B, Y)$. Since $\widetilde{\iota T}(a \otimes b) = \iota T(a, b) = \iota(\widetilde{T}(a \otimes b))$, it follows from Corollary 2.5 that $\iota\widetilde{T} \in \mathcal{I}(A\widehat{\otimes}_\pi B, Y)$. Injectivity of $\mathcal{I}$ yields that $\widetilde{T} \in \mathcal{I}(A\widehat{\otimes}_\pi B, E)$ and therefore $T \in \mathfrak{I}(A \times B, E)$. □

Corollary 2.7 *Let $\mathcal{I}$ be a surjective and closed operator ideal. A bilinear operator $T \in \mathfrak{B}(A \times B, E)$ belongs to the bideal $\mathfrak{I}$ if and only if there are a Banach space X, a bilinear operator $S \in \mathfrak{B}(A \times B, X)$ and a linear operator $R \in \mathcal{I}(X, E)$ such that $T = RS$.*

Proof The result is a consequence of Corollary 2.5 and [11, Proposition 3.2]. □

Corollary 2.7 shows that $\mathfrak{I}$ is the *composition ideal* generated by $\mathcal{I}$ (see [39, 7.3] and [11, §3]).

Next we consider the two concrete cases of bideals $\mathfrak{I}$ associated to compact operators and to weakly compact operators. For $\mathcal{I} = \mathcal{K}$ the ideal of compact linear operators, we designate by $\mathfrak{K}$ the corresponding bideal of bilinear operators. It follows from (2.1), (2.2) and Mazur theorem [33, V.2.6] that $T \in \mathfrak{K}(A \times B, E)$

if and only if $T(U_A \times U_B)$ is relatively compact in E. Other equivalent statements can be found in [8, Proposition 1].

To measure the distance from a bilinear operator $T \in \mathfrak{B}(A \times B, E)$ to $\mathfrak{K}$ it is natural to use the (ball) *measure of non-compactness* $\omega(T)$ defined by the infimum of the set of all $\sigma > 0$ for which there exists a finite set $\{w_1, \cdots, w_s\} \subseteq E$ such that

$$T(U_A \times U_B) \subseteq \bigcup_{j=1}^{s} \{w_j + \sigma U_E\}.$$

Proposition 2.8 *For the ideal $\mathcal{K}$ of compact operators and for any $T \in \mathfrak{B}(A \times B, E)$, we have*

$$\gamma_{\mathfrak{K}}(T) = \omega(T).$$

Proof Assume that there are $\sigma > 0$, a Banach space Z and a compact operator $R \in \mathcal{K}(Z, E)$ such that

$$T(U_A \times U_B) \subseteq R(U_Z) + \sigma U_E.$$

For any $\varepsilon > 0$, there is a finite set $\{w_1, \cdots, w_s\} \subseteq E$ such that $R(U_Z) \subseteq \bigcup_{j=1}^{s} \{w_j + \varepsilon U_E\}$. Hence

$$T(U_A \times U_B) \subseteq \bigcup_{j=1}^{s} \{w_j + (\sigma + \varepsilon) U_E\}.$$

This yields that $\omega(T) \leq \gamma_{\mathfrak{K}}(T)$.

In order to establish the converse inequality, suppose that $\sigma > 0$ and the finite set $\{w_1, \cdots, w_r\} \subseteq E$ satisfy that

$$T(U_A \times U_B) \subseteq \bigcup_{j=1}^{r} \{w_j + \sigma U_E\}.$$

Let $\{x_1, \cdots, x_n\}$ be all the linearly independent vectors among $\{w_1, \cdots, w_r\}$. So $\mathrm{sp}\{x_1, \cdots, x_n\} = \mathrm{sp}\{w_1, \cdots, w_r\}$. Using the Hahn-Banach theorem, we can find functionals $f_1, \cdots, f_n$ in the dual space E^* of E such that $f_j(x_k) = \delta_j^k$, $1 \leq j, k \leq n$, where δ_j^k is the Kronecker delta, and $x = \sum_{j=1}^{n} f_j(x) x_j$ for any $x \in \mathrm{sp}\{x_1, \cdots, x_n\}$ (see [40, Corollary IX.5.5]). Let X be the space E with the equivalent norm $\|x\|_X = (\max\{\|w_1\|_E, \cdots, \|w_r\|_E\})^{-1} \|x\|_E$. Put $R : X \longrightarrow E$ for the finite rank operator

$$Rx = \sum_{j=1}^{n} f_j(x) x_j\,, \quad x \in E.$$

Clearly $R \in \mathcal{K}(X, E)$. Moreover, $Rx = x$ for any $x \in \mathrm{sp}\{w_1, \cdots, w_r\}$. Therefore,

$$T(U_A \times U_B) \subseteq \bigcup_{j=1}^{r} \{R(w_j) + \sigma U_E\} \subseteq R(U_X) + \sigma U_E.$$

This yields that $\gamma_{\mathfrak{K}}(T) \leq \omega(T)$ and completes the proof. □

Now consider $\mathcal{I} = \mathcal{W}$ the ideal of weakly compact linear operators and let $\mathfrak{W}$ be the corresponding bideal. By (2.1), (2.2) and Krein-Šmulian theorem [33, Theorem V.6.4], we have that $T \in \mathfrak{W}(A \times B, E)$ if and only if $T(U_A \times U_B)$ is relatively weakly compact.

De Blasi [29] introduced a measure to mark the distance from a linear operator to $\mathcal{W}$. We can extend it naturally to deal with bilinear operators: If $T \in \mathfrak{B}(A \times B, E)$, we write $\eta(T)$ for the infimum of the set of all $\sigma > 0$ for which there is a weakly compact set W in E such that

$$T(U_A \times U_B) \subseteq W + \sigma U_E.$$

Another motivation for introducing $\eta(T)$ is a result of Grothendieck which says that a subset V of a Banach space X is relatively weakly compact if and only if for each $\varepsilon > 0$ there exists a weakly compact subset W of X satisfying that $V \subseteq W + \varepsilon U_X$ (see [1, Theorem 3.44]).

Proposition 2.9 *For the ideal $\mathcal{W}$ of weakly compact operators and for any $T \in \mathfrak{B}(A \times B, E)$, we have*

$$\gamma_{\mathfrak{W}}(T) = \eta(T).$$

Proof Clearly $\eta(T) \leq \gamma_{\mathfrak{W}}(T)$. On the other hand, assume that

$$T(U_A \times U_B) \subseteq W + \sigma U_E.$$

for some weakly compact set W in E and some $\sigma > 0$. Let $F = \overline{\mathrm{sp}\{W\}}$. Applying the Davis-Figiel-Johnson-Pelczyński factorization theorem (see [42, Theorem 286, p. 227] or [1, Theorem 5.37]), there is a reflexive Banach space Z and a bounded linear operator R from Z into F such that $W \subseteq R(U_Z)$. Therefore, $R \in \mathcal{W}(Z, E)$ satisfies that

$$T(U_A \times U_B) \subseteq R(U_Z) + \sigma U_E.$$

This yields that $\gamma_{\mathfrak{W}}(T) \leq \eta(T)$. □

We can also introduce the corresponding bideal $\mathfrak{I}$ to the ideals $\mathcal{I}$ of Banach-Saks operators, Rosenthal operators and Asplund operators by means of Definition 2.1. We denote them by $\mathfrak{BS}$, $\mathfrak{R}$ and $\mathfrak{A}$, respectively. Since any of these three ideals is surjective and closed, it follows from Corollary 2.5 that a bilinear operator $T \in \mathfrak{B}(A \times B, E)$ belongs to $\mathfrak{BS}$ (respectively, to $\mathfrak{R}$ or $\mathfrak{A}$) if and only if its linearization $\widetilde{T}$ is Banach-Saks (respectively, Rosenthal or Asplund) from $A \widehat{\otimes}_\pi B$ into E. For any of these three bideals $\mathfrak{I}$, we can use the measure $\gamma_{\mathfrak{I}}$ to quantify how far is a bilinear operator from $\mathfrak{I}$.

