Unitarizing infinite-range forces, graviton-graviton scattering, and the graviball

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Outline

§0 Preamble: Coulomb scattering

Partial-wave expansion:

$$\gamma = \frac{e_1 e_2 m}{p}$$

$$f_{N.R.}(p,\theta) = (2ip)^{-1}\sum_{\ell=0}^{\infty}(2\ell+1)\frac{\Gamma(1+\ell+i\gamma)}{\Gamma(1+\ell-i\gamma)}P_\ell(\cos\theta)$$

$$= -\frac{\gamma}{2p}\left(\sin\frac{\theta}{2}\right)^{-2-2i\gamma}\frac{\Gamma(1+i\gamma)}{\Gamma(1-i\gamma)}$$

⋆ There is no left-hand cut (LC)
⋆ Think of an expansion in powers of γ of the partial-wave amplitude (PWA) in red without LC, and then unitarize

It seems pretty familiar for the aficionados to Unitary ChPT

§1 Quantum gravity (QG), EFT, Unitarization

Confluence of S-matrix theory and Quantum Gravity (GV)

- **Unitarity and analyticity of the S-matrix**, dispersion relations (Gross,Jackiv,...)
- **Black holes as resonances**, transient states (Porto,Giddings, Bezrukov,Levkov, Sibiryakov,...)
- **Trans-planckian scattering** $Gs \gg 1$
  - **Eikonal approximation** (forward scattering) (Amati, Ciafaloni, Veneziano,'t Hooft, Verlinde,...)
  - **Classicalization** (molecular nature of black holes) (Dvali,Gómez,...)
- **On-shell amplitude program** (Bern,Dixon,Kosover,Carrasco,Arkani-Hamed,Alonso,Urbano,...)
- **Bootstrap** (Guerrieri, Penedones, Vieira,...)

We have developed the following corner

Scattering amplitudes in the low-energy EFT of gravity; derivative expansion→ loops are higher orders Donoghue, PRL72,2996(1994); arXiv:1702.00319[hep-ph]

$$S_{\rm grav} = \int d^4x\sqrt{-g}\left\{\Lambda_{\rm cc} + \frac{2}{\kappa^2}R + c_1 R^2 + c_2 R_{\mu\nu}R^{\mu\nu} + c_3 R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \dots\right\} \quad \kappa^2 = 32\pi G$$

Each R, $R_{\mu\nu}$, $R_{\mu\nu\alpha\beta} \sim p^2$ (two derivatives)
$i\partial_\alpha \sim p_\mu$ and so on

Each graviton: $G^{\frac{1}{2}}$ $\quad g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu}$

Einstein theory (for pure gravity): $c_1 = c_2 = \dots = 0$

$\Lambda_{\rm cc}$ is neglected (cosmological constant $-8\pi G\Lambda_{\rm cc}$)

EFT for $E \lesssim \Lambda$, e.g. $\Lambda_U \sim G^{-1/2} = 10^{19}$ GeV

(unitarity cutoff– more below)

Similarities with Chiral Perturbation Theory (ChPT), pion physics (QCD)

Massless pions (Chiral limit $m_q = 0$)

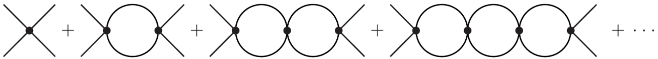
$i\partial_\mu \sim p_\mu$ Pure derivative expansion

EFT valid for $E < \Lambda$. Unitarity cutoff $\Lambda_U = 4\pi f_\pi \approx 1.2$ GeV
$\Lambda = M_\rho = 0.77$ GeV

σ of $f_0(500)$: Isoscalar Scalar $\pi\pi$ scattering is resonant

GKPY Equation García-Martín, Kaminski, Peláez, Ruiz de Elvira, PRL107,072001(2011)
$\sqrt{s_\sigma} = (457^{+14}_{-13} - i\,(297^{+11}_{-7}))$ MeV

One also generates the σ by unitarizing $I = J = 0$ $\pi\pi$ ChPT



General $\pi\pi$ interaction

NLO Unitarized ChPT, Albaladejo, JAO, PRD86,034003(2012)
$\sqrt{s_\sigma} = 458 \pm 14 - (261 \pm 17)\, i$ MeV

In an EFT the light degrees of freedom must be accounted for

$$\left|\frac{s_\sigma}{(4\pi f_\pi)^2}\right| = 0.22 \ll 1$$

This is an example of a Parametric enhancement JAO, Oset, PRD60,074023(1999)

LO 0^{++} $\pi\pi$ ChPT partial-wave amplitude (PWA), σ

$$T_{00}(s) = \frac{s - m_\pi^2/2}{f_\pi^2}$$

LO 1^{--} $\pi\pi$ ChPT PWA, $\rho(770)$

$$T_{11}(s) = \frac{s - 4m_\pi^2}{6 f_\pi^2}$$

Unitarized LO PWAs

Unitarity loop function

$$g(s;Q) = \frac{1}{16\pi^2} \log \frac{-s}{4Q^2}$$

Q is a momentum cutoff

$$T_{00}(s) = \left[\frac{f_\pi^2}{s - m_\pi^2} + g(s;Q)\right]^{-1} \quad T_{11}(s) = \left[\frac{6 f_\pi^2}{s - 4 m_\pi^2} + g(s;Q)\right]^{-1}$$

For the σ: $\underline{Q \approx 1 \text{ GeV}}$

For the ρ: $\underline{Q \approx 1 \text{ TeV}}$

Values taken from JAO,Oset,PRD60,074023(1999)

Note that Q only enters logarithmically

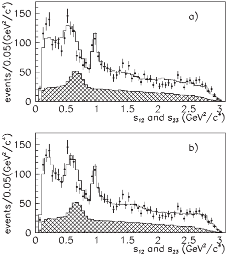


FIG. 2. s_{12} and s_{13} projections for data (error bars) and fast MC (solid line). The shaded area is the background distribution, (a) solution with the Fit 1, and (b) solution with Fit 2.

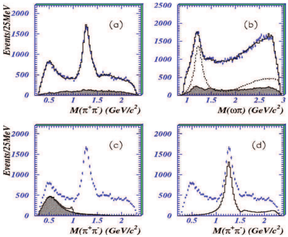


Fig. 2. Mass projections of data compared with the fit (histograms) using Eqs. (10)–(12) for the σ; the shaded region shows background estimated from sidebands. (a) and (b): $\pi\pi$ and $\omega\pi$ mass; the dashed curve in (b) shows the fitted $b_1(1235)$ signal (two charge combinations). (c) and (d): mass projections of 0^{++} and 2^{++} contributions to $\pi^+\pi^-$ from the fit; in (c), the shaded area shows the σ contribution alone, and the full histogram shows the coherent sum of σ and $f_0(980)$.

E791 PRD86,770(2001)

$D^+ \to \pi^-\pi^+\pi^+$

BES PLB598,149(2004)

$J/\Psi \to \omega\pi^+\pi^+$

The σ affects prominently low-energy scalar dynamics in QCD

$\star$ Vacuum, excitations of quark condensate: Scalar form factors of π
$\star$ π-nucleon σ term
$\star$ Large corrections to the current-algebraic prediction of 0^{++} $\pi\pi$ scattering lengths, and phase shifts in general
$\star$ Two-pion event distributions from heavy-meson decays
ETC Peláez, Phys.Rep.658,1(2016)

Is there a gravi-σ in the QG EFT that could affect so much (relatively) low-energy gravitational physics?

$I = J = 0$ $\pi\pi$ are attractive
$J = 0$ graviton-graviton interactions are also attractive

The Graviball

At LO Unitarized EFT-QG the pole position of the graviball and the σ are very similar relative to the cutoff of the EFTs

$$\left|\frac{s_P}{\Lambda_{\rm G}^2}\right| \approx \left|\frac{s_\sigma}{\Lambda_{\rm Hadron}^2}\right| = 0.22 \ll 1$$

PWAs do not converge to $\langle \mathrm{p}', \lambda_1' \lambda_1' | T | \mathrm{p}, \lambda_1 \lambda_2 \rangle$
Giddings, Porto, PRD81,025002(2010)

For Coulomb scattering Kang,Brown,PR128,2828(1962)

Lehmann ellipse

Effects dominated by some quantum numbers $\rightarrow$ selection of PWAs

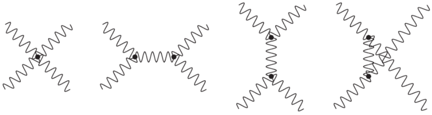
- Spectrum $[H, \mathrm{J}] = 0$
- Final-state interactions in multi-graviton production processes
- Interference with other forces, like Coulomb with strong interactions

§2 Graviton-graviton PWAs and IR divergences.

