# Jordan blocks and the Bethe ansatz: The eclectic spin chain as a limit

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# Non-Hermitian physics

Non-Hermitian systems are less studied than Hermitian ones (especially quantum ones, due to the Dirac-Von Neumann axioms), but this does not mean that they are not interesting.

Some interesting examples are

- Asymmetric transport
- Logarithmic conformal field theories.
- Electrical circuits with non-reciprocal devices (e.g. diodes or operational amplifiers).
- Etc.

# Eclectic spin chain

# Dilatation operators and spin chains

Let us consider the $\mathcal{N} = 4$ SYM theory with gauge group $SU(N)$ and focus on operators of the form $\mathrm{Tr}(X^N Z^{L-N})$ and permutations of it.

[Minahan, Zarembo, 2002] showed that the one-loop dimension of these operators can be computed using an effective $su(2)$ spin chain

$$\mathcal{N} = 4\, SYM \longleftrightarrow su(2) \text{ spin chain}$$

| | |
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| Scalar fields $X$, $Z$ | Spin up or down |
| Single trace operator | Closed spin chain |
| Anomalous dimension | Heisenberg Hamiltonian |

# Dilatation operators and spin chains

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More explicitly,

$$\Delta = \Delta_0 + g^2 H + \mathcal{O}(g^4) \ ,$$

where $H \propto \sum_{l=1}^{L} \vec{S}_l \cdot \vec{S}_{l+1} \propto \sum_{l=1}^{L} \mathbb{P}^{l,l+1}$.

# The finitely twisted Hamiltonian

However, we are more interested in a deformation of $\mathcal{N} = 4$ SYM called $\gamma_i$-deformation. [Fokken, Sieg, Wilhelm, 2013] proved that a similar relation holds, but with a deformed permutation.

# The finitely twisted Hamiltonian

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In particular we are interested on the dimension of single trace operators with three different kinds of scalar fields ($\Rightarrow su(3)$ spin chain). The effective Hamiltonian of the deformed theory is given by

$$\tilde{\mathbf{H}}_{(q_1,q_2,q_3)} = \sum_{l=1}^{L} \tilde{\mathbb{P}}^{l,l+1} \ ,$$

where $\tilde{\mathbb{P}}^{a,b}$ acts non-trivially on sites $a$ and $b$ as follows

$$\begin{aligned}
\tilde{\mathbb{P}}\left|11\right\rangle &= \left|11\right\rangle \ , & \tilde{\mathbb{P}}\left|22\right\rangle &= \left|22\right\rangle \ , & \tilde{\mathbb{P}}\left|33\right\rangle &= \left|33\right\rangle \ , \\
\tilde{\mathbb{P}}\left|12\right\rangle &= \frac{1}{q_3}\left|21\right\rangle \ , & \tilde{\mathbb{P}}\left|23\right\rangle &= \frac{1}{q_1}\left|32\right\rangle \ , & \tilde{\mathbb{P}}\left|31\right\rangle &= \frac{1}{q_2}\left|13\right\rangle \ , \\
\tilde{\mathbb{P}}\left|21\right\rangle &= q_3\left|12\right\rangle \ , & \tilde{\mathbb{P}}\left|32\right\rangle &= q_1\left|23\right\rangle \ , & \tilde{\mathbb{P}}\left|13\right\rangle &= q_2\left|31\right\rangle \ .
\end{aligned}$$

# Eclectic spin chain

The eclectic spin chain corresponds to the small coupling and strong twist limit of the $\gamma_i$-deformation.

If we substitute $q_i = \frac{\xi_i}{\epsilon}$ and compute the $\epsilon \to 0$ limit of the finitely twisted Hamiltonian

$$\hat{\mathbf{H}}_{\xi_1,\xi_2,\xi_3} = \lim_{\epsilon \to 0} \epsilon \tilde{\mathbf{H}}_{(\frac{\xi_1}{\epsilon},\frac{\xi_2}{\epsilon},\frac{\xi_3}{\epsilon})} = \sum_{l=1}^{L} \hat{\mathbb{P}}^{l,l+1} \ ,$$

where $\hat{\mathbb{P}}$ is a deformed permutation operator, whose only non-vanishing entries are

$$\hat{\mathbb{P}} \left|21\right\rangle = \xi_3 \left|12\right\rangle \ , \qquad \hat{\mathbb{P}} \left|32\right\rangle = \xi_1 \left|23\right\rangle \ , \qquad \hat{\mathbb{P}} \left|13\right\rangle = \xi_2 \left|31\right\rangle \ .$$

# This Hamiltonian is not diagonalisable

To show that this Hamiltonian is non-diagonalisable, we can check that there exist states $|\psi\rangle$ such that $(\hat{\mathbf{H}}_{\xi_1,\xi_2,\xi_3})^q|\psi\rangle \neq 0$ for any $q < k$ and $(\hat{\mathbf{H}}_{\xi_1,\xi_2,\xi_3})^q|\psi\rangle \neq 0$ for any $q \geq k$.

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If we consider the state consisting of only 1's as our vacuum, then the 2's will behave as right-moving excitations while the 3's will behave as left-moving excitations. In addition, if a 2 encounters a 3, they cannot pass through each other, forming an impenetrable wall.

By repeatedly applying the Hamiltonian to a state that contains the three kinds of fields, walls of the form ...222233333... will start forming. Once all the 2's and 3's are "locked" into a wall, the state will vanish if we apply the Hamiltonian another time.

We will denote by $L$ the number of fields of the operator/chain sites, by $M$ the number of 2's and 3's, and by $K$ the number of 3's.

# A goal, an example and a problem

# The goal

As with many problems in physics, we can try to find the (generalised) eigenstates of the eclectic spin Hamiltonian with only the usual methods and brute force, but (as expected) this approach becomes prohibitively difficult very soon.

Instead, we want to take advantage of the fact that the system is integrable at finite values of the $q_i$'s: We will compute the eigenvalues and eigenvectors using the Bethe ansatz and study their $q_i \to \infty$ limit.

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However, this task is not as easy as it seems. Let us illustrate it with an example.

# The example

Consider the finitely twisted Hamiltonian for the case of $L = 3$, $M = 2$ and $K = 1$ (one 1, one 2 and one 3), which can be represented as

$$\tilde{\mathbf{H}}_{(q_1,q_2,q_3)} = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & q_3 & q_1 & q_2 \\ 0 & 0 & 0 & q_2 & q_3 & q_1 \\ 0 & 0 & 0 & q_1 & q_2 & q_3 \\ \hline \frac{1}{q_3} & \frac{1}{q_2} & \frac{1}{q_1} & 0 & 0 & 0 \\ \frac{1}{q_1} & \frac{1}{q_3} & \frac{1}{q_2} & 0 & 0 & 0 \\ \frac{1}{q_2} & \frac{1}{q_1} & \frac{1}{q_3} & 0 & 0 & 0 \end{array}\right) .$$

In the strong twist limit, only the upper right block is non-trivial, so it is clearly non-diagonalisable.

