On the possibility of democratic redistribution in a two class society*

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Abstract

We investigate the possibility of democratic redistribution of resources in a society with some rich and some poor citizens called to vote over redistribution under majoritarian rule. To do so, we adopt a standard pivotal-voter model with costly voting, under the assumption that the poor citizens outnumber the rich. A redistribution trade-off emerges; despite the poor citizens outnumbering the rich, and thus "being stronger" in the democratic election, each single rich citizen has more at stake in the election than a single poor citizen, and thus her willingness to vote is greater. Multiplicity of equilibria, as standard in pivotal-voter models, arises. However, we impose an intuitive refinement, namely that voting probabilities are continuous in the cost of voting, and find that such refinement pins down a unique equilibrium, capable of characterizing the redistribution trade-off. The unique continuous equilibrium a key inequality threshold; if the number of poor citizens is lower than the square of the number of the rich citizens, then the poor citizens may vote in equilibrium and redistribution has a chance of winning, otherwise the poor citizens abstain with certainty and thus enter in a "poverty trap"—namely, redistribution is doomed to lose.

1 Introduction

Consider a stylized society with two "poor" citizens with 1 unit of resource each, and one "rich" citizen with 2 units of resources. Each citizen may cast a costly vote in a majoritarian election over whether to implement full redistribution of resources within the society. Full redistribution, resulting in 4/3 of resources for every citizen, would harm the rich citizen (losing 2/3) more than it would benefit a single poor citizen (gaining 1/3). Hence, the rich citizen has more at stake in the election than each of the two poor citizens. However, as a countervailing effect, if all citizens were to vote with the same exogenous probability, then full redistribution would be the most likely outcome of the election as there are more poor citizens than rich. We characterize this trade-off, arising in societies where the poor citizens outnumber the rich citizens—namely, the poor citizens are a more numerous electoral body than the rich, but each rich citizen has more at stake in the election.

We adopt a complete-information pivotal-voter model with costly voting.[1] This class of models is far from novel. The seminal contribution dates back to Palfrey and Rosenthal (1983),[2] where two

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[1]Our focus is different compared to other models in which there is voting over redistribution without cost of voting, as in for example Meltzer and Richard (1981). In their model, under voting, the level of redistribution is endogenously determined by the median voter and all citizens vote. We are more interested in what happens when the cost of voting is an issue, and how its magnitude affects the result of the redistribution election, in a model where voters vote only based on the possibility of being pivotal; for empirical evidence on how voters' voting decision depends on the probability of affecting the election outcome see Lyytikäinen and Tukiainen (2019).

[2]After the complete-information contribution by Palfrey and Rosenthal (1983), the same authors analyzed in 1985 a version of the model under private information on the cost of voting; see Palfrey and Rosenthal (1985). From then onwards, much of the literature has developed under private information, as the model is often more tractable and allows more elegant and neat results (e.g., Borgers, 2004; Taylor and Yildirim, 2010). Nevertheless, the general idea that voters compare the benefits of voting with its costs is older and has been long the interest of economists, dating back at least to Downs' (1957) seminal work. We consider only benefits that accrue from changing the policy to the voters' preferred outcome, despite

groups of citizens each preferring one of two alternatives simultaneously decide between abstaining or voting for their preferred alternative. The winner is decided by majority rule. Technical difficulties and multiplicity issues allowed Palfrey and Rosenthal to analyze only special cases.[3] The analysis of this model has been pushed forward by two other works. First, Nöldeke and Peña (2016) focus on the two groups having *symmetric* number of members and *symmetric* benefit from the favored alternative winning the election. Second, Mavridis and Serena (2018) focus on the two groups having *asymmetric* number of members and *symmetric* benefit from the favored alternative winning the election. In order to capture the motivating trade-off of poor-rich societies, we need a model with both *asymmetric* number of members and *asymmetric* benefits; the poor citizens outnumber the rich, but each rich citizen has more at stake in the election.[4]

After describing the model in Section 2, we fully characterize the simple three-poor-two-rich model in Section 3 where there are multiple equilibria.[5] We characterize analytically the equilibria where members of at least one group play a pure strategy in participation (Section 4.1, Section 4.2), and we characterize in part analytically and in part numerically the equilibria where members of both groups play a mixed strategy (Section 4.3). Our numerical analysis, which complements our analytical results, characterizes the redistribution trade-off. Despite the multiplicity of equilibria, only one equilibrium survives a novel and intuitive continuity refinement; that is, the equilibrium probability of voting is continuous in the cost of voting.[6] The continuity refinement turns out to single out a unique equilibrium in the general model with an arbitrary number of poor and rich citizens (see Section 5). We study the properties of the unique continuous equilibrium; if the society is sufficiently equal (namely, the number of poor citizens is *lower* than the square of the number of the rich citizens), then the poor citizens may vote and redistribution has a chance of winning, while in a sufficiently unequal society (namely, the number of poor citizens is *greater* than the square of the number of the rich citizens), the unique continuous equilibrium dictates that poor citizens abstain with certainty, and thus poor citizens are doomed to lose the election and redistribution is not implemented.

The above inequality threshold provides a clear-cut answer to the redistribution trade-off. Consider an increase in the number of poor citizens. On the one hand, it makes the individual resource under redistribution closer to the resources of a poor and further apart from those of a rich. This makes the stakes of the rich in the election increase, and that of the poor decrease.[7] On the other hand, an increase in the number of poor citizens makes the poor group bigger and thus stronger in the election; that is, if all citizens where to vote with the same exogenous probability full redistribution would be the most likely outcome of the election as there are more poor citizens than rich. All in all, the former effect of an increase in the number of poor citizens (making rich citizens vote "more") is *stronger* than the latter effect (making rich citizens relatively less numerous) the greater is the number of poor citizens; in fact, when the number of poor citizens is greater than the square of the number of the rich citizens, the former dominates the latter and in the unique continuous equilibrium poor citizens have no chance of winning the election. The opposite happens when the number of poor citizens is low, and hence poor citizens have a chance of democratically redistributing resources.

2 Model

There are two types $i \in \{m, n\}$ of citizens; $m > 1$ poor citizens and $n > 1$ rich citizens. Assume poor citizens are more numerous; $m > n$.[8] Citizens are simultaneously called to cast a vote between two alternatives, M and N. When we specialize the model to redistribution of resources, M is full redistribution and N is no redistribution of resources. Poor citizens prefer alternative M, in that if M wins (rather than loses) the individual payoff of a poor citizen increases by $\Delta\pi_m > 0$. Symmetrically, rich

a number of other benefits playing an important role in real-life; for instance, Wiese and Jong-A-Pin (2017) empirically examine the benefits arising from "expressive" motives of voting.

[3]To see these special cases, we refer to footnote 1 in Mavridis and Serena (2018).

[4]Palfrey and Rosenthal (1983) consider only the case of symmetric benefits.

[5]Palfrey and Rosenthal (1983) already noticed that the asymmetric-symmetric model suffers from a multiplicity issue.

[6]In fact, it would be hard to claim that negligible changes in the cost of voting could bring about drastic changes in the probabilities of voting. For example, if the voting center station moves slightly away from the home of a citizen, her probability of voting also changes negligibly.

[7]Acemoglu and Robinson (2000, 2012) examine a similar idea, in which if there are many poor citizens in a society the threat of revolution decreases since the spoils would be divided over a larger mass of citizens.

[8]By assuming $m > n > 1$ we avoid having to deal with trivial case distictions of $m = n$ or $n = 1$ throughout the paper.

citizens prefer alternative N, in that if N wins (rather than loses) the individual payoff of a rich citizen increases by $\Delta\pi_m > 0$. Citizens choose whether to vote for their preferred alternative or to abstain, since voting for the non-preferred alternative is strictly dominated. If a citizen casts a vote, she faces a cost of voting, $c > 0$. Thus, the increase in payoff –net of cost of voting– for a citizen i when her preferred alternative wins is $\Delta\pi_i - c$ if she voted, and $\Delta\pi_i$ if she did not vote. Citizens vote simultaneously and the winning alternative is decided by majority rule. Ties are broken by a fair coin toss.

(Running Redistribution Example) Throughout the paper, our running example is that of resource redistribution, where each poor citizen has an initial amount of resources equal to 1 and each rich citizen has an initial amount of resources equal to 2.[9] Under full redistribution each citizen ends up with the average resources in the society, which is:

$$\frac{2n+m}{n+m}$$

and under no redistribution everyone keeps her original amount of resources. Thus,

$$\begin{aligned} \Delta\pi_m &= \frac{2n+m}{n+m} - 1 = \frac{n}{n+m} && (1) \\ \Delta\pi_n &= 2 - \frac{2n+m}{n+m} = \frac{m}{n+m} && (2) \end{aligned}$$

Since the resources of the poor is less than the average resources, a poor m citizen would always prefer full redistribution (alternative M) and a rich n citizen would always prefer no redistribution (alternative N). Furthermore, notice from (1) and (2) that $\Delta\pi_n > \Delta\pi_m$; that is, a single rich citizen has more at stake in the election than a single poor citizen.[10]

We denote by p_i the probability of voting of citizen i. This probability maximizes her individual expected payoff, taking as given the choices of the other citizens. As in Palfrey and Rosenthal (1983), we consider Quasi-Symmetric Nash Equilibria (QSNE), that is, citizens of group i follow the same equilibrium strategy p_i^*. Hence, a QSNE is a pair (p_i^*, p_j^*) such that a citizen of group $i \in \{m, n\}$ would not want to deviate from p_i^* if she expects every other citizen of group i to also play p_i^* and all citizens of group j with $j \neq i$ to play p_j^*. A QSNE can be of one of the following three types:

1. "Pure": $(p_m^*, p_n^*) \in \{0,1\}^2$
2. "Partially Mixed": $p_m^* \in \{0,1\}, p_n^* \in (0,1)$ or $p_m^* \in (0,1), p_n^* \in \{0,1\}$
3. "Totally Mixed": $(p_m^*, p_n^*) \in (0,1)^2$.

Our asymmetric-asymmetric setting (see Introduction) will allow us to fully characterize the first two types of QSNE, and for sufficiently low (m,n) also the third. However, for arbitrary (m,n) we will tackle the characterization of the "Totally Mixed" equilibria partly analytically and partly numerically.