Tree-level amplitudes

$$|p_1,\lambda_1\rangle|p_2,\lambda_2\rangle \to |p_3,\lambda_3\rangle|p_4,\lambda_4\rangle$$

$$\lambda_i = \pm 2\,, \quad p = p_1 = -p_2\,, \quad p' = p_3 = -p_4$$



Born terms. Grisaru,van Nieuwenhuizen,Wu,PRD12,397(1975)

$$F_{22,22}(s,t,u) = F_{-2-2,-2-2}(s,t,u) = \frac{\kappa^2}{4}\frac{s^4}{stu}\,,$$

$$F_{-22,-22}(s,t,u) = F_{2-2,2-2}(s,t,u) = \frac{\kappa^2}{4}\frac{u^4}{stu}\,,$$

$$F_{2-2,-22}(s,t,u) = F_{2-2,2-2}(s,u,t) = F_{-22,2-2}(s,t,u) = \frac{\kappa^2}{4}\frac{t^4}{stu}$$

Related by parity and Bose-Einstein symmetry

Partial-wave amplitudes

$$\bar{T}^{(J)}_{\lambda'_1\lambda'_2,\lambda_1\lambda_2}(s) = {}_S\langle pJ,\lambda'_1\lambda'_2|T|pJ,\lambda_1\lambda_2\rangle_S$$
$$= \frac{1}{8\pi^2}\int_{-1}^{+1} d\cos\theta' d^J_{\lambda\lambda'}(\theta')\,{}_S\langle \mathrm{p}'_{xz},\lambda'_1\lambda'_2|T|p\mathrm{z},\lambda_1\lambda_2\rangle_S$$

S-matrix in partial waves:

$$\bar{S}^{(J)}(s) = 1 + i\frac{\pi 2^{|\lambda|/4}}{4}\bar{T}^{(J)}(s)$$
$$|\lambda| = |\lambda_1-\lambda_2| = |\lambda'_1-\lambda'_2| = 0\,,4$$

Symmetry property:

$$|pJM,\lambda_2\lambda_1\rangle_S = (-1)^J|pJM,\lambda_1\lambda_2\rangle_S$$

Selection rule:

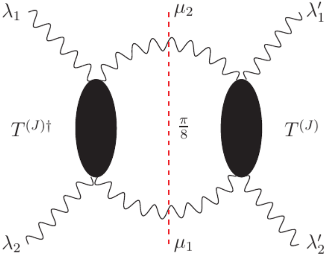
$$|pJm,\lambda_1\lambda_1\rangle_S = (-1)^J|pJM,\lambda_1\lambda_1\rangle_S \rightarrow \text{even } J$$

We only need to consider $T_{22,22}(s)$ and $T^{(J)}_{2-2,2-2}(s)$

Two-body unitarity

$$
{}_S\langle \mathrm{p}', \lambda_1'\lambda_2' | T | p\mathrm{z}, \lambda_1\lambda_2\rangle_S - {}_S\langle \mathrm{p}', \lambda_1'\lambda_2' | T^\dagger | p\mathrm{z}, \lambda_1\lambda_2\rangle_S
$$
$$
= \frac{i\theta(s)}{64\pi^2} \sum_{\mu_1,\mu_2} \int d\hat{\mathrm{q}}\, {}_S\langle \mathrm{p}', \lambda_1'\lambda_2' | T | \mathrm{q}, \mu_1\mu_2\rangle_S \, {}_S\langle \mathrm{q}, \mu_1\mu_2 | T^\dagger | p\mathrm{z}, \lambda_1\lambda_2\rangle_S
$$

Unitarity is most simply expressed in PWAs $\mathcal{O}(Gs)$



$$
\Im \bar{T}^{(J)}_{\lambda_1'\lambda_2',\lambda_1\lambda_2} = \frac{\pi}{8} \sum_{\mu_1,\mu_2} \bar{T}^{(J)}_{\lambda_1'\lambda_2',\mu_1\mu_2} \bar{T}^{(J)*}_{\mu_1\mu_2,\lambda_1\lambda_2}\, \theta(s) \ ,
$$

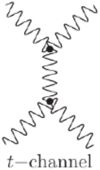
Infrared divergences

So far so good . . . but PWAs are IR divergent

$J = 0$ partial-wave projection of the Born term

$$F^{(0)}_{22,22}(s) = -\frac{\kappa^2 s^2}{16\pi^2}\int_{-1}^{+1}\frac{d\cos\theta}{t}$$
$$= -\frac{\kappa^2 s^2}{32\pi^2 \mathrm{p}^2}\left\{\log 2 - \lim_{\theta\to 0}\log(1-\cos\theta)\right\}$$
$$t = (\mathrm{p}-\mathrm{p}')^2 = -2\mathrm{p}^2(1-\cos\theta)$$

This is due to the exchange of a *virtual soft* graviton ($t \to 0$) in between two *external on-shell* graviton lines
Classification of Weinberg in PR140,B516(1965)



soft: $|t| << s$

Phase divergences

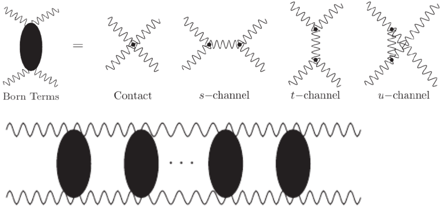
Dalitz,Proc.Roy.Soc.(London)206,509(1951)

Phase conjectured by Dalitz when studying the Born series up to 2nd order (e^2) for the scattering of a Dirac electron by a Yukawa/Coulomb potential

$$V(r) = \frac{e_1 e_2}{4\pi r} e^{-\mu r} \qquad \text{Phase} - \text{factor} = \exp\left\{\frac{i e_1 e_2}{2\pi \beta_{12}} \log \mu\right\}$$

Lorentz invariant relative velocity between particles a and b

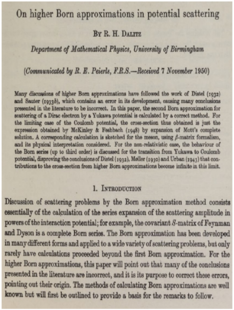
$$\beta_{ab} = \frac{\left[(p_a p_b)^2 - (m_a m_b)^2\right]^{1/2}}{p_a p_b}$$



Born Series

Extended up to 3rd order (e^3) for the non-relativistic case

Λ(1405), CDD poles, IR divergent phase,. . .



On higher Born approximations in potential scattering

By R. H. Dalitz

Department of Mathematical Physics, University of Birmingham

(*Communicated by R. E. Peierls, F.R.S.—Received* 7 *November* 1950)

Many discussions of higher Born approximations have followed the work of Distel (1932) and Sauter (1933*b*), which contains an error in its development, causing many conclusions presented in the literature to be incorrect. In this paper, the second Born approximation for scattering of a Dirac electron by a Yukawa potential is calculated by a correct method. For the limiting case of the Coulomb potential, the cross-section thus obtained is just the expression obtained by McKinley & Feshbach (1948) by expansion of Mott's complete solution. A corresponding calculation is sketched for the meson, using β-matrix formalism, and its physical interpretation considered. For the non-relativistic case, the behaviour of the Born series (up to third order) is discussed for the transition from Yukawa to Coulomb potential, disproving the conclusions of Distel (1932), Møller (1930) and Urban (1943) that contributions to the cross-section from higher Born approximations become infinite in this limit.

1. Introduction

Discussion of scattering problems by the Born approximation method consists essentially of the calculation of the series expansion of the scattering amplitude in powers of the interaction potential; for example, the covariant S-matrix of Feynman and Dyson is a complete Born series. The Born approximation has been developed in many different forms and applied to a wide variety of scattering problems, but only rarely have calculations proceeded beyond the first Born approximation. For the higher Born approximations, this paper will point out that many of the conclusions presented in the literature are incorrect, and it is its purpose to correct these errors, pointing out their origin. The methods of calculating Born approximations are well known but will first be outlined to provide a basis for the remarks to follow.

I spent a week discussing with Dalitz in a Frascati School about Castillejo-Dalitz-Dyson poles and my developments on the N/D method and CDD poles... he was surprised a young man knew at that time (1998) about the existence of the CDD poles, as Pennington told me. Dalitz offered me a postdoc to work with him in Oxford.