# Eigenvalues

This matrix can be diagonalised for generic values of the twist parameters, with eigenvalues

$$\lambda_1^{\pm} = \pm \sqrt{\sum_{i,j} \frac{q_i}{q_j}} \, ,$$

$$\lambda_2^{\pm} = \pm \frac{1}{\sqrt{2}} \sqrt{9 + \frac{(q_1 - q_2)(q_2 - q_3)(q_3 - q_1)}{q_1 q_2 q_3} - \sum_{i,j} \frac{q_i}{q_j}} \, ,$$

$$\lambda_3^{\pm} = \pm \frac{1}{\sqrt{2}} \sqrt{9 - \frac{(q_1 - q_2)(q_2 - q_3)(q_3 - q_1)}{q_1 q_2 q_3} - \sum_{i,j} \frac{q_i}{q_j}} \, .$$

In the strongly twisted limit, these three eigenvalues vanish, $E_i^{\pm} = \lim_{\epsilon \to 0} \epsilon \lambda_i^{\pm} = 0$.

# Eigenvectors

Let us focus on the eigenvectors associated to $\lambda_1^{\pm}$

$$v_1^{\pm} = \sqrt{\frac{(q_1+q_2+q_3)^2}{2(q_1+q_2+q_3)^2+3(\lambda_1^{\pm})^2}} \left( \frac{q_1+q_2+q_3}{\lambda_1^{\pm}}, \frac{q_1+q_2+q_3}{\lambda_1^{\pm}}, \frac{q_1+q_2+q_3}{\lambda_1^{\pm}}, 1, 1, 1 \right) .$$

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Substituting $q_i \to \frac{\xi_i}{\epsilon}$, both vectors become the same up to a sign in the limit of large twist

$$\lim_{\epsilon \to 0} v_1^{\pm} = \frac{\pm 1}{\sqrt{3}} (1,1,1,0,0,0) \ .$$

A similar situation occurs for the other four eigenvectors.

# Eigenvectors

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$$v_1^{\pm} = \sqrt{\frac{(q_1+q_2+q_3)^2}{2(q_1+q_2+q_3)^2+3(\lambda_1^{\pm})^2}} \left( \frac{q_1+q_2+q_3}{\lambda_1^{\pm}}, \frac{q_1+q_2+q_3}{\lambda_1^{\pm}}, \frac{q_1+q_2+q_3}{\lambda_1^{\pm}}, 1, 1, 1 \right) .$$

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A similar situation occurs for the other four eigenvectors.

Thus, out of the six linearly independent vectors that exist, we are only able to find three. Where are the remaining three?

# Generalised eigenvectors

The situation may seem dire, but we will show later that the key to finding the remaining vectors of the Hilbert space is to realise that they cannot be eigenvectors of the eclectic Hamiltonian, but generalised eigenvectors.

# Generalised eigenvectors

The situation may seem dire, but we will show later that the key to finding the remaining vectors of the Hilbert space is to realise that they cannot be eigenvectors of the eclectic Hamiltonian, but generalised eigenvectors.

After playing with the eigenvectors a bit, we find that the linear combination

$$\frac{v_1^+ + v_1^-}{|v_1^+ + v_1^-|} = \frac{1}{\sqrt{3}}(0,0,0,1,1,1) \ ,$$

is actually one of the missing vectors. In fact, we can check that it is a generalised eigenvector of rank 2 associated with $(1,1,1,0,0,0)$. We can do similar computations for the other pairs to recover the remaining two vectors.

# The problem

This is not a peculiarity of this example. [Ahn, Staudacher, 2020] found that generic eigenvectors at strong twist become states of the form

$$|\psi(k)\rangle = \sum_{l=1}^{L} e^{2\pi k l i/L} U^l |1, \dots, 1, 2, \dots, 2, 3, \dots, 3\rangle \ ,$$

where $U$ is the operator that shifts all positions to the left by one site.

They called these states **locked states** (as the relative positions of the excitations are frozen) and they are eigenvectors of the eclectic Hamiltonian. Although they only find eigenvectors of this type, they also show that those **cannot be the only** eigenvectors.

# Questions

This raises the following questions:

- Why are some eigenvectors missing?
- Can we find the remaining eigenvectors?
- Is there a way to find the generalised eigenvectors?

If we want construct a Bethe ansatz that works on non-diagonalisable systems, like the one we are considering, we have to answer these questions first.

# Exceptional points, coalescence and non-diagonalisable matrices

# Perturbation theory

Given two diagonalisable matrices, $A$ and $B$ such that $[A, B] \neq 0$, we can consider the matrix

$$H(\epsilon) = A + \epsilon B \ ,$$

with $\epsilon \in \mathbb{C}$, which will be non-Hermitian.

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It is well known that Hermitian matrices can have points $\epsilon_k$ such that two or more eigenvalues of $H(\epsilon_k)$ are **degenerate**, i.e. become equal. This is, of course, also true for non-Hermitian matrices.

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However, non-Hermitian matrices can have another type of degeneration. In particular, non-Hermitian matrices can have points $\epsilon_k$ such that two or more eigenvectors of $H(\epsilon_k)$ **coalesce**, i.e. become equal. These points are called **exceptional points**.

# Characteristic polynomials and eigenvectors

Before continuing, it is worth refreshing some concepts about Jordan normal forms.

Given a matrix $M$, we say that $\lambda_i$ is an eigenvalue of $M$ with **algebraic multiplicity** $n_i$ if it is a zero of degree $n_i$ of the characteristic polynomial of $M$, that is, if

$$\det(M - \lambda \mathbb{I}) \propto (\lambda - \lambda_i)^{n_i} \ ,$$

where $\mathbb{I}$ is the identity matrix. We say that $v_i$ is an eigenvector of $M$ associated with the eigenvalue $\lambda_i$ if

$$M v_i = \lambda_i v_i \ .$$

The total number of linearly independent vectors that fulfil this equation, i.e. the dimension of $\text{Ker}(M - \lambda_i I)$, is called **geometric multiplicity** of $\lambda_i$.

# Generalised eigenvectors

If the geometric multiplicity is smaller than the algebraic multiplicity, we say that the matrix is **non-diagonalisable** or **defective**.

In that case, we can "make up for the missing vectors" by introducing generalised eigenvectors. Given a defective eigenvalue $\lambda_i$ and an eigenvector $v_{i,\alpha}^{(1)}$ associated with it, we define the **generalised eigenvector of rank** $n$ as the vector fulfilling

$$(M - \lambda_i \mathbb{I}) v_{i,\alpha}^{(n)} = v_{i,\alpha}^{(n-1)} \ ,$$

where the index $\alpha$ labels the possible geometric multiplicity of the eigenvalue $\lambda_i$. The set of interlocked generalised eigenvectors is called **Jordan chain**.

We can check that a generalised eigenvector of rank $n$ satisfies the weaker condition

$$(M - \lambda_i \mathbb{I})^n v_{i,\alpha}^{(n)} = 0 \ .$$

# Jordan normal form

Generalised eigenvectors are all linearly independent, but not necessarily orthogonal. This is enough to write a similarity transformation that changes a given square matrix into a block diagonal matrix of the form, with blocks of the form

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & 0 & \dots \\ 0 & \lambda_i & 1 & 0 & \dots \\ 0 & 0 & \lambda_i & 1 & \dots \\ 0 & 0 & 0 & \lambda_i & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This matrix is called the **Jordan normal form** of $M$, and each block is called **Jordan block** or **Jordan cell**.

# A theorem

In this talk I will be making extensive use of the following theorem on the upper bound of generalised eigenvectors

**Theorem**

*The number of generalised eigenvectors of rank $l$ is never larger than the number of true eigenvectors, i.e.,*
$dim\left\{Ker(M-\lambda\mathbb{I})^l\right\} - dim\left\{Ker(M-\lambda\mathbb{I})^{l-1}\right\} \leq dim\left\{Ker(M-\lambda\mathbb{I})\right\}.$

*In fact,* $dim\left\{Ker(M-\lambda\mathbb{I})^l\right\} - dim\left\{Ker(M-\lambda\mathbb{I})^{l-1}\right\} \leq dim\left\{Ker(M-\lambda\mathbb{I})^n\right\} - dim\left\{Ker(M-\lambda\mathbb{I})^{n-1}\right\}$ *for any two* $n<l$.

