Disentangling the role of the exchange rate in oil-related scenarios for the European stock market *

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Abstract

To date, most analyses on spillovers from oil to stock market have been performed considering prices in domestic currency. This assumption implies merging the commodity risk with the exchange rate risk when oil and stocks are traded in different currencies. This article proposes incorporating explicitly the exchange rate in the analysis of international oil price shocks by using the convolution concept. I apply this framework to study the low quantile response of the European stock market under an oil stress scenario, without overlooking the role of the exchange rate. I capture the dependence structure between these three variables using a vine copula approach, while a Switching Markov technique allows for structural changes in this relationship. The empirical exercise shows that the same stress scenario could generate an opposite impact depending on the source of risk. This framework can improve our understanding of how the exchange rate interacts in global markets. Also, it contributes to reduce the uncertainty about the impact of foreign shocks where the exchange rate plays a relevant role.

Keywords: *Convolution, stress test, Exchange rate, spillover analysis*
JEL code: *E30, E37, E44, G10*

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Non-technical summary

To date, spillover analyses and stress test exercises translate international prices into domestic prices to build a distress scenario, merging the exchange rate risk and the asset risk. This paper proposes a new framework to design scenarios for stress testing in international markets without overlooking the role played by the exchange rate. This study uses the convolution concept to set the source of risk which is triggering the scenario, improving the design of tailor-made scenarios.

I perform an empirical exercise to quantify the response of the European stock market to extreme movements in oil prices in euros, which depends on the variable that leads the shock. This could be, for instance, extreme movements in oil prices triggered either by trade-related issues, which concern the commodity, or by unstable exchange rates. I employ weekly data from 2000 to 2018. The time series includes several crises where the markets have experienced great oscillations. The convolution concept is combined with a copula approach and a Switching Markov technique to gather features exhibited by the sample as asymmetric relationship, tail dependence or structural changes.

Results show that the losses in the European stock market might increase up to 30% under a distress scenario for oil prices depending on the source of risk that leads this event. Untangling which variable leads the scenario could be as important as the general oil movement in terms of the response of the European stock market. Higher losses are found in bearish oil scenarios where the shock originates in the oil market and in bullish oil scenarios where the depreciation of the Euro occurs. The decrease of oil demand in economic crises and the depreciation of domestic currency, owing to political uncertainty and weak economic fundamentals, may explain these results.

The proposed approach can improve our understanding of how exchange rate movements might affect stress tests in global markets. These findings call for a careful design of stress tests, incorporating the role played by the exchange rate in international shocks into the scenario to reduce the uncertainty about the response of the domestic economy and to perform a coherent impact assessment.

1 Introduction

Stress test analyses provide a deeper understanding of the interconnections across international markets in distress scenarios. The knowledge about the behaviour of financial variables in extreme scenarios is a fundamental cornerstone to build a resilient financial system and prevent from contagion spillovers. The exchange rate acts as a primary channel through which international markets connect to domestic economies. Financial variables must be denominated in the same currency to perform a stress test analysis due to magnitude issues, reflecting the actual price paid by domestic producers and consumers. This transformation implies merging two different sources of risk which may have an opposite effects on the domestic economy, increasing model risk in the stress test design. Taking into account the interaction of the exchange rate in the shock analysis reduces the uncertainty about the response of the domestic economy while improving the design of tailor-made scenarios.[1]

I propose using the convolution concept to incorporate the role of the exchange rate when estimating the effects of a distress scenario for global markets denominated in domestic currency. In particular, I employ this framework to analyse how the response of the European stock market to an oil-related scenario in euros could change, depending on the source of risk. The response of the European stock market is evaluated looking at a quantile of its conditioned distribution, i.e. the so-called Conditional Value-at-Risk ($CoVaR$). The focus on the tail of the distribution provides two main advantages compared to other statistical measures, e.g. conditioned mean response. First, it provides a more robust estimation to outliers than mean response results. Second, a focus on low percentiles is consistent with the assumption that economic agents are risk-averse, hence they are more interested in realising how adverse the behaviour could become than in knowing how the performance may be on average.

This article considers co-movements between oil and exchange rate returns when designing the stress test scenario for the stock market. Specifically, I characterize how the probability distribution of European stock market returns, conditioned on a given movement in oil prices, depends on whether the change comes from international prices or from exchange rate fluctuations. Several studies state that oil price movements are partially due to the currency movements (Basher and Sadorsky 2006, Samii and Clemenz 1988, Zhang et al. 2008) and also that stock market swings may be caused by exchange rate movements (Dominguez and Tesar 2006, Francis et al. 2006, He and Ng 1998, Jorion 1990). Likewise, extreme movements in oil prices could trigger trade imbalances leading to adjustments in exchange rates (Golub 1983, Krugman 1983) while oil spillovers to stock markets may appear due to the change in production cost and indirect effects on inflation rates (Arouri et al. 2011, Lee et al. 2012, Ojea Ferreiro 2019). Therefore, the complex network of connections between oil, exchange rates and stock markets implies the need of considering the simultaneous dependence between them. Overlooking one of these variables from the analysis could lead to misleading conclusions on the stock market exposure due to a fuzzy transmission channel. To my knowledge, Aloui and Ben Aïssa (2016) is the only article that considers the multivariate relationship between stock market, oil and exchange rate simultaneously. They employ a vine copula approach to estimate the joint distribution between the US stock market, the US-trade weighted exchange rate and oil returns using daily data. Their results from the Bai and Perron (2003) test indicate the presence of a structural change during the 2008 financial crisis. Reboredo and Ugolini (2016) and Ojea Ferreiro (2019) also find a structural change in the bivariate relationship between the stock market and oil returns, using the Kolmogorov Smirnov test and the estimation of Switching Markov models. Wang et al. (2013) point to a structural break in the relationship between stock markets and exchange rates. I use the copula vine approach to get the multivariate joint distribution between oil, the European stock market and the USDEUR exchange rate, while a Switching Markov technique allows for structural changes in this relationship. Given a distress scenario for oil denominated in euros, the convolution concept allows us to

[1]For instance, BCBS (2013) recommends analysing the bank position on a currency-by-currency basis for stress test purposes.

consider alternative combinations of events for the USDEUR exchange rate and oil returns that might lead to such a scenario, and evaluate the stock market implications of those alternative combinations. As we will see, the source of the shock in euro-denominated oil prices strongly conditions the effects on the stock market.

I use weekly data on oil, EUROSTOXX and USDEUR exchange rate from 2000 up to 2018 to perform this empirical exercise. Results indicate that diverse compositions of the oil shock have different effects on the lower quantiles of the European stock market. On the one side, highest 10% losses for the stock market could increase under a bearish oil scenario, led by trade-related pressures, up to 20% compared to the scenario where the source of risk is unknown. On the other side, losses in the European stock market could sharply increase up to 30% conditional on a bullish oil scenario led by a depreciation of the euro, compared to the same oil-related scenario where the triggering source is undefined. The findings indicate higher losses in the Value at Risk of the EUROSTOXX when a bearish oil-related scenario materializes compared to its unconditional Value at Risk. Nevertheless, the impact of a bullish oil-related scenario on the European stock market depends on the source of risk. Stock market responses to oil shocks present higher dispersion when a bearish scenario materializes compared to a bullish scenario. Empirical evidence shows an increase in the volatility of global markets jointly with a higher degree of co-movement and tail dependence across financial variables. The study identifies these periods: firstly, before 2003 at the same time of early 2000s recession; secondly, from 2008 to 2011, coinciding with the financial crisis and the beginning of European sovereign debt crisis; lastly, between 2014 to 2016, when 2010s oil glut occurs.

These findings have implications: firstly, for risk management, investors and traders, who are interested in portfolio strategies that reduce the exposure of their stock positions to commodity and exchange rate risk; secondly, for monetary and supervisory authorities, who need to build tailor-made stress test scenarios taking into account the role played by exchange rates; thirdly, for policy makers, who wish to understand the interactions between the main variables that drive the economy. Analysing the consequences of a distress scenario for international commodities in euros, rather than in US dollars, has also implications for the stability of prices for euro area producers and consumers.

The remainder of the article is laid out as follows: Section 2 presents three parts concerning the estimation. First Subsection 2.1 presents the copula concept and introduces the idea of convolution copula. Second, Subsection 2.2 refers to the modelling choice for marginal and joint distribution, paying special attention to the time-varying structure. Third, Subsection 2.3 focuses on the conditional quantile under a distress scenario, also known as Conditional Value at Risk ($CoVaR$). Section 3 presents the data employed for the empirical exercise in Section 4. Finally, Section 5 concludes.

2 Methodology

This section is divided in three parts. First, Subsection 2.1 presents a general and brief introduction of the copula and convolution concepts. The copula methodology is the backbone to model the joint dependence. This approach provides a great flexibility to model the joint distribution between oil, stock market and exchange rate, capturing tail behaviour and asymmetric dependence. Second, Subsection 2.2 studies the structure model that better fits the data. Recent literature points to a change in the dependence between these variables over time (Reboredo and Ugolini 2016, Ojea Ferreiro 2019, Reboredo 2012, Zhu et al. 2016, Aloui et al. 2013). A Switching Markov approach helps us to identify potential structural changes in volatility and dependence. A procedure similar to the one employed by Rodriguez (2007) and Hamilton and Susmel (1994) allows us to link the marginal behaviour for each variable to potential changes in the joint dependence. This method to define time-varying features of the model is robust to misspecification issues (Manner and Reznikova 2012) while providing higher flexibility than other dynamic models (Ojea Ferreiro 2019). Finally, Subsection 2.3 introduces how the conditional quantile under a distress scenario, i.e. the Conditional Value at Risk ($CoVaR$), is built. This risk measure indicates the quantile of the variable of interest in a stress test, where the triggering event is defined by a distress scenario for another variable. This assessment translates the complex linkages and connections between variables into potential losses.

2.1 Copula and convolution copula

The copula methodology allows for modelling marginal features and joint characteristics separately, which entails higher flexibility to gather complex patterns exhibited by financial data, like asymmetric relationship, joint tail dependence and non-linearities.[2] The Sklar (1959)'s theorem states that the joint cumulative probability can be expressed as the combination of the marginal cumulative distribution function and the copula function, which gathers the dependence characteristics across variables, i.e.

$$F(x,y) = C\left(F_X(x), F_Y(y)\right), \tag{1}$$

where F_k is the marginal cumulative distribution function of variable $k = X, Y$ and $C(\dots)$ is the copula function.