Phase divergences

This phase was demonstrated by Weinberg in *Infrared Photons and Gravitons*, PR140,B516(1965) as a corollary of his treatment of IR divergences in QED and gravity

"*the full effect of virtual infrared photons is to contribute to the S matrix for any process $\alpha \to \beta$ a factor*"

$$\frac{S_{\beta\alpha}}{S^0_{\beta\alpha}(\mathcal{L})} = \exp\left\{\frac{1}{2}\int_\mu^{\mathcal{L}} A(q)\right\}$$

The real part of $A(q)$ generates the IR divergence $(\mu/\mathcal{L})^{A/2}$.
This is cancelled by real soft-photon emission

Its imaginary part generates the Dalitz phase: "*so each different pair of particles in the initial or final state contributes to the S matrix a phase factor which for $\mu \ll \mathcal{L}$ may be written*"

$$\exp\left\{\frac{i}{4\pi}\frac{e_n e_m}{\beta_{nm}}\log\frac{\mu}{\mathcal{L}}\right\}$$

For gravity one also has the same rule

Every pair of particles in the initial or final state contributes with

$$\exp\left\{-i\frac{Gm_n m_m(1+\beta_{nm}^2)}{\beta_{nm}(1-\beta_{nm}^2)^{1/2}}\log\frac{\mu}{\mathcal{L}}\right\}$$

For graviton-graviton scattering (one pair in initial and final states)

$$S_c(s) = \lim_{m\to 0}\exp\left\{-i2\frac{Gm^2(1+\beta^2)}{\beta(1-\beta^2)^{1/2}}\log\frac{\mu}{\mathcal{L}}\right\}$$
$$= \exp\left\{-i2Gs\log\frac{\mu}{\mathcal{L}}\right\}$$

This is our first resummation taken from Weinberg, $\mathcal{L} \ll \sqrt{s}$

The phase does not depend on angle.
It is the same for all PWAs

Redefinition of the S matrix in PWAs

Comparing with Weinberg:

$$S_{\alpha\beta} \to \bar{S}^{(J)} \text{ and } S^0_{\alpha\beta} \to S^{(J)}$$

$$\bar{S}^{(J)} = S_c S^{(J)} = \exp\left\{-i2Gs\log\frac{\mu}{\mathcal{L}}\right\} S^{(J)}$$

Note: $S^{(J)}$ is unitary because only a phase factor has been introduced

$$S^{(J)}S^{(J)\dagger} = S^{(J)\dagger}S^{(J)} = I$$

$$\bar{S}^{(J)} = 1 + i\frac{\pi 2^{|\lambda|/4}}{4}\bar{T}^{(J)}$$

IR divergent

$$S^{(J)} = 1 + i\frac{\pi 2^{|\lambda|/4}}{4}T^{(J)}$$

IR finite

Redefinition of PW-projected Born terms

Up to $\mathcal{O}(G)$

$$S^{(J)} = S_c^{-1}\left(1 + i\frac{\pi 2^{|\lambda|/4}}{4}\bar{T}^{(J)}\right)$$

$$= 1 + i\frac{\pi 2^{|\lambda|/4}}{4}\underbrace{\left(\frac{8Gs}{\pi 2^{|\lambda|/4}}\log\frac{\mu}{\mathcal{L}} + F^{(J)}\right)}_{V^{(J)}} + \mathcal{O}(G^2)$$

Example: $J = 0$, $\lambda = 0$

$$F^{(0)}_{22,22} = -\frac{\kappa^2 s^2}{16\pi^2}\int_{-1}^{+1}\frac{d\cos\theta}{t-\mu^2} = \frac{\kappa^2 s}{8\pi^2}\log\left(1+\frac{4p^2}{\mu^2}\right)$$

$$\to \frac{8Gs}{\pi}\log\frac{2p}{\mu}$$

Thus

$$V^{(0)}_{22,22}(s) = \frac{8Gs}{\pi}\log\frac{2p}{\mathcal{L}}$$

Independence of J

$\lambda = 0$:

$$F^{(J)}_{22,22}(s) = -\frac{2Gs^2}{\pi}\int_{-1}^{+1} d\cos\theta \frac{P_J(\cos\theta)}{t-\mu^2}$$
$$= -\frac{2Gs^2}{\pi}\int_{-1}^{+1} d\cos\theta \frac{P_J(\cos\theta)-1}{t} + \frac{8Gs}{\pi}\log\frac{2p}{\mu}$$

$\lambda = 4$

$$F^{(J)}_{2-2,2-2}(s) = \frac{\kappa^2}{32\pi^2 s}\int_{-1}^{+1} d\cos\theta \frac{d^{(J)}_{44}(\theta)u^3}{t}$$
$$= \frac{G}{\pi s}\int_{-1}^{+1}\frac{d\cos\theta}{t}\left[d^{(J)}_{44}(\theta)u^3 + s^3\right] + \frac{4Gs}{\pi}\log\frac{2p}{\mu}$$

$$V^{(J)} = \frac{8Gs}{\pi 2^{|\lambda|/4}}\log\frac{\mu}{\mathcal{L}} + F^{(J)}$$

We know about $\mathcal{L}$

1. By dimensional analysis $\mathcal{L} \propto p$. It stems from

$$\int_{-1}^{+1} \frac{d\cos\theta}{-2\mathrm{p}^2(1-\cos\theta)-\mu^2}$$

2. $\mathcal{L} \ll p$

$$\mathcal{L} = \frac{\sqrt{s}}{a} = \frac{2p}{a}\ , \quad a \gg 1$$

$$V^{(0)}_{22,22} = \frac{8Gs}{\pi}\log a\ , \quad \log a = \mathcal{O}(1)$$

Ready to be unitarized

$\log a = \log a_{\rm phys} + \log a/a_{\rm phys}$ *The remnant dependence in the spectrum should decrease by including higher orders from the EFT input –exponentiation–*

§3 Unitarized PWAs

Unitarity of $S^{(J)}(s)$

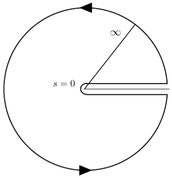
$$\Im \frac{1}{T^{(J)}} = -\frac{\pi 2^{|\lambda|/4}}{8}\theta(s)$$

IR-safe Born terms $V^{(J)}(s)$ do not have left-hand cut (LC)

Unitarized Born terms $\rightarrow T(s)$ only having right-hand cut (RC)

Our second resummation

JAO,Oller,PRD60,074023(1999); JAO,Meißner,PLB500,263(2001)



Contour of integration used for the dispersion of $T^{(J)}(s)^{-1}$

$$g(s) = c(s_0) - \frac{s - s_0}{8}\int_0^{\Lambda^2} \frac{ds'}{(s'-s)(s'-s)}$$

$$= c(s_0) + \frac{1}{8}\log\frac{-s}{s_0}$$

s_0: Subtraction point

$$T^{(J)}_{\lambda_1'\lambda_2',\lambda_1\lambda_2}(s) = \left[\frac{1}{R^{(J)}_{\lambda_1'\lambda_2',\lambda_1\lambda_2}(s)} + 2^{|\lambda|/4} g(s)\right]^{-1}$$

- $R^{(J)}(s)$ has no two-body RC, $\Im g(s) = -\frac{\pi}{8}\theta(s)$
- $R^{(J)}(s)$ is treated in perturbation theory
- $R^{(J)}(s)$ is obtained by matching with the EFT for graviton-graviton scattering

At LO

$$R^{(J)} = V^{(J)} + \mathcal{O}((Gs)^2)$$

$$T^{(J)}_{\lambda_1'\lambda_2',\lambda_1\lambda_2}(s) = \left[\frac{1}{V^{(J)}_{\lambda_1'\lambda_2',\lambda_1\lambda_2}(s)} + 2^{|\lambda|/4} g(s)\right]^{-1}$$

Naturalness to estimate $c(s_0)$

Compare $g(s)$ with $g_\Lambda(s)$ calculated with a cutoff

$$g_c(s) = -\frac{1}{8}\int_0^{\Lambda^2} \frac{ds'}{s'-s} = \frac{1}{8}\log\frac{-s}{\Lambda^2} + \mathcal{O}(s/\Lambda^2)$$

by taking $s_0 = \Lambda^2$ then

$$c(\Lambda^2) = 0$$

We will later isolate the dependence on Λ in s_P

Unitarity cutoff Λ_U

Expanding up to one loop $T^{(J)}(s)$

$$T^{(J)} = V^{(J)}\left(1 - \frac{V^{(J)}}{8}\log\frac{-s}{\Lambda^2}\right) + \mathcal{O}\left((Gs)^3\right)$$

If NLO/LO $\sim s/\Lambda_U^2$, from $J=0$

$$\Lambda_U^2 = \pi(G\log a)^{-1} \sim G^{-1} \ .$$

Perturbative-unitarity cutoff

Lee,Quigg,Thacker,PRD16,1519(1977)

$$T^{(0)}_{22,22} = -\frac{4i}{\pi}(\eta e^{2i\delta} - 1)\ , \quad \eta \in [0,1]$$

$$|\mathrm{Re}\ T^{(0)}_{22,22}| \leq \frac{4}{\pi},$$

For $J = 0$ this is violated for

$$s_{\rm pu} = \frac{1}{2}(G\log a)^{-1} \quad \& \ \Lambda^2 = \pi(G\log a)^{-1} = s_{\rm pu}2\pi$$

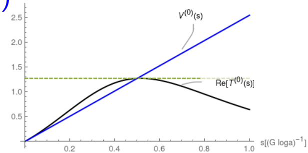
For ChPT

$$s_{\rm pu} = 8\pi f^2 \quad \& \ \Lambda_\chi^2 = (4\pi f)^2 = s_{\rm pu}2\pi$$

$$\sqrt{8\pi}f = 463 \text{ MeV} \approx M_\sigma$$

Comparing with unitarized $T^{(0)}(s)$

$$\Re T^{(0)}_{22,22}(s_{\rm pu}) \approx 0.95\,\Re V^{(0)}_{22,22}(s_{\rm pu})$$