Remark 2.10 Manzano, Rueda and Sánchez-Pérez [47, Definition 3.2] have proposed for marking the distance from a bilinear operator $T \in \mathfrak{B}(A \times B, E)$ to a bideal $\mathfrak{D}$ the functional

$$\Gamma(T) = \inf\{\sigma > 0 : T(U_A \times U_B) \subseteq S(U_X \times U_Y) + \sigma U_E,\ S \in \mathfrak{D}(X \times Y, E),\ X, Y \text{ any Banach spaces}\}.$$

Note that if $\mathcal{I}$ is a surjective and closed operator ideal and we consider the bideal $\mathfrak{I}$ then $\gamma_{\mathfrak{I}} = \Gamma$. Indeed, assume that

$$T(U_A \times U_B) \subseteq S(U_X \times U_Y) + \sigma U_E$$

for some $\sigma > 0$ and $S \in \mathfrak{I}(X \times Y, E)$. By Corollary 2.5, we know that $\widetilde{S} \in \mathcal{I}(X \widehat{\otimes}_\pi Y, E)$. Moreover, according to (2.1), $S(U_X \times U_Y) \subseteq \widetilde{S}(U_{X \widehat{\otimes}_\pi Y})$. Hence

$$T(U_A \times U_B) \subseteq \widetilde{S}(U_{X \widehat{\otimes}_\pi Y}) + \sigma U_E.$$

Therefore $\gamma_{\mathfrak{I}}(T) \leq \Gamma(T)$.

Conversely, suppose that

$$T(U_A \times U_B) \subseteq R(U_X) + \sigma U_E$$

with $R \in \mathcal{I}(X, E)$ and $\sigma > 0$. Then the bilinear operator $S : \mathbb{K} \times X \longrightarrow E$, $S(\lambda, x) = R(\lambda x)$, belongs to $\mathfrak{I}$ because it is the composition of the bounded bilinear operator $(\lambda, x) \longrightarrow \lambda x$ with R (one should have in mind Corollary 2.7). Since $S(U_{\mathbb{K}} \times U_X) = R(U_X)$, we get that

$$T(U_A \times U_B) \subseteq S(U_{\mathbb{K}} \times U_X) + \sigma U_E.$$

This gives that $\Gamma(T) \le \gamma_{\jmath}(T)$ and completes the proof of the equality $\Gamma(T) = \gamma_{\jmath}(T)$.

There is another functional that one can use to measure the distance of a bounded linear operator $S \in \mathcal{L}(E, F)$ to an operator ideal $\mathcal{I}$:

$$\begin{aligned}\beta_{\mathcal{I}}(S) =& \beta_{\mathcal{I}}(S : E \longrightarrow F)\\ =& \inf\{\sigma > 0 : \text{ there is a Banach space } Z \text{ and } R \in \mathcal{I}(E, Z)\\ & \text{such that } \|Sx\|_F \le \sigma\|x\|_E + \|Rx\|_Z,\ x \in E\}.\end{aligned}$$

It was introduced by Tylli [59]. It turns out that $\beta_{\mathcal{I}}(S) = 0$ if and only if S belongs to the closed injective hull of $\mathcal{I}$.

We also recall that an operator ideal $\mathcal{I}$ is called *symmetric* if $T^* \in \mathcal{I}(F^*, E^*)$ whenever $T \in \mathcal{I}(E, F)$ (see [50, 4.4]). Compact operators $\mathcal{K}$ and weakly compact operators $\mathcal{W}$ are examples of symmetric ideals.

Following Ramanujan and Schock [52], if $T \in \mathfrak{B}(A \times B, E)$ we define the *adjoint operator* $T^\times$ of T as the linear map $T^\times : E^* \longrightarrow \mathfrak{B}(A \times B)$ defined by

$$(T^\times f)(a, b) = f[T(a, b)],\ f \in E^*,\ (a, b) \in A \times B.$$

Next we study the relationship between $\beta_{\mathcal{I}}(T^\times)$ and $\gamma_{\jmath}(T)$.

Proposition 2.11 *Let $\mathcal{I}$ be a symmetric operator ideal and let $T \in \mathfrak{B}(A \times B, E)$ with A, B, E Banach spaces. Then*

$$\beta_{\mathcal{I}}(T^\times : E^* \longrightarrow \mathfrak{B}(A \times B)) \le \gamma_{\jmath}(T : A \times B \longrightarrow E).$$

Proof Suppose that there are $\sigma > 0$ and $R \in \mathcal{I}(Z, E)$ such that

$$T(U_A \times U_B) \subseteq R(U_Z) + \sigma U_E.$$

Since $\mathcal{I}$ is symmetric, we have that $R^* \in \mathcal{I}(E^*, Z^*)$. Take any $f \in E^*$. Given any $(a, b) \in U_A \times U_B$, there exist $w \in U_Z$ and $y \in U_E$ such that $T(a, b) = Rw + \sigma y$. Hence

$$|f[T(a, b)]| = |f(Rw) + \sigma f(y)| \le |f(Rw)| + \sigma|f(y)| \le |(R^* f)(w)| + \sigma\|f\|_{E^*} \le \|R^* f\|_{Z^*} + \sigma\|f\|_{E^*}.$$

Whence $\|T^\times f\|_{A\times B,\mathbb{K}} \le \|R^* f\|_{Z^*} + \sigma\|f\|_{E^*}$. This implies that $\beta_{\mathcal{I}}(T^\times) \le \gamma_{\jmath}(T)$. □

To show an inequality in the other direction, let $\tau_E : E \longrightarrow E^{**}$ be the natural embedding,

$$\tau_E(x) = \widehat{x} \quad \text{with} \quad \widehat{x}(f) = f(x) \quad \text{for} \quad f \in E^*.$$

Proposition 2.12 *Let $\mathcal{I}$ be a closed injective and symmetric operator ideal and let $T \in \mathfrak{B}(A \times B, E)$, with A, B, E Banach spaces. Then*

$$\gamma_{\jmath}(\tau_E T : A \times B \longrightarrow E^{**}) \le \beta_{\mathcal{I}}(T^\times : E^* \longrightarrow \mathfrak{B}(A \times B)).$$

Proof Let $W = \overline{T^\times(E^*)}$ which is a Banach space when endowed with the induced norm of $\mathfrak{B}(A \times B)$. We have that $(T^\times)^* \in \mathcal{L}(W^*, E^{**})$.