The proof of this theorem comes from noticing that, if $v^{(n)}$ is a generalised eigenvector of rank $n$, then $(M-\lambda\mathbb{I})^l v^{(n)}$ is either zero or a generalised eigenvector of rank $n-l$.

# Generalised eigenvectors as a limit

# Some explanations are in order

Now we can justify the peculiarities we saw in the previous example.

Let us consider an $N \times N$ matrix $M(\epsilon)$ such that is diagonalisable for almost all $\epsilon$, but it becomes similar to a single Jordan cell of size $N$ when $\epsilon = 0$.

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Let us consider an $N \times N$ matrix $M(\epsilon)$ such that is diagonalisable for almost all $\epsilon$, but it becomes similar to a single Jordan cell of size $N$ when $\epsilon = 0$.

If we consider the definition of eigenvector and apply the limit, we find

$$\lim_{\epsilon \to 0} [(M(\epsilon) - \lambda_i \mathbb{I}) v_i] = \left[ \lim_{\epsilon \to 0} (M(\epsilon) - \lambda_i \mathbb{I}) \right] \left[ \lim_{\epsilon \to 0} v_i \right] = 0 \ ,$$
$$\left[ M(0) - \mathbb{I} \lim_{\epsilon \to 0} \lambda_i \right] \left[ \lim_{\epsilon \to 0} v_i \right] = 0 \ .$$

This means that all eigenvectors of the diagonalisable matrix **become the only eigenvector** of the defective matrix $M(0)$. This is the mathematical explanation behind the coalescence of eigenvectors.

# That's good, but, where are the generalised eigenvectors?

If we want to find the generalised eigenvectors, we first have notice that

$$[M(\epsilon) - \lambda_1 \mathbb{I}][M(\epsilon) - \lambda_2 \mathbb{I}](\alpha_1 \, v_1 + \alpha_2 \, v_2) = 0 \ ,$$

holds for any constants $\alpha_i$.

If we take the limit of this expression and use that all eigenvalues have the same limit, we find

$$\left[M(0) - \mathbb{I} \lim_{\epsilon \to 0} \lambda_1\right] \left[M(0) - \mathbb{I} \lim_{\epsilon \to 0} \lambda_2\right] \left[\lim_{\epsilon \to 0} (\alpha_1 \, v_1 + \alpha_2 \, v_2)\right] =$$

$$= \left[M(0) - \mathbb{I} \lim_{\epsilon \to 0} \lambda_i\right]^2 \left[\lim_{\epsilon \to 0} (\alpha_1 \, v_1 + \alpha_2 \, v_2)\right] = 0 \ ,$$

which is the weaker definition of a generalised eigenvector of rank 2.

# But that is not enough

However, this is not enough to find the generalised eigenvector, as a general linear combination will always be an eigenvector

$$\lim_{\epsilon\to 0}[M(\epsilon)-\lambda_1\mathbb{I}](\alpha_1\, v_1+\alpha_2\, v_2)=\lim_{\epsilon\to 0}\alpha_2\,(\lambda_2-\lambda_1)\, v_2=0\ .$$

If we want to find a generalised eigenvector we need a linear combination that is not an eigenvector. This is accomplished if we choose both $\alpha_i$ to diverge as $(\lambda_2-\lambda_1)^{-1}$.

# Constructing the generalised eigenvector of rank 2

Taking into account that coalescing eigenvectors can still differ by a phase factor, we define the following linear combination

$$w_{ij} = \frac{v_i - \beta_{ji} v_j}{|v_i - \beta_{ji} v_j|} \qquad \text{with} \qquad \beta_{ji} = v_j^\dagger \cdot v_i \ ,$$

where $|v|^2 = v^\dagger \cdot v$ is the usual vector norm.

$\lim_{\epsilon \to 0} w_{ij}$ becomes a $\frac{0}{0}$ indeterminate form, which gives us the generalised eigenvector of rank 2 when computed using L'Hôpital's rule.

Similarly to what happens with eigenvectors, this process is independent of which vectors $v_i$ and $v_j$ we chose.

# Plus ultra

Similarly, to find the generalised eigenvector of rank $n$ we need to consider the linear combination of $n$ eigenvectors, as it fulfils the relation

$$\left[M(0) - \mathbb{I} \lim_{\epsilon\to 0} \lambda_i\right]^n \left[\lim_{\epsilon\to 0} \left(\sum_{i=1}^{n} \alpha_i \, v_i\right)\right] = 0 \ .$$

Therefore, we can recursively define the vectors

$$w_{ij}^{(n)} = \frac{w_{ji}^{(n-1)} - \beta_{kj}^{(n-1)} w_{ki}^{(n-1)}}{|w_{ji}^{(n-1)} - \beta_{kj}^{(n-1)} w_{ki}^{(n-1)}|} \qquad \text{with} \qquad \beta_{kj}^{(n-1)} = (w_{ji}^{(n-1)})^{\dagger} \cdot w_{ki}^{(n-1)} \ ,$$

where we fix $w_{ij}^{(0)} = v_i$. The limit of the vector $w_{ij}^{(n)}$ gives us a linear combination of generalised eigenvectors up to rank $n+1$ (as we are solving the weaker definition of generalised eigenvector).

# Properties of the $w_{ij}^{(k)}$

The vectors $w_{ij}^{(k)}$ are constructed in such a way that they are orthonormal for any value of $\epsilon$.

This, together with the fact that the limit of $w_{ij}^{(n)}$ gives us a linear combination of generalised eigenvectors up to rank $n+1$, is enough to show that the vectors $\{\lim_{\epsilon\to 0} v_i, \lim_{\epsilon\to 0} w_{ij}^{(1)}, \dots, \lim_{\epsilon\to 0} w_{ij}^{(N-1)}\}$ are equal to the generalised eigenvectors of $M(0)$ up to eigenvectors of lower rank.

Notice that these results extrapolate immediately to matrices with more than one Jordan cell as long as all the cells have different eigenvalues.

# More examples, more issues

# Can we extend the method?

We have established a method that gives us the generalised eigenvectors as long as all the Jordan cells have different eigenvalues. Now, we want to see what happens if we apply it to a matrix with two or more Jordan cells with the same eigenvalue.

Using [Ahn, Staudacher, 2020], we find that the eclectic spin chain with $L = 5$, $M = 3$ and $K = 1$ has two Jordan blocks of size five and one respectively, so we can use it as a testing ground.