Define A_i to be the probability that the vote of a citizen of group i is pivotal. A citizen of group i may cast a vote if:

$$A_i \frac{\Delta\pi_i}{2} \geq c. \tag{3}$$

Conditional on being pivotal, the extra utility of creating a tie (from 0 to $\frac{\Delta\pi_i}{2}$) or breaking a tie (from $\frac{\Delta\pi_i}{2}$ to $\Delta\pi_i$) is identical since the tie breaking rule is fair. This property explains the division by 2 in (3) and holds in general throughout the paper. The above inequality is identical to

$$A_i \geq \frac{2c}{\Delta\pi_i} \equiv B_i \tag{4}$$

[9] It will be clear that these resource level assumptions are qualitatively without loss of generality.

[10] Considering the extrema of no- and full-redistribution is, to some extent, without loss of generality. Since the resources of the poor are less than the average resources, the poor would like as much redistribution as possible and, at the same time, the rich would like as little redistribution as possible. Therefore, given a choice between any two proportional tax rates schemes, the poor would want to vote for the higher one and the rich for the lower one. In the end, there is no loss of generality to assume that the two tax rates that are competing are 0 (no redistribution) and 1 (full redistribution).

for $i \in \{m, n\}$. Note that allowing for asymmetric costs of voting is a straightforward generalization of our setting.

(Running Redistribution Example) Given the specifications (1) and (2), we can write

$$B_m = \frac{2c(n+m)}{n}, \quad (5)$$
$$B_n = \frac{2c(n+m)}{m}, \quad (6)$$

so that a poor citizen may cast a vote if $A_m \geq \frac{2c(n+m)}{n}$ and a rich citizen if $A_n \geq \frac{2c(n+m)}{m}$. Notice that $B_m > B_n$; as discussed in the Introduction, an individual rich has more at stake in the election than an individual rich.

In what follows we start the analysis focusing first on an easy non trivial case: $(m, n) = (3, 2)$. Then, we characterize the general (m, n) case.

3 Three-poor-two-rich society: $(m, n) = (3, 2)$

We start with a simple example where we assume that $m = 3$ and $n = 2$. A single citizen will analyze all the possible scenarios of what the other four citizens will do.

Focus on a poor citizen first. A single poor citizen knows that there are nine possible cases depending on whether the other four citizens (two rich and the other two poor citizens) vote or abstain. We consider the nine cases in the following table where $(\bar{m}, \bar{n})$ represent the number of citizens from each group that turn out to vote; that is, if $(\bar{m}, \bar{n}) = (0, 1)$, the other poor citizens do not vote, while one rich does and the other rich does not.

Poor citizen: pivotal if voting?	$(\bar{m}, \bar{n})$	Probability	$\frac{\Delta\pi_i}{2}$
Yes	(2,2)	$p_m^2 p_n^2$	$\frac{1}{5}$
Yes	(1,1)	$4p_m(1-p_m)p_n(1-p_n)$	$\frac{1}{5}$
Yes	(0,0)	$(1-p_m)^2(1-p_n)^2$	$\frac{1}{5}$
Yes	(0,1)	$2(1-p_m)^2 p_n(1-p_n)$	$\frac{1}{5}$
Yes	(1,2)	$2p_m(1-p_m)p_n^2$	$\frac{1}{5}$
No	(2,0)	$p_m^2(1-p_n)^2$	0
No	(2,1)	$2p_m^2 p_n(1-p_n)$	0
No	(1,0)	$2p_m(1-p_m)(1-p_n)^2$	0
No	(0,2)	$(1-p_m)^2 p_n^2$	0

In this table, every row corresponds to each possible case: the first three rows correspond to the cases where the vote of the poor citizen would break a tie. The fourth and fifth rows correspond to the case where her vote would create a tie. The last four rows correspond to the case where her vote would make no difference to the outcome of the election. The column Probability gives the probability that each case realizes. For example the $(2, 1)$ case realizes if the other two poor vote, one rich votes and the other rich does not. The single poor citizen examining this table knows that she is pivotal in fives cases out of nine. If she is not pivotal she would rather not vote, so as to save on the cost c. If she is pivotal she may want to vote, provided that the net utility from voting is positive.

Following the table above and the voting condition (3), a poor citizen would want to vote if:

$$\left[p_m^2 p_n^2 + 4p_m(1-p_m)p_n(1-p_n) + (1-p_m)^2(1-p_n)^2 + 2(1-p_m)^2 p_n(1-p_n) + 2p_m(1-p_m)p_n^2\right] \frac{1}{5} \geq c, \quad (7)$$

where the $\frac{\Delta\pi_i}{2} = \frac{1}{5}$ because;

1. if the poor votes and breaks a tie, she obtains the full redistribution outcome $7/5$ while not voting would give her the tie outcome $(1/2)(7/5) + (1/2)\,1$,

2. if the poor votes and creates a tie, she obtains the tie outcome $(1/2)\,(7/5)+(1/2)\,1$ while not voting would give her the no redistribution outcome 1,

and in both cases the difference in payoff is $1/5$.

A rich citizen solves a similar problem, considering the six corresponding cases. The resulting condition for the rich to want to vote is:

$$\left[3p_m^2(1-p_m)p_n + 3p_m(1-p_m)^2 p_n + 3p_m(1-p_m)^2(1-p_n) + (1-p_m)^3(1-p_n)\right]\frac{3}{10} \geq c. \qquad (8)$$

Note that for $c = 0$ both Inequalities (7) and (8) trivially hold, regardless of the probabilities, therefore all poor citizens vote and, as they outnumber rich citizens, full redistribution wins. On the other hand, for very high c neither inequality holds and no-one votes.

We now describe how a QSNE (p_m^*, p_n^*) is defined by Inequalities 7 and 8. If a pair (p_m^*, p_n^*) leads to Inequality 7 (8) to hold strictly then it must be that $p_m^* = 1$ ($p_n^* = 1$). If a pair (p_m^*, p_n^*) leads to Inequality 7 (8) to not hold then it must be that $p_m^* = 0$ ($p_n^* = 0$). These two cases form the four possible "Pure" QSNE; $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$. If a pair (p_m^*, p_n^*) leads to the two inequalities to hold with equality, then the pair (p_m^*, p_n^*) forms a "Totally Mixed" QSNE. If a pair (p_m^*, p_n^*) leads Inequality 7 to hold with equality, and Inequality 8 to hold with $<$ ($>$) then the pair is an equilibrium if $p_n^* = 0$ ($p_n^* = 1$). If a pair (p_m^*, p_n^*) leads Inequality 8 to hold with equality, and Inequality 7 to hold with $<$ ($>$) then the pair is an equilibrium if $p_m^* = 0$ ($p_m^* = 1$). These last two cases form "Partially Mixed" QSNE.

Such reasoning leads to the general solution depicted in Figure 1, where we adopted the notation consistent with our *(Running Redistribution Example)*; namely, $B_n = B$ and $B_m = \frac{m}{n}B$, so as to have only one parameter B simplifying the graphical exposition (see (1) and (2)).[11]

For any B we examine all the types of equilibria: "Pure", "Partially Mixed" and "Totally Mixed". We find that for low B there are three different types of equilibria, which are depicted in the first row of Figure 1, while for larger B we only have the equilibrium depicted in the second row. The unique equilibrium for B sufficiently large corresponds to the characterization in Propositions 1 to 5 (Subsections 4.1 and 4.2).

The maximum values of B for existence of each equilibrium in the first row are the following: $B_1 = 1/2$, $B_2 = 30/49$, $B_3 = 30/49$. The minimum value of B for existence of the equilibrium in the second row is $B_4 = 1/2$, with $B_1 = B_4 < B_2 = B_3$. Now, take for example $B = 1/3$. There are three equilibria: respectively, one in which both types of citizens are playing a mixed strategy , with the rich voting with a much higher probability than the poor; one that both are playing a mixed strategy but now the poor are voting with higher probability than the rich; and one that the rich vote for sure and the poor play a mixed strategy. On the other hand for larger B (but not too large), say $B = 2/3$, there is only one type of equilibrium, the one where the rich are playing a mixed strategy, and the poor abstain for sure. And, naturally, if B is very high (i.e., greater than 1), the cost of voting is much greater than the benefit, thus no-one votes.[12]

Continuity. Beginning from a high B we will have to be on the "Pure" equilibrium in the bottom panel of the Figure. As B decreases and gets lower than 1, a unique equilibrium exists which is the "Partially Mixed" equilibrium which is drawn in the same panel. As we keep decreasing B, we move along the two curves of the "Partially Mixed" equilibrium until we reach $B_2 = B_3 = 30/49$, where the "Totally Mixed 2" equilibrium (second panel) and the Partially Mixed equilibrium in which the rich are voting with certainty (third panel) start being defined; however moving to any of these equilibria would involve an upwards jump in both probabilities of voting. However, exactly where the "Partially Mixed" equilibrium of the fourth panel stops being defined, is where the "Totally Mixed 1" equilibrium starts being defined (since $B_1 = B_4$), and on top of that the probabilities change continuously from one equilibrium to the other. As intuition suggests, we would expect a small decrease in costs or benefits to incur a reasonably small effect on the probabilities of voting. For this reason, we consider the smooth transition of the unique continuous equilibrium more plausible than the discontinuous behavior of the

[11] In this $(m,n) = (3,2)$ case, this notation implies $B = \frac{10}{3}c$.

[12] The probability of being pivotal is at most 1, and thus the voting condition (4) is never satisfied.