Let $R : A \times B \longrightarrow W^*$ be the bilinear operator given by $R(a, b)(T^\times f) = f[T(a, b)]$ where $f \in E^*$. For any $a \in A$, $b \in B$ we have

$$\begin{aligned}\|R(a, b)\|_{W^*} =& \sup\{|R(a, b)(T^\times f)| \,:\, \|T^\times f\|_{\mathfrak{B}(A\times B)} \le 1\}\\ =& \sup\{|f[T(a, b)]| \,:\, \|T^\times f\|_{\mathfrak{B}(A\times B)} \le 1\}\\ =& \sup\{|(T^\times f)(a, b)| \,:\, \|T^\times f\|_{\mathfrak{B}(A\times B)} \le 1\} \le \|a\|_A\|b\|_B.\end{aligned}$$

Hence $\|R\|_{A\times B,W^*} \le 1$.

The following factorization holds

$$\begin{array}{ccccc} A\times B & \xrightarrow{\;T\;} & E & \xrightarrow{\;\tau_E\;} & E^{**}\,. \\ & {\scriptstyle R}\searrow & & \nearrow{\scriptstyle (T^\times)^*} & \\ & & W^* & & \end{array}$$

Indeed, for any $f \in E^*$ we have

$$\big((T^\times)^* R(a,b)\big)(f) = R(a,b)(T^\times f) = f[T(a,b)] = \widehat{T(a,b)}(f) = \tau_E(T(a,b))(f).$$

Whence, by Lemma 2.3/(i) and (iii) and [41, Proposition 1.2], we derive that

$$\gamma_{\mathfrak{I}}(\tau_E T) = \gamma_{\mathfrak{I}}\left((T^\times)^* R\right) \le \gamma_{\mathcal{I}}\left((T^\times)^*\right) = \beta_{\mathcal{I}}(T^\times).$$

□

The following extension of the measure $\beta_{\mathcal{I}}$ has been proposed by Manzano, Rueda and Sánchez-Pérez [46] for the case of a bideal $\mathfrak{D}$: For $T \in \mathfrak{B}(A\times B, E)$,

$$\begin{aligned} \beta_{\mathfrak{D}}(T) &= \beta_{\mathfrak{D}}(T : A\times B \longrightarrow E) \\ &= \inf\Big\{\sigma > 0 : \text{ there is a Banach space } Z \text{ and } R \in \mathfrak{D}(A\times B, Z) \text{ such that} \\ &\qquad \|\sum_{j=1}^m T(a_j,b_j)\|_E \le \sigma \sum_{j=1}^m \|a_j\|_A \|b_j\|_B + \|\sum_{j=1}^m R(a_j,b_j)\|_Z, \\ &\qquad\qquad \text{for all } m\in\mathbb{N},\ a_j\in A,\ b_j\in B,\ 1\le j\le m\Big\}. \end{aligned}$$

Next we show that $\mathfrak{I}$ can be also characterized in terms of $\beta_{\mathfrak{I}}$.

Proposition 2.13 *Let $\mathcal{I}$ be a surjective and closed operator ideal and let $T \in \mathfrak{B}(A\times B, E)$ with A, B, E Banach spaces. Then*

$$\beta_{\mathfrak{I}}(T : A\times B \longrightarrow E) = \beta_{\mathcal{I}}(\widetilde{T} : A\widehat{\otimes}_\pi B \longrightarrow E).$$

Proof Assume that there is a Banach space Z and $R \in \mathfrak{I}(A\times B, Z)$ such that

$$\|\sum_{j=1}^m T(a_j,b_j)\|_E \le \sigma \sum_{j=1}^m \|a_j\|_A \|b_j\|_B + \|\sum_{j=1}^m R(a_j,b_j)\|_Z$$

for all $m\in\mathbb{N}$, $a_j\in A$, $b_j \in B$ and $1\le j\le m$. Take any $x\otimes y \in A\otimes B$ and any representation $x\otimes y = \sum_{j=1}^m a_j\otimes b_j$ with $a_j\in A$, $b_j\in B$. We have

$$\begin{aligned} \|\widetilde{T}(x\otimes y)\|_E &= \|\sum_{j=1}^m T(a_j,b_j)\|_E \le \sigma \sum_{j=1}^m \|a_j\|_A\|b_j\|_B + \|\sum_{j=1}^m R(a_j,b_j)\|_Z \\ &= \sigma \sum_{j=1}^m \|a_j\|_A\|b_j\|_B + \|\widetilde{R}(x\otimes y)\|_Z. \end{aligned}$$

Therefore

$$\|\widetilde{T}(x\otimes y)\|_E \le \sigma\pi(x\otimes y) + \|\widetilde{R}(x\otimes y)\|_Z.$$

Take now any $u \in A\widehat{\otimes}_\pi B$. It follows from [32, Proposition 9, p. 227] that there exist sequences $(u_n) \subseteq A \otimes B$ such that $(u_n) \to u$ in $A\widehat{\otimes}_\pi B$. Whence,

$$\|\widetilde{T}u\|_E = \lim_{n\to\infty} \|\widetilde{T}(u_n)\|_E \le \sigma \lim_{n\to\infty} \pi(u_n) + \lim_{n\to\infty} \|\widetilde{R}(u_n)\|_Z = \sigma\pi(u) + \|\widetilde{R}u\|_Z.$$

Since R belongs to $\mathfrak{I}$, according to Corollary 2.5, the linear operator $\widetilde{R}$ belongs to $\mathcal{I}$. Therefore, we derive that $\beta_{\mathcal{I}}(\widetilde{T}) \le \beta_{\mathfrak{I}}(T)$.

Conversely, suppose that there is a Banach space Z and $R \in \mathcal{I}(A\widehat{\otimes}_\pi B, Z)$ such that

$$\|\widetilde{T}z\|_E \le \sigma\pi(z) + \|Rz\|_Z,\ z \in A\widehat{\otimes}_\pi B.$$

Then the bilinear operator $S : A \times B \longrightarrow Z$, defined by $S(a,b) = R(\chi(a,b)) = R(a \otimes b)$, belongs to $\mathfrak{I}$. Moreover, for any $m \in \mathbb{N}$, $(a_j)_{j=1}^m \subseteq A$ and $(b_j)_{j=1}^m \subseteq B$, we have

$$\begin{aligned}\|\sum_{j=1}^m T(a_j,b_j)\|_E &= \|\widetilde{T}\big(\sum_{j=1}^m a_j \otimes b_j\big)\|_E \le \sigma\pi\big(\sum_{j=1}^m a_j \otimes b_j\big) + \|R\big(\sum_{j=1}^m a_j \otimes b_j\big)\|_Z \\ &\le \sigma \sum_{j=1}^m \|a_j\|_A \|b_j\|_B + \|\sum_{j=1}^m S(a_j,b_j)\|_Z.\end{aligned}$$

This yields that $\beta_{\mathfrak{I}}(T) \le \beta_{\mathcal{I}}(\widetilde{T})$ and completes the proof. □

Combining Proposition 2.13 with Corollary 2.5 we obtain the characterization of $\mathfrak{I}$ in terms of $\beta_{\mathfrak{I}}$:

Corollary 2.14 *Let $\mathcal{I}$ be a surjective injective closed operator ideal and let $T \in \mathfrak{B}$. Then*

$$T \in \mathfrak{I} \ \text{ if and only if } \ \beta_{\mathfrak{I}}(T) = 0.$$

Subsequently, we write id_G for the identity operator of the Banach space G.