If we focus on states of zero momentum, the Hamiltonian of the finitely twisted case can be written as

$$\tilde{\mathbf{H}}_{(q_1,q_2,q_3)} = \begin{pmatrix} 2 & q_3 & q_1 & q_2 & 0 & 0 \\ \frac{1}{q_3} & 0 & q_3 & q_3 & q_1+q_2 & 0 \\ \frac{1}{q_1} & \frac{1}{q_3} & 1 & 0 & q_3 & q_1 \\ \frac{1}{q_2} & \frac{1}{q_3} & 0 & 1 & q_3 & q_2 \\ 0 & \frac{1}{q_1}+\frac{1}{q_2} & \frac{1}{q_3} & \frac{1}{q_3} & 0 & q_3 \\ 0 & 0 & \frac{1}{q_1} & \frac{1}{q_2} & \frac{1}{q_3} & 2 \end{pmatrix} .$$

# Simpler matrix

However, this matrix is very difficult to diagonalise analytically. For the sake of this example, we will consider a simpler matrix

$$\tilde{H}^{(1)} = \begin{pmatrix} 2 & q_3 & q_1 & q_2 & 0 & 0 \\ 0 & 0 & q_3 & q_3 & q_1 + q_2 & 0 \\ 0 & 0 & 1 & 0 & q_3 & q_1 \\ 0 & 0 & 0 & 1 & q_3 & q_2 \\ 0 & 0 & 0 & 0 & 0 & q_3 \\ 0 & 0 & 0 & 0 & \frac{1}{q_3} & 2 \end{pmatrix} .$$

If when we substitute $q_i \to \frac{\xi_i}{\epsilon}$ and compute the $\epsilon \to 0$ limit, we will obtain the same strongly twisted Hamiltonian.

# The eigenvectors...

The eigenvectors of this matrix are

$$
\begin{aligned}
v_1 &= (1, 0, 0, 0, 0, 0) \ , \\
v_2 &= (\xi_3, -2\epsilon, 0, 0, 0, 0) \ , \\
v_3 &= \left(\xi_3^2 + \xi_2 \epsilon, -2\xi_3 \epsilon, 0, -\epsilon^2, 0, 0\right) \ , \\
v_4 &= \left(\xi_3^2 + \xi_1 \epsilon, -2\xi_3 \epsilon, 0, -\epsilon^2, 0, 0\right) \ , \\
v_5 &= \left(\xi_3^4 - \frac{1+2\sqrt{2}}{2+\sqrt{2}}(\xi_1+\xi_2)\xi_3^2\epsilon + \frac{2-\sqrt{2}}{2+\sqrt{2}}(\xi_1^2+\xi_2^2)\frac{\epsilon^2}{2}, -\frac{4+3\sqrt{2}}{2+\sqrt{2}}\xi_3^3\epsilon + \frac{8+5\sqrt{2}}{2+\sqrt{2}}(\xi_1+\xi_2)\xi_3\frac{\epsilon^2}{2},\right. \\
&\quad \left. \frac{\xi_3^2\epsilon^2}{2} + \frac{1-\sqrt{2}}{2}\xi_1\epsilon^3, \frac{\xi_3^2\epsilon^2}{2} + \frac{1-\sqrt{2}}{2}\xi_2\epsilon^3, -\frac{1+\sqrt{2}}{2+\sqrt{2}}\xi_3\epsilon^3, \frac{\epsilon^4}{2+\sqrt{2}}\right) \ , \\
v_6 &= \left(\xi_3^4 - \frac{1-2\sqrt{2}}{2-\sqrt{2}}(\xi_1+\xi_2)\xi_3^2\epsilon + \frac{2+\sqrt{2}}{2-\sqrt{2}}(\xi_1^2+\xi_2^2)\frac{\epsilon^2}{2}, -\frac{4-3\sqrt{2}}{2-\sqrt{2}}\xi_3^3\epsilon + \frac{8-5\sqrt{2}}{2-\sqrt{2}}(\xi_1+\xi_2)\xi_3\frac{\epsilon^2}{2},\right. \\
&\quad \left. \frac{\xi_3^2\epsilon^2}{2} + \frac{1+\sqrt{2}}{2}\xi_1\epsilon^3, \frac{\xi_3^2\epsilon^2}{2} + \frac{1+\sqrt{2}}{2}\xi_2\epsilon^3, -\frac{1-\sqrt{2}}{2-\sqrt{2}}\xi_3\epsilon^3, \frac{\epsilon^4}{2-\sqrt{2}}\right) \ .
\end{aligned}
$$

# ... and their limits

It is easy to see that all the vectors coalesce to $\hat{u}_1 = (1, 0, 0, 0, 0, 0)$ at the exceptional point. In addition, if we construct the following five vectors

$$w_{i,1}^{(1)} = \frac{v_i - (v_i \cdot v_1) v_1}{\epsilon} \ ,$$

we get that they all become proportional to $\hat{u}_2 = (0, 1, 0, 0, 0, 0)$ in the limit of vanishing $\epsilon$. If we consider now the following four vectors

$$w_{i,2}^{(2)} = \frac{w_{i,1}^{(1)} - (w_{2,1}^{(1)} \cdot w_{i,1}^{(1)}) w_{2,1}^{(1)}}{\epsilon} \ ,$$

we find different linear combinations of $\hat{u}_3$ and $\hat{u}_4$ when $\epsilon$ approaches 0. This should not be possible.

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After some algebra, we find that the vector $(0, \xi_2 - \xi_1, \xi_3, -\xi_3, 0, 0)$ is indeed another eigenvector of the strongly twisted Hamiltonian.

# ... and their limits

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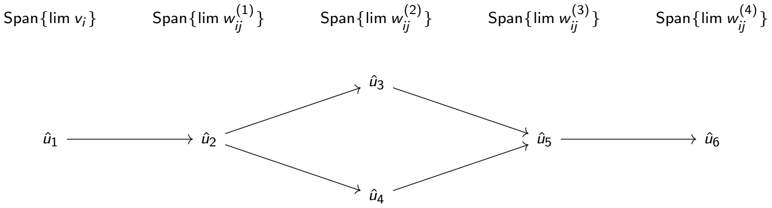
If we continue the process, we find

$$\lim_{\epsilon\to 0} w^{(3)} = (0,0,0,0,1,0) = \hat{u}_5 \ , \quad \lim_{\epsilon\to 0} w^{(4)} = (0,0,0,0,0,1) = \hat{u}_6 \ .$$

# A diagram

We can condense all the information about generalised eigenvectors into the following diagram



# Information from the diagram

The diagram seems to indicate a Jordan block of size 5 and a Jordan block of size 1.

We can check if this is correct by using the definition of generalised eigenvector and the information from the diagram. After some tedious algebra

$$\hat{\mathbf{H}}^5 \frac{2\xi_3^3\hat{u}_6 - 3\xi_3^2(\xi_1+\xi_2)\hat{u}_5 + 2\xi_3(\xi_1^2+\xi_2^2+3\xi_1\xi_2)(\hat{u}_3+\hat{u}_4) - 2(\xi_1^3+\xi_2^3+4\xi_1^2\xi_2+4\xi_1\xi_2^2)\hat{u}_2}{4\xi_3^7} =$$

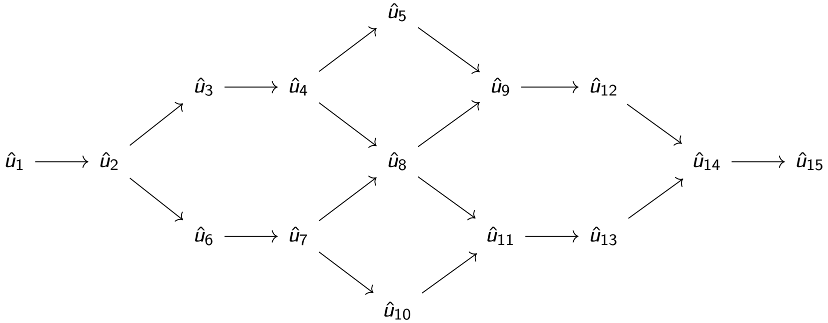
$$\hat{\mathbf{H}}^4 \frac{2\xi_3^2\hat{u}_5 - \xi_3(3\xi_1+\xi_2)\hat{u}_4 - \xi_3(\xi_1+3\xi_2)\hat{u}_3 + (\xi_1^2+\xi_2^2+6\xi_1\xi_2)\hat{u}_2}{4\xi_3^5} =$$

$$\hat{\mathbf{H}}^3 \frac{\xi_3(\hat{u}_3+\hat{u}_4) - (\xi_1+\xi_2)\hat{u}_2}{2\xi_3^3} = \hat{\mathbf{H}}^2\frac{\hat{u}_2}{\xi_3} = \hat{\mathbf{H}}\hat{u}_1 = 0 \ ,$$

$$\hat{\mathbf{H}}[\xi_3(\hat{u}_3-\hat{u}_4) - (\xi_1-\xi_2)\hat{u}_2] = 0 \ .$$

# $L = 7$, $M = 3$, $K = 1$

This problem keeps getting worse as we increase the number of states. If we analyse the case with $L = 7$, we find the following diagram




Thus, we conclude that this phenomenon is not an accident but a feature.