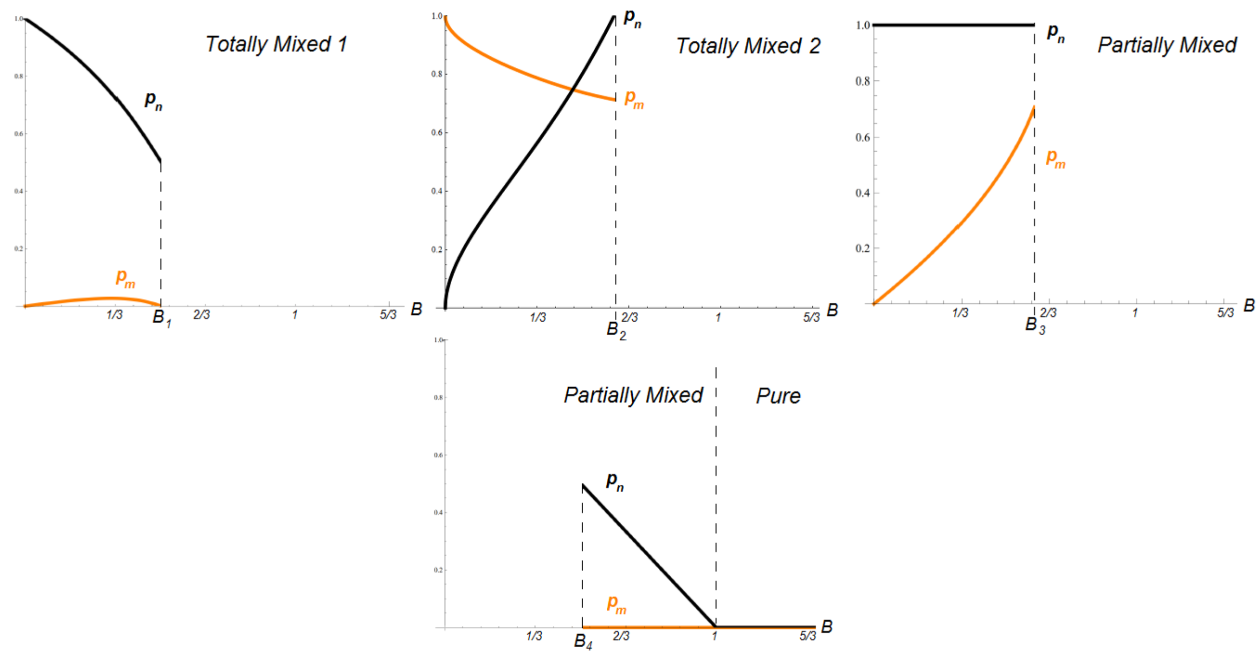

Figure 1: First row, first panel: "Totally Mixed 1". First row, second panel: "Totally Mixed 2". First row, third panel: "Partially Mixed". Bottom panel: "Partially Mixed" continuously connected to the "Pure".

other two equilibria, and thus our continuity refinement selects the "Totally Mixed 1" together with the equilibria in the fourth panel. [13]

Numerical simulations in Section 4.3 will show that the uniqueness of a continuous equilibrium is a property holding, not only for $(m, n) = (3, 2)$, but for general (m, n). We will focus on the unique continuous equilibrium and run comparative statics to shed light on the redistribution trade-off spelled out in the Introduction and on how it depends on the number of poor and rich citizens in the society. We will also provide the intuition for our results, and derive an inequality cut-off describing when full redistribution has or does not have a chance of winning the election.

4 General (m, n)

The probability of being pivotal is crucial for the voting/abstention choice in (4). While we discussed it in the previous section for the special case of $(m, n) = (3, 2)$, it is useful to provide the expression for a general pair (m, n). The following table shows the calculations that a single poor citizen must make when she knows that she faces $m - 1$ poor citizens and n rich with $(\bar{m}, \bar{n})$ signifying how many of the rest actually vote in each instance.

For economy of space we have omitted the cases that the poor citizen is not pivotal. The resulting expression for the pivotal probability is as follows:

$$A_i = \sum_{s=0}^{n} \binom{i-1}{s} \binom{j}{s} p_i^s (1-p_i)^{i-s-1} p_j^s (1-p_j)^{j-s} + \sum_{s=0}^{n-1} \binom{i-1}{s} \binom{j}{s+1} p_i^s (1-p_i)^{i-s-1} p_j^{s+1} (1-p_j)^{j-s-1} \tag{9}$$

for $i, j \in \{m, n\}$, $i \neq j$.

[13] Formally, we use a standard definition of continuity (see for example page 943 of Mas-Colell et al. 1995). This definition means that, for all $i, j \in \{m, n\}$, $i \neq j$, there is a single continuous selection of the equilibrium correspondences $p_i^*(B_i, B_j)$ mapping (B_i, B_j) to equilibrium probabilities of voting.

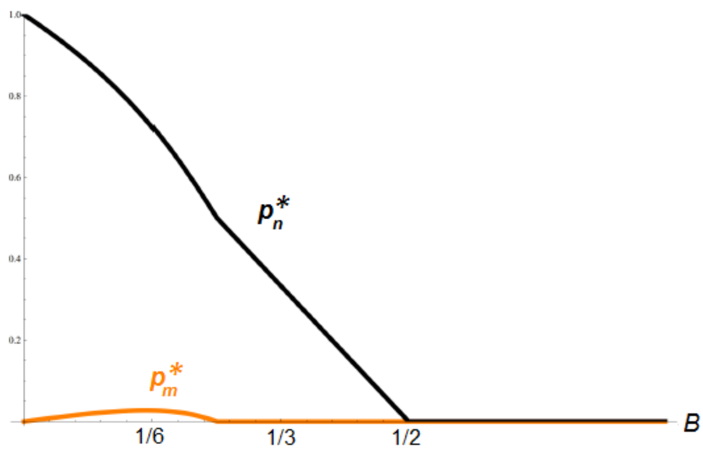

Figure 2: Unique continuous equilibrium for $m = 3$ and $n = 2$.

Poor citizen: pivotal if voting?	$(\bar{m}, \bar{n})$	Probability	$\Delta\pi_i$
Yes	(n,n)	$\frac{(m-1)!}{n!(m-1-n)!} p_p^n (1-p_p)^{m-n-1} p_r^n$	$\frac{n}{n+m}$
Yes	(n-1,n)	$\frac{(m-1)!}{(n-1)!(m-n)!} p_p^{n-1} (1-p_p)^{m-n} p_r^n$	$\frac{n}{n+m}$
Yes	(n-1,n-1)	$\frac{(m-1)!}{(n-1)!(m-n)!} n p_p^{n-1} (1-p_p)^{m-n} p_r^{n-1} (1-p_r)$	$\frac{n}{n+m}$
Yes	(n-2,n-1)	$\frac{(m-1)!}{(n-2)!(m-n+1)!} n p_p^{n-2} (1-p_p)^{m-n+1} p_r^{n-1} (1-p_r)$	$\frac{n}{n+m}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
Yes	(0, 1)	$n(1-p_p)^{m-1} p_r (1-p_r)^{n-1}$	$\frac{n}{n+m}$
Yes	(0, 0)	$(1-p_p)^{m-1}(1-p_r)^n$	$\frac{n}{n+m}$
No	All other cases		0

With the general expression for A_i in our hands, we can analyse the QSNE of the general voting game, given the relative costs of voting B_i and B_j, and classify them according to their type: "Pure", "Partially Mixed" or "Totally Mixed". As we will see, the first two types (i.e., equilibria in which some citizens play pure strategies) can be fully characterized, as we do in Subsections 4.1 and 4.2, while the last type of equilibria will be tackled in Subsection 4.3.

4.1 "Pure" equilibria and "Partially Mixed" equilibria with abstention

If $A_i < B_i$, citizen i's dominant strategy is to abstain. On the other hand if $A_i > B_i$ citizen i's dominant strategy is to vote. When $A_i = B_i$ citizen i is indifferent between voting and abstaining; therefore this is a necessary condition for citizen i to employ a mixed strategy. From this condition we see that B_i can be interpreted as the minimum A_i for citizen i to be willing to vote. Clearly if $B_i > 1$, citizen i abstains.

Proposition 1. *If $B_i \geq 1$ then $p_i^* = 0$.*

Proof. Trivial generalization of Proposition 1 in Mavridis and Serena (2018). □

The previous proposition shows that if relative costs of voting B_i are high enough for citizens of both groups, the only equilibrium that exists is the "Pure" one in which nobody votes.

Obviously, the situation described in Proposition 1 is not very interesting. Therefore, next we allow one of the two relative costs of voting to be low enough such that citizens from one of the two groups might consider voting; in other words, $B_i \geq 1$ only for citizens i. Then Proposition 2 yields a simple and unique characterization of j's equilibrium strategy:

Proposition 2. *For $B_i \geq 1$ and $B_j \in (0,1)$ the unique QSNE is that $p_i^* = 0$ and $p_j^* = 1 - B_j^{\frac{1}{j-1}}$, for all $i, j \in \{m, n\}$, $i \neq j$.*[14]

Proof. By Proposition 1, $p_i^* = 0$. Suppose $p_j^* = 0$. Then any single j citizen would have an incentive to deviate and vote for sure in order to single-handedly decide the election in favor of the j-group. Thus $p_j = 0$ is not an equilibrium. On the other hand, suppose $p_j^* = 1$. This means that j group wins for sure with a margin of j votes. Then any single j-citizen would have an incentive not to pay the cost without affecting the outcome. Thus $p_j = 1$ is not an equilibrium. Therefore $p_j^* \in (0,1)$. Plugging $p_i^* = 0$ in A_j (see equation Equation 9) we have $A_j = (1-p_j)^{j-1}$, and since (4) must hold with equality for citizens j to mix, we have:

$$(1-p_j)^{j-1} = B_j$$

or equivalently:

$$p_j = 1 - B_j^{\frac{1}{j-1}}.$$

□

Note that this proposition is not paralleled by any result in Mavridis and Serena (2018), which assume that $B_i = B_j$.

The two previous propositions examine cases in which citizens from at least one group find it too costly to vote, no matter what citizens from the other group do. These cases gave rise to two types of equilibria; a "Pure" equilibrium in which everybody's strictly dominant strategy is not to vote (Proposition 1), and a "Partially Mixed" equilibrium in which citizens from one group have a strictly dominant strategy not to vote and citizens from the other group play a mixed strategy (Proposition 2).

Comparative Statics. For $x \in (0,1)$ the expression $1 - x^{\frac{1}{j-1}}$ is strictly decreasing in x. Therefore $p_j^* = 1 - B_j^{\frac{1}{j-1}}$ is strictly decreasing in c_j and strictly increasing in $\Delta\pi_j$. Higher individual cost-payoff ratio results in j-citizens voting with lower probability.

In the remaining of this subsection we examine what happens when citizens from neither group have a strictly dominant strategy to abstain, ie. what happens when $B_m < 1$ and $B_n < 1$. Under these conditions citizens of both groups may vote with positive probability. This causes strategic interactions that may generate multiple equilibria.