Recall that an operator ideal $\mathcal{I}$ is said to have the *factorization property* if for every operator $R \in \mathcal{I}(E,F)$ there exists a Banach space G with $id_G \in \mathcal{I}(G,G)$ and operators $T_1 \in \mathcal{L}(E,G)$ and $T_2 \in \mathcal{L}(G,F)$ such that $R = T_2T_1$.

Ideals $\mathcal{W}$, $\mathcal{BS}$, $\mathcal{R}$, $\mathcal{A}$ have the factorization property (see [3, 5, 28, 43]).

We close this section with a result on the structure of $\mathfrak{I}$ when $\mathcal{I}$ has the factorization property. It is an easy consequence of Corollary 2.7.

Corollary 2.15 *Let $\mathcal{I}$ be a surjective closed ideal with the factorization property. The necessary and sufficient condition for $T \in \mathfrak{B}(A \times B, E)$ to belong to $\mathfrak{I}$ is that there are a Banach space G with $id_G \in \mathcal{I}(G,G)$ and operators $T_1 \in \mathfrak{B}(A \times B, G)$ and $T_2 \in \mathcal{L}(G,E)$ such that $T = T_2T_1$.*

3 Real interpolation

By a *Banach couple* $\bar{A} = (A_0, A_1)$ we mean two Banach spaces A_0, A_1 which are continuously embedded in the same Hausdorff topological vector space. Let $A_0 + A_1$ be their sum and let $A_0 \cap A_1$ be their intersection. The norms of these spaces are

$$\begin{aligned}\|a\|_{A_0+A_1} &= \inf\Big\{\|a_0\|_{A_0} + \|a_1\|_{A_1} : a = a_0 + a_1\ ,\ a_j \in A_j\Big\}, \\ \|a\|_{A_0\cap A_1} &= \max\Big\{\|a\|_{A_0}\ ,\ \|a\|_{A_1}\Big\}.\end{aligned}$$

Given any $t > 0$, we may equivalently renorm $A_0 + A_1$ by the *Peetre's K-functional*

$$K(t,a) = K(t,a; A_0, A_1) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1,\ a_j \in A_j\}.$$

For $1 \leq q < \infty$ and $0 < \theta < 1$, the *real interpolation space* realized as a K-space in discrete form $\bar{A}_{\theta,q} = (A_0, A_1)_{\theta,q}$ consists of all $a \in A_0 + A_1$ having a finite norm

$$\|a\|_{\theta,q} = \left(\sum_{m=-\infty}^{\infty} \left(2^{-\theta m} K(2^m, a)\right)^q \right)^{\frac{1}{q}}.$$

We refer to [45] and the monographs [4, 6, 9, 12, 13, 58] for properties of the real interpolation space $(A_0, A_1)_{\theta,q}$. These spaces also make sense for $q = \infty$ and they can be equivalently defined using integrals instead of series, but we are not going to use it here.

There is a constant $C_{\theta,q} > 0$ such that

$$K(t, a) \leq C_{\theta,q}\, t^{\theta}\, \|a\|_{\theta,q}, \;\; a \in (A_0, A_1)_{\theta,q}, \;\; t > 0, \quad \text{(see [9, Theorem 3.1.2])}.$$

Hence, spaces $(A_0, A_1)_{\theta,q}$ are of *class* $\mathcal{C}_K(\theta; \bar{A})$ according to [9, Definition 3.5.1] or [58, Definition 1.10.1].

Let $\bar{B} = (B_0, B_1)$ and $\bar{E} = (E_0, E_1)$ be other Banach couples. We write $T \in \mathfrak{B}(\bar{A} \times \bar{B}, \bar{E})$ if T is a bilinear operator defined on $(A_0 + A_1) \times (B_0 + B_1)$ with values in $E_0 + E_1$ such that the restrictions $T : A_j \times B_j \longrightarrow E_j$ are bounded for $j = 0, 1$. If $T \in \mathfrak{B}(\bar{A} \times \bar{B}, \bar{E})$ then the *bilinear interpolation theorem* establishes that the restriction

$$T : (A_0, A_1)_{\theta,p} \times (B_0, B_1)_{\theta,q} \longrightarrow (E_0, E_1)_{\theta,r}$$

is bounded provided that $0 < \theta < 1$ and $1 \leq p, q, r \leq \infty$ and $1/p + 1/q = 1 + 1/r$ (see [37, 45]).

Subsequently, if A is a Banach space and $\lambda > 0$, we write λA for the space A normed by $\lambda \| \cdot \|_A$.

Let $(E_m)_{m \in \mathbb{Z}}$ be a sequence of Banach spaces and let $(\lambda_m)_{m \in \mathbb{Z}}$ be a sequence of non-negative scalars. For $1 \leq q < \infty$, we denote by $\ell_q(\lambda_m E_m)$ the vector valued ℓ_q-space defined by

$$\ell_q(\lambda_m E_m) = \left\{ x = (x_m) : x_m \in E_m \quad \text{and} \quad \|x\|_{\ell_q(\lambda_m E_m)} = \left(\sum_{m=-\infty}^{\infty} (\lambda_m \|x_m\|_{E_m})^q \right)^{\frac{1}{q}} < \infty \right\}.$$

We denote by $Q_k : \ell_q(\lambda_m E_m) \longrightarrow \lambda_k E_k$ the projection $Q_k(x_m) = x_k$, and by $P_n : \lambda_n E_n \longrightarrow \ell_q(\lambda_m E_m)$ the embedding $P_n x = (\delta_m^n x)$ where δ_m^n is the Kronecker delta.

For $1 < r < \infty$, we say that an operator ideal $\mathcal{I}$ satisfies the Σ_r-*condition* if for any sequences of Banach spaces (E_m), (F_m) and any $T \in \mathcal{L}(\ell_r(E_m), \ell_r(F_m))$, it follows from $Q_k T P_n \in \mathcal{I}(E_n, F_k)$ for any n, k, that $T \in \mathcal{I}(\ell_r(E_m), \ell_r(F_m))$.

Heinrich [43] has shown that $\mathcal{W}$, $\mathcal{BS}$, $\mathcal{R}$ and $\mathcal{A}$ satisfy the Σ_r-condition for $1 < r < \infty$ (see also [17]). On the contrary, the ideal $\mathcal{K}$ does not satisfy the Σ_r-condition for any $1 < r < \infty$ because $id_{\ell_r} \notin \mathcal{K}(\ell_r, \ell_r)$.

Saying that $\mathcal{I}$ satisfies the Σ_r-condition means that the operator $T \in \mathcal{L}(\ell_r(E_m), \ell_r(F_m))$ belongs to $\mathcal{I}$ if and only if all elements of its matrix representation belong to $\mathcal{I}$. Such an ideal must be closed (see [43] or [17]).

As we show next, for some ideals $\mathcal{I}$ satisfying the Σ_r-condition, one can characterize when an interpolated bilinear operator belongs to $\mathfrak{I}$.