# Understanding the issue

# Why are eigenvectors misplaced?

If we come back to the relation

$$\left[ M(0) - \mathbb{I} \lim_{\epsilon \to 0} \lambda_i \right] \left[ \lim_{\epsilon \to 0} v_i \right] = 0 \ ,$$

we should notice that this condition is sufficient but not necessary. This means that the limit of an eigenvector is always an eigenvector of the defective matrix, but not all eigenvectors of the defective matrix have to be the limit of eigenvectors.

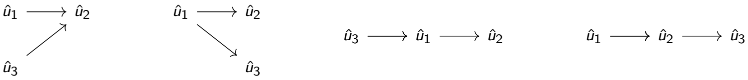
For the distinguishable case both $\lim_{\epsilon \to 0} w_{ij}^{(n-1)}$ and $\text{Ker}\{(M(0) - \lambda \mathbb{I})^n\} / \text{Ker}\{(M(0) - \lambda \mathbb{I})^{n-1}\}$ were one-dimensional and we could identify both spaces. Now we cannot do it.

# Four examples

How the Jordan chains mix depends on how we approach the exceptional point. Consider these four matrices, which become the same at the exceptional point

$$\begin{pmatrix} 0 & 1 & \epsilon^2 \\ \epsilon^2 & \epsilon^4 & \epsilon^2 \\ 0 & 0 & \epsilon^6 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & \epsilon^2 \\ \epsilon^2 & \epsilon^4 & \epsilon \\ 0 & 0 & \epsilon^6 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & \epsilon^2 \\ \epsilon^5 & 0 & 0 \\ \epsilon & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & \epsilon^2 \\ 0 & \epsilon^4 & \epsilon^2 \\ 0 & 0 & \epsilon^6 \end{pmatrix} .$$

The results of applying our recipe to each matrix can be summarised in the following diagrams



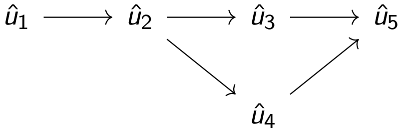

In these four examples we see that the Jordan chain of one eigenvector can appear wherever it wants, depending on how we approach the exceptional point.

# A fifth example

This mixing can be even more confusing with larger Jordan chains. Consider the matrix

$$\begin{pmatrix} \epsilon & 1 & \epsilon & 0 & 0 \\ 0 & \epsilon^4 & 1 & \epsilon & 0 \\ 0 & 0 & \epsilon^7 & \epsilon^9 & 0 \\ 0 & 0 & 0 & \epsilon^6 & 1 \\ 0 & 0 & 0 & 0 & \epsilon^8 \end{pmatrix} .$$

the diagram representing the structure of the vectors we obtain takes the form




So, we cannot be sure if $\hat{u}_5$ is a generalised eigenvector of rank 1, 2 or 4.

We call this effect **chain mixing** for obvious reasons.

# But there is hope

Despite the chain mixing, the procedures still conserve two important properties:

1. It is complete, in the sense that it gives us all the generalised eigenvectors of $M(0)$.
2. If a generalised eigenvector of rank $n$ appears at the $m$-th step, necessarily an eigenvector of rank $n-1$ should have appeared at the $m-1$-th step.

# Coordinate Bethe Ansatz at finite twist for $K = 1$

# The twist and the nested Bethe ansatz

Now we want to apply the procedure to the eclectic spin chain for any values of $L$ and $M$, but we will focus exclusively in the $K = 1$ case.[1]

First we have to diagonalise the finite twist spin chain. This can be done using a slightly modified version of the Nested Coordinate Bethe Ansatz of [Sutherland, 1985], as we have to deal with the twists.

[1] General $K$ coming soon.

# The twist and the nested Bethe ansatz

Now we want to apply the procedure to the eclectic spin chain for any values of $L$ and $M$, but we will focus exclusively in the $K = 1$ case.[1]

First we have to diagonalise the finite twist spin chain. This can be done using a slightly modified version of the Nested Coordinate Bethe Ansatz of [Sutherland, 1985], as we have to deal with the twists.

For example, the ansatz for the case of two different excitations takes the form

$$|\psi_{23}(p_1,p_2)\rangle = \sum_{1\leq n_1<n_2\leq L} \left[ A_{23} e^{i(p_1 n_1 + p_2 n_2)} \frac{q_3^{n_1}}{q_2^{n_2}} + \tilde{A}_{23} e^{i(p_1 n_2 + p_2 n_1)} \frac{q_3^{n_1}}{q_2^{n_2}} \right] S_{n_1}^{2,+} S_{n_2}^{3,+} |0\rangle$$

$$+ \sum_{1\leq n_1<n_2\leq L} \left[ A_{32} e^{i(p_1 n_1 + p_2 n_2)} \frac{q_3^{n_2}}{q_2^{n_1}} + \tilde{A}_{32} e^{i(p_1 n_2 + p_2 n_1)} \frac{q_3^{n_2}}{q_2^{n_1}} \right] S_{n_1}^{3,+} S_{n_2}^{2,+} |0\rangle \ ,$$

[1] General $K$ coming soon.

# Bethe equations

Imposing periodicity of this ansatz, we get the following Bethe matrix equations

$$e^{ip_1 L}\begin{pmatrix} q_3^L A_{22} \\ q_3^L A_{23} \\ q_2^{-L} A_{32} \\ q_2^{-L} A_{33} \end{pmatrix} = \begin{pmatrix} \tilde{A}_{22} \\ \tilde{A}_{23} \\ \tilde{A}_{32} \\ \tilde{A}_{33} \end{pmatrix} = S(p_1, p_2)\begin{pmatrix} A_{22} \\ A_{23} \\ A_{32} \\ A_{33} \end{pmatrix} , \qquad e^{ip_2 L}\begin{pmatrix} q_3^L \tilde{A}_{22} \\ q_2^{-L} \tilde{A}_{23} \\ q_3^L \tilde{A}_{32} \\ q_2^{-L} \tilde{A}_{33} \end{pmatrix} = \begin{pmatrix} A_{22} \\ A_{23} \\ A_{32} \\ A_{33} \end{pmatrix} = S(p_2, p_1)\begin{pmatrix} \tilde{A}_{22} \\ \tilde{A}_{23} \\ \tilde{A}_{32} \\ \tilde{A}_{33} \end{pmatrix} .$$

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Which generalises to

$$e^{ip_k L} q_3^{(3-f_k)L} q_2^{(2-f_k)L} = S(p_k, p_{k+1}) \dots S(p_k, p_M) S(p_k, p_1) \dots S(p_k, p_{k-1}) \,,$$

where $f_k$ is the flavour of the $k$-th excitation.