Postdiction of the σ

Chiral limit

$$V_{\pi\pi}^{(0)} = \frac{s}{f_\pi^2}$$

$$T_{\pi\pi;II}^{(0)} = \left[\frac{f_\pi^2}{s} + \frac{1}{(4\pi)^2}\log\frac{-s}{\Lambda^2} - i\frac{1}{8\pi}\right]^{-1}$$

$$\Lambda = 4\pi f_\pi$$

$$x_\sigma = \frac{s_\sigma}{\Lambda^2}$$

Secular equation

$$\frac{1}{x_\sigma} + \log(-x_\sigma) - i2\pi = 0$$

$$x_\sigma \simeq -i\frac{2}{3\pi} = -i\,0.20$$

Numerically, $x_\sigma = 0.07 - i\,0.20$

Numerically, $x_\sigma = 0.07 - i\,0.20$

Physical π mass, GKPY: $x_\sigma = 0.09 - i\,0.20$

Not bad! Back envelope calculation 

§4 Prediction of the graviball

Blas, Martin Camalich, JAO, PLB827, 136991 (2022)

$$V^{(0)}_{22,22}(s) = \frac{8Gs}{\pi}\log a$$

$$T^{(0)}_{22,22;II}(s) = \left[\frac{\pi}{8Gs\log a} + \frac{1}{8}\log\frac{-s}{\Lambda^2} - i\frac{\pi}{4}\right]^{-1}$$

$$\omega = \frac{\Lambda^2}{\Lambda_U^2} = \Lambda^2\frac{G\log a}{\pi}$$

Secular equation

$$\frac{1}{\omega x} + \log(-x) - i2\pi = 0$$

$$x = \frac{s_P}{\Lambda^2}$$

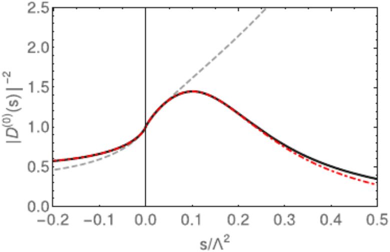
$$\omega = 1 \to x \simeq -i\frac{2}{3\pi} = -i0.20 \qquad x \sim \frac{1}{\omega}$$

Numerically, $x = 0.07 - i\,0.20$

$x = 0.07 - i\,0.20 \rightarrow$ Resonant shape peaks at surprisingly low values of s

Omnès function $\Omega^{(J)}(s) = 1/D^{(J)}(s)$

$$\Omega^{(J)}(s) = \left[1 + V^{(J)}(s)g(s)\right]^{-1} = T^{(J)}(s)/V^{(J)}(s)$$



At $s \simeq 0.2$ then $1 \sim V^{(0)}g$ (graviball)
Dashed line: Perturbative $|1 - V^{(0)}(s)g(s)|^2$
Solid line: $|D^{(0)}(s)|^{-2}$

Analogous to $|\Omega^{(0)}_{\pi\pi}(s)|^2$ driving e.g. final-state interactions in $D^+ \rightarrow \pi^+\pi^+\pi^-$ [E791]

Include light degrees of freedom in EFT

We have a parametric enhancement in the EFT both for the σ and graviball

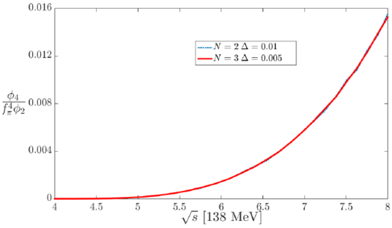
One has to account for such light resonances $|s/\Lambda^2| \approx 0.2$

Unitarized EFT is a way to accomplish this by resumming $(s/\Lambda^2)^n$ to account for two-body unitarity along the RC: Unitarity and analyticity

Suppression of phase space for massless multi-particle states Salas-Bernández, Llanes-Estrada, JAO, Escudero-Pedrosa,SciPost Phys.11,020(2021)

Phase space of n massless particles pions

$$\phi_n = \frac{s^{n-2}}{2(4\pi)^{2n-3}(n-1)!(n-2)!}$$



$\sqrt{s} \in [0.55, 1.1]$ GeV , 4π & 2π

Width of the graviball

Interesting application of $D^{(0)}(s)$

The pole $s_P \simeq -i\frac{2}{3\pi}\Lambda^2$ is almost purely imaginary

It is very far from a narrow-resonance case $s_R \simeq M_R^2 - iM_R\Gamma_R$

Goldberger,Watson,PR136,B1472(1964) calculate Γ_P in terms of $D^{(0)}(s)$

Probability of decay of the graviball into 2 gravitons after a time t

$$A(t) = \int_0^\infty dE \frac{B(E)}{|D^{(0)}(E)|^2} e^{-iEt}$$
$$A(0) = 1$$

$B(E)$ has no RC, it is smoother and we factor it out

$$A(t) = \frac{1}{\int_0^\infty \frac{dE}{|D^{(0)}(E)|^2}} \int_0^\infty \frac{dE}{|D^{(0)}(E)|^2} e^{-iEt}$$

Width: $A(1/\Gamma) = \exp(-1)$

$$\Gamma_P = 0.23\,\Lambda$$

For the σ, $\Gamma_\sigma = 267$ MeV instead of $-2\Im\sqrt{s_\sigma} = 500 - 600$ MeV

Making lighter the graviball

By increasing the number N of fields with $m^2 \ll G^{-2}$
The number of channels in the intermediate increases as $\sim N$
The unitarity loop function $g(s) \to Ng(s)$

$$T^{(0)}_{22,22}(s) \approx \left[\frac{\pi}{8Gs\log a} + \frac{N}{8}\log\frac{-s}{\Lambda^2}\right]^{-1}$$

Secular equation

$$\frac{1}{\omega N x_N} + \log(-x_N) - i2\pi = 0$$

Solution

$$x_N \approx -i\frac{2}{3\pi N} + i\frac{\log N}{10\pi N}$$

Gravity interactions between fields are attractive

Scenarios with a large number N of light fields:

Dvali,Fortschr.Phys.58,528(2010); Dvali,Redi,PRD77,045027(2008);

Arkani-Hamed,Cohen,D'Agnolo,Hook,Kim,Pinner,PRL117,251801(2016);

Extra Dimensions Arkani-Hamed,Dimopoulos,Dvali,PLB429,263(1998);

Antoniadis,Arkani-Hamed,Dimopoulos,Dvali,PLB436,257(1998);

Csaki,arXiv:0806.3801[hep-th]

Monomials involving three or four Riemannian tensors

$\{R^3\}$: Six derivatives; $\{R^4\}$: Eight derivatives

$\{R^3\}$ gives vanishing contributions to $F_{22,22}$

van Nieuwenhuizen,Wu, J.Math.Phys.18,182(1977)

$\{R^4\}$ contributions to $F_{22,22}$ evaluated in Huber, Brandhuber, De Angelis, Travaglini, PRD102,046014(2020) with spinor formalism

$$F_{R^4;22,22}(s,t,u) = \frac{\widetilde{\beta}\kappa^2}{\pi} s^4$$

$$V^{(0)}_{R^4;22,22}(s) = \frac{8\widetilde{\beta}}{\pi^2} s^4 \ , \quad \widetilde{\beta} \sim \Lambda^{-6}$$

$$V^{(0)}_{R^4;22,22}(s) = \frac{32 s^4}{\pi \Lambda^8 \log a}$$

Interaction kernel in the unitarization formula

$$R^{(0)}_{22,22}(s) = \frac{8s}{\Lambda^2} + \frac{32s^4}{\pi\Lambda^8} \ , \quad \log a = 1$$

Secular equation $s = s_P/\Lambda^2$

$$(x + 4x^4/\pi)^{-1} + \log(-x) - i\,2\pi = 0$$
$$x = 0.07 - i\,0.21 \ , \quad 3\% \text{ of deviation}$$

In agreement with the estimate $|x|^3 \sim 1\%$ for a N3LO correction (s^4&s)

Expected leading corrections are NLO loop ones $|x| \sim 0.2$ (there are no counterterms at NLO in pure gravity)

§5 Solvable toy model: AC scattering

I invented this toy model and called it AC from A: Adler, C: Coulomb

$$V(r) = \frac{\alpha}{r}\frac{E^2}{M^2} \ , \quad E = \frac{p^2}{2m}$$

It can be solved exactly from Coulomb scattering

$$\alpha \to \alpha E^2/M^2$$

E only enters parametrically in the Schrödinger equation

AC-model asymptotic wave function

$$u_\ell^{AC} \sim A \sin\left(pr - \ell\frac{\pi}{2} + \sigma_\ell^{AC}(p) + \frac{p^4}{(2mM)^2}\gamma \log 2pr\right)$$

$$\sigma_\ell^{AC}(p) = \arg\Gamma\left(1 + \ell - i\frac{p^4}{(2mM)^2}\gamma\right) \ , \quad \gamma = \frac{\alpha m}{p}$$

AC-model S matrix, $S_\ell^{AC}(p)$

$$S_\ell^{AC}(p) = e^{2i\sigma_\ell^{AC}} = \frac{\Gamma\left(1 + \ell - i\gamma p^4/(2mM)^2\right)}{\Gamma\left(1 + \ell + i\gamma p^4/(2mM)^2\right)}$$