It is easy to see that when $B_m < 1$ and $B_n < 1$ no "Pure" equilibria exist.

Proposition 3. *For $B_i < 1$ and $B_j < 1$, no "Pure" QSNE exist, for all $i, j \in \{m, n\}$ and $i \neq j$.*

Proof. Trivial generalization of Proposition 2 in Mavridis and Serena (2018). □

After proving that for $B_m < 1$ and $B_n < 1$ no "Pure" equilibria exist, the next proposition establishes that for $B_m < 1$ and $B_n < 1$ a "Partially Mixed" equilibrium does exist.

Proposition 4. *For $B_i < 1$ and $B_j < 1$, there exists a "Partially Mixed" QSNE with $p_i^* = 0$ and $p_j^* = 1 - B_j^{\frac{1}{j-1}}$, for all $i, j \in \{m, n\}$ and $i \neq j$ if and only if $B_i \geq \underline{B_i} \equiv jB_j - (j-1)B_j^{\frac{j}{j-1}}$.*

Proof. An equilibrium where $p_i^* = 0$ implies: $A_j = (1-p_j)^{j-1}$ and $A_i = (1-p_j)^j + jp_j(1-p_j)^{j-1}$. The former means that a citizen j is pivotal only if none of her groupmates vote (her vote breaks the tie in which nobody votes). The latter means that a citizen i is pivotal if none of j citizens vote or if only one of them votes. In order for the i-citizens to not want to vote we must have:

$$A_i \leq B_i,$$

or equivalently

$$(1-p_j)^j + jp_j(1-p_j)^{j-1} \leq B_i,$$

and similarly, for the j-group citizen to play a mixed strategy we must have:

$$(1-p_j)^{j-1} = B_j, \tag{10}$$

[14] As a reminder, in the beginning of this section we have assumed $m > n > 1$. It is easy to see that for $j = 1$ the unique equilibrium is: $p_i^* = 0$ and $p_j^* = 1$

dividing the two conditions and rearranging we get:

$$\begin{aligned} 1 - p_j + jp_j &\leq \frac{B_i}{B_j} \\ (j-1)p_j &\leq \frac{B_i}{B_j} - 1 \end{aligned} \tag{11}$$

Isolate p_j in (10) and plug it in (11) to get

$$(j-1)\left(1 - B_j^{\frac{1}{j-1}}\right) \leq \frac{B_i}{B_j} - 1 \tag{12}$$

Or equivalently,

$$\begin{aligned} -B_j^{\frac{j}{j-1}} &\leq \frac{B_i - jB_j}{j-1} \\ B_i &\geq jB_j - (j-1)B_j^{\frac{j}{j-1}} \equiv \underline{B_i} \end{aligned} \tag{13}$$

□

$\underline{B_i}$ is an increasing bijection from $[0,1]$ to $[0,1]$, such that if $B_j = 0$, $\underline{B_i} = 0$, and if $B_j = 1$, $\underline{B_i} = 1$.

Note that the equilibria pinned down by Proposition 2 and Proposition 4 are essentially the same, the difference being that Proposition 2 provides the range of B_i's for which the equilibrium is unique, and Proposition 4 provides the range of B_i's for which that equilibrium continues to exist although not necessarily uniquely. This is an important finding in terms of uniqueness of a continuous equilibrium. In fact, first notice that Proposition 4 gives us for every $B_j \in (0,1)$ the lowest B_i for existence of the "Partially Mixed" equilibrium $p_i^* = 0$ and $p_j^* = 1 - B_j^{\frac{1}{j-1}}$. This satisfies the system $A_i \leq B_i$ and $A_j = B_j$, which implies that the system $A_i = B_i$ and $A_j = B_j$ is also satisfied (A_i's are continuous in p_i's). Therefore, exactly at those values of (B_i, B_j) there must be a "Totally Mixed" equilibrium. In our *(Running Redistribution Example)*, as discussed in Section 3 we set $B_n = B$ and $B_m = \frac{m}{n}B$. Numerical simulations show that that "Totally Mixed" equilibrium survives from that value of B all the way down to 0.

Proposition 4 is silent with respect to which of the two groups will be playing a mixed strategy and which will not be voting. What it says is that if $B_m < 1$ and $B_n < 1$ it can be either that the m citizens do not vote and the n citizens play a mixed strategy, or that the n citizens do not vote and the m citizens play a mixed strategy. The next proposition shows that for a given pair (B_i, B_j) these two "Partially Mixed" equilibria of Proposition 4 cannot co-exist; in other words, for a given pair (B_i, B_j) we either have m citizens not voting and n playing a mixed strategy *or* n citizens not voting and m playing a mixed strategy (but not both).

Proposition 5. *For $B_i < 1$ and $B_j < 1$, $B_i \geq \underline{B_i}$ and $B_j \geq \underline{B_j}$ are mutually exclusive, for all $i, j \in \{m, n\}$ and $i \neq j$.*

Proof. Suppose not and consider the $(B_i, B_j)-$space. We first show that $\underline{B_i} > B_j$, or equivalently:

$$\begin{aligned} (j-1)B_j &> (j-1)B_j^{\frac{j}{j-1}} \\ 1 &> B_j^{\frac{1}{j-1}}. \end{aligned}$$

For the same reason we also have $\underline{B_j} > B_i$. Then, $B_i \geq \underline{B_i}$ and $\underline{B_i} > B_j$ imply $B_i > B_j$, while $B_j \geq \underline{B_j}$ and $\underline{B_j} > B_i$ imply $B_j > B_i$ leading to a contradiction. □

4.2 "Partially Mixed"equilibria with voting

We are left to analyze the "Partially Mixed" equilibria in which citizens of one group are voting with certainty and the others are playing a mixed strategy. There cannot be an equilibrium in which the majority group votes with certainty and the minority group plays a mixed strategy (see Mavridis and Serena (2018) for a proof). Therefore the equilibria left to analyze are of the type $p_n^* = 1$ and $p_m^* \in (0,1)$. The next four propositions will establish conditions for this type of equilibrium to exist, and Proposition 10 will characterize the equilibria behavior.

Proposition 6. *There exists a $\hat{B} < 1$ such that if both $B_m \leq \hat{B}$ and $B_n \leq \hat{B}$ hold there exists a unique "Partially Mixed" equilibrium of the form $p_n^* = 1$ and $p_m^* \in (0,1)$.*

Proof. It follows from Proposition 3 in Mavridis and Serena (2018). However we repeat some key steps of their proof that will help the proofs of the following propositions.

When $p_n^* = 1$ we have:

$$A_m = \binom{m-1}{n} p_m^n (1-p_m)^{m-n-1} + \binom{m-1}{n-1} p_m^{n-1} (1-p_m)^{m-n}$$

and

$$A_n = \binom{m}{n-1} p_m^{n-1} (1-p_m)^{m-n+1} + \binom{m}{n} p_m^n (1-p_m)^{m-n}.$$

A_m is strictly increasing in p_m in $(0, \hat{p}_m)$ and strictly decreasing otherwise, with

$$\hat{p}_m = \frac{n(n-1)}{n(n-1) + \sqrt{n(n-1)(m-n)(m-n-1)}}.$$

We furthermore know that $A_m \geq A_n$ if and only if $p_m \leq p_m^{**}$, with

$$p_m^{**} = \frac{n(n-1)}{n(n-1) + \sqrt{n(n-1)(m-n)(m-n+1)}}.$$

The cut-off for costs is therefore $\hat{B} = A_m|_{p_m = p_m^{**}}$. □

Hence the ranking between B_m and B_n does not affect the existence and uniqueness of the equilibrium of the form $p_n^* = 1$ and $p_m^* \in (0,1)$, as long as both are below $\hat{B}$.

Proposition 7. *If $B_n > \hat{B}$ or if $B_m > B^{max}$, $B^{max} = A_m|_{p_m = \hat{p}_m}$, no "Partially Mixed" equilibrium of the form $p_n^* = 1$ and $p_m^* \in (0,1)$ exists.*

Proof. We will prove that A_n is strictly increasing in p_m for $p_m < p_m^{**}$ and strictly decreasing for $p_m > p_m^{**}$. Take the derivative of A_n with respect to p_m. This derivative is greater than zero when:

$$\binom{m}{n-1} \left((n-1)p_m^{n-2}(1-p_m)^{m-n+1} - (m-n+1)p_m^{n-1}(1-p_m)^{m-n}\right) + \binom{m}{n}\left(np_m^{n-1}(1-p_m)^{m-n} - (m-n)p_m^n(1-p_m)^{m-n-1}\right) \geq 0$$

Simplifying this expression we get:

$$m(m-2n+1)p_m^2 + 2pn(n-1) - n(n-1) \leq 0,$$

which is precisely Inequality (5) of Mavridis and Serena (2018). Using a similar analysis as in that paper we see that for $p_m < p_m^{**}$ A_n is strictly increasing and for $p_m > p_m^{**}$ it is strictly decreasing. This implies that A_n is strictly increasing when $A_n > A_m$ and strictly decreasing when $A_n < A_m$, which further implies that the point of intersection between A_m and A_n is the maximum of A_n. Therefore, $\hat{B} = A_m|_{p_m = p_m^{**}}$, and if $B_n > \hat{B}$ this would necessarily imply $A_n < B_n$. A similar argument follows for $B_m > B^{max}$, $B^{max} = A_m|_{p_m = \hat{p}_m}$. □

Naturally if the cost of voting is too high for any group then nobody from that group will vote, and no "Partially Mixed" equilibrium of the form $p_i^* = 1$ and $p_j^* \in (0,1)$ exists. The next two propositions examine the cases when the cost of voting for the minority is relatively small and the cost of voting for the majority relatively large.

