

# Tetraquarks in the covariant approaches of QFT (to QCD)

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- Tetraquarks need relativistic description in the QFT
- In general, four-body system where transitions to two-body states are allowed, like in  $2q2\bar{q} \rightarrow q\bar{q}$
- Frameworks:
  - (i) time-ordered perturbation theory (TOPT)
  - (ii) equal time quasipotential approach
  - (iii) 4D covariant approach based on the analysis of Feynman diagrams

- History of the covariant equations for tetraquarks:
  - (i) 1992, Kvi., Khvedelidze, *Theor. Math. Phys.* **90**, 62, 4-body covariant equations without annihilation (like a 4 quark system, no transition to  $2q$ )
  - (ii) 2012, Heupel, Eichmann, Fischer, *Phys.Lett.* **B718**, 545, numerical solution of the 4-body covariant equations
  - (iii) 2014, Kvi., Blankleider, *Phys.Rev.* **D90**, 045042, inclusion of the effect of quark-antiquark annihilation, coupling to the 2-body, quark-antiquark, channel
  - (iv) 2020, Santowsky, Eichmann, Fischer, Wallbott, Williams, *Phys.Rev.* **D102**, 5, another, "phenomenological", attempt to include the coupling to the 2-body,  $q\bar{q}$ , channel
  - (v) 2022, Kvi., Blankleider, *Phys.Rev.* **D106**, 5, the general exact equations for tetraquarks

- 4-body covariant equations like a 4 quark system,  
$$\Psi = G_0^{(4)} K^{(4)} \Psi$$

- Challenges in derivation:

- (i) The problem of overcounting even for no coupling to two-body channel -sum of pair interaction kernels overcounts therefore **subtractions** are needed

$$K^{(4)} = \sum K_{ij}^{(2)} - K_{12}^{(2)} K_{34}^{(2)} - K_{13}^{(2)} K_{24}^{(2)} - K_{14}^{(2)} K_{23}^{(2)},$$

\*Such subtractions are not canceled within  $K^{(4)}$ , cancelation happens between different iterations of  $K^{(4)}$ .

$$K^{(4)} + K^{(4)} K^{(4)} = \dots - K_{12}^{(2)} K_{34}^{(2)} + K_{12}^{(2)} K_{34}^{(2)} + K_{34}^{(2)} K_{12}^{(2)}$$

\*Unusual structure of  $K^{(4)}$  for using Faddeev-Yakubovsky rearrangement

- (ii) In the 3D quasipotential and TOPT approaches 4q kernels even cannot be expressed in terms of 2q kernels  $K_{ij}^{(2)}$

- To rearrange the equation  $\Psi = G_0^{(4)} K^{(4)} \Psi$  break the kernel into three parts,  $K^{(4)} = K_1 + K_2 + K_3$ , where

$$K_3 = K_{12}^{(2)} + K_{34}^{(2)} - K_{12}^{(2)} K_{34}^{(2)},$$

$$K_1 = K_{13}^{(2)} + K_{24}^{(2)} - K_{13}^{(2)} K_{24}^{(2)}, \quad (K_2 \text{ similarly}),$$

and define  $T_i$ ,  $T_i = K_i + K_i G_0^{(4)} T_i$  with the solutions,

$$T_3 = T_{12}^{(2)} + T_{34}^{(2)} + T_{12}^{(2)} T_{34}^{(2)},$$

$$T_1 = T_{13}^{(2)} + T_{24}^{(2)} + T_{13}^{(2)} T_{24}^{(2)}, \quad (T_2 \text{ similarly}),$$

where  $T_{ij}^{(2)}$  are two-body off-shell scattering amplitudes,

$$T_{ij}^{(2)} = K_{ij}^{(2)} + K_{ij}^{(2)} G_0^{(2)} T_{ij}^{(2)}.$$

Then the rearranged equation is

$$\Psi = \sum_i \Psi_i. \text{ where } \Psi_i = G_0^{(4)} T_i \sum_{j \neq i} \Psi_j$$