The conditional copula $C_{y|x}\left(F_Y(y)|F_X(x) = x_k\right)$ expresses the conditional distribution function of a variable Y given a realization for variable X (Joe 1996). Conditional copulas are essential for the simulation process and for the construction of complex models, such as vine copulas. The conditional copula is the results of the partial derivative of the copula function with respect to one of its input factors, i.e.

$$\begin{aligned} F(y|X=x) &= C_{y|x}\left(u_y|u_x\right) \\ &= \frac{\partial C\left(u_x, u_y\right)}{\partial u_x}, \end{aligned} \tag{2}$$

where $u_x = F_X(x)$ and $u_y = F_Y(y)$.

The concept of copula convolution (C-convolution) appears when the interest of the analysis lies in the distribution of a variable $Z = X + Y$, where X and Y are not independent (Cherubini et al. 2004). The distribution of Z in terms of the joint distribution of X and Y is

$$F_Z(z) = \int_0^1 C_{y|x}\left(F_Y\left(z - F_X^{-1}(u)\right)|u\right) du, \tag{3}$$

where marginal characteristics and dependence features are combined to get the distribution of $Z = X+Y$. We can use $F_Y \overset{C}{*} F_X$ to express that the distribution of variable Z (F_Z) is the results of the convolution of the distributions of X (F_X) and Y (F_Y).

Cherubini et al. (2004) show that the C-convolution is closed with respect to mixtures of copula functions. If $C(u_x, u_y) = \pi A(u_x, u_y) + (1-\pi)B(u_x, u_y)$ where A, B are copula functions and $\pi \in [0,1]$, then

$$\begin{aligned} F_Y \overset{C}{*} F_X &= F_Y \overset{\pi A + (1-\pi)B}{*} F_X \\ &= \pi F_Y \overset{A}{*} F_X + (1-\pi) F_Y \overset{B}{*} F_X. \end{aligned} \tag{4}$$

[2]See, for instance, Joe et al. (2010), Nikoloulopoulos et al. (2012), Kim et al. (2013).

The implications for modelling the time-varying dependence given by a Switching Markov process are direct. The copula and the marginal distributions functions in Equation (3) are assumed to be absolutely continuous, so the probability density function of variable $Z = X + Y$ is

$$f_Z(z) \quad = \quad \int_0^1 c_{X,Y}\left(u, F_Y\left(z - F_X^{-1}(u)\right) f_y\left(z - F_X^{-1}(u)\right)\right) du, \tag{5}$$

where f_y refers to the probability density function of variable Y and $c_{X,Y}(\dots)$ is the density copula between X and Y, i.e. the derivative of the copula function with respect to all its inputs.

The oil log-return denominated in euros is the sum of the logarithmic change of the oil denominated in US dollars and the logarithmic change in the exchange rate USDEUR[3]. Hence, the financial variable of oil denominated in euros is the result of the convolution of two stochastic processes. The goal of this article is to assess the impact of a distress scenario for oil in euros on the European stock market depending on which variable of the convolution leads the shock, i.e. commodity shock or exchange rate shock.

2.2 Model and estimation

This section is divided into two stages. First, I present the marginal model and the dependence structure across variables. Then, in a second stage, the focus is on the estimation process. The marginal model takes into account a possible switch in the market stability, using a $SWARCH$ model to gather potential structural breaks, i.e. the transition probability to move between a tranquil and a distress state is the same for all the assets but their parameters are not. This assumption is supported by the figures provided in Section 3, where a simple rolling windows approach shows an increase in volatility between 2009 and 2014 for all the assets, while their correlation drastically changed. I impose a two-state model, which keeps the model tractable and makes easier the interpretation of the state. Changes in dependence across variables would happen together with volatility switches in the marginal models. The high-volatility state could be seen as an instability period for trade, which would lead to a change in the relationship between markets. This way of linking the states between the marginal distributions and the dependence structure would allow us to reduce significantly the numbers of parameters providing a parsimonious model, making easier the estimation of a high-dimensional model. Furthermore, there is evidence in literature regarding the link between the change between low-volatile periods and high-volatile periods and the shift in dependence across assets. Edwards and Susmel (2001) find evidence of volatility co-movements across Latin American countries, Boyer et al. (2006) link high-volatility periods to an increase in co-movement across markets and Baele (2005) indicates a contagion effect between US market and European equity indices during high-volatility periods.

2.2.1 Joint model

Marginal model. The aim of this section is to select a parsimonious representation for the model, allowing for changes across possible regimes while keeping the model tractable. A specification in which all the parameters change with each regime would be numerically unwieldy and over-parametrized. I consider potential structural changes in key parameters for the marginal distribution, i.e. changes in variance, which would be related to changes in dependence.
I characterise the marginal densities of the stock (s), oil (o) and exchange rate (c) returns by an $ARMA(p,q)$ model, i.e.

$$r_{k,t} = \underbrace{\phi_{k,0} + \sum_{j=1}^{p} \phi_{k,j} r_{k,t-j} + \sum_{i=1}^{q} \psi_{k,i}\, \epsilon_{k,t-i}}_{\mu_{k,t}} + \epsilon_{k,t}, \quad k = s, o, c \tag{6}$$

[3]The euro is the quote currency and the US dollar is the base currency.

where p and q are non-negative integers, $\phi_{k,j}$ and $\psi_{k,i}$ are respectively the autoregressive (AR) and the moving average (MA) parameters and $\epsilon_{k,t} = \sigma_{k,t} z_{k,t}$. $z_{k,t}$ is a Gaussian variable with zero mean and unit variance, i.e. the probability density function of $z_{k,t}$ is

$$f(z_{k,t}) = \frac{1}{\sqrt{2\pi}} \exp(-z_{k,t}^2/2). \tag{7}$$

The variance of $\epsilon_{k,t}$ has dynamics given by a Markov Switching Autoregressive Conditional Heteroskedasticity model ($SWARCH(K,Q)$)[4]. The presence of structural breaks in variance might explain the high persistence found in ARCH models (Lamoureux and Lastrapes 1990, Hwang and Valls Pereira 2008). The structural changes during the estimation period might explain also the kurtosis presented in the financial returns (Leon Li and Lin 2004). I employ the model specification by Hamilton and Susmel (1994) where the variance of $\epsilon_{k,t}$ can be divided into two components, i.e.

$$\sigma_{k,t}^2 = \kappa_{k,s_t} h_{k,t}, \tag{8}$$

where κ_{k,s_t} is a scale parameter of the variance depending on the state at time t. $s_t = l$ refers to the regime l at time t where $l = 1, \dots K$. The regimes are not directly observable but the probability of being on them can be implicitly estimated. The probability of switching across regimes evolves according to a first order Markov Chain of size K where K represents the number of states or regimes. κ_{k,s_t} is normalized at unity at state 1 ($s_t = 1$) while for the remainder states is higher than one. $h_{k,t}$ follows a ARCH(q) process, i.e.

$$h_{k,t} = \alpha_{k,0} + \sum_{q=1}^{Q} \alpha_{k,q} \left(\frac{\epsilon_{k,t-q}^2}{\kappa_{k,s_{t-q}}} \right) \tag{9}$$

where $\alpha_{k,0}$ and $\alpha_{k,q}$ are the ARCH parameters, which must be higher than zero. Note that when $s_t = 1$, $\kappa_{k,s_t} = 1 \quad \forall k$, i.e. the combination of a low-volatility regime in one market and a distress state in another market is not allowed.

I assume two states to keep the model tractable, i.e. $K = 2$, while $Q = 1$ so a $SWARCH(2,1)$ is employed to model the variance of the financial returns. It is worth noting that there are $K(Q+1)$ potential realizations of the variance at time t, because Equation (9) depends on the Q most recent $\epsilon_{k,t-q}^2$ standardized by $\kappa_{k,s_{t-q}}$ for $q = 1, \dots, Q$. Each state of the Switching Markov process has an economic interpretation. State 1 indicates a period of low volatility, which can be linked to tranquil periods. On the other side, State 2 presents a high-volatility period, where there is uncertainty about the future performance of assets. The uncertainty would lead to a change in the relationship between the variables, i.e. co-movement in distress periods would present stronger tail dependence due to contagion across assets, while in tranquil times the relationship might be diverse (Reboredo (2012), Ojea Ferreiro (2019)).

Dependence structure. Complex multivariate data can be modelled using bivariate copula in a hierarchical way like bricks of a more elaborate building. The graphical representation of these constructions are the vines. Depending on the pair-copula decomposition we could talk about Canonical vine copulas (C-Vine) or Drawable vine copulas (D-Vine) . C-Vine copulas have a star structure while D-Vine copulas have a path structure. Figure 1 represents the graph-based tree structure of the copula decomposition of three assets (1, 2 and 3). The left figure shows the construction under a C-Vine copula while the right figure represents a D-Vine copula structure. As a matter of fact, in a three-dimensional case the copula decomposition is both a C-Vine and D-Vine. Note that the tree under the left copula structure is equivalent to the right panel in Figure 1.

I start modelling the joint dependence as a truncated vine, assuming that the joint dependence could be

[4] K refers to the number of states and Q indicates the lags of the $ARCH(Q)$ model.

explained through a common exposure to the exchange rate. This structure for the vine copula is based on the key role that the exchange market plays between the stock market and the international commodity market. Indeed, an exchange market is a *conditio sine qua non* for the stability in international trade markets and the economic growth in stock markets. Oil and stock returns are assumed conditionally independent once the dependence through exchange rate is taken into account. Following Figure 1, this assumption implies that the link in T_2 step does not exist. In a second stage this assumption is relaxed, studying the complete vine structure as a natural extension of the truncated vine approach. This is the expected way to study the relationship because the structure chosen in the T_2 step depends on the structure in the T_1 step.