# Results of the NCBA

Let me skip the details of the computations and go straight to the answer.

The ansatz for general $M$ and $K=1$ takes the form

$$|\psi\rangle = \sum_{k=1}^{M} \sum_{\sigma \in S_M} \psi_k(\sigma) e^{i\sum_j p_{\sigma(j)} n_j} \frac{q_3^{\sum_j n_j}}{(q_2 q_3)^{n_k}} S_{n_1}^{2+} \cdots S_{n_{k-1}}^{2+} S_{n_k}^{3+} S_{n_{k+1}}^{2+} \cdots S_{n_M}^{2+} |0\rangle \, .$$

In terms of the rapidity variable $e^{ip_i} = \frac{u_i}{u_i+1}$ we have

$$\psi_{k-l}(Id.) = \left[ \frac{1}{(q_1 q_2 q_3)^l} \prod_{j=0}^{l-1} \frac{u_{k-j} - \bar{x}}{u_{k-j-1} - \bar{x} + 1} \right] \psi_k(Id.) \, ,$$

with the other coefficients fixed by periodicity.

# Bethe equations

The rapidities $u_i$ and the auxiliary rapidity $\bar{x}$ have to satisfy the Bethe equations

$$\frac{(q_2 q_3)^L}{(q_1 q_2 q_3)^M} \prod_{j=1}^{M} \frac{\bar{x} - u_k}{\bar{x} - u_k - 1} = 1 \ ,$$

$$\frac{q_3^L}{q_1 q_2 q_3} \frac{\bar{x} - u_k}{\bar{x} - u_k - 1} \prod_{j \neq k} \frac{u_k - u_j + 1}{u_k - u_j - 1} = \left( \frac{u_k + 1}{u_k} \right)^L \ .$$

# Coalescence of Bethe vectors for $M = 2$

# Rapidities

Analysing the Bethe equations, we find that the solutions behave as

$$u_1 \approx \epsilon^{\alpha} u_- \ , \qquad u_2 \approx -1 + \epsilon^{\alpha} u_+ \ , \qquad \bar{x} \approx u_2 + \epsilon^{\gamma} \hat{v} \ ,$$

where $\alpha = \frac{L-3}{L-1}$ and $\gamma = 2L - 6$.

The Bethe equations become

$$u_-^L = \frac{\xi}{\xi_3^L}(u_- - u_+) \ , \quad (-u_+)^L = \frac{\xi}{\xi_2^L}(u_- - u_+) \ , \quad \hat{v} = -\frac{2\xi_1^L}{\xi^{L-2}} \ ,$$

where $\xi = \xi_1 \xi_2 \xi_3$.

# Coefficients

The coefficients satisfy

$$\frac{\psi_1(Id.)}{\psi_2(Id.)} = \frac{1}{q_1 q_2 q_3} \frac{u_2 - \bar{x}}{u_1 - \bar{x} + 1} \ .$$

So, substituting the behaviour of the rapidities

$$\begin{pmatrix} \psi_2(Id.) \\ \psi_1(Id.) \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{q_1 q_2 q_3} \frac{-\hat{v}\epsilon^{\gamma}}{2+(u_- - u_+)\epsilon^{\alpha} - \hat{v}\epsilon^{\gamma}} \end{pmatrix} \approx \begin{pmatrix} 1 \\ -\frac{\hat{v}}{2\xi}\epsilon^{\gamma+3} \end{pmatrix}$$

$$\begin{pmatrix} \psi_1(\tau) \\ \psi_2(\tau) \end{pmatrix} = \begin{pmatrix} \frac{\epsilon^{-\alpha}+u_- - u_+}{q_1 q_2 q_3 u_- - q_1 q_2 q_3 u_+} + \frac{-\hat{v}\epsilon^{\gamma-\alpha}}{q_1 q_2 q_3 (u_- - u_+)\left(2 - u_- \epsilon^{\alpha} + u_+ \epsilon^{\alpha} - \hat{v}\epsilon^{\gamma}\right)} \\ \frac{\epsilon^{-\alpha}}{u_- - u_+} - \frac{-\hat{v}\epsilon^{\gamma-\alpha}(-1+u_-\epsilon^{\alpha} - u_+\epsilon^{\alpha})}{(u_- - u_+)\left(2 - u_-\epsilon^{\alpha} + u_+\epsilon^{\alpha} - \hat{v}\epsilon^{\gamma}\right)} \end{pmatrix} \approx \begin{pmatrix} \frac{1}{\xi(u_- - u_+)}\epsilon^{3-\alpha} \\ \frac{1}{u_- - u_+}\epsilon^{-\alpha} \end{pmatrix}$$

# How do they behave?

$$\psi_2(Id.)e^{i(p_1 n_1+p_2 n_2)}\frac{q_3^{n_1}}{q_2^{n_2}} \sim \epsilon^{(1-\alpha)(n_2-n_1)} ,$$

$$\psi_2(\tau)e^{i(p_1 n_2+p_2 n_1)}\frac{q_3^{n_1}}{q_2^{n_2}} \sim \epsilon^{(1+\alpha)(n_2-n_1)-\alpha} ,$$

$$\psi_1(Id.)e^{i(p_1 n_1+p_2 n_2)}\frac{q_3^{n_2}}{q_2^{n_1}} \sim \epsilon^{\gamma+3-(1+\alpha)(n_2-n_1)} ,$$

$$\psi_1(\tau)e^{i(p_1 n_2+p_2 n_1)}\frac{q_3^{n_2}}{q_2^{n_1}} \sim \epsilon^{3-\alpha-(1-\alpha)(n_2-n_1)} ,$$

so we can ignore the second and third contribution, as they are always subleading.

# Leading wavefunction

Putting everything together, we find

$$
\begin{aligned}
|\psi\rangle &\approx \sum_{n_1<n_2}\left[\psi_2(Id.)e^{i(p_1 n_1+p_2 n_2)}\frac{q_3^{n_1}}{q_2^{n_2}}S^{2,+}_{n_1}S^{3,+}_{n_2}+\psi_1(\tau)e^{i(p_1 n_2+p_2 n_1)}\frac{q_3^{n_2}}{q_2^{n_1}}S^{3,+}_{n_1}S^{2,+}_{n_2}\right]|0\rangle \\
&\approx \sum_{n_1<n_2}\left[\frac{(\xi_3 u_-)^{n_1}}{(-\xi_2 u_+)^{n_2}}\epsilon^{(1-\alpha)(n_2-n_1)}S^{2,+}_{n_1}S^{3,+}_{n_2}+\frac{1}{\xi(u_- - u_+)}\frac{(\xi_3 u_-)^{n_2}}{(-\xi_2 u_+)^{n_1}}\epsilon^{3-\alpha-(1-\alpha)(n_2-n_1)}S^{3,+}_{n_1}S^{2,+}_{n_2}\right]|0\rangle \\
&\approx \epsilon^{1-\alpha}\left[\sum_{n_1=1}^{L-1}\frac{-1}{\xi_2 u_+}\left(-\frac{\xi_3 u_-}{\xi_2 u_+}\right)^{n_1}S^{2,+}_{n_1}S^{3,+}_{n_1+1}\right]|0\rangle+\frac{\epsilon^{1-\alpha}}{\xi(u_- - u_+)}\frac{(\xi_3 u_-)^{L}}{(-\xi_2 u_+)^{1}}S^{3,+}_{1}S^{2,+}_{L}|0\rangle\ .
\end{aligned}
$$