- Three goals achieved. **No analog in QM for two of them:**
  - (i) kernels  $T_i$  are expressed in terms of two-body t-matrices  $T_{ij}^{(2)}$   
(usual result of Faddeev rearrangement in Quantum Mechanics)
  - (ii) got rid of **subtraction** terms present before rearrangement  
(**subtractions do not exist in the 4-body potential of QM**)
  - (iii) In the 3D quasipotential and TOPT approaches the rearrangement is **necessary** at least for the kernels to be expressed in terms of functions defined in the two-body subsystems  $T_1 + T_2 + \int dz T_1(E - z) T_2(z)$   
(such a problem does not exist in Quantum Mechanics, equations' kernels are clear **before rearrangement as well**, the sum of pair interaction potentials)

# 1992, Kvi., Khvedelidze, Theor. Math. Phys. **90**, 62, 4-body covariant equations

- That time (in 1992) we even **did not dream** that this sort of equations could be solved numerically, we have been attracted only by the mathematical challenges they pose. But, alas, **in 20 years since then they are being solved!** (see next slides)
- (i) This fact,  
(ii) the development of methods of numerical solutions,  
(iii) the well elaborated input functions offered on the market  
(iv) massive calculations carried out by now  
**revived our interest**

- The first results for tetraquarks in a covariant continuum approach based on our equation of 1992 for four quarks, but applied to a system

of 2 quarks+2 antiquarks,  $2q2\bar{q}$ ,

$$\Psi_i = G_0^{(4)} T_i \sum_{j \neq i} \Psi_j,$$

where

$$T_3 = T_{12}^{(2)} + T_{34}^{(2)} + T_{12}^{(2)} T_{34}^{(2)},$$

$$T_1 = T_{13}^{(2)} + T_{24}^{(2)} + T_{13}^{(2)} T_{24}^{(2)}, \quad (T_2 \text{ similarly}),$$

- This approach is "similar in spirit to the quark-diquark model of the nucleon" elaborated in G. Eichmann, I R. Alkofer, et al., Phys. Rev. C 79 (2009) 012202, perhaps in earlier work

- Using 4q eqs for the system of  $2q+2\bar{q}$  implies that in the  $2q2\bar{q}$  Green function  $G^{(4)}$   $q\bar{q}$  annihilation diagrams are neglected, i.e. its  $q\bar{q}$ -irreducible part  $G_{ir}^{(4)}$  is considered, see below
- This is basically equivalent to 4-body QM with relativistic kinematics, quark number non-conservation is not yet accounted for
- Further approximations are made to reduce  $2q2\bar{q}$  space to  $MM + D\bar{D}$ , i.e. to come to a picture of tetraquark as composed of two mesons,  $MM$ , or of diquark-antidiquark,  $D\bar{D}$ :  
(i)  $T_3 \sim T_{12}^{(2)} T_{34}^{(2)}$ ,  $T_1 \sim T_{13}^{(2)} T_{24}^{(2)}$ , ( $T_2$  similarly),

(ii) reduction to a two-body problem proceeds by assuming that the two-body T-matrices  $T_{ij}^{(2)}$  are dominated by meson and diquark

pole contributions,  $T_{ij}^{(2)} = i\Gamma_{ij}D_{ij}\bar{\Gamma}_{ij}$ ,

where  $D_{ij}(P_{ij}) = 1/(P_{ij}^2 - m_{ij}^2)$  is the propagator for the bound particle (diquark, antidiquark, or meson) in the two-body channel  $ij$

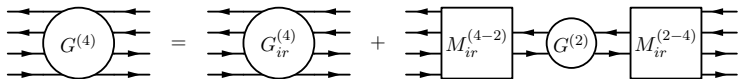
- **Physical results in Phys.Lett. B718, 545 (2012):**

For the lightest scalar tetraquark a mass of the order of 400 MeV and a wave function dominated by the pion-pion constituents is found.