Numerical solution of the AC scattering model

Screened potential

$$V_R(r) = \frac{\alpha}{r}\frac{E^2}{M^2}\theta(R-r)$$

Solve this finite-range potential and match with the asymptotic $u_\ell^{AC}(r)$ at $r = R$ R.Landau,QM Vol.2,Wiley-VCH,1995

From the calculated phase shifts $\delta_\ell(p)$ one can get $\sigma_\ell^{AC}(p)$

$$\delta_\ell(p) = \sigma_\ell^{AC} + \frac{p^4}{(2mM)^2}\gamma \log 2pR$$

The diverging Coulomb-like phase $\frac{p^4}{(2mM)^2}\gamma \log 2pR$ appears

It cancels with that from $\delta_\ell(p)$ to produce the finite $\sigma_\ell^{AC}(p)$

This method can be applied to solve numerically for the strong phase shifts in the presence of a Coulomb potential

One can also proceed similarly in momentum space with a Lippmann-Schwinger equation

Fourier transform

$$F^{AC}(p) = \frac{p^4}{(2mM)^2}\frac{4\pi\alpha}{q^2}(1-\cos qR)\ , \quad q^2 = 2p^2(1-\cos\theta)$$

S matrix

$$\bar{S}_\ell^{AC}(p) = 1 + i\frac{p}{m\pi}\bar{T}_\ell$$

From $\delta_\ell(p) = \sigma_\ell^{AC}(p) + \frac{p^4}{(2mM)^2}\gamma \log 2pR$

$$S_c^{AC} = e^{2i\frac{p^4}{(2mM)^2}\gamma \log 2pR}$$

At $\mathcal{O}(\alpha)$ implies this contribution to the Born-term

$$\delta F_\ell^{AC}(p^2) = \frac{p^2\pi\alpha}{2(mM)^2}\log 2pR$$

Independent of PWA (ℓ)

By redefining the S matrix

$$S_\ell^{AC}(p) = S_c^{AC}(p)^{-1}\bar{S}_\ell^{AC}(p)$$

IR finite projected Born terms $V_\ell^{AC}(p^2)$

$$F_0^{AC}(p) = \frac{1}{2}\int_{-1}^{+1} d\cos\theta \frac{p^4\pi\alpha}{(mM)^2 q^2}(1-\cos qR)$$
$$= \frac{p^2\pi\alpha}{2(mM)^2}\left(\gamma_E + \log 2pR\right) + \mathcal{O}(R^{-2})$$

$$V_0^{AC}(p^2) = F_0^{AC}(p) - \delta F_0^{AC}(p) = \frac{p^2\pi\alpha}{(mM)^2}\frac{\gamma_E}{2}$$

Unitarizing

$V_0^{AC}(p^2)$ has no LC.

The *exact* $S_\ell^{AC}(p)$ has only RC

$$S_\ell^{AC}(p) = e^{2i\sigma_\ell^{AC}} = \frac{\Gamma\left(1+\ell - i\gamma p^4/(2mM)^2\right)}{\Gamma\left(1+\ell + i\gamma p^4/(2mM)^2\right)}$$
$$S_\ell^{AC}(p) = 1 + i\frac{mp}{\pi}V_\ell^{AC}(p^2) + \mathcal{O}(\alpha^2)$$

With the IR regulator $\mu \to 0^+$

Invoking Weinberg's resummation for IR soft virtual photons

$$S_c(p) = \exp\left\{ i2\alpha \frac{p^4}{(2mM)^2} \gamma \log \frac{\mathcal{L}}{\mu} \right\}$$

$$F_0^{AC}(p) = \frac{p^2 \pi \alpha}{(mM)^2} \log \frac{2p}{\mu}$$

$$V_0^{AC}(p^2) = \frac{p^2 \pi \alpha}{(mM)^2} \log a$$

$F_0^{AC}(p)$ and $V_0^{AC}(p^2)$ are proportional to those in gravity

$\log a \to \gamma_E/2 \approx 0.3$ by comparing with the exact IR-finite S-wave projected Born term V_0^{AC}

We have larger $\log a$ when considering the spectrum (non-perturbative physics).

Unitarization of Coulomb-AC

The Sugawara-Kanazawa theorem (SK) PR123,1895(1961) can be applied to Coulomb scattering
$\alpha \to \alpha(E/M)^2$ to get AC scattering

$$V_0^C(p^2) = \frac{4\pi\alpha}{p^2}\log a$$

$$S_\ell^C(p) = \frac{\Gamma(1+\ell-i\gamma)}{\Gamma(1+\ell+i\gamma)} \underbrace{\longrightarrow}_{p\to\infty} 1 + i\frac{mp}{\pi}V_\ell^C + \mathcal{O}(p^{-3})$$

SK applied to with only RC

$$f(p^2) = \bar{f}(\infty) + \frac{1}{\pi}\int_0^\infty dk^2 \frac{\Delta f(k^2)}{k^2-p^2}$$

$$\frac{1}{p^2 T_0^C(p^2)} = \frac{1}{4\pi\alpha\log a} - \frac{m}{2\pi^2}\int_0^\infty dk^2 \frac{1}{k(k^2-p^2)}$$

$$T_0^C(p^2) = \left[\frac{p^2}{4\pi\alpha \log a} - i\frac{m\sqrt{p^2}}{2\pi} \right]^{-1}$$

$$T_0^{AC}(p^2) = \left[\frac{(mM)^2}{p^2\pi\alpha \log a} - i\frac{mp}{2\pi} \right]^{-1}$$

Note: Coulomb is a high-energy perturbative EFT, AC is a low-energy perturbative EFT

AC Spectrum

$$S_\ell^{AC}(p) = e^{2i\sigma_\ell^{AC}} = \frac{\Gamma\left(1+\ell - i\gamma p^4/(2mM)^2\right)}{\Gamma\left(1+\ell + i\gamma p^4/(2mM)^2\right)}$$

Exact pole positions

$$p(\nu) = (-i)^{1/3}\lambda(\nu)\ , \quad \nu = \ell + n$$

$$\lambda(\nu) = \left\{ \frac{4mM^2}{\alpha}(1+\ell+n) \right\}^{1/3}$$

$$p_1(\nu) = i\lambda(\nu)$$

$$p_2(\nu) = e^{-i\pi/6}\lambda(\nu)$$

$$p_3(\nu) = -e^{i\pi/6}\lambda(\nu)$$

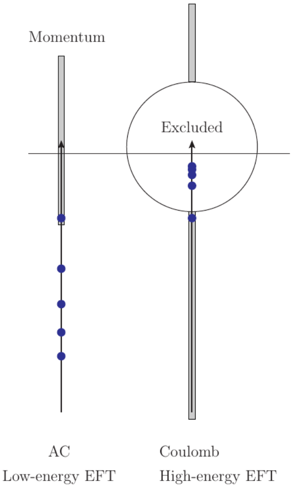
A bound state and a resonance

AC scattering

$$p_1(\nu) = i\left\{\frac{4mM^2}{\alpha}(1+\nu)\right\}^{1/3}$$

Coulomb scattering

$$p(\nu) = \frac{im\alpha}{1+\nu}\ , \quad \nu = \ell + n$$



AC scattering

$$p_1(\nu) = i\left\{\frac{4mM^2}{\alpha}(1+\nu)\right\}^{1/3}$$

Coulomb scattering

$$p(\nu) = \frac{im\alpha}{1+\nu}\ , \quad \nu = \ell + n$$

From the unitarized expression

$$T_0^{AC}(p^2) = \left[\frac{(mM)^2}{p^2\pi\alpha\log a} - i\frac{m\sqrt{p^2}}{2\pi}\right]^{-1}$$

$$p(\nu) = (-i)^{\frac{1}{3}}\left[\frac{2mM^2}{\alpha\log a}\right]^{\frac{1}{3}}$$

The exact result for AC (and Coulomb) is reproduced with $\log a = 1/2 = \mathcal{O}(1)$

Inclusion of higher orders

$$S_\ell^{AC}(p) = e^{2i\sigma_\ell^{AC}} = \frac{\Gamma\left(1+\ell - i\gamma p^4/(2mM)^2\right)}{\Gamma\left(1+\ell + i\gamma p^4/(2mM)^2\right)}$$

$$S_\ell^{AC}(p) = 1 + i\frac{mp}{\pi}\underbrace{\left[V_\ell^{AC}(p^2)^{-1} - i\frac{mp}{2\pi}\right]^{-1}}_{T_\ell^{AC}(p^2)}$$

$$V_\ell^{AC}(p^2) = \frac{2i\pi}{mp}\frac{\Gamma(1+\ell+i\frac{p^3\alpha}{4mM^2}) + \Gamma(1+\ell-i\frac{p^3\alpha}{4mM^2})}{\Gamma(1+\ell+i\frac{p^3\alpha}{4mM^2}) - \Gamma(1+\ell-i\frac{p^3\alpha}{4mM^2})}$$

$$\delta = \frac{p^3\alpha}{4mM^2(1+\nu)}\,, \quad V_\ell^{AC} = -\frac{2\pi}{mp}v(\delta)$$

$$v(\delta) = (1+\nu)\psi_0(1+\ell)\delta + \frac{(1+\nu)^3}{6}[2\psi_0(1+\ell)^3 - \psi_2(1+\ell)]\delta^3$$
$$+ \frac{(1+\nu)^5}{120}[16\psi_0(1+\ell)^5 - 20\psi_0(1+\ell)^2\psi_2(1+\ell) + \psi_4(1+\ell)]\delta^5$$
$$+ \mathcal{O}(\delta^7)$$

$$T_{0;N}^{AC} = \left[\frac{1}{V_{0;N}^{AC}} - i\frac{mp}{2\pi}\right]^{-1}$$
$$\frac{1}{v(\delta)} + i = 0$$