Theorem 3.1 *Let $1 < r < \infty$ and let $\mathcal{I}$ be an injective and surjective operator ideal which satisfies the Σ_r-condition. Let $0 < \theta < 1$ and $1 \leq p, q < \infty$ with $1/p + 1/q = 1 + 1/r$. Suppose that $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, $\bar{E} = (E_0, E_1)$ are Banach couples and let $T \in \mathfrak{B}(\bar{A} \times \bar{B}, \bar{E})$. Then a necessary and sufficient condition for*

$$T : (A_0, A_1)_{\theta,p} \times (B_0, B_1)_{\theta,q} \longrightarrow (E_0, E_1)_{\theta,r} \;\; \textit{to belongs to} \;\; \mathfrak{I} \tag{3.1}$$

is that

$$T : (A_0 \cap A_1) \times (B_0 \cap B_1) \longrightarrow E_0 + E_1 \;\; \textit{belongs to} \;\; \mathfrak{I}. \tag{3.2}$$

Proof If we assume (3.1), then (3.2) follows from the bounded continuous embeddings $A_0 \cap A_1 \hookrightarrow (A_0, A_1)_{\theta,p}$, $B_0 \cap B_1 \hookrightarrow (B_0, B_1)_{\theta,q}$ and $(E_0, E_1)_{\theta,r} \hookrightarrow E_0 + E_1$.

In order to show that (3.2) is sufficient for statement (3.1), first observe that, by the bilinear interpolation theorem, we have that $T \in \mathfrak{B}(\bar{A}_{\theta,p} \times \bar{B}_{\theta,q}, \bar{E}_{\theta,r})$. Hence, according to Corollary 2.5, it suffices to show that

$$\widetilde{T} : (A_0, A_1)_{\theta,p} \widehat{\otimes}_\pi (B_0, B_1)_{\theta,q} \longrightarrow (E_0, E_1)_{\theta,r} \text{ belongs to } \mathcal{I}. \tag{3.3}$$

For $m \in \mathbb{Z}$, let F_m be the space $E_0 + E_1$ normed by $2^{-\theta m} K(2^m, \cdot)$. The operator $\iota : (E_0, E_1)_{\theta,r} \longrightarrow \ell_r(F_m)$ defined by $\iota(x) = (\cdots, x, x, x, \cdots)$, is an isometric embedding. By the injectivity of $\mathcal{I}$, to establish (3.3) it suffices to check that

$$\iota\widetilde{T} : (A_0, A_1)_{\theta,p} \widehat{\otimes}_\pi (B_0, B_1)_{\theta,q} \longrightarrow \ell_r(F_m) \text{ belongs to } \mathcal{I}. \tag{3.4}$$

Furthermore, since $\mathcal{I}$ satisfies the Σ_r-condition, statement (3.4) will follow if

$$\widetilde{T} : (A_0, A_1)_{\theta,p} \widehat{\otimes}_\pi (B_0, B_1)_{\theta,q} \longrightarrow E_0 + E_1 \text{ belongs to } \mathcal{I}. \tag{3.5}$$

And, by Corollary 2.5, condition (3.5) holds provided that

$$T : (A_0, A_1)_{\theta,p} \times (B_0, B_1)_{\theta,q} \longrightarrow E_0 + E_1 \text{ belongs to } \mathfrak{I}. \tag{3.6}$$

By assumption

$$T : (A_0 + A_1) \times (B_0 + B_1) \longrightarrow E_0 + E_1 \text{ is bounded}$$

and

$$T : (A_0 \cap A_1) \times (B_0 \cap B_1) \longrightarrow E_0 + E_1 \text{ belongs to } \mathfrak{I}.$$

Moreover, the ideal $\mathfrak{I}$ is surjective in the sense of [47, Definition 2.3] and the space $(A_0, A_1)_{\theta,p}$ is of class $\mathcal{C}_K(\theta; \bar{A})$ and $(B_0, B_1)_{\theta,q}$ is of class $\mathcal{C}_K(\theta; \bar{B})$. Consequently, applying [47, Corollary 4.7], we get that

$$T : (A_0, A_1)_{\theta,p} \times (B_0, B_1)_{\theta,q} \longrightarrow E_0 + E_1 \text{ belongs to } \mathfrak{I}.$$

This is (3.6) and therefore (3.1) follows. □

Theorem 3.1 for $\mathcal{I} = \mathcal{W}$ is known. It was proved by Cobos and Fernández-Cabrera [16].

As a direct consequence of Theorem 3.1 we obtain the following interpolation result for bilinear operators belonging to $\mathfrak{I}$.

Corollary 3.2 *Let $1 < r < \infty$ and let $\mathcal{I}$ be an injective and surjective operator ideal which satisfies the Σ_r-condition. Let $0 < \theta < 1$ and $1 \leq p, q < \infty$ with $1/p + 1/q = 1 + 1/r$. Suppose that $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, $\bar{E} = (E_0, E_1)$ are Banach couples and let $T \in \mathfrak{B}(\bar{A} \times \bar{B}, \bar{E})$. If $T \in \mathfrak{I}(A_j \times B_j, E_j)$ for $j = 0$ or 1, then $T \in \mathfrak{I}(\bar{A}_{\theta,p} \times \bar{B}_{\theta,q}, \bar{E}_{\theta,r})$.*

Besides $\mathcal{W}$, Corollary 3.2 (and Theorem 3.1) applies for $\mathcal{I} = \mathcal{BS}$, $\mathcal{R}$, $\mathcal{A}$. They show interpolation properties for bilinear operators belonging to $\mathfrak{BS}$, $\mathfrak{R}$, $\mathfrak{A}$.

Sometimes we do not have that $T \in \mathfrak{B}(\bar{A} \times \bar{B}, \bar{E})$, but only that T is defined on $(A_0 \cap A_1) \times (B_0 \cap B_1)$ with values in $E_0 \cap E_1$ and that there are constants $M_j > 0$ such that

$$\|T(a, b)\|_{E_j} \leq M_j \|a\|_{A_j} \|b\|_{B_j},\ a \in A_0 \cap A_1,\ b \in B_0 \cap B_1,\ j = 0, 1. \tag{3.7}$$

In what follows, we write $\mathfrak{C}(\bar{A} \times \bar{B}, \bar{E})$ for the collection of all those bilinear operators T satisfying (3.7).

If $\bar{A} = (A_0, A_1)$ is a Banach couple and A is a Banach space with $A_0 \cap A_1 \hookrightarrow A$, we write A° for the closure of $A_0 \cap A_1$ in A. Clearly (A_0°, A_1°) is also a Banach couple. As one can see in [9] or [58], $(A_0^\circ, A_1^\circ)_{\theta,q} = (A_0, A_1)_{\theta,q}$. Moreover, $A_0 \cap A_1$ is dense in $(A_0, A_1)_{\theta,q}$ if $1 \leq q < \infty$, so $(A_0, A_1)^\circ_{\theta,q} = (A_0, A_1)_{\theta,q}$.