Let us consider a three dimensions vector with joint distribution $F(x_1, x_2, x_3)$ to motivate how to model the multivariate structure. The Sklar (1959)'s theorem from Equation (1) can be rewritten in a three-dimension space as

$$F(x_1, x_2, x_3) = C\left(F(x_1), F(x_2), F(x_3)\right), \tag{10}$$

where subscripts of the cumulative distribution functions were omitted to avoid cumbersome notation. The joint density function expressed in terms of copulas and marginal densities is

$$f(x_1, x_2, x_3) = c\left(F(x_1), F(x_2), F(x_3)\right) f(x_1) f(x_2) f(x_3), \tag{11}$$

where factorizing recursively we obtain

$$f(x_1, x_2, x_3) = f(x_1) f(x_2|x_1) f(x_3|x_1, x_2), \tag{12}$$

where the subscripts of density functions were also omitted. Equation (12) can be rewritten using Bayes' theorem as

$$\begin{aligned} f(x_1, x_2, x_3) &= f(x_1) \frac{f(x_2, x_1)}{f(x_1)} \frac{f(x_3, x_2|x_1)}{f(x_2)} \\ &= f(x_1) c\left(F(x_2), F(x_1)\right) f(x_2) c\left(F(x_3|x_1), F(x_2|x_1)\right) f(x_3|x_1), \end{aligned} \tag{13}$$

where $f(x_3|x_1) = \frac{f(x_3,x_1)}{f(x_1)}$, which in terms of copulas is $f(x_3|x_1) = c\left(F(x_3), F(x_1)\right) f(x_3)$. Joe (1996) demonstrates that $F(x_j|x_k) = P(X_j < x_j|X_k = x_k)$ for $j, k = 1, 2, 3 \quad j \neq k$ is expressed by the conditional copula, i.e. $C\left(F(x_j)|F(x_k)\right) = \frac{\partial C(F(x_j),F(x_k))}{\partial F(x_k)}$.
To sum up, the joint density distribution under the vine approach can be expressed as

$$f(x_1, x_2, x_3) = \underbrace{c\left(F(x_2), F(x_1)\right) c\left(F(x_3), F(x_1)\right) c\left(F(x_3|x_1), F(x_2|x_1)\right)}_{c(F(x_1),F(x_2),F(x_3))} f(x_1) f(x_2) f(x_3) \tag{14}$$

In the current study, x_1 represents the returns of the exchange rate $USDEUR$ (r_c), while x_2 and x_3 represent oil and stock returns respectively (r_o, r_s). Observe that $c\left(F(x_3|x_1), F(x_2|x_1)\right) = 1$ in the case of a truncated vine approach. I choose between a set of copulas that present different features in terms of tail dependence, i.e. the probability of having very extreme realizations for one market given very extreme realizations for another market. Gaussian copula does not present tail dependence but it allows for positive and negative association, Student t copula also allows for positive and negative association but it presents symmetric tail dependence. Gumbel and Clayton copulas allow only for positive asymmetric association, while Clayton copula has lower tail dependence, Gumbel copula has upper tail dependence. The 90 degrees rotated version of Clayton and Gumbel allows for gathering negative association and asymmetric tail dependence. Further information about these copulas is provided in Appendix D.
I use graphical tools as bivariate histograms and analytical tools as the Akaike Information Criterion Corrected for small-sample bias ($AICC$) to choose a suitable copula structure that fits the true data dependence. $AICC$ is chosen because of being the principal indicator for selection copulas in the conditional

risk measure literature[5], i.e

$$AICC = 2k\frac{T}{T-k-1} - 2\log(\hat{L}),$$

where T is the sample size, k is the number of estimated parameters and $\hat{L}$ is the Log-likelihood value. Minimum $AICC$ value indicates the best copula fit.

2.2.2 Switching Markov and estimation procedure

This subsection presents first the Switching Markov specification that rules a shift in the variance of each variable and dependence across them. Then in a second stage, I propose to use a EM algorithm (Hamilton 1990) for the estimation process, which allows for decompositing the optimization problem in a set of simpler problems where the transition probability of the Markov Chain and the parameters within each regime are not estimated at the same time. The EM algorithm simplifies the computational challenge of maximizing numerically an likelihood surface plagued with multiple local optimum as happens in switching models.

Switching Markov specification Let us define Ψ as a vector 2x2 that gathers the conditional joint density function of $r_{o,t}$, $r_{c,t}$, $r_{s,t}$ given by a low-volatile or high-volatile regime at t and $t-1$, where the relationship across variables might change, i.e.

$$\Psi = \begin{bmatrix} f\left(r_{o,t}, r_{c,t}, r_{s,t}; \Theta_{s_t=1,s_{t-1}=1}\right) & f\left(r_{o,t}, r_{c,t}, r_{s,t}; \Theta_{s_t=1,s_{t-1}=2}\right) \\ f\left(r_{o,t}, r_{c,t}, r_{s,t}; \Theta_{s_t=2,s_{t-1}=1}\right) & f\left(r_{o,t}, r_{c,t}, r_{s,t}; \Theta_{s_t=2,s_{t-1}=2}\right) \end{bmatrix}, \tag{15}$$

where $\Theta_{s_t,s_{t-1}}$ is the vector of parameters under the regime s_t at time t and regime s_{t-1} at time $t-1$. Note that s_{t-1} is only considered for the variance given by the $SWARCH(2,1)$, while the dependence across variables only depends on the current state s_t.
The conditional densities depend only on the current regime s_t and the previous regime s_{t-1}, i.e.

$$f\left(r_{o,t}, r_{c,t}, r_{s,t}; I_{t-1}, s_t = j, s_{t-1} = i; \Theta_{s_t,s_{t-1}}\right) = f\left(r_{o,t}, r_{c,t}, r_{s,t}; I_{t-1}, s_t = j, s_{t-1} = i, s_{t-2} = k ...; \Theta_{s_t,s_{t-1}}\right),$$

for $i, j = 1, 2$ and I_{t-1} refers to the information set at $t-1$. I assume that the evolution of s_t follows a first order Markov chain independent from past observations, i.e.

$$p_{ij} = P(s_t = j|s_{t-1} = i) = P(s_t = i|s_{t-1} = j, s_{t-2} = k, I_{t-1}), \tag{16}$$

for $i, j, k = 1, 2$.
The transition matrix defined by the Markov Chain is

$$P = \begin{bmatrix} p_{11} & 1-p_{22} \\ 1-p_{11} & p_{22} \end{bmatrix}, \tag{17}$$

where each column i indicates the probability of remaining on the state i (p_{ii}) or moving to state j (p_{ij}) conditioned to the fact that we are currently at state i for $i, j = 1, 2$ and $i \neq j$. Obviously, $p_{ii} + p_{ij} = 1$ because only two states exist. That is the reason why p_{ij} is presented as $1 - p_{ii}$.
Let us assume that the set of parameters Θ are known. Let us gather the probability assigned to the observation at time t of being the result of regime j, i.e. $P(s_t = j|I_t; \Theta)$, in a vector $\hat{\xi}_{t|t}$,

$$\hat{\xi}_{t|t} = \left[P(s_t = 1|I_t; \Theta), P(s_t = 2|I_t; \Theta)\right]'.$$

[5]Among others Brechmann and Schepsmeier (2013), Reboredo and Ugolini (2015a), Reboredo and Ugolini (2015b), Reboredo and Ugolini (2016), Rodriguez (2007), Reboredo (2011) and Ojea Ferreiro (2018)

$\hat{\xi}_{t|t}$ comprises the inference about the regime at time t given the information available at that period. The probability assigned to the observation at time $t+1$ of being the result of regime j given the information at time t is collected in vector $\hat{\xi}_{t+1|t}$,

$$\hat{\xi}_{t+1|t} = [P(s_{t+1} = 1|I_t;\theta), P(s_{t+1} = 2|I_t;\theta)]'.$$

$\hat{\xi}_{t+1|t}$ is the probability forecast of being in the next period $t+1$ at each regime given the information available at t. The forecast probability for the next period is obtained as the product of the inference probability by the transition matrix, i.e.

$$\hat{\xi}_{t+1|t} = P\hat{\xi}_{t|t}.$$

The link between $\hat{\xi}_{t|t}$ and $\hat{\xi}_{t+1|t}$ is obtained by the updated probabilities, including the new available information through Bayes' theorem, i.e.

$$P(s_t = j|I_t;\Theta) = \frac{\sum_{i=1}^{2} P(s_t = j, s_{t-1} = i|I_{t-1};\Theta) f(r_{o,t}, r_{c,t}, r_{s,t}|I_{t-1};\Theta_{s_t=j,s_{t-1}=i})}{L_t(r_{o,t}, r_{c,t}, r_{s,t}; I_{t-1},\Theta)}, \quad (18)$$

where $P(s_t = j, s_{t-1} = i|I_{t-1};\Theta) = P(s_{t-1} = i|I_{t-1};\Theta)p_{ij}$ and $L_t(r_{o,t}, r_{c,t}, r_{s,t}; I_{t-1},\Theta)$ is the likelihood function at time t. To get the likelihood at time t we have to assess the sum of the product of the joint density conditioned to the occurrence of each possible set of states at t and $t-1$ by their probability given the information set at $t-1$, i.e.

$$L_t(r_{o,t}, r_{s,t}, r_{c,t}; I_{t-1},\Theta_t) = \sum_{j=1}^{2}\sum_{i=1}^{2} f(r_{o,t}, r_{s,t}, r_{c,t}|\Theta_{s_t=j,s_{t-1}=i}, I_{t-1})P(s_t = j, s_{t-1} = i|I_{t-1}), \quad (19)$$

where $\Theta_{s_t=j,s_{t-1}=i}$ stands for the set of parameters of the joint distribution at regime j at time t and regime i at time $t-1$. Rewriting Equation (18), that connects $\hat{\xi}_{t|t}$ and $\hat{\xi}_{t+1|t}$, in a matrix form

$$\hat{\xi}_{t|t} = \frac{(P \odot [\hat{\xi}_{t-1|t-1}, \hat{\xi}_{t-1|t-1}] \odot \Psi)' 1_2}{1_2'\{(P \odot [\hat{\xi}_{t-1|t-1}, \hat{\xi}_{t-1|t-1}] \odot \Psi)' 1_2\}},$$

where Ψ was defined in Equation (15) while $\odot$ represent the element-wise product.
To start the iteration we need a value for $\hat{\xi}_{1|0}$, for which I use the unconditional probabilities of each state that can be expressed in a matrix form as

$$\hat{\xi}_{1|0} = \pi = (A'A)^{-1}A'(0,0,1)'$$

where

$$A = \begin{bmatrix} I_2 - P \\ 1_2' \end{bmatrix} = \begin{bmatrix} 1-p_{11} & p_{22}-1 \\ p_{11}-1 & 1-p_{22} \\ 1 & 1 \end{bmatrix}.$$

and I_N is the identity matrix of size NxN and 1_N is a $(Nx1)$ vector of ones. To finish this subsection I present the Kim (1994)'s algorithm for smoothed inferences, which are used to present the probabilities of being in each state at each time t given the complete information of the sample T, i.e.

$$\hat{\xi}_{t|T} = \hat{\xi}_{t|t} \odot \left\{P'\left[\hat{\xi}_{t+1|T}(\div)\hat{\xi}_{t+1|t}\right]\right\},$$

where $\odot$ and $(\div)$ represent the element-wise product and division respectively. Taking into account that current set of parameters depends on the state at t and $t-1$, we can rewrite previous equation as

$$\hat{\xi}_{t|T} = 1_2' \underbrace{\left\{[\hat{\xi}_{t|t}, \hat{\xi}_{t|t}]' \odot P \odot \left[\hat{\xi}_{t+1|T}(\div)\hat{\xi}_{t+1|t}, \hat{\xi}_{t+1|T}(\div)\hat{\xi}_{t+1|t}\right]\right\}}_{\xi_{t+1,t|T}},$$

where $\xi_{t+1,t|T} = \begin{bmatrix} P(s_{t+1}=1, s_t=1|I_T;\Theta) & P(s_{t+1}=1, s_t=2|I_T;\Theta) \\ P(s_{t+1}=2, s_t=1|I_T;\Theta) & P(s_{t+1}=2, s_t=2|I_T;\Theta) \end{bmatrix}$.
The smoothed probability of being in state j at t and in state i at $t-1$ is

$$P(s_t = j, s_{t-1} = i|I_T;\Theta) = \frac{P(s_t = j|I_T;\Theta)}{P(s_t = j|I_{t-1};\Theta)} p_{ij} P(s_{t-1} = i|I_{t-1};\Theta), \tag{20}$$

for $t > 1$.