	$N=1$	$N=3$	$N=5$	Exact
p_1 [Λ]	$i\,0.860$	$i\,0.729$	$i\,0.717$	$i\,0.716$
p_2 [Λ]	$0.745 - i\,0.430$	$0.631 - i\,0.364$	$0.621 - i\,0.358$	$0.620 - i\,0.358$

Lightest pole positions in units of $\Lambda = (2\pi m M^2/\alpha\gamma_E)^{1/3}$ of the S-wave PWA $T_{0;N}^{AC}(p^2)$ for the AC model

§6 Estimate for log a, $d > 4$ scattering

$d \geq 5$ there are no IR divergences

Extrapolate from $d \simeq 5$ to $d = 4$ as smooth as possible. Optimized perturbation theory.

$$V^{(0),d}_{22,22}(s) = \frac{\Gamma(d/2-2)}{\Gamma(d-3)} \frac{4Gs}{\pi} \left\{ \frac{s}{4\pi\mu_f^2} \right\}^{\frac{d-4}{2}}$$

$$T^{(0),d}_{22,22}(s) = \left[V^{(0),d}_{22,22}(s)^{-1} + \frac{1}{8} \log \frac{-s}{\Lambda^2} \right]^{-1}$$

$$G_d = G\mu_f^{4-d}$$

Unitarity cutoff

$$\Lambda_d^2 = \left((4\pi\mu_f^2)^{\frac{d-4}{2}} 2\pi G^{-1} \frac{\Gamma(d-3)}{\Gamma(d/2-2)} \right)^{\frac{2}{d-2}}$$

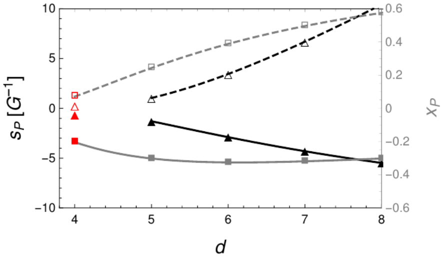
$\Lambda_d^2 \to 0$ for $d \to 4$ because IR divergences at $d = 4$ (Weinberg's resummation becomes increasingly important)

The graviball remains for $d > 4$

$s_P = x\Lambda_d^2$

Secular equation

$$(\omega x)^{1-d/2} + \log(-x) - i2\pi = 0$$



For definiteness
$\omega = 1,\ 2\mu_f^2 G = 1$

Real part: Dashed lines
Imaginary part: Solid lines
$\log a = 1$ for $d = 4$

$\Re s_P > -\Im s_P$ already for $d \gtrsim 5$
$|x| < 1$ but $|s_P| > G^{-1}$ (trans-planckian)

The σ remains for $d > 4$

No IR $\rightarrow s_\sigma$ can be smoothly extrapolated to $d = 4$

$$V_0^{\pi\pi,d}(s) = \frac{\Gamma(d/2-1)}{\Gamma(d-2)} \frac{s}{f_\pi^2} \left(\frac{s}{4\pi\mu_f^2}\right)^{\frac{d-4}{2}}$$

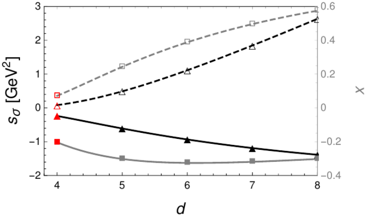
$$f_\pi^d = f_\pi \mu_f^{\frac{d-4}{2}}$$

Unitarity cutoff

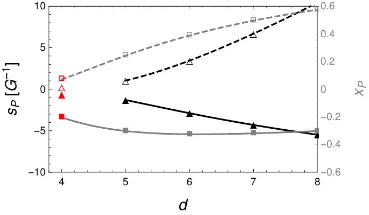
$$\Lambda_d^2 = \left((\pi\mu_f^2)^{\frac{d}{2}-2}(4\pi f_\pi)^2 \frac{\Gamma(d-2)}{\Gamma(\frac{d}{2}-1)}\right)^{\frac{2}{d-2}}$$

Secular equation

$$x^{1-\frac{d}{2}} + \log(-x) - i2\pi = 0 \ .$$



$$\mu_f^2 = 2\pi^{\frac{1}{2}} f_\pi$$



$$\omega = 1, \ 2\mu_f^2 G = 1$$

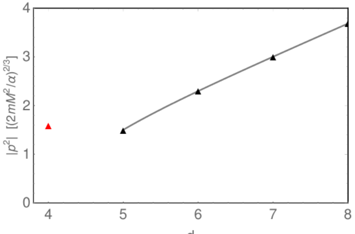
$J = 0$ AC scattering for $d > 4$

$$F_0^{AC,d}(p^2) = \frac{\alpha p^{d-6}}{\mu_f^{d-4}\pi^{d/2-3}} \frac{\Gamma(d/2-2)}{\Gamma(d-3)}$$

$$\Lambda'_d = (\pi\mu_f^2)^{\frac{1}{2}} \left(\frac{8\pi m M^2/\alpha}{(\pi\mu_f^2)^{\frac{3}{2}}} \frac{\Gamma(d-3)}{\Gamma(d/2-2)} \right)^{\frac{1}{d-1}}$$

$$x_d^{d-1} + \frac{i}{\pi} = 0$$

$$p^2(d) = \pi\mu_f^2 \left(-i \frac{8 m M^2/\alpha}{(\pi\mu_f^2)^{\frac{3}{2}}} \frac{\Gamma(d-3)}{\Gamma(d/2-2)} \right)^{\frac{2}{d-1}}$$



$$\mu_f = \frac{1}{\sqrt{\pi}} \left(\frac{2mM^2}{\alpha} \right)^{\frac{1}{2}}$$

Coulomb scattering with $d \geq 4$

For Coulomb scattering one has the transition for high- to low-energy EFT for increasing d

$$F_0^{C,d}(p^2) = \frac{\alpha\gamma^{4-d}}{\pi^{d/2-3}p^2}\frac{\Gamma(d/2-2)}{\Gamma(d-3)}$$

$$p^{d-6}$$

Secular equation

$$x^{d-5} = -\frac{i}{\pi}$$

No solution for $d = 5$

$d = 4$ bound state, $d = 6$ virtual state

Maximal-stability estimate of $\log a$

We evaluated the Weinberg's exponent in Dimensional Regularization

$$\int \frac{d^d q}{(2\pi)^d} B(q) = -4\pi G s^2 i \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2+i\varepsilon)(p_1\cdot q - i\varepsilon)(p_2\cdot q + i\varepsilon)}$$
$$\to -\frac{Gs}{\pi}\left(-\frac{s}{4\pi}\right)^{-\epsilon}\frac{\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}\Gamma(-\epsilon)^2$$

$$S_c^{\mathrm{dr}}(s) = \exp\left[iGs\left(-\frac{1}{\epsilon} + \log\frac{s}{\mu_h^2} + \gamma_E - \log(4\pi)\right)\right]$$

$$\log a = \frac{\mu_h}{\mu_f}$$
$$y = \mu_f^2 G$$
$$s \to s_P$$

Exponent of $S_c^{\mathrm{dr}}(s)$

$$iGs\left(\frac{2}{d-4}+\gamma_E-2\log a+\frac{2}{d-2}\log\frac{\Gamma(d-3)}{2y\Gamma(d/2-2)}+\log x_d\right)$$

Critical density $d_c < 5$: The pole term is as important as the rest of terms

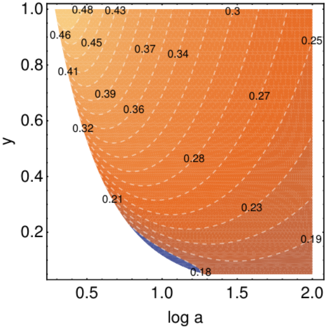
$$\frac{2}{d_c-4}=\left|2\log a-\gamma_E-\frac{2}{d_c-2}\log\frac{\Gamma(d_c-3)}{2y\Gamma(d_c/2-2)}-\log x_d\right|$$

$d_c(\log a, y) < 5$, and define the "distance"

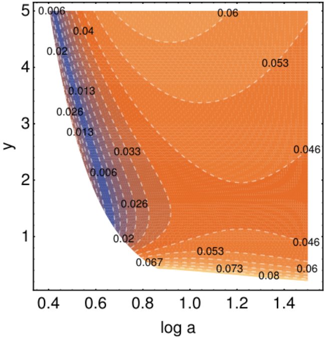
$$r(\log a, y)=\frac{|\Lambda_{d_c}^2-\Lambda^2|}{\Lambda^2}$$

which should be minimized to enhance smoothness in the transition from $d=5$ to $d=4$

Graviball: $\log a \approx 1$



Ac model: $\log a \approx 1/2\,.$



In agreement with the exact result

This procedure of maximal smoothness is reminiscent of

i) Dimensional continuation in statistical mechanics, where typically only a few terms in the density expansion are available and from which the optimal solution is sought.