Proposition 8. *Let $m = n+1$ and $B_m \in [\hat{B}, B^{max})$. Then there exists a unique "Partially Mixed" equilibrium given by $p_n^* = 1$ and $p_m^* \in (0,1)$ if $A_n|_{p_m = p_m^*} \geq B_n$. Let $m = n+1$ and $B_m = B^{max}$ then there is no "Partially Mixed" equilibrium.*

Proof. The first part of the proof follows from Proposition 3 in Mavridis and Serena (2018). Notice that if $n = m-1$ then $\hat{p}_m = 1$ and in this case A_m is strictly increasing for $p_m \leq 1$. Because of this $A_m = B_m$ has a unique solution $p_m^* > p_m^{**}$. The equilibrium exists as long as $A_n|_{p_m = p_m^*} \geq B_n$. For the second part notice that when $B_m = B^{max}$ then necessarily $p_m^* = 1$. But this means that $A_n|_{p_m=1} = 0$, which means that nobody from the minority group would vote. □

The next proposition considers the $m > n+1$ case.

Proposition 9. *Let $m > n+1$ and $B_m \in [\hat{B}, B^{max})$. There are two equilibria $p_n^* = 1$ and $p_{m1}^* \in (0,1)$, and $p_n^* = 1$ and $p_{m2}^* \in (0,1)$ with $p_{m1} < p_{m2}$ if $B_n < A_n|_{p_m = p_{m2}^*}$, one equilibrium $p_n^* = 1$ and $p_{m1}^* \in (0,1)$ if $B_n \in [A_n|_{p_m = p_{m1}^*}, A_n|_{p_m = p_{m2}^*})$ and no equilibrium otherwise. Let $m > n+1$ and $B_m = B^{max}$. Then there exists one equilibrium $p_n^* = 1$ and $p_m^* \in (0,1)$ if $B_n < A_n|_{p_m = p_m^*}$.*

Proof. The first part of the proof follows from Proposition 3 in Mavridis and Serena (2018). If $m > n+1$ then A_m reaches a maximum at $\hat{p}_m < 1$ and at $p_m = 1$ A_m becomes zero. Since $\hat{B} = A_m|_{p_m = p_m^{**}}$ and $p_m^{**} < \hat{p}_m$, $A_m = B_m$ has two roots, which we will call p_{m1}^* and p_{m2}^* and without loss of generality we will assume that $p_{m1}^* < p_{m2}^*$. It follows from Proposition 3 in Mavridis and Serena (2018) that if $B_n > A_n|_{p_m = p_{m1}^*}$ no equilibrium exists. If $B_n \in [A_n|_{p_m = p_{m1}^*}, A_n|_{p_m = p_{m2}^*})$ then only the equilibrium $p_n^* = 1$,$p_{m1}^* \in (0,1)$ exists. Otherwise, if $B_n < A_n|_{p_m = p_{m2}^*}$, there are two equilibria $p_n^* = 1$, $p_{m1}^* \in (0,1)$ and $p_n^* = 1$, $p_{m2}^* \in (0,1)$. For the second part notice that since B_m equals the unique maximum of A_m there is only one solution of $B^{max} = A_m$. □

Comparative Statics. We provide comparative statics on p_m^* of the "Partially Mixed" equilibrium of the form $p_n^* = 1$ and $p_m^* \in (0,1)$.

Proposition 10. *In the intervals for which A_m is increasing, p_m^* is increasing in B_m and decreasing otherwise. For $A_m \leq (\geq) A_n$, p_m^* is decreasing (increasing) in m. Finally if $m(m-2n-1)p_m^2 + 2n(n-1)p_m - n(n-1) \leq (\geq) 0$ then p_m^* is increasing (decreasing) in n.*

Proof. The three parts follow from Proposition 3 in Mavridis and Serena (2018). The only part that needs some further analysis is the third one. Notice that this quadratic inequality follows from Proposition 3 in Mavridis and Serena (2018) and that the roots of the quadratic equation $m(m-2n-1)p_m^2 + 2n(n-1)p_m - n(n-1) = 0$ are given by:

$$\tilde{p}_m = \frac{n(n-1)}{n(n-1) \pm \sqrt{n(n-1)(n(n-1) + m(m-2n-1))}}.$$

We discard the root with the negative sign in front of the square root because it becomes negative if $m-2n-1 > 0$ and greater than one otherwise. Let $m-2n-1 > 0$. If $p_m \leq \tilde{p}_m$ then $m(m-2n-1)p_m^2 + 2n(n-1)p_m - n(n-1) \leq 0$ holds and p_m^* is increasing in n. On the other hand if $m-2n-1 < 0$ the function $f(p_m) = m(m-2n-1)p_m^2 + 2n(n-1)p_m - n(n-1)$ is strictly concave and has a unique maximum. At the unique maximum the value of the function is: $n(n-1) + m(m-2n-1)$, which is greater than zero as long as $n(n-1) \geq m(2n+1-m)$. In this case if $p_m \leq \tilde{p}_m$ then $m(m-2n-1)p_m^2 + 2n(n-1)p_m - n(n-1) < 0$ holds and p_m^* is increasing in n. Otherwise if $n(n-1) < m(2n+1-m)$ then $m(m-2n-1)p_m^2 + 2n(n-1)p_m - n(n-1) < 0$ always holds and p_m^* is increasing in n. □

In this section we derived the full characterization of the "Pure" and "Partially Mixed" equilibria. Several differences emerge from the setting with symmetric benefits analyzed in Mavridis and Serena

(2018). While in the symmetric setting of Mavridis and Serena (2018) the "Partially Mixed" is unique, in the present asymmetric setting there are two "Partially Mixed" (as clear from the propositions of this section, and also already from Figure 1). However, only one connects continuously with the unique "Pure". Furthermore, the same "Partially Mixed" also connects continuously to one "Totally Mixed", as we anticipated in Section 3 and we will further discuss in the next Section. Finally, while Mavridis and Serena (2018) analysed the "Totally Mixed" only numerically, we derive an analytical result in Proposition 11. The Proposition verifies Mavridis and Serena (2018)'s conjectures, arising from their numerical results.

4.3 "Totally Mixed" equilibria and continuity refinement

Analytical findings. Propositions 1 to 10 fully characterized the "Pure" and "Partially Mixed" equilibria of the voting game. We are left to analyze the "Totally Mixed" equilibria. As discussed earlier in the paper, the mixing conditions for the m citizens is defined by:

$$A_m = B_m \tag{14}$$

and the mixing condition for the n citizens is defined by:

$$A_n = B_n \tag{15}$$

Since we have imposed the condition that all citizens within a group employ the same strategy, it suffices to focus on the mixing condition of an m citizen for every p_n, and that of an n citizen for every p_m, and analyze the intersections between these two in the (p_m, p_n)-space. These intersections are "Totally Mixed" equilibrium pairs (p_m^*, p_n^*), which are what we are after in this Subsection.

In contrast with the "Pure" and "Partially Mixed" cases, it is challenging to derive a general algebraic solution for equilibrium strategies of the "Totally Mixed" case, as expressions (14) and (15) are a system of two polynomial equations of arbitrary power. Instead, in order to analyze them we will use a number of indirect results about the space these equilibria lie on. For this it is useful to distinguish among the three cases: $B_m = B_n$, $B_m > B_n$, and $B_m < B_n$. Thus, by (14) and (15), these translate into $A_m = A_n$, $A_m > A_n$, and $A_m < A_n$. Analyzing $A_m = A_n$ will greatly help the analysis of the other two cases.

The set of points for which $A_m = A_n$ is depicted by the two black lines of Figure 3; the increasing and the decreasing one. These two black lines divide the (p_m, p_n)-space in four regions depending on the ranking of A_m and A_n. Keep in mind that the two mixing conditions are defined in the same space; dividing the space in these four regions will help us analyze how the intersections of the two mixing conditions behave.[15]

In Appendix A we characterize the set $A_m = A_n$, through a series of lemmas. First, we find the four points where these two lines touch the edges of the (p_m, p_n)-space (Lemma 1). Second, we characterize the decreasing line connecting the top-left corner with the bottom-right corner (Lemma 2). Finally we characterize the increasing line (Lemmas 3 and 4).[16] We summarize this series of lemmas in Proposition 11.

Proposition 11. *The only points* $(p_m, p_n) \in (0, 1)^2$ *satisfying condition*

$$A_m = A_n \tag{16}$$

are the points along the line $p_m + p_n = 1$ *and along a continuous line that goes from* $(0, 0)$ *to* $(p_m^{***}, 1)$ *where* $p_m^{***} = \frac{n(n-1)}{n(n-1)+\sqrt{n(n-1)(m-n)(m-n+1)}}$.

Proof. See Appendix A. □

[15] At this stage it is interesting to compare our analysis with the one of Palfrey and Rosenthal (1983). In particular in Section 6 of Palfrey and Rosenthal (1983), they discuss "totally quasi-symmetric equilibria", which are what we call "Totally Mixed" equilibria. However they analyze two special cases, which in our notation are: i) $p_m = p_n$ and $m = n$ and ii) $p_m + p_n = 1$. In terms of our Figure 3 it means that they analyze equilibria that might arise along the two diagonals (in the case of the 45-degree line, they also assume $m = n$).

[16] Note that, as it will be explained in more detail later, the fact that the increasing line is in fact increasing is not needed. What is only needed is that it crosses the decreasing line once.

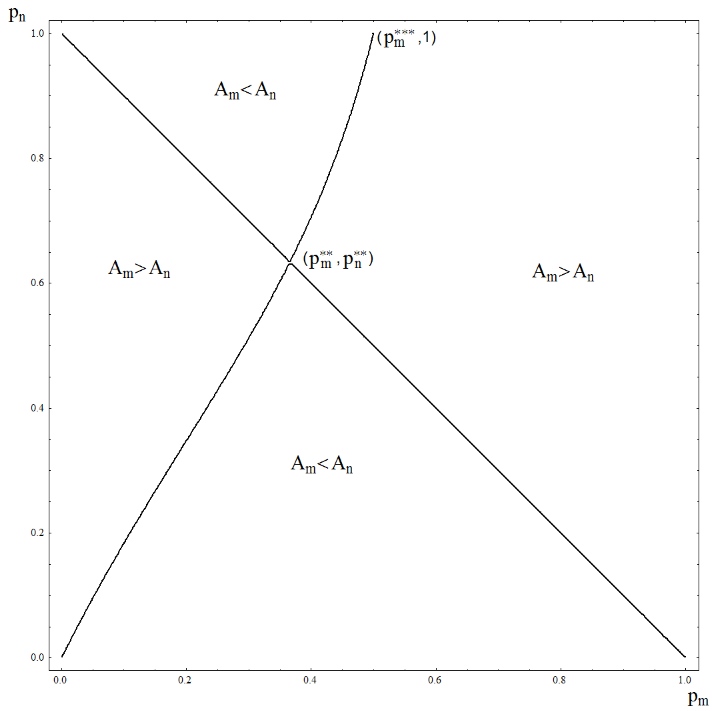

Figure 3: Set of points in the (p_m, p_n)-space according to whether $A_m \gtreqless A_n$ when $m = 3$ and $n = 2$.