Estimation procedure. I employ the EM algorithm, proposed by Hamilton (1990), to obtain the maximum likelihood estimates for our model, which are subject to a discrete shift. There are several reasons that motivate the use of EM algorithm instead of using the full maximum likelihood estimation. First, the maximization of a likelihood function with respect to a great number of unknown parameters implies a computational challenge due to the possible existence of multiple local optimum, specially in switching models. Second, It provides numerical robustness over other methods of optimization like Newton-Raphson where, if the likelihood surface is not concave, might arrive to a local maxima/minima (Dempster et al. 1977). The EM algorithm is numerically stable as the result of dividing the optimization problem into a sequence of simpler optimization problems where the probabilities of switching between regimes and the estimates within each regime are not jointly estimated. I use a large number of starting values for the EM algorithm to ensure an improvement in efficiency. The EM algorithm has been employed already in copula-based models with Switching Markov dynamics by Stöber and Czado (2014) and Chollete et al. (2009).

To implement the EM algorithm, first compute the smoothed probabilities (Expectation step or $E-step$) as shown by Kim (1994)'s algorithm. Then, employ these probabilities to reweigh the observed data and maximize the reweighed log-likelihood to generate new estimates (Maximization step or $M-step$). Employ the new estimates to reassess the smoothed probabilities in an iterative process. The EM algorithm is an analytic solution to a sequence of optimization problems, where the solution in the $n+1$ iteration increases the value of the log-likelihood function in relation to the estimates in the n iteration, achieving in the limit a optimum of the log-likelihood function.[6]

Steps to perform the EM algorithm

- $E-Step$: Inference the expected values of the state process given the observation vector, i.e. assess the conditional probabilities for the process being in a certain regimen at time t and $t-1$ given the full sample. Equation (20) provides

$$P(s_t = j, s_{t-1} = i|I_T) \quad \text{for} \ \ i, j = 1, 2$$

- $M-step$: Maximize the expected log-likelihood function using the smoothed probabilities to obtain new and more exact ML estimates, i.e. instead of maximize $\sum_{t=1}^{T} log\left(L_t(r_{o,t}, r_{c,t}, r_{s,t}; I_{t-1}, \Theta)\right)$, we maximize

$$\sum_{t=1}^{T}\sum_{j=1}^{2}\sum_{i=1}^{2} log\left(f(r_{o,t}, r_{s,t}, r_{c,t}|\Theta_{s_t=j,s_{t-1}=i})\right) P(s_t = j, s_{t-1} = i|I_T),$$

  where $P(s_t = j, s_{t-1} = i|I_T)$ was obtained in the previous step. Notice that we are maximizing the expected conditional log-likelihood, but not the log-likelihood. We use the new estimates to update

[6]Alternatively, we can see the new estimates in the following iteration of the EM algorithm as the results of the sum of the weighted conditions over all possible states. In other words, the EM algorithm "replaces" the unobserved scores by their expectation given the estimated parameter vector in the previous iteration.

the smooth probabilities and the expected conditional log-likelihood to be maximized, we repeat the iterative algorithm until some convergence criteria are meet, e.g. in terms of the new estimates

$$|\Theta^{n+1} - \Theta^{n}| < \varepsilon,$$

where ε has a small value, e.g. $\varepsilon = 10^{-4}$.

The EM algorithm prevents from estimating at the same time the parameters within each state and the transition matrix between states, which simplifies the maximization problem. Reparametrizations are used to guarantee that all iterates are in the parameter space. For instance instead of looking for values of $\kappa_{k,s_t=2}$, I obtain the optimal estimate for a parameter x such that $exp(x) + 1 = \kappa_{k,s_t=2}$. Hamilton and Susmel (1994) also employ this kind of transformations to estimate the parameters of its $SWARCH$ model. The transition probabilities between states for iteration n are obtained from

$$p_{ij}^{n} = \frac{\sum_{t=2}^{T} P(s_t = j, s_{t-1} = i | I_T; \Theta^{n-1})}{\sum_{t=2}^{T} P(s_{t-1} | I_T; \Theta^{n-1})}, \tag{21}$$

Further information regarding the EM algorithm for Switching Markov models can be found in Hamilton (1990) and Janczura and Weron (2012) among others.

2.3 Untangling the oil shock to the European stock market into commodity and exchange rate risk

The actual oil price that European firms have to cope with is the product of the oil price in USD by the exchange rate $USDEUR$[7]. The actual exposure to swings in oil prices is the sum of the logarithmic changes in oil and in the exchange rate. The convolution of the distribution of oil and exchange rate log-returns is the distribution of the oil log-returns denominated in euros. Ojea Ferreiro (2019) analyses the impact of a oil shock denominated in euros into an extreme quantile of the European stock market using the Conditional Value-at-Risk ($CoVaR$) (Adrian and Brunnermeier (2016), Girardi and Ergün (2013)). The $CoVaR$ measure indicates a quantile of the European stock market returns given a sharp change in oil prices. The change in oil prices denominated in euros (r_{oe}) may come from different sources, i.e. commodity risk, exchange rate risk or a combination of both. Following Ojea Ferreiro (2019), the bearish $CoVaR_{s|oe}(\alpha, \beta)$ of the stock returns would be obtained implicitly from

$$\begin{aligned} P\left(r_s < CoVaR_{s|oe} | r_{oe} < VaR_{oe}(\alpha)\right) &= \frac{P\left(r_s < CoVaR_{s|oe}, r_{oe} < VaR_{oe}(\alpha)\right)}{P(r_{oe} < VaR_{oe}(\alpha))} \\ &= \beta, \end{aligned} \tag{22}$$

where $P(r_{oe} < VaR_{oe}(\alpha)) = \alpha$.
Following Equation (3), $r_{oe,t}^{*} = VaR_{oe,t}(\alpha)$ is obtained from

$$\begin{aligned} F_{oe,t}(r_{oe,t}^{*}) &= \int_0^1 C_{o|c,t}\left(F_{o,t}\left(r_{oe,t}^{*} - F_c^{-1}(u)\right) | u\right) du \\ &= \alpha. \end{aligned} \tag{23}$$

We have infinitive combinations of exchange rate returns and oil returns denominated in US dollars such that

$$r_{oe}^{*} = r_c + r_o,$$

[7] Note that $USDEUR$ indicates how many euros are exchanged by one US dollar.

but notice that not all the combinations are equally probable[8] nor their implications for the conditional distribution of stock returns would be the same. Given a quantile q_c of the distribution of the exchange rate returns, there is an unique quantile q_o of the oil returns in US dollars such that $VaR_{oe}(\alpha) = F_c^{-1}(q_c) + F_o^{-1}(q_o)$. Actually, conditioning to the oil returns in euros being below a quantile α and the exchange rate $USDEUR$ being below a percentile q_c is the same than conditioning to the exchange rate returns being below a quantile q_c and to the oil denominated in US dollars such that its convolution would be below the quantile α, i.e.

$$r_{oe}^* \geq F_c^{-1}(q_c) + r_o,$$

hence oil returns denominated in dollars should be below

$$r_o \leq r_{oe}^* - F_c^{-1}(q_c)$$

which in terms of quantiles would be

$$\begin{aligned} P\left(r_o \leq r_{oe}^* - F_c^{-1}(q_c)\right) &= F_o(r_{oe}^* - F_c^{-1}(q_c)) \\ &= q_o. \end{aligned} \tag{24}$$

Consequently, a different response of the stock market returns might occur given the same scenario for oil returns in euros but different distress in the exchange rate returns. Incorporating the role of the exchange rate in the oil-related scenario helps us to generate tailor-made stress test where the distress in global market is tangled with the evolution of exchange markets.

$CoVaR_{s|oe}(\alpha, \beta)$ in Equation (22) transforms into $CoVaR_{s|oe,c}(\alpha, q_c, \beta)$ when the exchange rate is also considered in the scenario, getting

$$\begin{aligned} P\left(r_s < CoVaR_{s|oe} | r_{oe} < VaR_{oe}(\alpha), r_c < VaR_c(q_c)\right) &= \frac{P\left(r_s < CoVaR_{s|oe,c}, r_o < VaR_{oe}(\alpha), r_c < VaR_c(q)\right)}{P(r_{oe} < VaR_{oe}(\alpha), r_c < VaR_c(q_c))} \\ &= \beta, \end{aligned}$$

Equation (24) implies an equivalence between $CoVaR_{s|oe,c}(\alpha, q_c, \beta)$ and $CoVaR_{s|o,c}(q_o, q_c, \beta)$. Using this equivalence we can obtain at each time t the upper threshold of the quantile of the oil returns in US dollars such that for a certain upper threshold of the quantile of the exchange rate, the sum of returns is at or below the quantile α of the oil denominated in euros. Setting a scenario for the exchange rate to compute $CoVaR$ provides useful information about the magnitude of the impact assessment depending on the source of risk that triggers the movement of oil prices.