Zinn-Justin, Scholarpedia,5,8346(2010); Nishida, Son, PRL97,050403(2006)

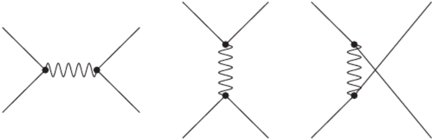
By minimizing $r(\log a, y)$ one is reducing the needed number of terms in an expansion of $V^{(0;d)}_{22,22}(s)$ in powers of $d-5$ up to $d=4$.
Higher-order terms are increasingly more sensitive to the singularity at $d=4$ of $\Gamma(d/2-2)$.

ii) Optimized perturbation theory: The principle of minimal sensitivity to fix scale ambiguities in perturbation theory to improve its convergence properties.

Stevenson, PRD23,2916(1981); Brodsky, Lepage, Mackenzie, PRD28,228(1983); Su, Commun.Theor.Phys.57,409(2012)

§7 Spinless scalar-scalar scattering

Kabar,Ortiz,NPB388,570(1992) studied scalar-scalar scattering for $s \gg G^{-1}$ in the Eikonal approximation≡'t Hooft, Veneziano, Verlinde *et al* approaches



$s-$channel $t-$channel $u-$channel

Born terms

Eikonal approximation $s,\ -u \gg -t$ Forward scattering

$$F_\chi(s,t,u) = -\frac{16\pi G\gamma(s)}{t}$$

Full Born terms

$$F(s,t,u) = \frac{4\pi G}{stu}\left[s^4 + t^4 + u^4 - 4m^2(2(s^3+t^3+u^2) - stu)\right.$$
$$\left. -36m^4(st+su+tu) + 256m^8\right]$$

Partial-wave projection and IR divergences

$$F(s,t,u)=-16\pi G\gamma(s)\left(\frac{1}{t}+\frac{1}{u}\right)+8\pi G\frac{tu-2m^4}{s}-16\pi G(s-3m^2)$$

$$\gamma(s)=\frac{s^2}{2}+m^4-2m^2s$$

$$F^{(\ell)}(s)=\frac{1}{4}\int_{-1}^{+1}d\cos\theta\, F(s,t',u')P_\ell(\cos\theta)$$

$$=\frac{16\pi G(1+(-1)^\ell)\gamma(s)}{s-4m^2}\log\frac{2p}{\mu}-4\pi G(1+(-1)^\ell)\gamma(s)\int_{-1}^{+1}d\cos\theta\frac{P_\ell-1}{t'}$$

$$+\frac{2\pi G}{s}\int_{-1}^{+1}d\cos\theta P_\ell(\cos\theta)t'u'-8\pi G\left(s-3m^2+\frac{m^4}{s}\right)\delta_{\ell 0}$$

Weinberg's phase factor (the same for all ℓ)

$$S_c(s)=\exp\left\{-i2\frac{Gm^2(1+\beta^2)}{\beta(1-\beta^2)^{1/2}}\log\frac{\mu}{\mathcal{L}}\right\}\ ,\ \beta=\frac{\sqrt{1-4m^2/s}}{1-2m^2/s}$$

$$S_c(s) = 1 + i\frac{p}{4\pi\sqrt{s}}\delta F(s) + \mathcal{O}(G^2)$$

$$\delta F(s) = -\frac{16\pi G(1+(-1)^\ell)\gamma(s)}{s-4m^2}\log\frac{\mu}{\mathcal{L}}$$

$$V^{(\ell)}(s) = F^{(\ell)}(s) - \delta F(s) \to \log\frac{2p}{\mathcal{L}} \equiv \log a$$

$$V^{(\ell)}(s) = \frac{16\pi G(1+(-1)^\ell)\gamma(s)}{s-4m^2}\log a - 4\pi G(1+(-1)^\ell)\gamma(s)\int_{-1}^{+1} d\cos\theta \frac{P_\ell - 1}{t'} + \frac{2\pi G}{s}\int_{-1}^{+1} d\cos\theta P_\ell(\cos\theta)t'u' - 8\pi G\left(s - 3m^2 + \frac{m^4}{s}\right)\delta_{\ell 0}$$

Novelties for $m \neq 0$ compared to hadron-hadron or graviton-graviton scattering:

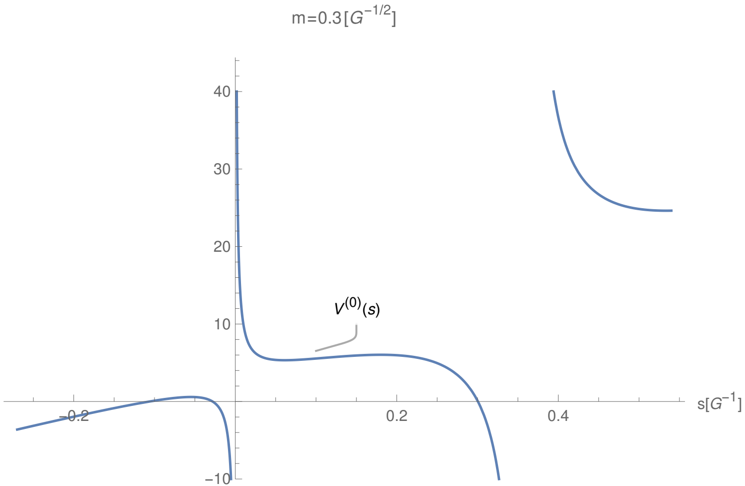
$$\underbrace{V^{(\ell)}(s = 4m^2) = \infty}_{\text{Associated to IR divergent structure}}$$

In addition, for $m > 0$

$$V^{(2)}(s=0) = \underbrace{V^{(0)}(s=0)}_{\text{Weird}} = \infty$$

Datta,Gabrielli,Mele *Violation of angular momentum selection rules in quantum gravity* PLB579,189(2004)

No $s = 0$ pole remains after unitarizing $J = 0$



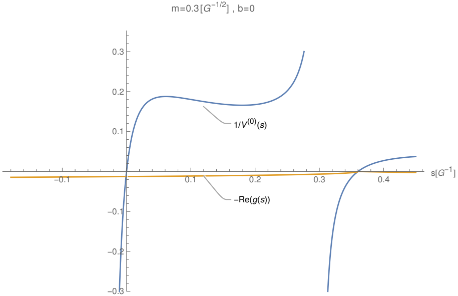
Potential $V^{(0)}(s)$. An involved structure results even at LO

Unitarization

$$T^{(0)}(s) = \left[\frac{1}{V^{(0)}(s)} + g(s)\right]^{-1}$$

$$\frac{3}{2\pi} V^{(0)}(s) = 4m^4 \left(\frac{1}{s} + \frac{12 \log a}{s - 4m^2}\right) + s(-11 + 24 \log a) + 28m^2$$

$$g(s) = \frac{1}{(4\pi)^2} \left(b + \sigma(s) \log \frac{\sigma(s)+1}{\sigma(s)-1}\right)$$

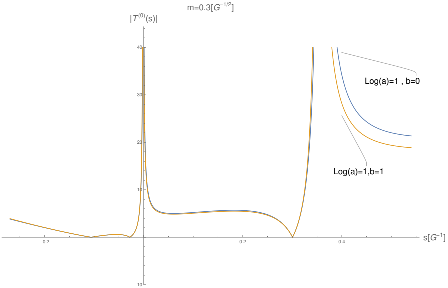


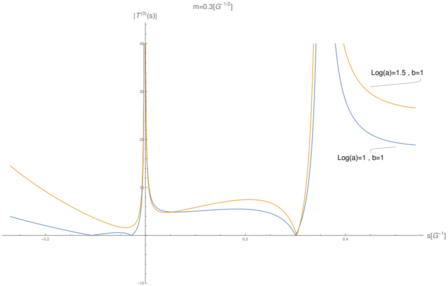
$b = 0 \to g(4m^2) = 0$: a bound state at $s = 4m^2$ and $s \lesssim 0$

$b > 0$ and $\mathcal{O}(1)$: The same pattern but with deeper bound states

$b < 0$ and $\mathcal{O}(1)$: Resonance $s \gtrsim 4m^2$ in the 1st RS #. Excluded