Both results are in agreement with a meson molecule picture for the  $f_0(600)$ .

The results suggest the presence of a potentially narrow all-charm tetraquark in the mass region 5 – 6 GeV

- So far mathematically pure four-quark system was considered.
- But in Quantum Field Theory (QFT) the number of particles is not conserved.
- The  $2q2\bar{q}$  Green function  $G^{(4)}$  in terms of its  $q\bar{q}$ -irreducible part  $G_{ir}^{(4)}$  and its  $q\bar{q}$ -reducible part  $M_{ir}^{(4-2)} G^{(2)} M_{ir}^{(2-4)}$ .  
The exact relation:



- That  $2q2\bar{q}$  system couples to  $q\bar{q}$  (via  $M_{ir}^{(2-4)}$ ) makes a tetraquark a more complicated object than defined in QM as a bound state of 4 quarks

# 2014, Kvi., Blankleider, Phys.Rev. **D90**, 045042, simplest coupling to 2-body, quark-antiquark, channel

- First attempt to account for coupling to two-body  $q\bar{q}$  state  
2014, Kvi., Blankleider, Phys.Rev. **D90**, 045042

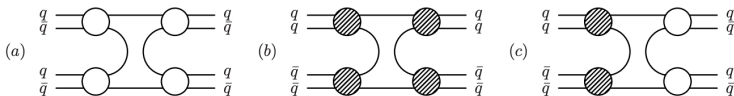


FIG. 1. Examples of terms involving  $q\bar{q}$  annihilation contributing to the tetraquark amplitude within a model involving only meson, diquark and antidiquark constituents: (a)  $MM$  scattering, (b)  $D\bar{D}$  scattering, (c)  $D\bar{D} \leftarrow MM$  transition. Such terms are not taken into account when the covariant four-body equations of KK [3] are used to describe the  $2q2\bar{q}$  system.



- this **disconnected** diagrams are used as (**unusual!**) parts of the input  $2q$  T-matrices, but in a sophisticated way to avoid non-physical/inexistent contributions to the four-body eqs

- Equations with minimal coupling to  $q\bar{q}$  channel were obtained

$$\begin{aligned}
 \text{---} \circ \Phi_M &= \frac{1+\mathcal{P}}{2} \text{---} \circ \Phi_M - 2 \text{---} \circ \Phi_D + \text{---} \circ \Gamma^* \\
 \text{=} \circ \Phi_D &= - \text{=} \circ \Phi_M + \text{=} \circ \Gamma^* \\
 \text{=} \circ \Gamma^* &= \frac{1}{2} \text{=} \circ \Phi_M + \text{=} \circ \Phi_D
 \end{aligned}$$

Timidly criticised for disconnected  $q\bar{q}$  T-matrix with  $\delta$ -function.

Criticism is unfair (wrong) since no kernel of the final eqs involves  $\delta$ -function, although used in the derivation

2020, Santowsky, Eichmann, Fischer, Wallbott, Williams, Phys.Rev.**D102**, 5, "phenomenological" attempt to include coupling to  $q\bar{q}$  channel

- Phenomenologically motivated equations ("phenomenological" is a magic word sometimes used to "justify" equations even if inconsistent with the rules of QFT):

$$\begin{aligned}
 \Phi_1 &= \text{tree} + \text{loop} \\
 \Phi_3 &= \text{tree} + \text{loop} \\
 \Gamma^* &= K^{(2)}\Gamma^* + K^{(2)}\Phi_1 + K^{(2)}\Phi_3
 \end{aligned}$$

where  $K^{(2)}$  is one-gluon-exchange  $q\bar{q}$  kernel

- These eqs are erroneous because correspond to a two-body BSE with **reducible** kernel,

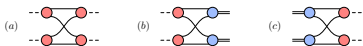
2020, Santowsky, Eichmann, Fischer, Wallbott, Williams,  
 Phys.Rev.**D102**, 5, "phenomenological" attempt to include  
 coupling to  $q\bar{q}$  channel