Vine structure We could express the $CoVaR_{s|oe,c}(\alpha, q_c, \beta)$ given the chosen vine structure as

$$\frac{\int_0^{q_c} C_{s,o|c}\left(C_{s|c}(F_s(CoVaR_{s|oe,c})|u), C_{o|c}(q_o|u)\right) du}{C_{o,c}(q_o, q_c)} = \beta. \tag{25}$$

where q_o is given by Equation (24). To compared these results with the one obtained without any information about the exchange market, we combine Equation (3) and Equation (25) to get

$$\begin{aligned} CoVaR_{s|oe}(\alpha, \beta) &= \frac{\int_0^{1} C_{s,o|c}\left(C_{s|c}(F_s(CoVaR_{s|oe})|u), C_{o|c}(F_o\left(VaR_{oe}(\alpha) - F_c^{-1}(u)\right)|u)\right) du}{\alpha} \\ &= \beta, \end{aligned} \tag{26}$$

where $VaR_{oe}(\alpha)$ is obtained from the convolution of the exchange rate and the oil in USD following Equation (23). Appendix E provides information about how to build the $CoVaR$ measure using copulas conditioned to a bullish oil-related scenario.

[8]This would be only in the case of independent variables.

3 Data

I employ weekly data of the European stock market, the $USDEUR$ exchange rate and oil prices from 07 January 2000 to 07 September 2018. I obtain weekly returns from the log difference between two consecutive Fridays. The time series includes several crises during this period, e.g. the dot-com crisis, the 2008 financial crisis and the European debt crisis, where both oil prices and exchange rates experienced great oscillations.
Concerning commodity prices, I use the Europe Brent crude oil spot price sourced from the US Energy Information Agency (http://www.eia.doe.gov), which is the main benchmark to settle the price of light crudes. Brent crude oil is denominated in US dollars per barrel. The $USDEUR$ exchange rate is obtained from the European Central Bank Statistical Data Warehouse (https://sdw.ecb.europa.eu). Regarding the European stock market, I employ the EUROSTOXX index from Datastream.

Table 2 shows some descriptive statistics for the full sample and two sub-sample that correspond to the pre-crisis and the post-crisis periods. It indicates a clear change in higher moments, i.e. skewness and kurtosis, and in the relationship between variables. Figure 2 shows the evolution of correlation and volatility using a rolling window approach with a five-year length. The charts depict two set of evidence. First, there is a general shift in correlation across the variables between the period $2009-2014$. Second, this period coincides with a general change in the volatility level of those markets. This evidence indicates that a Switching Markov model, where variance and dependence move together across regimes, could explain the dynamic shown by the data. Figure 3 indicates an excess of kurtosis and the presence of left skewness in the histogram, which could be explained by a discrete switch in variance (Leon Li and Lin 2004).

[Insert Table 2 here]
[Insert Figure 2 here]
[Insert Figure 3 here]

4 Results

This section presents the results obtained from the estimation process in a first stage. The implications of the stress test for the conditional quantile of the Eurostoxx are analysed in a second stage.

4.1 Model estimates

Regarding the model choice, this section goes from the simplest model to the most sophisticated one. Simplest model, i.e. a truncated vine structure using Gaussian copulas, provides useful information concerning the data fit to implement further improvements. Performing a likelihood ratio test against the Student copula provides essential information regarding the importance of tail dependence in the model structure. The analysis using graphical tools help us to infer the actual dependence between the percentiles of the variables given by their estimated marginal distributions. The analysis obtained from the simplest model would point to a truncated model where the dependence could be different between states. This intermediate model, where the truncated vine structure could be non-elliptic and different between states, is the cornerstone to build more complex structures. Indeed, following Figure 1, the copula choice in step T_2 depends on the copula choice in step T_1, i.e. the truncated vine. The analysis of the conditional distribution of oil and stock returns given the exchange rate returns would give us an idea about the dependence between oil and stock returns once considered a common exposure to the exchange rate. This analysis would lead to the last model, the most complete one, to get a comprehensive idea about the links between these three key variables in the economy.

4.1.1 Simplest model: truncated vine structure using elliptical copulas

I present first the results for the elliptical models. Table 3 reports the estimate of the model, where the exchange rate is linked to oil in USD and EUROSTOXX, but EUROSTOXX and oil in USD are not directly connected, i.e. a truncated vine structure. Left table presents the results under Gaussian assumptions while right table shows the estimates under Student copula. The link between EUROSTOXX and the $USDEUR$ exchange rate is quite weak, $|\rho_{2,3}| < 0.1$, while the relationship between Oil in USD and $USDEUR$ is statistically significant and negative in both regimes. Hence, there is a link between the increases in oil prices and the appreciation of the euro against the US dollar. The table also shows the likelihood ratio statistic between the Student model and the Gaussian model. Its p-value is lower than 5%, indicating the importance of the tail dependence to explain the relationship between the set of variables.

Figure 4 presents the histogram and the likelihood under the Gaussian distribution where the variance within each state might differ following the $SWARCH$ model. The excess of kurtosis in the Gaussian distribution could be explained by a realization from a Gaussian distribution with higher variance. This feature of $SWARCH$ models was already underscored by Leon Li and Lin (2004).

Figure 5 presents the unconditional coverage backtesting test proposed by Kupiec (1995). The x-axis shows different quantiles of the marginal distribution chosen as threshold to count exceedances. The right axis presents the p-value where the null hypothesis is that $\alpha 100\%$ of the sample is below the threshold shown by the $VaR(\alpha)$. Left axis indicates the number of exceedances. Black line presents the current number of exceedances while the red lines are the bounds at 10%, 5% and 1% under the null hypothesis. These charts help us to check how well the model suits the data. The subgraph related to oil returns indicates that our model presents less outliers than observed in the data for quantiles between 0.45 and 0.15, but the model fits well the tail below quantile 0.15. On the other side, the model fits well EUROSTOXX distribution above quantile 0.05. The $USDEUR$ returns is fitted well by our model, even for extreme quantiles the p-value is higher than 0.05.

Figure 6 presents the conditional coverage backtesting test proposed by Christoffersen (1998), where the null hypothesis is that VaR violations are independent while the alternative hypothesis is that VaR violations follows a first order Markov Chain. Right axis shows the p-value of the Christoffersen (1998)'s test while left axis presents the number of exceedances. Left axis presents the number of observation. Red solid line presents the number of observations without exceedances at t and $t-1$. Red dashed line shows the number of pairwise observations where we have an exceedance at t but not at $t-1$ while the black dotted line shows the opposite. Red dotted line shows the number of pairwise observations with two consecutive exceedances. The p-value is higher than 0.10 for most of the quantiles. Hence, there is no evidence of a clustering of exceedances.

Finally, Figure 7 presents the bivariate histogram between oil in USD and $USDEUR$ returns (top figures) and the bivariate histogram between $USDEUR$-EUROSTOXX (bottom figures). The probability integral transform is chosen from state j if the smoothed probability of being at regime j is higher than 90% where $j = 1, 2$. The oil in USD - $USDEUR$ relationship presents a cluster of data in high quantiles of oil returns and to a lesser extent in the opposite tail. These features could be explained by a Student or a 90° rotated Clayton. The oil in USD-$USDEUR$ link shows some degree of higher dependence in high quantiles of exchange rate and low quantiles of oil returns under state 2. Gaussian copula or a 90° rotated Gumbel might fit well the data as potential copulas. The $USDEUR$-EUROSTOXX link is quite homogeneously distributed under state 1, so a Gaussian or independent copula could match the data, while there is a higher dependence in high quantiles of exchange rates returns and low quantiles of EUROSTOXX returns under state 2, which could be consistent with a 90° Gumbel copula. These potential copulas are analysed and compared in the next subsection.

[Insert Table 3 here]
[Insert Figure 4 here]
[Insert Figure 5 here]
[Insert Figure 6 here]
[Insert Figure 7 here]

4.1.2 Intermediate model: truncated vine structure

Table 4 shows the AICC values for the potential copulas indicated by Figure 7. Lowest value indicates the best copula fit for the truncated vine structure. According to $AICC$ results, the best fit is provided by the Student copula (state 1) and the Gaussian copula (state 2) for the Oil-$USDEUR$ link and the independence copula (state 1) and the 90° rotated Clayton (state 2) for the EUROSTOXX-$USDEUR$ dependence. Table 5 indicates the estimates for the best copula model within the truncated vine structure. Figures 8 and 9 present the uncoverage and coverage backtesting test for the $CoVaR(\alpha, \beta)$ of oil returns (top figure) and EUROSTOXX (bottom figure) given that the exchange rate returns are below its $VaR(\alpha)$. X-axis shows the joint probability of observing an exceedance, i.e. $\alpha\beta$, where $\alpha = \beta$. Figure 8 and 9 indicate that the copula choice meets the criteria in terms of number of exceedances and the independence of these VaR violations.[9]

Table 6 presents the results of the independence test based on the empirical Kendall's τ. The conditional distribution of EUROSTOXX and Oil in USD given exchange rate returns are assumed to be independent by the truncated vine structure. This hypothesis is rejected at 1% significance level. Hence, the vine structure should include a direct link between oil and EUROSTOXX returns, even once the exchange rate connection is taken into account. The copula choice for this conditional dependence between oil in USD and EUROSTOXX is studied in the next section.

[Insert Table 4 here]
[Insert Table 5 here]
[Insert Figure 8 here]
[Insert Figure 9 here]
[Insert Table 6 here]

4.1.3 Advance model: vine structure

Figure 10 shows the conditional bivariate histogram given the exchange rate returns under the truncated vine structure. The conditional copula is set to be obtained from state j if the probability of being at state j is higher than 90%. There is a higher dependence between low quantiles of oil returns and high quantiles of EUROSTOXX returns under state 1. A Clayton copula could fit the lower tail dependence presented under state 2. Table 7 presents the values of the Akaike Information Criterion with a correction for small sample size (AICC) for a set of models where the Clayton copula defines the dependence between oil in USD and EUROSTOXX conditional on the exchange rate under state 2, while under state 1 i consider 90° rotated Clayton, Gaussian and the independent copula, i.e. the product of the copula inputs. The best fit according to the AICC value is given by the Gaussian copula under state 1 and the Clayton copula under state 2.

Table 8 presents the estimates of the model and their standard deviations. The link between oil and exchange rate presents tail dependence with negative association under state 1, while EUROSTOXX and

[9]Further information on how to build backtesting tests for the $CoVaR$ can be found in Appendix A of Ojea Ferreiro (2018).

exchange rate are independent and the relationship between oil and EUROSTOXX is not statistically significant at 5% significant level. The variance multiplies by two for oil and exchange rate returns under the distress regime, while the variance of EUROSTOXX is three times higher within this state. The relationship between oil in US dollars and the exchange rate does not exhibit tail dependence under state 2 while the link between oil in US dollars and EUROSTOXX presents lower tail dependence.