Using the Bethe equation $(-\xi_2 u_+)^L = \xi(u_- - u_+)$, we can combine the two terms into

$$
\frac{|\psi\rangle}{\epsilon^{1-\alpha}} \approx \frac{-1}{\xi_2 u_+}\sum_{n_1=1}^{L}\left(-\frac{\xi_3 u_-}{\xi_2 u_+}\right)^{n_1}S^{2,+}_{n_1}S^{3,+}_{n_1+1}|0\rangle = |\psi^{(1)}(u_-/u_+)\rangle\ ,
$$

# The eigenvector

$$\frac{|\psi\rangle}{\epsilon^{1-\alpha}} \approx \frac{-1}{\xi_2 u_+} \sum_{n_1=1}^{L} \left( -\frac{\xi_3 u_-}{\xi_2 u_+} \right)^{n_1} S_{n_1}^{2,+} S_{n_1+1}^{3,+} |0\rangle = |\psi^{(1)}(u_-/u_+)\rangle \ ,$$

This wavefunction has the form of a locked state, so it is an eigenstate of the eclectic Hamiltonian.

In addition, it only depends on the rapidities through the ratio $\frac{u_-}{u_+}$. Written in momenta, the wavefunction depends only on the total momentum $p_1 + p_2$. The prefactor $\frac{1}{\xi_2 u_+}$ is only a normalisation that we can erase.

# The generalised eigenvector of rank 2

To find the generalised eigenvector of rank 2, we can consider the linear combination

$$e^{ip_1}|\psi(p_1,p_2)\rangle - e^{ip_1'}|\psi(p_1',p_2')\rangle \ ,$$

with $p_1+p_2=p_1'+p_2'$, that is, the linear combination of two coalescing vectors.

$$e^{ip_1}|\psi(p_1,p_2)\rangle - e^{ip_1'}|\psi(p_1',p_2')\rangle \approx \epsilon^{2-\alpha}\sum_{n_1=1}^{L-2}\left[\frac{u_-}{(-\xi_2 u_+)^2}\left(-\frac{\xi_3 u_-}{\xi_2 u_+}\right)^{n_1} - \frac{u_-'}{(-\xi_2 u_+')^2}\left(-\frac{\xi_3 u_-'}{\xi_2 u_+'}\right)^{n_1}\right]\cdot$$
$$\cdot S^{2,+}_{n_1}S^{3,+}_{n_1+2}|0\rangle + \epsilon^{3-(1-\alpha)(L-2)}\left[\frac{u_-}{\xi(u_- - u_+)}\frac{(\xi_3 u_-)^L}{(-\xi_2 u_+)^2} - \frac{u_-'}{\xi(u_-' - u_+')}\frac{(\xi_3 u_-')^L}{(-\xi_2 u_+')^2}\right]S^{3,+}_2 S^{2,+}_L|0\rangle +$$
$$+\epsilon^{3-(1-\alpha)(L-2)}\left[\frac{u_-}{\xi(u_- - u_+)}\frac{(\xi_3 u_-)^{L-1}}{(-\xi_2 u_+)^1} - \frac{u_-'}{\xi(u_-' - u_+')}\frac{(\xi_3 u_-')^{L-1}}{(-\xi_2 u_+')^1}\right]S^{3,+}_1 S^{2,+}_{L-1}|0\rangle$$
$$\approx \epsilon^{2-\alpha}\frac{u_-}{u_+}\,\frac{u_+' - u_+}{\xi_2^2 u_+ u_+'}\sum_{n_1=1}^{L}\left(-\frac{\xi_3 u_-}{\xi_2 u_+}\right)^{n_1} S^{2,+}_{n_1}S^{3,+}_{n_1+2}|0\rangle = \epsilon^{2-\alpha}|\psi^{(2)}(u_-/u_+)\rangle \ .$$

which also only depends on $\frac{u_-}{u_+}$, proportional to the total momentum.

# The generalised eigenvector of rank 3

Similarly, we construct the generalised eigenvector of rank 3 as

$$\frac{e^{i(p_1+p_1')}}{e^{ip_1'}-e^{ip_1}}\left(e^{ip_1}|\psi(p_1,p_2)\rangle - e^{ip_1'}|\psi(p_1',p_2')\rangle\right) - \frac{e^{i(p_1+p_1'')}}{e^{ip_1''}-e^{ip_1}}\left(e^{ip_1}|\psi(p_1,p_2)\rangle - e^{ip_1''}|\psi(p_1'',p_2'')\rangle\right)$$

$$\approx \text{const.}\ \epsilon^{3-\alpha}\sum_{n_1=1}^{L}\left(-\frac{\xi_3 u_-}{\xi_2 u_+}\right)^{n_1} S^{2,+}_{n_1} S^{3,+}_{n_1+3}|0\rangle = \epsilon^{3-\alpha}|\psi^{(3)}(u_-/u_+)\rangle \ .$$

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$$\approx \text{const. } \epsilon^{3-\alpha}\sum_{n_1=1}^{L}\left(-\frac{\xi_3 u_-}{\xi_2 u_+}\right)^{n_1} S^{2,+}_{n_1}S^{3,+}_{n_1+3}|0\rangle = \epsilon^{3-\alpha}|\psi^{(3)}(u_-/u_+)\rangle \ .$$

In general, at every step we find only one vector of the form

$$|\psi^{(k)}(u_-/u_+)\rangle = \text{const. } \sum_{n_1=1}^{L}\left(-\frac{\xi_3 u_-}{\xi_2 u_+}\right)^{n_1} S^{2,+}_{n_1}S^{3,+}_{n_1+k}|0\rangle \ .$$

Thus, we can claim that we have a single Jordan chain of length $L-1$ for every allowed value of the total momentum.

# Coalescence of Bethe vectors for $M = 3$

# The case of $M = 3$

The case of $M = 3$ has some additional subtleties, but it is mostly similar.

$$\frac{|\psi\rangle}{\epsilon^{3-2\alpha}} \approx \frac{\xi_3(\tilde{u}'_- - \tilde{u}_-)}{\xi_2^2 u_+^2} \sum_{n_1=1}^{L} \left( -\frac{\xi_3^2 u_-^2}{\xi_2 u_+} \right)^{n_1} S_{n_1}^{2,+} S_{n_1+1}^{2,+} S_{n_1+2}^{3,+} |0\rangle \ ,$$

$$\frac{e^{i(2p_1+p_2)}|\psi(p_1, p_2, p_3)\rangle - e^{i(2p'_1+p'_2)}|\psi(p'_1, p'_2, p'_3)\rangle}{\epsilon^4} \approx \text{const.} \sum_{n_1=1}^{L} \left( -\frac{\xi_3^2 u_-^2}{\xi_2 u_+} \right)^{n_1} S_{n_1}^{2,+} S_{n_1+2}^{2,+} S_{n_1+3}^{3,+} |0\rangle \ .$$

# Where everything changes

However, everything changes when we try to compute the generalised eigenvector of rank 3.

In that case, we find the linear combination

$$a_1 \left[ \sum_{n_1=1}^{L} \left( -\frac{\xi_3^2 u_-^2}{\xi_2 u_+} \right)^{n_1} S_{n_1}^{2,+} S_{n_1+3}^{2,+} S_{n_1+4}^{3,+} |0\rangle \right] + a_2 \left[ \sum_{n_1=1}^{L} \left( -\frac{\xi_3^2 u_-^2}{\xi_2 u_+} \right)^{n_1} S_{n_1}^{2,+} S_{n_1+1}^{2,+} S_{n_1+3}^{3,+} |0\rangle \right] ,$$

where $a_1$ and $a_2$ depend on the states we choose. This is an indication that we have an eigenvector in disguise.