$|T^{(0)}(s)|$ varying b and $\log a$ →Smooth dependence





$$b = 0, \log a = 1.0$$
$$s_B = 4m^2 = 0.360$$

$$b = 1, \log a = 1.0$$
$$s_B = 0.354$$

$$b = 1, \log a = 1.5$$
$$s_B = 0.351$$

Eikonal amplitude

Kabat,Ortiz,NPB388,570(1992)

$$f(k,t) = \frac{\alpha}{2p}\left(\frac{4p^2}{-t}\right)^{1-i\eta}\frac{\Gamma(1-i\eta)}{\Gamma(1+i\eta)}$$

$$\eta(s) = 2G\frac{\gamma(s)}{s\sigma(s)}$$

Poles ($|s| \gg G^{-1}$)

$$1 - i\eta(s) = 0,\ -1,\ -2\ \ldots$$

First Riemann sheet poles: Bound states

$x = n^2/8m^4$, $n \in \mathbb{N}$

$$s = 2m^2\left\{1 \pm \sqrt{\frac{1}{2} - x + x\sqrt{1+\frac{1}{x}}}\right\}$$

$$\in [0,(2-\sqrt{2})]m^2 \cup [2+\sqrt{2},4]m^2$$

It resembles the pattern obtained with the unitarization of the Born terms in the unitary (low-energy) EFT approach

Second Riemann sheet poles: $x = n^2/8m^4$, $\quad n \in \mathbb{N}$

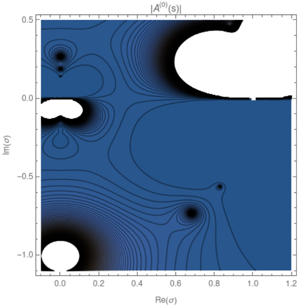
$$s_P = 2m^2 \left\{ 1 \pm \sqrt{\frac{1}{2} - x - x\sqrt{1 + \frac{1}{x}}} \right\}$$

- $1 \leq n < Gm^2$ Virtual-state poles
  $s \in [2 - \sqrt{2}, 2 + \sqrt{2}]m^2$
- $n > Gm^2 \gg 1$ Resonance poles
- The transition happens for $m^2 = G^{-1}$ and $n = 1$

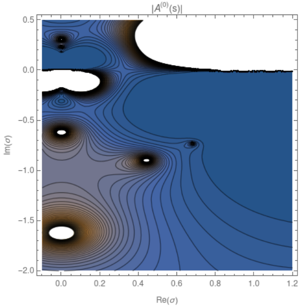
For $m^2 < G^{-1}$ there are only resonance poles

But, unless $n \gg 1$, this is out of the eikonal approach which requires $|s_P| \sim m^2 \gg G^{-1}$

Poles in the $\sigma \equiv \sqrt{1 - \frac{4m^2}{s}}$ plane, $m^2 \geq G^{-1} \to$ Resonance and virtual-state poles



$m^2 = 1.0\ G^{-1}$



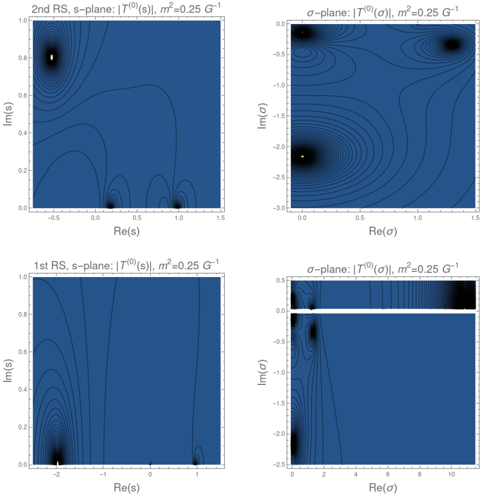
$m^2 = 1.5\ G^{-1}$

$1 \leq n \leq [m^2]$ Virtual-state poles
$n > [m^2]$ Resonance poles

One can see in all cases a clear influence of the poles on the physical axis $\sigma \in [0, 1]$

Unitarized $J = 0$ PWA: $m^2 = 0.25\, G^{-1}$

Poles in the s- and σ-planes



σ-plane: 1st and 2nd RS's

Clear impact of the poles on the physical values $\sigma \in [0, 1]$

Mistaken arguments to neglect poles in the 2nd Riemann sheet in Kabat,Ortiz,NPB388,570(1992)

1.- "These second-sheet poles do not correspond to physical states. They correspond to non-normalizeable (exponentially growing at infinity) solutions to the Klein-Gordon equation (4.3)."

This happens for any virtual or resonance states, thus all resonances are fallacious

Resonances are normalizable Gamow states: Hernández,Mondragón,PRC29,722(1984) Proper analytical extrapolation from p ($\Im p > 0$) to $p_R(\Im p_R < 0)$

$$\left[\int_0^\infty dk\, u(p;k)^2\right]_{p_R} = \left[\int_0^\infty dr\, u(p;r)^2\right]_{p_R} = 1$$

By inserting particle number operators in QFT JAO,ANP396,429(2018)

`https://www.youtube.com/watch?v=Qu_Y1qOX8g0&t=1339s`

2.-"THE REASON IS THAT CONTINUING TO THE SECOND SHEET IN THE AMPLITUDE (4.5) CHANGES $\eta(s) \to -\eta(s)$, WHICH IS EQUIVALENT STAYING ON THE FIRST SHEET BUT REPLACING $G \to -G$. THIS CHANGES THE SIGN OF THE POTENTIAL (4.8) AND LEADS TO THE POLES (4.7)"

It refers to $\frac{\Gamma(1-i\eta)}{\Gamma(1+i\eta)}$ and $\eta(s) = 2G\frac{\gamma(s)}{s\sigma(s)}$ with $\sigma(s) \to -\sigma(s)$ in the 2nd Riemann sheet (RS).

But we go for $s > 4m^2$ from $s + i\epsilon$ (1st RS) to $s - i\epsilon$ (2n RS) in a continuous way, no flip of sign at at all!

3.- "Similar second-sheet poles arise in non-relativistic scattering from a repulsive Coulomb potential [26]."

Detail, $\alpha \to -\alpha$ implies a Repulsive Potential in the 1st or Physical RS

The bound states turn into virtual state poles

$$f_{N.R.}(p,\theta) = -\frac{\gamma}{2p}\left(\sin\frac{\theta}{2}\right)^{-2-2i\gamma} \frac{\Gamma(1-i\alpha m/p)}{\Gamma(1+i\alpha m/p)} ,$$
$$p_P = i\frac{\alpha m}{n} , \quad n \in \mathbb{N}$$

§8 Summary and outlook

1. Formalism for unitarizing forces of infinite range
2. Graviton-graviton scattering
3. Coulomb scattering and AC scattering
4. Scalar-scalar scattering
5. Removing of a global phase factor S_c
6. IR safe partial-wave projected Born terms or LO EFT amplitudes
7. Standard unitarization techniques from hadron physics satisfying two-body unitarity and analyticity (RC)
8. Suppression of multi-graviton intermediate states for $s < G^{-1}$

9. Graviball at $s_P = (\varkappa - i\,\frac{2}{3\pi})\Lambda^2$, $\Lambda^2 \simeq \pi G^{-1}$ and $\varkappa \ll 1$
10. Its resonance effects peak at $s \ll \Lambda^2$
11. Large corrections to S-wave graviton-graviton scattering calculated in perturbation theory from EFT
12. Graviball becomes lighter and narrower when more light fields are included, $s_P \sim 1/N$. It is robust under these effects
13. Graviball persits for $d > 4$
14. Determination of $\log a$ by requiring "maximum smoothness" in a dimensional expansion from $d = 5$ to $d = 4$
15. Our method reproduces the first poles in AC scattering and the deepest bound state for Coulomb scattering

16. Study of Scalar-scalar scattering within the same framework
17. 1st RS. Bound states
18. Report on the Eikonal approach results. Similarities.
19. Virtual and Resonance poles
20. Criticism to Kabat,Ortiz NPB388,570(1992) mistaken reasons to reject 2nd RS poles

21. Unitarized NLO EFT amplitudes, or from other theories like string theory; coupled channels by including extra matter fields, etc
22. Apply the method to hadron physics, $\pi^+\pi^-$, $\pi^\pm p$ or $K^\pm p$
23. Primordial GWs, reduction of tensor perturbations and associated B-modes from inflationary models
24. Modification of gravitational physics at low scales in scenarios with $N \gg 1$ (μm with a UV cutoff around TeV scale).

8. *Guerrieri, Penedones, Vieira, PRL127,081601(2021):* S-matrix **Bootstrap**. $d = 10$ maximal supergravity. Unitarity bound demanded→ derive lower bound for leading Wilson coefficient of UV completeness (R^4). Connect with the $J^{PC} = 0^{++}$ graviball, which dominates a spectral function

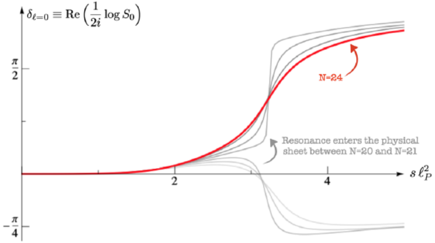


Figure: $s_P \simeq 3.2 + 0.3\,i$ GeV^{-1} $(d = 10)$ "It is therefore tempting to identify the graviball as the first excited string state."