[**Insert Figure 10 here**]
[**Insert Table 7 here**]
[**Insert Table 8 here**]

4.2 Stress test for the Eurostoxx given a distress scenario for oil returns in euros and the role of the exchange rate.

This subsection starts looking at the conditional distribution of the exchange rate returns under different scenarios for oil in euros. Lighter colours in Figure 11 indicates a higher probability for those values of the exchange rate returns. The distribution exhibits skewness features depending on the scenario. The exchange rate presents a positive skewness given a bearish scenario for oil prices in euros, while a negative skewness is exhibited in the opposite scenario. This empirical feature matches what we could expect from the economic theory. Indeed, under a scenario of *low-priced* oil, the demand would increase, so the supply of euros and the demand of US dollars increases in the exchange markets. The higher pressure on the foreign currency would implya depreciation of the euro against the US dollar, explaining the positive skewness under this scenario.
To observe how well the distribution of oil returns in euros is fitted by the convolution function in Equation (23), Figure 12 plots the oil returns in euros together with its $5-th$ and $10-th$ percentiles obtained from the convolution. The VaR adapts to the changes in volatility indicating an adequate fit for the empirical data.

[**Insert Figure 11 here**]
[**Insert Figure 12 here**]

Figure 13 shows the combination of quantiles (top) / returns (bottom) of oil returns in US dollars and $USDEUR$ that provides the $VaR(\alpha)$ of the oil returns in euros. Note that bottom figure is a straight line, because the oil return in US dollar is a linear function given a $VaR(\alpha)$ of the oil returns in euros and a value for the exchange rate returns. The changes over time are due to the changes in the $VaR(\alpha)$ of the oil in EUR. Note that the relationship is not linear when we are dealing with quantiles (top chart).
Figure 14 shows the distribution of the CoVaR over the sample 2000-2018. Left chart presents a scenario where the oil in euros is below its $10-th$ percentile ($\alpha = 0.1$) while the right chart shows a scenario where oil in euros is above its $90-th$ percentile ($\alpha = 0.9$). X-axis compares the same scenario depending on the upper (left chart) or lower (right chart) threshold for the quantile of the exchange rate (q_c). On the one side, left figure shows a scenario where oil prices denominated in euros experience a downward movement and the $USDEUR$ returns are below its q_c100-th percentile. On the other side, right graph presents a scenario where oil prices denominated in euros experience a upward movement and the $USDEUR$ returns are above its q_c100-th percentile. Label C in the x-axis refers to the convolution of oil and the exchange rate, i.e. without doing any assumption regarding the exchange rate following Equation (26). The bearish $CoVaR$ of the EUROSTOXX returns presents higher dispersion over time than the bullish $CoVaR$. For both scenarios $CoVaR$ increases for higher quantiles of the exchange rate returns. This implies than bearish $CoVaR$ where the main source of risk is the movement in oil prices denominated in US dollar and bullish $CoVaR$ where the source of risk comes from the exchange rate are the most volatile scenarios.

[**Insert Figure 13 here**]
[**Insert Figure 14 here**]

Figure 15 presents the Value at Risk of EUROSTOXX (black dashed line), the Value at Risk of the EUROSTOXX under a distress scenario for the oil price in euros (red solid line) and its range of potential values depending on the source of risk that triggers the distress scenario (grey area). Left figure shows a bearish scenario for oil denominated in euros, i.e. below its 10-th percentile, while the right chart indicates a bullish scenario, i.e. oil returns in euros above its 90-th percentile. The response of the EUROSTOXX VaR might be different depending on the source of the shock, i.e. from the exchange rate or from the oil price. Grey areas show the how response of EUROSTOXX could vary under the same scenario for oil in EUR depending on the source of the shock. This allow us to build a range of uncertainty about the impact assessment depending on the source of risk.[10] On the one side, the bearish $CoVaR$ is lower than the VaR of EUROSTOXX returns no matter which is the source of risk, although the magnitude of the difference between $CoVaR$ and VaR might vary. On the other side, bullish $CoVaR$ returns are higher than the VaR returns of EUROSTOXX, but this could change depending on the source of risk that triggers the scenario.

[**Insert Figure 15 here**]

Let us assume that the scenario considered by the VaR or the $CoVaR$ of the EUROSTOXX materializes. Then, given that $r_{s,t} = log(P_{s,t}) - log(P_{s,t-1})$, the losses of a 100EUR portfolio would be $100(1-\exp(VaR))$ and $100(1-\exp(CoVaR))$ respectively. Figures 16 and 17 show in the right axis the losses on a 100EUR portfolio when the distress scenario materializes. Grey line indicates the losses in the portfolio when the $CoVaR$ scenario occurs. The $CoVaR$ losses come from a downward movement of the oil price in euros in Figure 16 while in Figure 17 come from an upward movement. The $CoVaR$ is obtained setting the source of risk unknown using Equation (26). Black dashed line indicates the losses that comes from the VaR of the EUROSTOXX returns. Grey areas indicate periods where the smoothed probabilities of being at the high-volatility state are higher than 90%. This regime is identified in three main periods: before 2003, coinciding with the dot-com crisis; between 2008 to 2011, when the financial crisis and the European sovereign debt crisis occur; and between 2014 to 2016, matching with the oil glut period.

Left axis presents how the nominal losses on the EUR 100 portfolio changes in percentage depending on which is the main source of risk leading the change, compared to the $CoVaR$ losses with an undefined triggering variable. The losses decrease between $1\% - 9\%$ compared to the bearish $CoVaR$ if the triggering shock in the downward movement in oil prices comes from the exchange rate. Indeed, the 10% highest EUROSTOXX losses under a bearish scenario alleviate if the appreciation of the euro is triggering the scenario. On the other side, $CoVaR$ losses increase between $4\% - 20\%$ when oil movements are generating the downward trend of oil price in euros. The bearish $CoVaR$ triggered by the exchange rate depicts a scenario where the appreciation of the euro indicates a high foreign demand of European goods, which appreciates the domestic currency. The bearish $CoVaR$ triggered by trade-related issues might be related to an economic slump scenario where the oil demand decreases, coinciding with higher losses in the VaR of EUROSTOXX. Regarding the bullish scenario for oil returns in euros, $CoVaR$ losses decrease around a 4% when the shock is led by the oil returns in US dollars. This could be explained by the fact that economies in the expansion phase of the economic cycle present a high demand of energy products (Fernández Casillas et al. 2012). Losses increase between $3\% - 30\%$ when the depreciation of the euro explains the bullish trend of oil in euros. The depreciation of the euro could be indicative of an economic crisis in the euro area.

The findings in this section prove that the same scenario for oil in euros might describe very different economic frameworks depending on the source of this shock. Hence, identifying the source of risk that leads the distress scenario is relevant to build tailor-made stress tests and to get a better understanding about the relationship between variables in extremes scenarios.

[10] To build the range of *uncertainty* I choose a set of quantiles of the exchange rate returns (q_c) from 10^{-8} to $1 - 10^{-8}$.

[Insert Figure 16 here]
[Insert Figure 17 here]

5 Conclusion

A given scenario for oil prices in euros is consistent with different scenarios of the USDEUR exchange rate, conditioning the response of the European stock market to the energy-related scenario. However, the literature has overlooked this feature when performing spillover analyses until now. This article suggests combining the vine copula approach with the convolution concept, getting the most out of financial data to design stress test scenarios where the global markets and the exchange rate interact to define the distress event. The vine copula approach allows for modeling complex multivariate distributions while the convolution copula reflects the interaction between oil prices and the exchange rate. This framework allows for considering tailor-made scenarios, reducing the uncertainty regarding the role of the exchange rate in the distress scenario. I perform an empirical exercise using weekly returns of EUROSTOXX, Brent oil in US dollars and the exchange rate $USDEUR$ for the period 2000-2018 to check how the response of the European stock market could change depending on the role played by the exchange rate in the distress scenario. A simple rolling windows analysis indicates a switch in the variance and correlation parameters of these variables. I employ a $SWARCH$ model where the parameters of the copula and the variance switch jointly across regimes. I combine the convolution concept with the vine copula methodology to get the analytical expression for the $CoVaR$. The EM algorithm provides the estimates of the model following an iterative process. This approach distinguishes the main drivers that create a distress scenario for oil prices in euros. These components are the volatility of oil returns denominated in USD, the volatility of the $USDEUR$ exchange rate returns, and the co-movement between oil prices and the exchange rate.

Findings show the importance of the tail dependence and the need of modelling the relationship between oil in USD and stock market, even once the common exposure to the exchange rate has been taken into account. Empirical evidence shows an increase in the volatility of global markets jointly with a higher degree of co-movement and tail dependence across financial variables. These structural changes have been identified before 2003, between 2008-2011 and between 2014-2016. These periods coincide with the early 2000s recession, the financial crisis with the consequent European sovereign debt crisis, and the 2010s oil glut. A diverse compositions of the oil shock could have different effects on the lower quantiles of the European stock market. Results indicate higher losses for the European stock market conditioned to a bearish oil-related scenario compared to the unconditional VaR, while under a bullish oil-related scenario these losses might be higher or lower than the unconditional VaR losses depending on the source of risk that triggers the scenario. Stock responses to oil shocks present a higher dispersion under a bearish scenario than under a bullish oil-related scenario. Defining the triggering variable in a distress scenario for oil in euros is crucial to design a coherent stress test. I show that the VaR response of EUROSTOXX could be different depending on which is the source of risk that generates the scenario for the oil price denominated in euros. Stock losses, in a bearish oil scenario, sharply increase if the triggering risk is a trade-related issue, while exchange rate risk might exacerbate stock losses under a bullish oil scenario. The decrease of oil demand in economic crises and the depreciation of the domestic currency, due to political uncertainty and weak economic fundamentals, may explain these results.

The proposed approach can improve our understanding of how the exchange rate movements might affect stress test exercises in global markets. Possible extensions could consider interactions of exchange rate to foreign economies where there is a significant exposure. For instance, Spanish financial institutions have a great exposure to Latin American countries. Analysing the response of these financial firms to shocks in these countries depending on the source of the shock, i.e. foreign stock markets or exchange rates, will be useful to build better hedging strategies, increasing the resilience of the financial sector to instabilities in this region.