- Indeed write the above eqs in compact form,

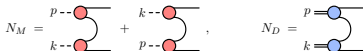
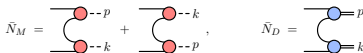
$$\Phi = VG_0^{(4)}\Phi + NG_0^{(2)}\Gamma^*$$

$$\Gamma^* = K^{(2)}G_0^{(2)}\Gamma^* + K^{(2)}G_0^{(2)}\bar{N}G_0^{(4)}\Phi$$

where the kernel  $V$  in the  $MM, D\bar{D}$  space are



and  $N, \bar{N}$  are



2020, Santowsky, Eichmann, Fischer, Wallbott, Williams,  
Phys.Rev.**D102**, 5, "phenomenological" attempt to include  
coupling to  $q\bar{q}$  channel

- Now it is easy to extract from them the two-body BS eq for the two-body,  $q\bar{q}$ , wave function  $\Gamma^*$ ,

$$\Gamma^* = [K^{(2)} + K^{(2)} G_0^{(2)} \bar{N} G_{ir}^{(4)} M] G_0^{(2)} \Gamma^* \quad (\text{R})$$

where  $G_{ir}^{(4)} = G_0^{(4)} + G_0^{(4)} V G_{ir}^{(4)}$

The red two-body propagator  $G_0^{(2)}$  implies  $q\bar{q}$  reducibility of the BS kernel.

- Some reducible kernels are legitimate, like e.g. Faddeev's ones in that they produce the same solutions as an irreducible one. We have shown that the kernel (R) **does not fit** into the general form of legitimate kernels.

# 2020, Santowsky, Eichmann, Fischer, Wallbott, Williams, Phys.Rev.**D102**, 5, "phenomenological" attempt to include coupling to $q\bar{q}$ channel

- The difference from the correct eqs **cannot be justified as an approximation** because the kernel  $K^{(2)} G_0^{(2)} \bar{N}$  is more complicated (involves extra 4D integral) than the correct one  $\bar{N}$
- To see the error in terms of missing physics transform the wrong eq into the form

$$\Gamma^* = \left[ 1 - K^{(2)} G_0^{(2)} \right]^{-1} K^{(2)} G_0^{(2)} \bar{N} G_{ir}^{(4)} N G_0^{(2)} \Gamma^*$$

$$\Rightarrow \Gamma^* = T^{(2)} G_0^{(2)} \bar{N} G_{ir}^{(4)} N G_0^{(2)} \Gamma^*$$

where  $T^{(2)}$  is t-matrix corresponding to the kernel  $K^{(2)}$ ,

$$T^{(2)} = K^{(2)} + K^{(2)} G_0^{(2)} T^{(2)}$$

The correct form would be

$$\Gamma^* = \left[ 1 + T^{(2)} G_0^{(2)} \right] \bar{N} G_{ir}^{(4)} N G_0^{(2)} \Gamma^*$$

# 2020, Santowsky, Eichmann, Fischer, Wallbott, Williams, Phys.Rev.**D102**, 5, "phenomenological" attempt to include coupling to $q\bar{q}$ channel

- Attempts to compare equations by one (or even few) observable they produce are **meaningless**. Nevertheless let us see one of the attempts in the literature:
- To see how this difference affects the tetraquark mass assume that its value is close to the pole position of  $G_{ir}^{(4)}$  (calculations confirm). Then we can use pole approximation for  $G_{ir}^{(4)} \sim i\Psi_0\bar{\Psi}_0/(E - M_0)$  which allows to solve the eq algebraically:

$$E = M_0 + \bar{\Psi}_0 N G_0^{(2)} T^{(2)} G_0^{(2)} \bar{N} \Psi_0 \sim 456 \quad \text{incorrect}$$

$$E = M_0 + \bar{\Psi}_0 N \left[ G_0^{(2)} + G_0^{(2)} T^{(2)} G_0^{(2)} \right] \bar{N} \Psi_0 \sim 484 \quad \text{correct}$$