This result characterizes analytically what Mavridis and Serena (2018) found numerically, and described in their Figure 2. Furthermore, notice that in their Figure 2, the mixing conditions always cross in the set $A_m = A_n$, because they are interested only in the symmetric case of $B_m = B_n$. On the contrary, we are interested in the more general case where $B_m \gtreqless B_n$. For what concerns our *(Running Redistribution Example)*, as can be seen in (5) and (6), we have $B_m > B_n$, and thus all equilibria lie in one of the two regions where $A_m > A_n$ of Figure 3.

Numerical findings. Our numerical exercise shows that the two mixing conditions (14) and (15) cross at most once in each of the four regions delimited by the set of points such that $A_m = A_n$ (see Figure 3). This implies that for every pair (B_m, B_n), we always have at most two "Totally Mixed" equilibria, one where $p_m^* + p_n^* < 1$ which we name "Totally Mixed 1", and one where $p_m^* + p_n^* > 1$ which we name "Totally Mixed 2".[17] Including the "Partially Mixed" equilibrium previously characterized, this shows that, all in all, we have at most three equilibria.

The crucial feature that emerges from our numerical exercise is that exactly one "Totally Mixed" equilibrium satisfies the continuity refinement mentioned in Section 3. We now deliver the graphical intuition about uniqueness and continuity.

In Figure 4 we depict in the (p_m, p_n)-space the conditions (14) and (15), where the red (blue) lines represent the mixing condition correspondence $A_n = B_n$ ($A_m = B_m$) for a $n-$ ($m-$) citizen for four values of B, $\{0.166, 0.333, 0.5, 0.61\}$.[18] In particular, we choose the third value to be the minimum B such that the "Totally Mixed 1" equilibrium disappears (bottom-left panel), and the fourth value to be the minimum B such that even the "Totally Mixed 2" equilibrium disappears. The black lines in Figure 4 are the set of (p_m, p_n) satisfying $A_m = A_n$, as in Figure 3.

The equilibrium on the right side of the panel is the "Totally Mixed 2" equilibrium, which exists as long as $B \leq 0.61$. This equilibrium cannot clearly be continuously connected to the "Partially Mixed"

[17] These definitions of "Totally Mixed 1" and "Totally Mixed 2" are in line with those of Section 3.

[18] Notice that Figure 4 is reminiscent to Figure 2 in Mavridis and Serena (2018). However their setting with symmetric payoffs yielded equilibria always lying along the increasing or decreasing line. In our setting with asymmetric payoffs the equilibria do not lie on these two lines.

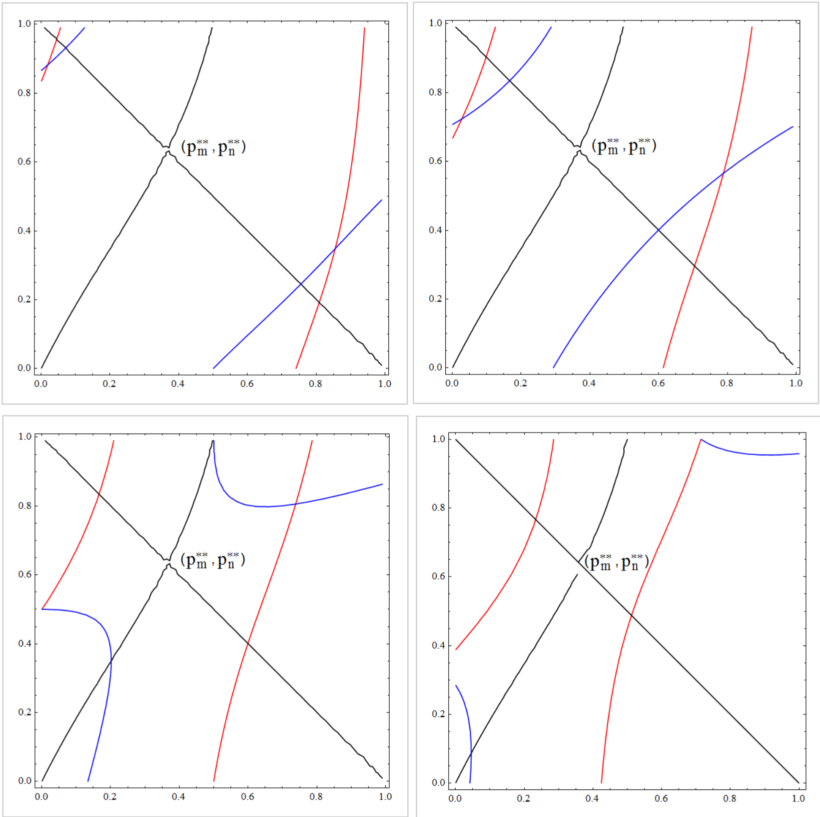

Figure 4: Mixing conditions correspondences in the (p_m, p_n)-space respectively for $B = \{0.166, 0.333, 0.5, 0.61\}$.

equilibrium, as we need p_i^* to go to 0 and at the same time p_j^* to be interior.[19] For this to happen we need that $p_m^* + p_n^* < 1$, which contradicts "Totally Mixed 2", which hence is ruled out by the continuity refinement.

The equilibrium on the left side of the panel is the "Totally Mixed 1" equilibrium, which exists as long as $B \leq 0.5$. In particular, it converges to the "Partially Mixed" equilibrium $(p_m^*, p_n^*) = (0, 0.5)$ as $B \to 0.5$. Our numerical exercise shows that this continuous connection between the "Partially Mixed" and the "Totally Mixed 1" is a general property. We exploit the uniqueness of equilibrium under the continuity refinement in order to discuss comparative statics and characterization of the *(Running Redistribution Example)*, to which we dedicate the next Section, so as to shed light on the redistribution trade-off spelled out in the Introduction.

5 Application - voting over redistribution of resources

From Proposition 4 we know that the "Partially Mixed" $p_m^* = 0$ and $p_n^* = 1 - B_n^{\frac{1}{n-1}}$ exist if and only if $B_m > nB_n - (n-1)B_n^{\frac{n}{n-1}}$. Recall from (5) and (6) that $B_m = 2c(n+m)/n$ and $B_n = 2c(n+m)/m$. Finally, recall that the notation for our *(Running Redistribution Example)* is $B_n = B$ and $B_m = \frac{m}{n}B$. Plugging these expressions into

$$B_m \geq nB_n - (n-1)B_n^{\frac{n}{n-1}}, \tag{17}$$

[19] In terms of Figure 3 and Figure 4, equilibria must converge to the horizontal or vertical axis. This happens in the third panel of Figure 4.

we obtain

$$\begin{aligned}\left(m-n^2\right)B+n(n-1)B^{\frac{n}{n-1}} &> 0\\ B &> \left(\frac{n^2-m}{n^2-n}\right)^{n-1}.\end{aligned}$$

And in fact, the lowest value of B for which the "Partially Mixed" exists is 0 when $n^2 \leq m$, while when $n^2 > m$ it is $\left(\frac{n^2-m}{n^2-n}\right)^{n-1}$, which decreases in m. In fact, as we increase m, we see that such threshold moves to the left in the graph.

Notice that the "Partially Mixed" $p_m^* = 0$ and $p_n^* = 1 - B_n^{\frac{1}{n-1}}$ exists until $p_n^* \searrow 0$, which implies $1 - B_n^{\frac{1}{n-1}} = 0 \iff B_n = B = 1$. Hence, the highest B where the "Partially Mixed" exists is always equal to $B = 1$. Moreover, the mixing equilibrium probability $p_n^* = 1 - B^{\frac{1}{n-1}}$ in the "Partially Mixed" has the same graph in the four plots, as the functional form is the same and n does not change. Only its existence region becomes bigger as m increases.

Therefore, if the "Partially Mixed" still exists as B goes to 0 (equivalently, c goes to 0), then the "Partially Mixed" and "Pure" together form the unique equilibrium without any need for continuity, and the poor citizens will never vote for any $c > 0$. Notice that the second term of the right-hand side of (17) goes to 0 faster than the other terms in the inequality as B's go to zero, and thus in the limit it is negligible. Then, sufficiently close to 0, we are left with only $B_m > nB_n$. By plugging the expressions for B_m and B_n we get:

$$m \geq n^2. \tag{18}$$

This is a necessary and sufficient condition for $p_m^* = 0$ to hold in the unique equilibrium for any $c > 0$. Also, it has a nice interpretation. If the society is sufficiently equal ($m < n^2$), the poor might vote and redistribution has a chance of winning. However, in a sufficiently unequal society ($m \geq n^2$), the poor are doomed to lose the election.

Comparative Statics. How does a change of m affect (p_m^*, p_n^*)? We answer with the support of Figure 5. We fix $n = 3$, and set m so as to initially have a very equal society ($m = 4$, top-left), and gradually increase the inequality ($m = 5$, top-right, and $m = 6$, bottom-left), until we hit inequality $m = n^2$ ($m = 9$, bottom-right). When we hit this inequality threshold condition (18) is satisfied and the "Totally Mixed 1" equilibrium (which survives continuity) disappears, and we are left with the "Partially Mixed" and "Pure" only.[20]

Consider $n = 3$ and $m = 4$. Since the society has (slightly) more poor than rich citizens, the average resource level is closer to the resources of a poor citizen than to the resources of a rich citizen, thus if full redistribution wins, the individual loss of a single rich citizen is greater than the individual gain of a single poor citizen. For this reason, an n rich citizen is willing to vote for greater B's than an m poor citizen. In other words, a rich has more at stake than a poor, and thus is willing to face a greater cost of voting. Therefore, p_n^* turns positive for greater values of B than p_m^* does, as we can see in Figure 5.