These results have consequences, firstly, for risk managers, investors and traders, interested on portfolio strategies to control the exposure of its stock positions to commodity and exchange rate risk; secondly, for regulatory authorities and supervisors, who seek tailor-made stress test scenarios that take into account the expected performance of the exchange rates; thirdly, for policy makers, who wish to understand the interactions between the main variables that drive the economy; lastly, for monetary authorities, who are interested to quantify the stock losses if scenarios of unstable energy prices materialize.

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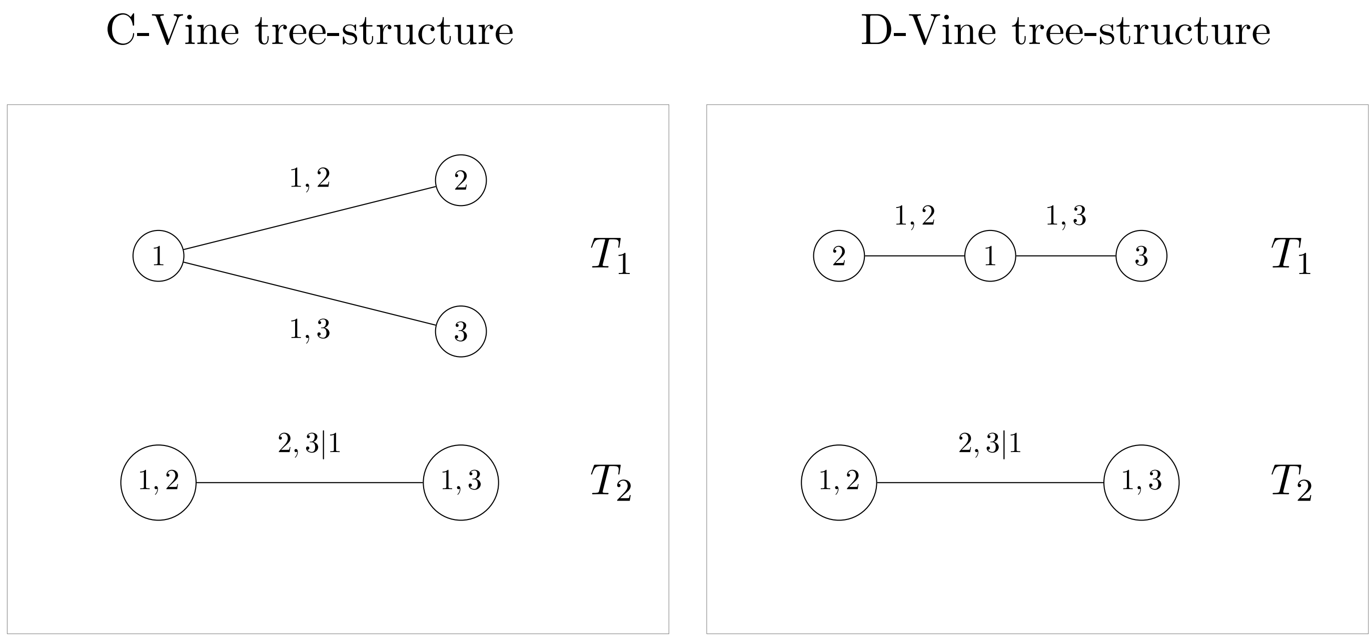
Appendices

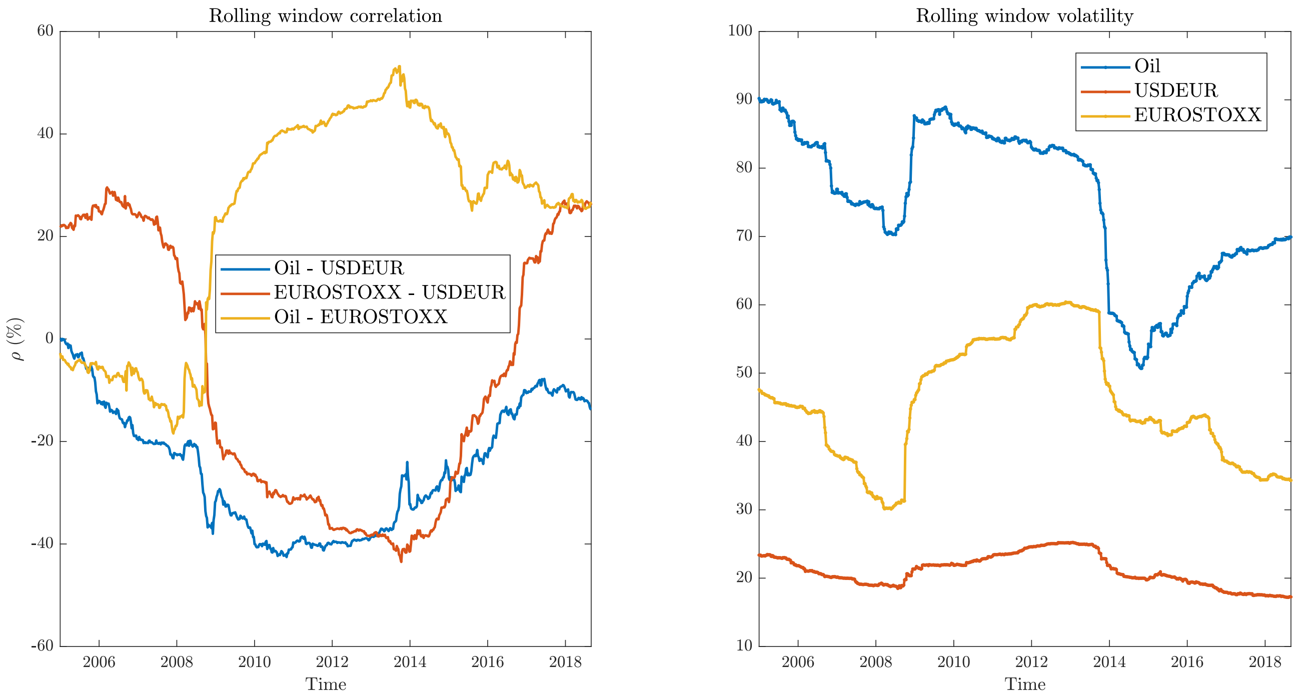
A Figures

Figure 1: Example of a three-dimensional C- (left-top panel), D-vine (right-top panel) with edge indices.



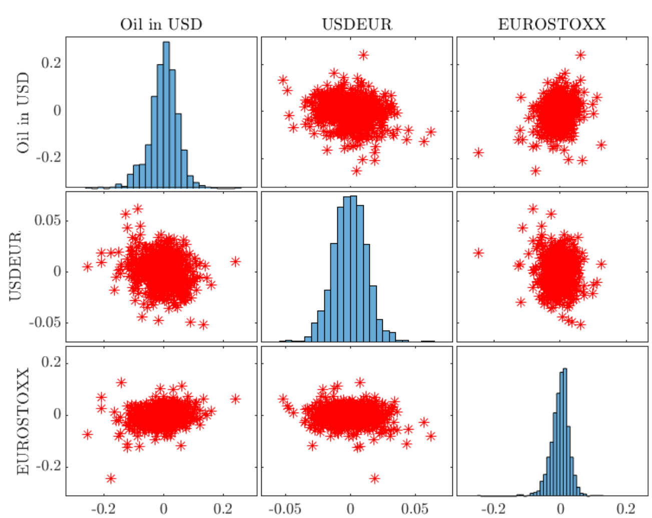
Structure graphs gives the representation of the joint probability density function in the form of a nested set of trees (T_1, T_2). Each node corresponds to a density distribution, each edge corresponds to a pair-copula density and the edge label corresponds to the subscript of the pair-copula density. distribution. Note that C-Vine and D-Vine in this example show the same way of decomposing the density. Under the vine structure, variable 1 is connected to variable 2 and 3 in a first stage (T_1). Variable 2 and 3 are connected through the relationship that both have with variable 1 in T_1, and conditioned to the value of variable 1 they present an additional link between them in the second stage (T_2). Note that if the model is limited up to T_1, variable 2 and 3 would be unconditionally dependent through variable 1 but conditioned independent given a realization of variable 1.

Figure 2: Time-varying correlation and volatility.



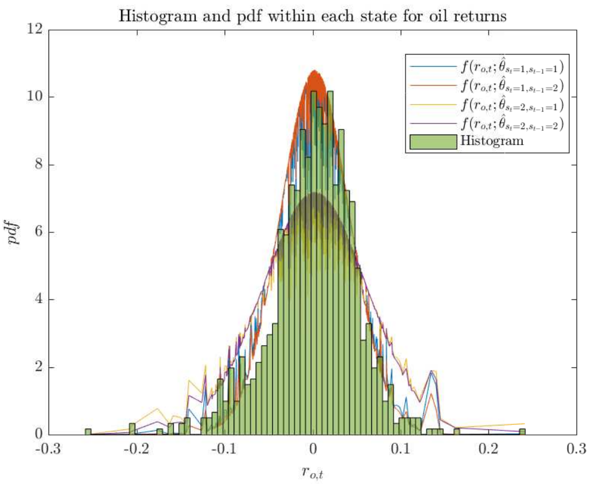
These figures show the evolution of the correlation and volatility using a rolling window with a window length of five years, i.e. at each time t I assess the correlation and the volatility of the weekly returns between $t-260$ and t. The figures depict two set of evidence. First, there is a general shift in correlation across the variables between the period $2009-2014$. Second, this period coincides with a general change in the volatility level of those markets. This evidence indicates that a Switching Markov model, where variance and dependence move together across regimes, might explain the dynamic shown by the data. Volatility value is obtained annualizing the standard deviation shown in percentage, i.e. standard deviation is multiplied by $\sqrt{52}100$.

Figure 3: Histogram and scatter plots for the bivariate relationships.

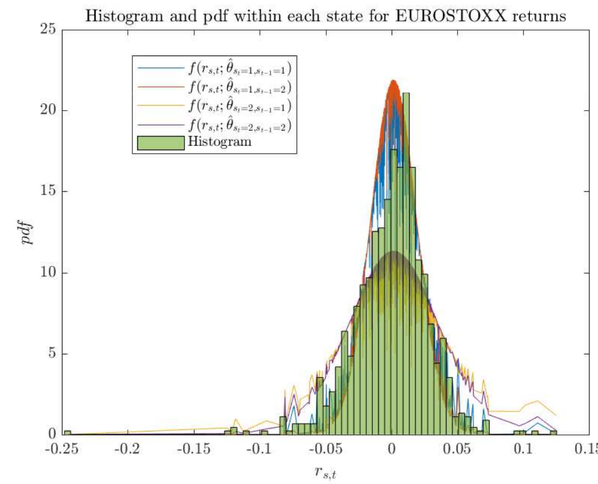


This figure shows the histogram for each variable and the scatter plot between each pair of variables. Concerning the histograms, they indicate an excess of kurtosis and the presence of left skewness which could be explained by a discrete switch in variance.