$M_0 \sim 403$  is the "bare" mass, i.e. generated by purely 4q eqs,  $\bar{\Psi}_0 \dots \Psi_0$  are contributions from coupling to 2q states

2020, Santowsky, Eichmann, Fischer, Wallbott, Williams,  
Phys.Rev.**D102**, 5, "phenomenological" attempt to include  
coupling to  $q\bar{q}$  channel

- So the incorrect eqs miss the 2q contribution part  
 $\bar{\Psi}_0 N G_0^{(2)} \bar{N} \Psi_0$  where quarks **do not interact** with each other.
- Santowsky et al calculations show that this missing part makes  
50% contribution to the effect of coupling to 2q states

- Derivation of dynamical eqs in QFT for "given number,  $n$ , of particles" is based on exposing the  $n$ -particle "off-shell" states in the Green function. For example, in the two-body case one exposes 2-body states to derive the **seminal BSE**,

$$T = K + K G_0^{(2)} K + K G_0^{(2)} K G_0^{(2)} K + \dots$$

$$T = K + K G_0^{(2)} T$$

where  $K$  (sum of all, so called,  **$q\bar{q}$  irreducible diagrams**) involves all but two-body states, which are exposed in the two-body propagator  $G_0^{(2)}$ .

- This derivation works only for the **two-body BSE**. In the case of more than two particles,  $n > 2$ , iterations of the  $n$ -particle irreducible diagrams are **overcounted**. This problem needs a **subtle treatment**

- To derive 4-body eqs we expose 4-body states in the two-body irreducible kernel  $K$ ,  $K^{(2)} = \Delta + K_{4q-red}^{(2)} = \Delta + \bar{N}G_{ir}^{(4)}N$  where  $\Delta$  is  $2q2\bar{q}$  irreducible, all  $2q2\bar{q}$  reducible diagrams are contained in  $\bar{N}G_{ir}^{(4)}N$ , namely in  $G_{ir}^{(4)}$ .
- **Be more clear\*\*\*** So the complicated problem of overcounting inherent to the full 4-body Green function is reduced to much simpler one of its  $2q$  irreducible part,  $G_{ir}^{(4)}$ .

$$G_{ir}^{(4)} = G_0^{(4)} + G_0^{(4)} K^{(4)} G_{ir}^{(4)} \quad (A)$$

$$= G_0^{(4)} + G_0^{(4)} K^{(4)} G_0^{(4)} + G_0^{(4)} K^{(4)} G_0^{(4)} K^{(4)} G_0^{(4)} + \dots \quad (B)$$

where overcounting is avoided by only subtractions in

$$K^{(4)} = \sum K_{ij}^{(2)} - K_{12}^{(2)} K_{34}^{(2)} - K_{13}^{(2)} K_{24}^{(2)} - K_{14}^{(2)} K_{23}^{(2)},$$

- One could use the two-body eq for tetraquark,

$$\Gamma^* = K^{(2)} G_0^{(2)} \Gamma^*$$

where the kernel  $K^{(2)}$  is complicated by the infinite series (B).

$$K^{(2)} = \Delta + \bar{N} G_{ir}^{(4)} N$$

$$G_{ir}^{(4)} = G_0^{(4)} + G_0^{(4)} K^{(4)} G_0^{(4)} + G_0^{(4)} K^{(4)} G_0^{(4)} K^{(4)} G_0^{(4)} + \dots \quad (\text{B})$$

- This complication is handled by using the equation

$$G_{ir}^{(4)} = G_0^{(4)} + G_0^{(4)} K^{(4)} G_{ir}^{(4)}$$

to derive coupled channel set of 2q-4q equations with a simpler kernel  $K^{(4)}$ .