If $T \in \mathfrak{C}(\bar{A} \times \bar{B}, \bar{E})$, using (3.7) one can check that T may be uniquely extended to a bounded bilinear operator $T : A_j^\circ \times B_j^\circ \longrightarrow E_j^\circ$ with $\|T\|_{A_j^\circ \times B_j^\circ, E_j^\circ} \leq M_j$ for $j = 0, 1$. Furthermore, see [38, Theorem 4.1], the bilinear interpolation theorem still holds if $T \in \mathfrak{C}(\bar{A} \times \bar{B}, \bar{E})$ with the effect that if $0 < \theta < 1$, $1 \leq p, q, r < \infty$ and $1/p + 1/q = 1 + 1/r$ then T may be uniquely extended to a bounded bilinear operator from $\bar{A}_{\theta,p} \times \bar{B}_{\theta,q}$ to $\bar{E}_{\theta,r}$.

We complete the paper with an interpolation formula for the measure $\gamma_{\mathfrak{I}}(T)$ when the ideal $\mathcal{I}$ is symmetric and $T \in \mathfrak{C}$. For this aim, we recall that if (E_0, E_1) is a Banach couple, $0 < \theta < 1$ and $1 \leq r < \infty$, then the space $(E_0, E_1)_{\theta,r}$ can be described by using the J-functional

$$J(t, x) = J(t, x; E_0, E_1) = \max\{\|x\|_{E_0} \, , \, t\|x\|_{E_1}\} \, , \, x \in E_0 \cap E_1 .$$

In fact, $x \in (E_0, E_1)_{\theta,r}$ if and only if x can be represented in the form $x = \sum_{m=-\infty}^{\infty} x_m$ (convergence in $E_0 + E_1$) with $(x_m) \subseteq E_0 \cap E_1$ and $\Big(\sum_{m=-\infty}^{\infty} \big(2^{-\theta m} J(2^m, x_m)\big)^r \Big)^{1/r} < \infty$. Moreover

$$\|x\|_{\theta,r;J} = \inf \left\{ \Big(\sum_{m=-\infty}^{\infty} \big(2^{-\theta m} J(2^m, x_m)\big)^r \Big)^{1/r} \, : \, x = \sum_{m=-\infty}^{\infty} x_m \right\}$$

defines an equivalent norm to $\|\cdot\|_{\theta,r}$. We write $(E_0, E_1)_{\theta,r;J}$ for the real interpolation space realized by the J-functional.

Note that condition $\Big(\sum_{m=-\infty}^{\infty} \big(2^{-\theta m} J(2^m, x_m)\big)^r \Big)^{1/r} < \infty$ implies that the series $\sum_{m=-\infty}^{\infty} x_m$ is convergent in $E_0 + E_1$ because we have

$$\begin{aligned}
\sum_{m=-\infty}^{\infty} K(1, x_m) &\leq \sum_{m=0}^{\infty} 2^{-m} J(2^m, x_m) + \sum_{m=-1}^{-\infty} J(2^m, x_m) \\
&\leq \Big(\sum_{m=0}^{\infty} 2^{-(1-\theta)mr'} \Big)^{1/r'} \Big(\sum_{m=0}^{\infty} (2^{-\theta m} J(2^m, x_m))^r \Big)^{1/r} \\
&+ \Big(\sum_{m=-1}^{-\infty} 2^{\theta m r'} \Big)^{1/r'} \Big(\sum_{m=-1}^{-\infty} (2^{-\theta m} J(2^m, x_m))^r \Big)^{1/r} < \infty
\end{aligned}$$

where $1/r + 1/r' = 1$.

Let G_m be the space $E_0 \cap E_1$ normed by $J(2^m, \cdot)$. It is not hard to check that

$$\omega : \ell_r(2^{-\theta m} G_m) \longrightarrow (E_0, E_1)_{\theta,r;J} \, , \ \ \omega(x_m) = \sum_{m=-\infty}^{\infty} x_m \text{ (convergence in } E_0 + E_1),$$

is a metric surjection.

Lemma 3.3 *If $\bar{E} = (E_0, E_1)$ is a Banach couple, then $(E_0 + E_1)^\circ = E_0^\circ + E_1^\circ$.*

Proof Since $E_j \hookrightarrow E_0 + E_1$, it is clear that $E_j^\circ \hookrightarrow (E_0 + E_1)^\circ$ for $j = 0, 1$. Hence, $E_0^\circ + E_1^\circ \hookrightarrow (E_0 + E_1)^\circ$. To check the converse embedding, first note that if $x \in E_0 \cap E_1$ and $x = x_0 + x_1$ with $x_j \in E_j$, then $x_j \in E_0 \cap E_1$ for $j = 0, 1$. Whence $\|x\|_{E_0^\circ + E_1^\circ} \leq \|x\|_{E_0 + E_1}$ and so $(E_0 + E_1)^\circ \hookrightarrow E_0^\circ + E_1^\circ$. □

According to [9, Theorem 2.7.1], the couple of dual spaces $(E_0^{\circ *}, E_1^{\circ *})$ is a Banach couple and

$$\Big(E_0^\circ + E_1^\circ \, , \, K(2^m, \cdot; E_0^\circ, E_1^\circ) \Big)^* = \Big(E_0^{\circ *} \cap E_1^{\circ *} \, , \, J(2^{-m}, \cdot; E_0^{\circ *}, E_1^{\circ *}) \Big). \tag{3.8}$$

Theorem 3.4 *Let $0 < \theta < 1$, $1 \le p, q < \infty$, $1 < r < \infty$ with $1/p + 1/q = 1 + 1/r$ and let $\mathcal{I}$ be an injective, surjective and symmetric operator ideal satisfying the $\Sigma_{r'}$-condition with $1/r + 1/r' = 1$. Assume that $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$, $\bar{E} = (E_0, E_1)$ are Banach couples and that $T \in \mathfrak{C}(\bar{A} \times \bar{B}, \bar{E})$. Then T may be uniquely extended to a bounded bilinear operator from $\bar{A}_{\theta,p} \times \bar{B}_{\theta,q}$ to $\bar{E}_{\theta,r}$ satisfying that*

$$\gamma_\jmath\big(\tau T : \bar{A}_{\theta,p} \times \bar{B}_{\theta,q} \longrightarrow \ell_r(2^{-\theta m} F_m^{**})\big) \le C\, \gamma_\jmath(T : A_0^\circ \times B_0^\circ \longrightarrow E_0^\circ)^{1-\theta} \gamma_\jmath(T : A_1^\circ \times B_1^\circ \longrightarrow E_1^\circ)^{\theta},$$

where the constant $C > 0$ is independent of T. Here $F_m = \big(E_0^\circ + E_1^\circ\, ,\, K(2^m, \cdot)\big)$ and $\tau = \tau_{\ell_r(2^{-\theta m} F_m)}\iota$.