Consider an increase of m ($n = 3$ and $m = 5$, or 6). This has the effect of sharpening the asymmetry in willingness to face the cost of voting between rich and poor: in fact, now, p_n^* turns positive for even greater B's (the rich has even more at stake to lose in case of full redistribution), while p_m^* turns positive for even lower B's (the poor has even less at stake to win in case of full redistribution). This widens the "Partially Mixed" region (see Figure 5).

If the increase in m reaches the inequality threshold when $m = n^2$ ($n = 3$ and $m = 9$), the poor has so little at stake that she is nowhere willing to face the cost of voting with positive probability in equilibrium.[21] A further increase in m would still imply $p_m^* = 0$ everywhere, and further increases the willingness to vote of the rich (i.e., p_n^* increases for any given B, and p_n^* turns positive for greater B's).

This could be interpreted as a "poverty trap": the greater is the share of poor in a society, the less likely is redistribution of resources to be the outcome of a democratic process (and if $m \geq n^2$ this probability is zero).

[20] This possibility that members of the majority never vote in the unique continuous equilibrium did not emerge from the discussion of the special case of $m = 3$ and $n = 2$ in Figure 1. In fact, when $m = 3$ and $n = 2$, $m < n^2$.

[21] Remember that if $m \geq n^2$ the equilibrium is unique without the need for continuity selection.

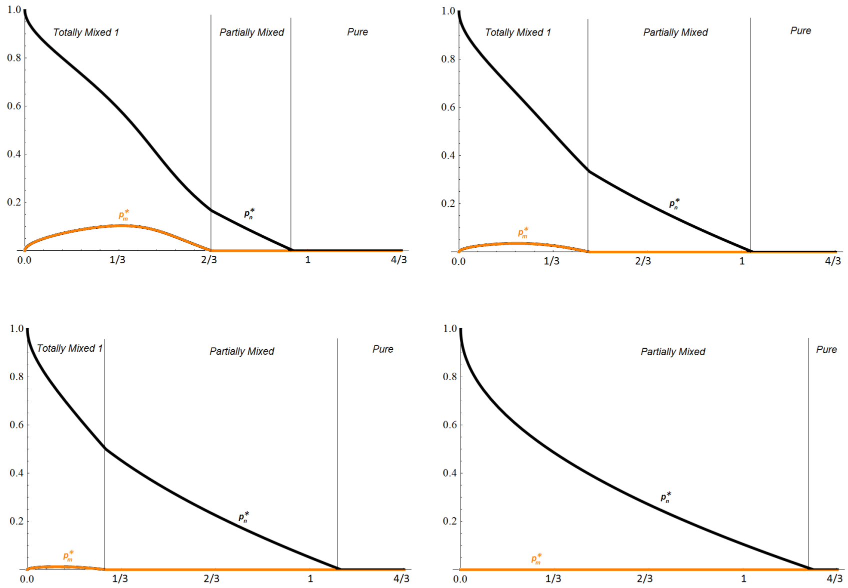

Figure 5: Effects of increasing m, keeping $n = 3$, on voting probabilities in the unique continuous equilibrium. First row left panel $m = 4$, right panel $m = 5$. Second row, left panel $m = 6$ right panel $m = 9$.

Other applications. Beside the redistribution example other applications fit our *asymmetric-asymmetric* setting. Our complete information setting springs from Palfrey and Rosenthal (1983), however we depart from them in that a citizen's benefit of having the favorite alternative win can be asymmetric according to whether the citizen supports one or the other alternative, meaning that the personal benefit can differ between citizens supporting one alternative or the other. On top of that, we allow for asymmetric group sizes. For instance, think of a university faculty consisting of several economists and a few lawyers, all called to vote over who to hire between two job market candidates: an economist and a lawyer.[22] Both economists and lawyers are better-off if the newly hired candidate is of their same type. Furthermore, the benefit for a lawyer from hiring another lawyer is greater than the one for an economist from hiring another economist because of the asymmetric size of the two groups; that is, since lawyers are fewer, having another lawyer in the department sharply increases each lawyer's coauthoring possibilities, whereas the benefit for an economist from having a new economist in the faculty is lower because they are already plenty. In other words, the benefit is asymmetric across citizens of different groups. Another example of asymmetric benefit across groups of asymmetric size is the following. Residents of two neighborhoods are called to vote over the location of a new school in one of the two neighborhoods. In neighborhood 1 there is already a school, in neighborhood 2 there is none: thus, despite the fact that each resident strictly prefers the school to be located in her neighborhood, residents in neighborhood 1 "care less" than residents in neighborhood 2 about the location of the school since there is already a school in neighborhood 1. Both these settings (faculty and neighborhoods) are captured by our *asymmetric-asymmetric* setup which, to the best of our knowledge, has not been studied before. Finally, on the contrary of the already studied *symmetric-symmetric* and *asymmetric-symmetric* setups, we deploy a continuity refinement, pinning down a unique equilibrium which we analysed.

[22] The cost of voting in this case is the opportunity cost of showing up to vote that day instead of, for example, being on vacation or doing research.

Appendix

We prove Proposition 11 by way of the following lemmata. See Figure 3.

Lemma 1. *The only points satisfying (16) and* $(p_m, p_n) \in \{0,1\}^2$ *are:* $(0,0)$, $(0,1)$, $(1,0)$, *and* $(p_m^{***}, 1)$, *with* $p_m^{***} = \frac{n(n-1)}{n(n-1)+\sqrt{n(n-1)(m-n)(m-n+1)}}$. *Also,* $p_m^{***} = 1$ *iff* $m = n$.

Proof. By continuity of A_m and A_n in p_m and p_n, in order to analyze the behavior of A_m and A_n in $(p_m, p_n) \in \{0,1\}^2$ we can compute the following limits for A_m

$$\begin{aligned}
\lim_{p_m \to 0} A_m &= np_n(1-p_n)^{n-1} + (1-p_n)^n \\
\lim_{p_m \to 1} A_m &= \begin{cases} p_n^n + np_n^{n-1}(1-p_n) & \text{if } m = n \\ \binom{m-1}{n} p_n^n & \text{if } m = n+1 \\ 0 & \text{if } m > n+1 \end{cases} \\
\lim_{p_n \to 0} A_m &= (1-p_m)^{m-1} \\
\lim_{p_n \to 1} A_m &= \binom{m-1}{n-1} p_m^{n-1}(1-p_m)^{m-n} + \binom{m-1}{n} p_m^n (1-p_m)^{m-n-1}
\end{aligned}$$

and for A_n

$$\begin{aligned}
\lim_{p_m \to 0} A_n &= (1-p_n)^{n-1} \\
\lim_{p_m \to 1} A_n &= \begin{cases} p_n^{n-1} & \text{if } m = n \\ 0 & \text{if } m > n \end{cases} \\
\lim_{p_n \to 0} A_n &= m(1-p_m)^{m-1} + (1-p_m)^m \\
\lim_{p_n \to 1} A_n &= \binom{m}{n} p_m^n (1-p_m)^{m-n} + \binom{m}{n-1} p_m^{n-1}(1-p_m)^{m-n+1}
\end{aligned}$$

From the above,
- if $p_m = 0$, (16) holds iff $p_n = 0$ or $p_n = 1$
- if $p_m = 1$, (16) holds iff $p_n = 0$ or $p_n = 1$ and $m = n$
- if $p_n = 0$, (16) holds iff $p_m = 0$ or $p_m = 1$
- if $p_n = 1$, (16) is equivalent to

$$\begin{aligned}
\binom{m-1}{n-1} p_m^{n-1}(1-p_m)^{m-n} + \binom{m-1}{n} p_m^n (1-p_m)^{m-n-1} &= \binom{m}{n} p_m^n (1-p_m)^{m-n} + \binom{m}{n-1} p_m^{n-1}(1-p_m)^{m-n+1} \\
\binom{m-1}{n-1}(1-p_m) + \binom{m-1}{n} p_m &= \binom{m}{n} p_m(1-p_m) + \binom{m}{n-1}(1-p_m)^2 \qquad (19)
\end{aligned}$$

If $m = n$, (19) boils down to

$$1 - p_m = p_m(1-p_m) + m\,(1-p_m)^2$$

whose unique solution is $p_m = 1$.
If $m > n$, (19) boils down to

$$\frac{(1-p_m)}{m-n} + \frac{p_m}{n} = \frac{mp_m(1-p_m)}{n(m-n)} + \frac{m(1-p_m)^2}{(m-n)(m-n+1)}$$

Solving the simple polynomial in the last expression we see that p_m^{***} is indeed one of its two roots (the second root has to be discarded since it is greater than 1). □

Lemma 2. *Equation* $p_m + p_n = 1$ *solves* $A_m = A_n$ $\forall (p_m, p_n) \in [0,1]^2$.

Proof. From (9) plug A_m and A_n into (16), simplify for $(1-p_m)^m(1-p_n)^n$, and use $p_n = 1-p_m$ to obtain

$$\sum_{s=0}^{n}\binom{m-1}{s}\binom{n}{s}p_m^{s-s}(1-p_m)^{-s-1+s} + \sum_{s=0}^{n-1}\binom{m-1}{s}\binom{n}{s+1}p_m^{s-s-1}(1-p_m)^{-s-1+s+1}$$
$$= \sum_{s=0}^{n-1}\binom{m}{s}\binom{n-1}{s}p_m^{s-s-1}(1-p_m)^{-s+s} + \sum_{s=0}^{n-1}\binom{m}{s+1}\binom{n-1}{s}p_m^{s+1-s-1}(1-p_m)^{-s-1+s}$$

$$p_m\sum_{s=0}^{n}\binom{m-1}{s}\binom{n}{s} + (1-p_m)\sum_{s=0}^{n-1}\binom{m-1}{s}\binom{n}{s+1}$$
$$= (1-p_m)\sum_{s=0}^{n-1}\binom{m}{s}\binom{n-1}{s} + p_m\sum_{s=0}^{n-1}\binom{m}{s+1}\binom{n-1}{s}$$

$$p_m\sum_{s=0}^{n}\binom{m-1}{s}\binom{n}{n-s} + (1-p_m)\sum_{s=0}^{n-1}\binom{m-1}{s}\binom{n}{n-s-1}$$
$$= (1-p_m)\sum_{s=0}^{n-1}\binom{m}{s}\binom{n-1}{n-s-1} + p_m\sum_{s=0}^{n-1}\binom{m}{s+1}\binom{n-1}{n-s-1}$$
$$p_m\binom{m+n-1}{n} + (1-p_m)\binom{m+n-1}{n-1} = (1-p_m)\binom{m+n-1}{n-1} + p_m\binom{m+n-1}{n}$$
$$0 = 0$$

where in the second-to-last step we used the symmetry rule for binomial coefficients, and in the last step we used Vandermonde's identity.[23] □

Next we characterize the set of points $A_m = A_n$ that are depicted by an increasing line in the (p_m, p_n)-space by means of two lemmas. In Lemma 3 We show that there exists a point (p_m^{**}, p_n^{**}) along the decreasing line which divides the neighborhoods of the decreasing line into two parts:

1. The first part is the one connecting (p_m^{**}, p_n^{**}) and $(1,0)$, where we prove that increasing p_m (i.e., moving to the right of the line), increases A_m *more* than A_n. Since exactly along the line $A_m = A_n$, this result implies that to the right of the segment connecting (p_m^{**}, p_n^{**}) and $(1,0)$ we have $A_m > A_n$ and to its left we have $A_m < A_n$.
2. The second part is the one connecting $(0,1)$ and (p_m^{**}, p_n^{**}), where we prove that increasing p_m (i.e., moving to the right of the line), increases A_m *less* than A_n. Since exactly along the line, $A_m = A_n$, this result implies that to the right of the segment connecting $(0,1)$ and (p_m^{**}, p_n^{**}) we have that $A_m < A_n$ and to its left $A_m > A_n$.