# 2022, Kvi., Blankleider, Phys.Rev. **D106**, 5, the general exact equations for tetraquarks

- Indeed the 2q eq  $\Gamma^* = \left[ \Delta + \bar{N} G_{ir}^{(4)} N \right] G_0^{(2)} \Gamma^*$  can be written as  $\Gamma^* = \Delta G_0^{(2)} \Gamma^* + \bar{N} G_0^{(4)} \Phi$ , where  $\Phi = G_0^{(4)-1} G_{ir}^{(4)} N G_0^{(2)} \Gamma^*$ . This relation defines  $\Phi$  as the 4q component of the tetraquark. Using the eq for  $G_{ir}^{(4)}$  we get the eq

$$\Phi = K^{(4)} G_0^{(4)} \Phi + N G_0^{(2)} \Gamma^*,$$

which in combination with  $\Gamma^* = \Delta G_0^{(2)} \Gamma^* + \bar{N} G_0^{(4)} \Phi$ , represents the set of eqs with "simple" kernels  $K^{(4)}$ ,  $N$ ,  $\Delta$ ,  $\bar{N}$ .

- In the above mentioned pole approximation for input 2q t-matrices these 4q eqs become 2-body eqs in  $q\bar{q}$ -MM space,

$$\Phi = V G_0^M \Phi + N G_0^{(2)} \Gamma^*.$$
$$\Gamma^* = \Delta G_0^{(2)} \Gamma^* + \bar{N} G_0^M \Phi$$

where the 4q propagator  $G_0^{(4)}$  is replaced by two-body 2M and  $D\bar{D}$  propagators; 4q kernel  $K^{(4)}$  is replaced by 2 meson and  $D\bar{D}$  kernels.

# 2022, Kvi., Blankleider, Phys.Rev. **D106**, 5, the general exact equations for tetraquarks

- The final equations in diagrams:

$$\begin{aligned}
 \text{Diagram 1: } \Phi_M &= \frac{1+\mathcal{P}}{2} \text{ (diagram with two red nodes) } - 2 \text{ (diagram with two red and two blue nodes) } + \text{ (diagram with two red nodes and } \Gamma^* \text{) } \\
 \text{Diagram 2: } \Phi_D &= - \text{ (diagram with two blue and two red nodes) } + \text{ (diagram with two blue nodes and } \Gamma^* \text{) } \\
 \text{Diagram 3: } \Gamma^* &= \text{ (diagram with } \Delta \text{ and } \Gamma^* \text{) } + \frac{1}{2} \text{ (diagram with two red nodes and } \Phi_M \text{) } + \text{ (diagram with two blue nodes and } \Phi_D \text{) }
 \end{aligned}$$

These are **general exact** four-body eqs (even if  $N, K^{(4)}$  are **approximate** in the case where 4q-2q transitions are allowed. Here is important that  $\Delta$  (and  $N$ ) are clearly specified:  $\Delta$  is the sum of all  $q\bar{q}$  irreducible diagrams except those included in  $\bar{N}G_{ir}^{(4)}N$ .

- **\*\*\*Be more clear** The transition to the 4-body eq is not only a technical trick to have simpler kernels; in the case of existence of a tetraquark it is related to specific properties of 4q reducible part in 2q kernel, **which is a crucial physics point**

- Universe 7 (2021) 4, 94:  
"there are **significant disagreements** between different theoretical approaches. Indeed, Refs. [...] predict heavy tetraquark masses **below or slightly above the thresholds** of the decays to two quarkonia and, thus, stable or significantly suppressed against fall-apart decays with a very narrow decay width. ... other approaches predict such tetraquark masses **significantly above** these thresholds and, thus, they can be observed only as broad resonances."  
Therefore it is important to compare theoretical basis of these approaches.
- A universal set of equations is derived which produces all of these approaches in different approximations. Exposing three body  $Mq\bar{q}, \bar{D}qq, D\bar{q}\bar{q}$  states is inevitable to compare them on the theoretical foundation level.

For example, the structure of tetraquarks can be studied by calculating **currents**. To construct them the "gauging equations method" should be applied to the tetraquark eqs (they are convenient for this procedure) just as it has been **successfully** done in the quark-diquark picture for nucleon

THANK YOU!