Proof By the bilinear interpolation theorem [38, Theorem 4.1], the operator T may be uniquely extended to a bounded bilinear operator

$$T : (A_0, A_1)_{\theta,p} \times (B_0, B_1)_{\theta,q} \longrightarrow (E_0, E_1)_{\theta,r} = (E_0^\circ, E_1^\circ)_{\theta,r}.$$

Let $F_m = \big(E_0^\circ + E_1^\circ\, ,\, K(2^m, \cdot\,; E_0^\circ, E_1^\circ)\big)$ and let

$$\iota : (E_0^\circ, E_1^\circ)_{\theta,r} \longrightarrow \ell_r(2^{-\theta m} F_m), \quad \iota(x) = (\cdots, x, x, x, \cdots),$$

which is an isometric embedding. The composed operator

$$\tau = \tau_{\ell_r(2^{-\theta m} F_m)}\iota : (E_0^\circ, E_1^\circ)_{\theta,r} \longrightarrow \ell_r(2^{-\theta m} F_m)^{**} = \ell_r(2^{-\theta m} F_m^{**}),$$

$\tau(x) = (\cdots, \widehat{x}, \widehat{x}, \widehat{x}, \cdots)$, is also an isometric embedding.

Put $G_m = \big(E_0^{\circ *} \cap E_1^{\circ *}\, ,\, J(2^m, \cdot\,; E_0^{\circ *}, E_1^{\circ *})\big)$. Using (3.8), we obtain that

$$\ell_r(2^{-\theta m} F_m)^* = \ell_{r'}(2^{\theta m} F_m^*) = \ell_{r'}(2^{\theta m} G_{-m}), \quad 1/r + 1/r' = 1.$$

Note that $\eta : \ell_{r'}(2^{\theta m} G_{-m}) \longrightarrow \ell_{r'}(2^{-\theta m} G_m)$, $\eta(f_m) = (f_{-m})$, is an isometric isomorphism.

For any $(a, b) \in \bar{A}_{\theta,p} \times \bar{B}_{\theta,q}$ and any $(f_m) \in \ell_{r'}(2^{\theta m} G_{-m})$, we have

$$\begin{aligned}(\iota T)^\times (f_m)(a, b) &= (f_m)[(\iota T)(a, b)] = (f_m)(\cdots, T(a, b), T(a, b), T(a, b), \cdots) \\ &= \sum_{m=-\infty}^{\infty} f_m(T(a, b)) = \omega(\eta(f_m))(T(a, b)) = (T^\times \omega\eta(f_m))(a, b)\end{aligned}$$

where

$$\omega : \ell_{r'}(2^{-\theta m} G_m) \longrightarrow (E_0^{\circ *}, E_1^{\circ *})_{\theta, r'; J}, \;\; \omega(f_m) = \sum_{m=-\infty}^{\infty} f_m \;\; \text{(convergence in } E_0^{\circ *} + E_1^{\circ *})$$

is a metric surjection. So

$$(\iota T)^\times = T^\times \omega\eta. \tag{3.9}$$

According to Proposition 2.12, (3.8) and (3.9), we get

$$\begin{aligned}&\gamma_\jmath\big(\tau T : \bar{A}_{\theta,p} \times \bar{B}_{\theta,q} \longrightarrow \ell_r(2^{-\theta m} F_m^{**})\big) \\ &\qquad = \gamma_\jmath\big(\tau_{\ell_r(2^{-\theta m} F_m)}\iota T : \bar{A}_{\theta,p} \times \bar{B}_{\theta,q} \longrightarrow \ell_r(2^{-\theta m} F_m^{**})\big) \\ &\qquad \le \beta_{\mathcal{I}}\big((\iota T)^\times : \ell_{r'}(2^{\theta m} F_m^*) \longrightarrow \mathfrak{B}(\bar{A}_{\theta,p} \times \bar{B}_{\theta,q})\big) \\ &\qquad = \beta_{\mathcal{I}}\big(T^\times \omega\eta : \ell_{r'}(2^{\theta m} G_{-m}) \longrightarrow \mathfrak{B}(\bar{A}_{\theta,p} \times \bar{B}_{\theta,q})\big).\end{aligned}$$

We can estimate $\beta_{\mathcal{I}}\big(T^{\times}\omega\eta\big)$ with the help of the factorization

$$\begin{array}{ccc} \ell_{r'}(2^{\theta m}G_{-m}) & \xrightarrow{\quad T^{\times}\omega\eta \quad} & \mathfrak{B}(\bar{A}_{\theta,p}\times\bar{B}_{\theta,q}). \\ & {\scriptstyle \eta}\searrow \qquad \nearrow{\scriptstyle T^{\times}\omega} & \\ & \ell_{r'}(2^{-\theta m}G_m) & \end{array}$$

Since $\|\eta\|_{\ell_{r'}(2^{\theta m}G_{-m}),\ell_{r'}(2^{-\theta m}G_m)} = 1$, we derive that

$$\gamma_{\jmath}\big(\tau T : \bar{A}_{\theta,p}\times\bar{B}_{\theta,q} \longrightarrow \ell_r(2^{-\theta m}F_m^{**})\big) \le \beta_{\mathcal{I}}\big(T^{\times}\omega : \ell_{r'}(2^{-\theta m}G_m) \longrightarrow \mathfrak{B}(\bar{A}_{\theta,p}\times\bar{B}_{\theta,q})\big).$$

Let $X_j = \mathfrak{B}(A_j^{\circ}\times B_j^{\circ})$, $j=0,1$, and consider the Banach couple $\bar{X}=(X_0,X_1)$. Since $T : A_j^{\circ}\times B_j^{\circ} \longrightarrow E_j^{\circ}$ boundedly, we have that $T^{\times} : E_j^{\circ *} \longrightarrow X_j$ is bounded for $j=0,1$. Besides

$$\omega : \ell_1(G_m) \longrightarrow E_0^{\circ *} \quad \text{and} \quad \omega : \ell_1(2^{-m}G_m) \longrightarrow E_1^{\circ *}$$

are bounded. Whence

$$T^{\times}\omega : \ell_1(2^{-jm}G_m) \longrightarrow X_j \quad \text{boundedly for} \quad j=0,1.$$

By [15, Thorem 3.1],

$$\big(\ell_1(G_m), \ell_1(2^{-m}G_m)\big)_{\theta,r'} = \ell_{r'}(2^{-\theta m}G_m). \tag{3.10}$$

Therefore, using the interpolation theorem for linear operators, we obtain that

$$T^{\times}\omega : \ell_{r'}(2^{-\theta m}G_m) \longrightarrow (X_0,X_1)_{\theta,r'} \quad \text{is bounded.}$$

Furthermore, by [48, Theorem 2.1] or [20, Lemma 4.1],

$$(X_0,X_1)_{\theta,r'} \hookrightarrow \mathfrak{B}(\bar{A}_{\theta,p}\times\bar{B}_{\theta,q}).$$

By the factorization

$$\begin{array}{ccc} \ell_{r'}(2^{-\theta m}G_m) & \xrightarrow{\quad T^{\times}\omega \quad} & \mathfrak{B}(\bar{A}_{\theta,p}\times\bar{B}_{\theta,q}) \\ & {\scriptstyle T^{\times}\omega}\searrow \qquad \nearrow & \\ & (X_0,X_1)_{\theta,r'} & \end{array}$$

we derive that

$$\gamma_{\jmath}\big(\tau T : \bar{A}_{\theta,p}\times\bar{B}_{\theta,q} \longrightarrow \ell_r(2^{-\theta m}F_m^{**})\big) \le C_1\,\beta_{\mathcal{I}}\big(T^{\times}\omega : \ell_{r'}(2^{-\theta m}G_m) \longrightarrow (X_0,X_1)_{\theta,r'}\big).$$