# Counting linearly independent vectors

We would like to have a way to find how many linearly independent vectors we find at a given step.

Turns out that this is not a difficult task. The behaviour at $\epsilon = 0$ of the wavefunction is controlled by the momentum factors

$$e^{i(p_1 n_1 + p_2 n_2 + p_3 n_3)} \frac{q_3^{n_1} q_3^{n_2}}{q_2^{n_3}} \sim \epsilon^{(\alpha-1)[(n_1-n_3)+(n_2-n_3)]} .$$

Thus, terms with the same $(n_1 - n_3) + (n_2 - n_3)$ will appear at the same step of the computation.

Therefore, the number of linearly independent vectors will be equal to the number of integer solutions of $(n_1 - n_3) + (n_2 - n_3) = n$.

# System of Diophantine equations

In addition, to solving $(n_1 - n_3) + (n_2 - n_3) = n$, we need to maintain the ordering $1 \leq n_1 < n_2 < n_3 \leq L$.

The computation simplifies if we introduce the variables $x_1 + 1 = n_2 - n_1$ and $x_2 + 1 = n_3 - n_2$, giving us

$$x_1 + 2x_2 = n - 3 = \Delta \, , \qquad x_1 \geq 0 \, , \qquad x_2 \geq 0 \, , \qquad x_1 + x_2 + 2 \leq L - 1 \, .$$

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$$x_1 + 2x_2 = n - 3 = \Delta \ , \qquad x_1 \geq 0 \ , \qquad x_2 \geq 0 \ , \qquad x_1 + x_2 + 2 \leq L - 1 \ .$$

If we also introduce $x_0$ such that $x_0 + x_1 + x_2 + 2 = L - 1$, we can write the system of equations

$$\begin{cases} x_0 + x_1 + x_2 = L - 3 \\ x_1 + 2x_2 = \Delta \\ x_j \geq 0 \end{cases} .$$

# $L = 7,\ M = 3,\ K = 1$

| $\Delta$ | $(x_0, x_1, x_2)$ | Number of sol. |
|---|---|---|
| 0 | $(4, 0, 0)$ | 1 |
| 1 | $(3, 1, 0)$ | 1 |
| 2 | $(2, 2, 0)$, $(3, 0, 1)$ | 2 |
| 3 | $(1, 3, 0)$, $(2, 1, 1)$ | 2 |
| 4 | $(0, 4, 0)$, $(1, 2, 1)$, $(2, 0, 2)$ | 3 |
| 5 | $(0, 3, 1)$, $(1, 1, 2)$ | 2 |
| 6 | $(0, 2, 2)$, $(1, 0, 3)$ | 2 |
| 7 | $(0, 1, 3)$ | 1 |
| 8 | $(0, 0, 4)$ | 1 |

Thus, we conjecture Jordan cells of sizes 1, 5 and 9.

# Coalescence of Bethe vectors for general $M$

# Mixed flavour $S$-matrix

The case of general $M$ follows the same structure. The momenta behave as

$$e^{i\sum_i p_i n_i}\frac{\prod_i q_3^{n_i}}{q_2^{n_M} q_2^{n_M}} \sim \epsilon^{(\alpha-1)(\sum_i n_i)-(\alpha+\beta-2)n_M} = \epsilon^{(\alpha-1)\sum_i (n_i - n_M)} ,$$

where we have used that $(M-K)(\alpha-1) = K(\beta-1)$ to make the final rewriting.

Therefore, the number of linearly independent vectors at a given step is equal to the number of integer solutions of $\sum_i (n_M - n_i) = n$, with restriction $1 \leq (n_M - n_{M-1}) < \cdots < (n_M - n_2) < (n_M - n_1) \leq L-1$.

# System of linear Diophantine equations

Performing the substitution $n_{i+1} - n_i = x_i + 1$ and writing the inequality $n_M - n_1 \leq L - 1$ as the equality $x_0 + n_M - n_1 = L - 1$, we map the problem to a system of linear Diophantine equations

$$\begin{cases} \sum_{j=0}^{M-1} x_j = L - M \\ \sum_{j=0}^{M-1} (j x_j) = \Delta \\ x_j \geq 0 \end{cases} .$$

# Number of solutions

To find the number of solutions, we have to realise that the set of integers $(x_0 + 1, x_1, \ldots, x_{M-1})$ is a solution for length $L$ if $(x_0, x_1, \ldots, x_{M-1})$ is a solution for length $L - 1$.

Thus, we can find the number of solutions for $L$ by adding the number of solutions with length $L$ and $x_0 = 0$ to the number of solutions for $L - 1$.

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Thus, we can find the number of solutions for $L$ by adding the number of solutions with length $L$ and $x_0=0$ to the number of solutions for $L-1$.

If we write the system of Diophantine equations with $x_0=0$ (and subtract the first and second equations), we find

$$\begin{cases} \sum_{j=1}^{M-1} x_j = L-M \\ \sum_{j=1}^{M-1} [(j-1)x_j] = \Delta + M - L \\ x_j \geq 0 \end{cases},$$

which is exactly the system we would find for length $L-1$ and $M-1$ excitations.

# The recurrence relation

Thus, the number of solutions for $\Delta$, $L$ and $M$ is equal to the number of solutions for $\Delta$, $L-1$ and $M$ plus the number of solutions for $\Delta + M - L$, $L-1$ and $M-1$.

This is more clear if we define the generating function

$$F(L,M,x) = \sum_{\Delta=0}^{\infty} x^{\Delta} \#\{\text{Solutions for given values of } L, M, \Delta\} \ .$$

There, it becomes the recurrence relation

$$F(L,M,x) = F(L-1,M,x) + x^{L-M} F(L-1,M-1,x) \ ,$$

supplemented by the initial condition $F(L,2,x) = \sum_{j=0}^{L-2} x^j$ from the $M=2$ case.

# Gaussian polynomials

There is a special function with the properties we need, the **Gaussian polynomials** or **q-deformed binomial coefficients**

$$F(L, M, x) = \binom{L-1}{M-1}_x = \prod_{k=1}^{M-1} \frac{1-x^{L-k}}{1-x^k} \ .$$

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Thus, we can find the number of solutions by reading the coefficients of these polynomials.

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We can extract some general properties from it. For example, the longest Jordan cell has to be of size $(L-M)(M-1)+1$. After some careful computations, we even can check that chain mixing does not spoil this result.

# Conclusions

# Results

- We have constructed a reliable method that uses the eigenvectors of a diagonalisable matrix to compute the generalised eigenvectors of the non-diagonalisable matrix that arises at an exceptional point.
- The method works straightforwardly for matrices with distinguishable Jordan cells. However, it needs some additional work for matrices with degenerate Jordan cells.
- We applied this method to the specific case of the eclectic spin chain of [Ipsen, Staudacher, Zippelius, 2019]. We were able to find the Jordan cell structure for the case of $K = 1$.[2]
- Our results match those of [Ahn, Corcoran, Staudacher, 2021], which were computed by other means.
- It would be interesting to apply to other integrable systems, e.g., those in [De Leeuw, Pribytok, Ryan, 2019].

[2] General $K$ coming soon.