Lemma 3. *There exists a unique pair $(p_m^{**}, p_n^{**}) \in (0,1)^2$ with $p_m^{**} + p_n^{**} = 1$ such that*

$$\left.\frac{\partial A_m}{\partial p_m}\right|_{p_m+p_n=1} > \left.\frac{\partial A_n}{\partial p_m}\right|_{p_m+p_n=1} \quad \textit{iff } p_m > p_m^{**} \ \textit{(or equivalently } p_n < p_n^{**}\textit{)}$$

Also, if $m = n$, then $p_m^{**} = p_n^{**} = \frac{1}{2}$, and if $m > n$, then $p_m^{**} \in (0, \frac{1}{2})$ and $p_n^{**} \in (\frac{1}{2}, 1)$. In particular,

$$p_m^{**} = \frac{n(n-1)}{n(n-1) + \sqrt{m(m-1)n(n-1)}} \quad \textit{and } p_n^{**} = 1 - p_m^{**}$$

[23] Vandermonde's identity states that $\binom{m+n}{r} = \sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k}$ for $m, n, r \in \mathbb{N}_0$.

Proof. For notation simplicity and for the sake of space we define the following

$$\begin{aligned}\tilde{p}_{s,m} &= \left(\frac{p_m}{1-p_m}\right)^s\\ \tilde{p}_{s,n} &= \left(\frac{p_n}{1-p_n}\right)^s\end{aligned}$$

Then,

$$\left.\frac{\partial A_m}{\partial p_m}\right|_{p_m+p_n=1} > \left.\frac{\partial A_n}{\partial p_m}\right|_{p_m+p_n=1}$$

$$\sum_{s=0}^{n}\binom{m-1}{s}\binom{n}{s}\tilde{p}_{s,m}\tilde{p}_{s,n}\frac{s+p_m}{p_m(1-p_m)^2}+\sum_{s=0}^{n-1}\binom{m-1}{s}\binom{n}{s+1}\tilde{p}_{s,m}\tilde{p}_{s,n}\frac{s+p_m}{p_m(1-p_m)^2}\frac{p_n}{1-p_n}\Bigg|_{p_m+p_n=1} >$$
$$\sum_{s=0}^{n-1}\binom{m}{s}\binom{n-1}{s}\tilde{p}_{s,m}\tilde{p}_{s,n}\frac{s}{p_m(1-p_m)(1-p_n)}+\sum_{s=0}^{n-1}\binom{m}{s+1}\binom{n-1}{s}\tilde{p}_{s,m}\tilde{p}_{s,n}\frac{s+1}{(1-p_m)^2(1-p_n)}\Bigg|_{p_m+p_n=1}$$

by noticing that $p_m + p_n = 1$ implies $\tilde{p}_{s,m}\tilde{p}_{s,n} = 1$ the above inequality simplifies to

$$\sum_{s=0}^{n}\binom{m-1}{s}\binom{n}{s}\frac{s+p_m}{p_m(1-p_m)^2}+\sum_{s=0}^{n-1}\binom{m-1}{s}\binom{n}{s+1}\frac{s+p_m}{p_m^2(1-p_m)} >$$
$$\sum_{s=0}^{n-1}\binom{m}{s}\binom{n-1}{s}\frac{s}{p_m^2(1-p_m)}+\sum_{s=0}^{n-1}\binom{m}{s+1}\binom{n-1}{s}\frac{s+1}{p_m(1-p_m)^2}$$

$$\sum_{s=0}^{n}\binom{m-1}{s}\binom{n}{s}p_m(s+p_m)+\sum_{s=0}^{n-1}\binom{m-1}{s}\binom{n}{s+1}(1-p_m)(s+p_m) >$$
$$\sum_{s=0}^{n-1}\binom{m}{s}\binom{n-1}{s}(1-p_m)s+\sum_{s=0}^{n-1}\binom{m}{s+1}\binom{n-1}{s}p_m(s+1)$$

Note that some summands in the above inequality contain s only in the binomial coefficients. By applying to these terms the same procedure at the end of Lemma 2 (i.e. symmetry rule for binomial coefficients and Vandermonde's identity), we get

$$p_m\sum_{s=0}^{n}\binom{m-1}{s}\binom{n}{s}s+p_m^2\binom{m+n-1}{n}+(1-p_m)\sum_{s=0}^{n-1}\binom{m-1}{s}\binom{n}{s+1}s+p_m(1-p_m)\binom{m+n-1}{n-1} >$$
$$(1-p_m)\sum_{s=0}^{n-1}\binom{m}{s}\binom{n-1}{s}s+p_m\sum_{s=0}^{n-1}\binom{m}{s+1}\binom{n-1}{s}s+p_m\binom{m+n-1}{n}$$

$$p_m\sum_{s=0}^{n}\binom{m-1}{s}\binom{n}{s}s+(1-p_m)\sum_{s=0}^{n-1}\binom{m-1}{s}\binom{n}{s+1}s >$$
$$(1-p_m)\sum_{s=0}^{n-1}\binom{m}{s}\binom{n-1}{s}s+p_m\sum_{s=0}^{n-1}\binom{m}{s+1}\binom{n-1}{s}s+p_m(1-p_m)\left[\binom{m+n-1}{n}-\binom{m+n-1}{n-1}\right]$$

we now analyze the summations left (containing s not only in the binomial coefficient), and use the fact that $\sum_{s=0}^{n}\binom{m}{s}\binom{n}{s}s = n\binom{m+n-1}{n}$ and that $\sum_{s=0}^{n}\binom{m}{s}\binom{n}{s+1}s = m\binom{m+n-1}{n-2}$,[24] and get

$$\begin{aligned}
&np_m\binom{m+n-2}{n} + (m-1)(1-p_m)\binom{m+n-2}{n-2} > \\
&(n-1)(1-p_m)\binom{m+n-2}{n-1} + (n-1)p_m\binom{m+n-2}{n} + p_m(1-p_m)\left[\binom{m-n-1}{n} - \binom{m-n-1}{n-1}\right]
\end{aligned}$$

$$\begin{aligned}
&(m-1)(1-p_m)\binom{m+n-2}{n-2} > \\
&\quad (n-1)(1-p_m)\binom{m+n-2}{n-1} - p_m\binom{m+n-2}{n} + p_m(1-p_m)\left[\binom{m-n-1}{n} - \binom{m-n-1}{n-1}\right]
\end{aligned}$$

and simplifying by $\frac{(m+n-2)!}{(m-2)!(n-2)!}$ we get

$$\begin{aligned}
\frac{1-p_m}{m} &> \frac{1-p_m}{m-1} - \frac{p_m}{n(n-1)} + p_m(1-p_m)\frac{(m-n)(m+n-1)}{m(m-1)n(n-1)} \\
-n(n-1)(1-p_m) &> -m(m-1)p_m + p_m(1-p_m)(m-n)(m+n-1) \\
(m-n)(m+n-1)p_m^2 + 2n(n-1)p_m - n(n-1) &> 0 \\
p_m &> \frac{n(n-1)}{n(n-1)+\sqrt{m(m-1)n(n-1)}} = p_m^{**}
\end{aligned}$$

If $m = n$ it is trivial to see that $p_m^{**} = \frac{1}{2}$. But notice also that p_m^{**} decreases in m, and hence by $m > n$, $p_m^{**} \in (0, \frac{1}{2})$ and $p_n^{**} \in (\frac{1}{2}, 1)$. □

Lemma 4 concludes the characterization of the increasing line.

Lemma 4. *There exists a unique and continuous line in the (p_m, p_n)-space which satisfies $A_m = A_n$ and connects $(0,0)$ and $(p_m^{***}, 1)$ Furthermore, this line crosses the $p_m + p_n = 1$ line once at (p_m^{**}, p_n^{**}).*

Proof. Lemma 2 establishes that the decreasing line connects two out of the four points satisfying $A_m = A_n$ along the edges. The line connecting the remaining two points is continuous and by Lemma 3 crosses the decreasing line once, at (p_m^{**}, p_n^{**}). □

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$$\begin{aligned}
\sum_{s=0}^{n}\binom{m}{s}\binom{n}{s}s &= \sum_{s=0}^{n}\binom{m}{s}\frac{n!}{s!(n-s)!}s \\
&= \sum_{s=0}^{n}\binom{m}{s}\frac{n!}{(s-1)!(n-s)!} = \sum_{s=0}^{n}\binom{m}{s}\frac{n!}{(s-1)!(n-1-s+1)!} \\
&= \sum_{s=0}^{n}\binom{m}{s}\frac{(n-1)!}{(s-1)!((n-1)-(s-1))!}n = n\binom{m+n-1}{n}
\end{aligned}$$

Where the last equality follows from Valdemore's identity. The calculations for the other summation are similar.

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