Next we use (3.10) and the description of the real interpolation space by means of the J-functional to establish that

$$\beta_{\mathcal{I}}\big(T^{\times}\omega : \ell_{r'}(2^{-\theta m}G_m) \longrightarrow (X_0,X_1)_{\theta,r'}\big) \le C_2\,\beta_{\mathcal{I}}(T^{\times} : E_0^{\circ *} \longrightarrow X_0)^{1-\theta}\,\beta_{\mathcal{I}}(T^{\times} : E_1^{\circ *} \longrightarrow X_1)^{\theta}. \tag{3.11}$$

Take any $\sigma_j > \beta_{\mathcal{I}}(T^{\times} : E_j^{\circ *} \longrightarrow X_j)$. By the definition of $\beta_{\mathcal{I}}(T^{\times})$, there are Banach space Z_j and operators $R_j \in \mathcal{I}(E_j^{\circ *}, Z_j)$ such that

$$\|T^{\times}f\|_{X_j} \le \sigma_j\|f\|_{E_j^{\circ *}} + \|R_j f\|_{Z_j}\ ,\ f\in E_j^{\circ *}\ ,\ j=0,1.$$

Pick $k\in\mathbb{Z}$ with $2^{k-1}\le \sigma_1/\sigma_0 < 2^k$. Given any $(f_m)\in\ell_{r'}(2^{-\theta m}G_m)$ let $d_m = 2^{-\theta m}J(2^m, f_m; E_0^{\circ *}, E_1^{\circ *})$. Since

$$2^{-\theta}2^{-\theta(m-k)}(\sigma_1/\sigma_0)^{-\theta}J(2^{m-k}\sigma_1/\sigma_0, f_m) \le d_m,$$

we have that

$$2^{-\theta}2^{(j-\theta)(m-k)}(\sigma_1/\sigma_0)^{j-\theta}d_m^{-1}f_m \in U_{E_j^{\circ *}}\ ,\ j=0,1.$$

Therefore

$$\|T^{\times}f_m\|_{X_j} \leq 2^{\theta}2^{(\theta-j)(m-k)}\sigma_0^{1-\theta}\sigma_1^{\theta}d_m + \|R_jf_m\|_{Z_j}\ ,\ j=0,1. \tag{3.12}$$

For $m\in\mathbb{Z}$, let $W_m=(Z_0\times Z_1)_{\ell_1}$ be the product of Z_0 and Z_1, normed by $\|(z_0,z_1)\|_1=\|z_0\|_{Z_0}+\|z_1\|_{Z_1}$. Consider the operator $R:\ell_{r'}(2^{-\theta m}G_m)\longrightarrow \ell_{r'}(W_m)$ defined by

$$R(f_m)=\Big(2^{-\theta(m-k)}R_0f_m\ ,\ 2^{(1-\theta)(m-k)}R_1f_m\Big).$$

For any $(f_m)\in\ell_{r'}(2^{-\theta m}G_m)$, we have

$$\begin{aligned}\|R(f_m)\|_{\ell_{r'}(W_m)} &= \Big(\sum_{m=-\infty}^{\infty}\Big(2^{-\theta(m-k)}\|R_0f_m\|_{Z_0}+2^{(1-\theta)(m-k)}\|R_1f_m\|_{Z_1}\Big)^{r'}\Big)^{\frac{1}{r'}}\\ &\leq \max\Big\{2^{\theta k},2^{(\theta-1)k}\Big\}\max\Big\{\|R_0\|_{E_0^{\circ *},Z_0}\ ,\ \|R_1\|_{E_1^{\circ *},Z_1}\Big\}\\ &\quad\times\Big(\sum_{m=-\infty}^{\infty}\Big(2^{-\theta m}\|f_m\|_{E_0^{\circ *}}+2^{(1-\theta)m}\|f_m\|_{E_1^{\circ *}}\Big)^{r'}\Big)^{\frac{1}{r'}}\\ &\leq 2\max\Big\{2^{\theta k},2^{(\theta-1)k}\Big\}\max\Big\{\|R_0\|_{E_0^{\circ *},Z_0}\ ,\ \|R_1\|_{E_1^{\circ *},Z_1}\Big\}\|(f_m)\|_{\ell_{r'}(2^{-\theta m}G_m)}.\end{aligned}$$

Therefore R is bounded. For any $n,m\in\mathbb{Z}$, we have

$$(Q_mRP_n)(f)=\delta_n^m\Big(2^{-\theta(n-k)}R_0f\ ,\ 2^{(1-\theta)(n-k)}R_1f\Big).$$

So, $Q_mRP_n\in\mathcal{I}(G_n,W_m)$. Since $\mathcal{I}$ satisfies the $\Sigma_{r'}$-condition, we derive that

$$R\in\mathcal{I}\big(\ell_{r'}(2^{-\theta m}G_m),\ell_{r'}(W_m)\big).$$

Finally, given any $(f_m)\in\ell_{r'}(2^{-\theta m}G_m)$, using (3.12), we obtain

$$\begin{aligned}\|(T^{\times}\omega)(f_m)\|_{(X_0,X_1)_{\theta,r'}} &\leq \Big(\sum_{m=-\infty}^{\infty}\Big(2^{-\theta(m-k)}J(2^{m-k},T^{\times}f_m;X_0,X_1)\Big)^{r'}\Big)^{\frac{1}{r'}}\\ &\leq \Big(\sum_{m=-\infty}^{\infty}\Big(\max_{j=0,1}\Big\{2^{\theta}\sigma_0^{1-\theta}\sigma_1^{\theta}d_m+2^{(j-\theta)(m-k)}\|R_jf_m\|_{Z_j}\Big\}\Big)^{r'}\Big)^{\frac{1}{r'}}\\ &\leq \Big(\sum_{m=-\infty}^{\infty}\Big(2^{\theta}\sigma_0^{1-\theta}\sigma_1^{\theta}d_m\Big)^{r'}\Big)^{\frac{1}{r'}}\\ &\quad+\Big(\sum_{m=-\infty}^{\infty}\Big(\|2^{-\theta(m-k)}R_0f_m\|_{Z_0}+\|2^{(1-\theta)(m-k)}R_1f_m\|_{Z_1}\Big)^{r'}\Big)^{\frac{1}{r'}}\\ &\leq 2^{\theta}\sigma_0^{1-\theta}\sigma_1^{\theta}\|(f_m)\|_{\ell_{r'}(2^{-\theta m}G_m)}+\|R(f_m)\|_{\ell_{r'}(W_m)}.\end{aligned}$$

This yields that

$$\beta_{\mathcal{I}}\big(T^{\times}\omega:\ell_{r'}(2^{-\theta m}G_m)\longrightarrow(X_0,X_1)_{\theta,r'}\big)\leq C_2\sigma_0^{1-\theta}\sigma_1^{\theta}$$

and establishes (3.11). Applying Proposition 2.11 we complete the proof. □

Writing down Theorem 3.4 for $\mathcal{I} = \mathcal{W}$, the ideal of weakly compact operators, we obtain a quantitative version in terms of the measure $\gamma_{\mathfrak{W}} = \eta$ of the bilinear interpolation result established by the authors in [22, Theorem 3.1].

Conflict of Interest The authors declare no conflict of interest.

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