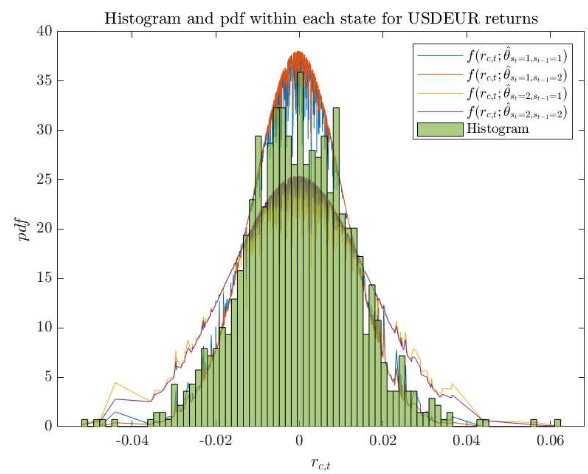
Figure 4: Histogram and Marginal distribution within each state



(a) Oil returns



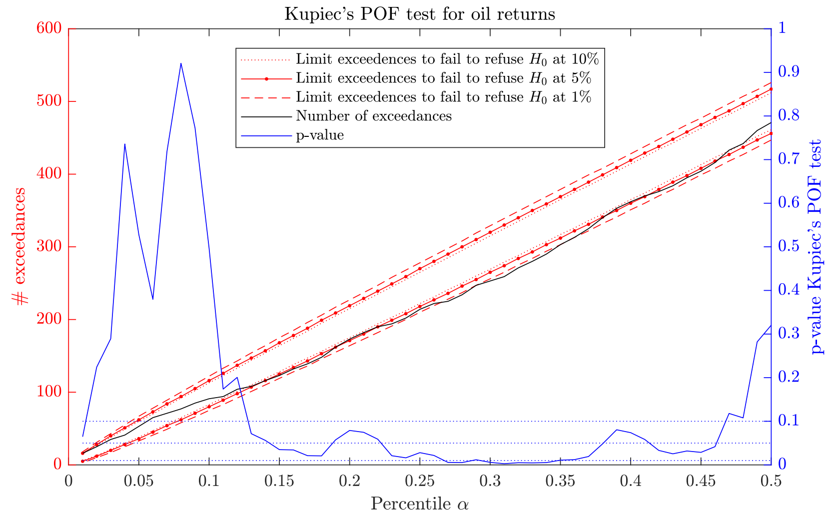
(b) EUROSTOXX returns



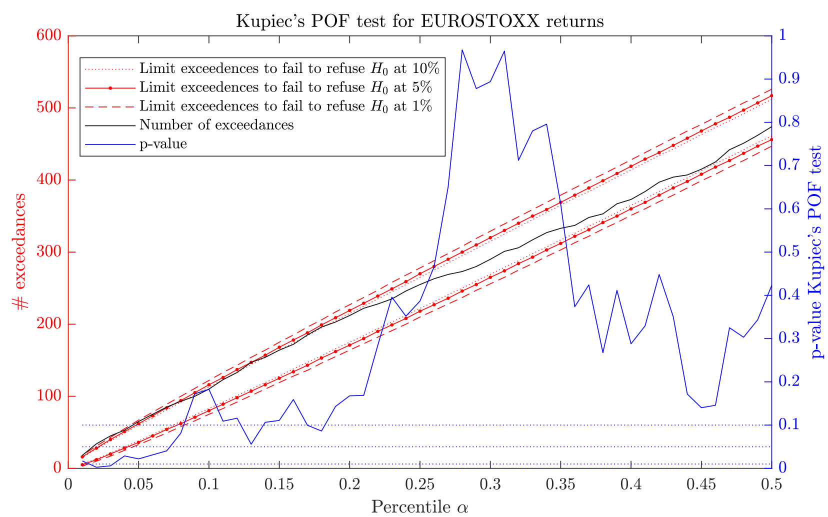
(c) USDEUR returns

The histogram (green bars) is scaled to be equivalent to the probability distribution function within each state. Althought at time t we have only 2 states we have four pdf because the current variance according to the SWARCH model in the equation (9) depends on the state at t and the state at $t-1$. Note that higher moments can be obtained given higher probability to the distribution with higher dispersion for extreme values.

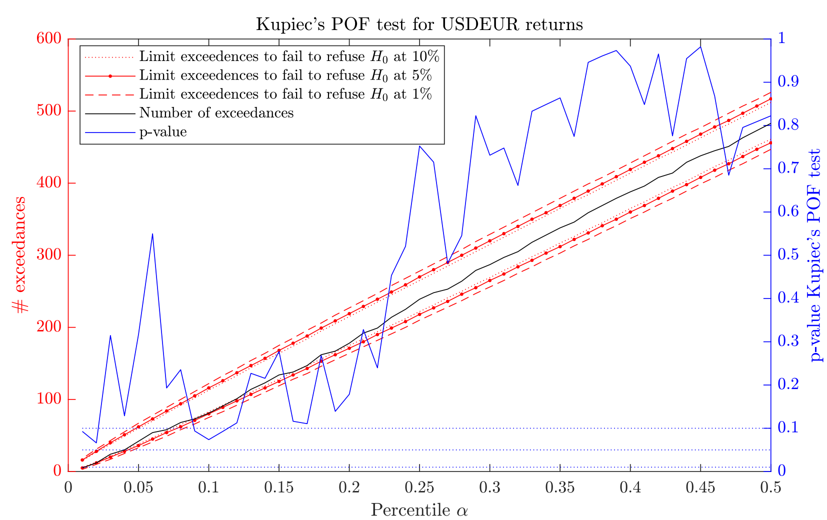
Figure 5: Kupiec's POF test



(a) Oil returns



(b) EUROSTOXX returns

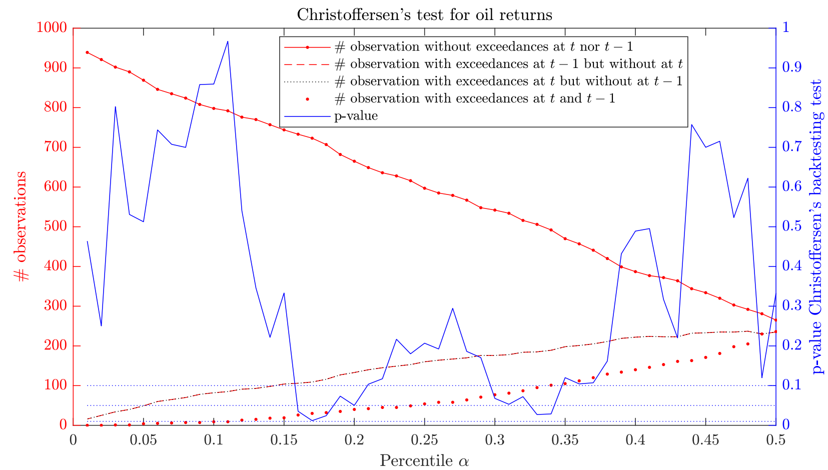


(c) USDEUR returns

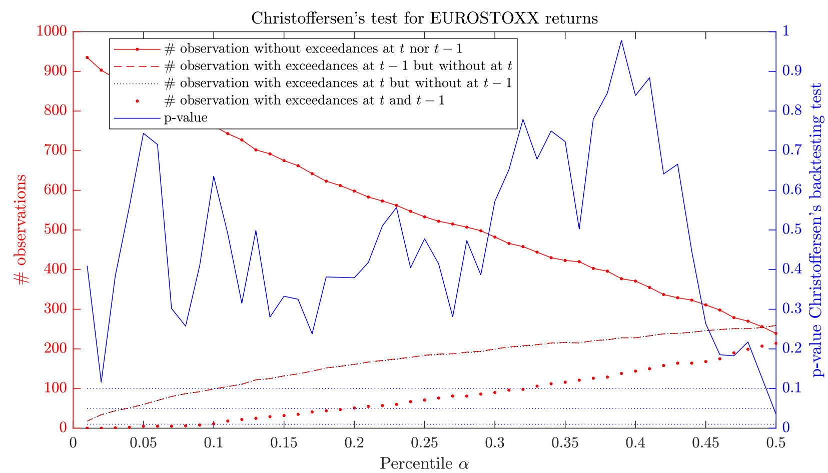
These figures present the unconditional coverage backtesting test proposed by Kupiec (1995) to check the number of exceedances of a VaR with a α% significance level (x-axis).

Right axis shows the p-value of the Kupiec (1995)'s test while left axis presents the number of exceedances. Confidence intervals for the null hypothesis are presented in the red lines for the 1%,5% and 10% significance level. Black line presents the current number of exceedances.

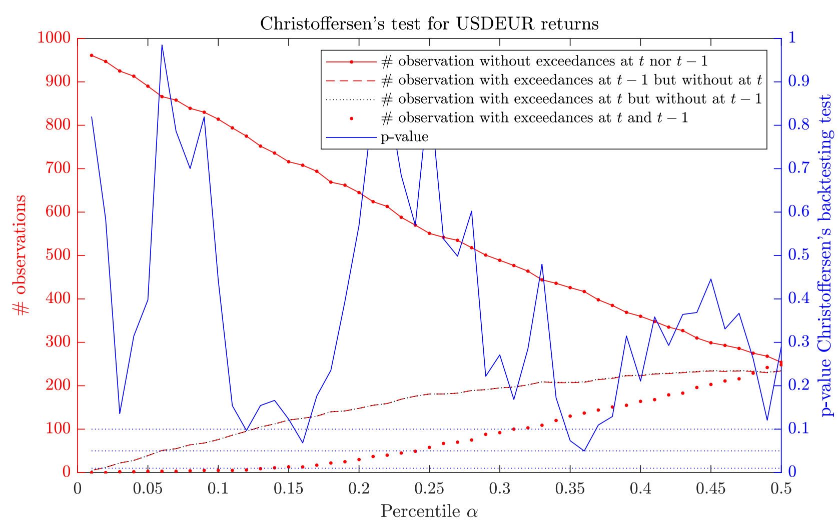
Figure 6: Christoffersen test



(a) Oil returns



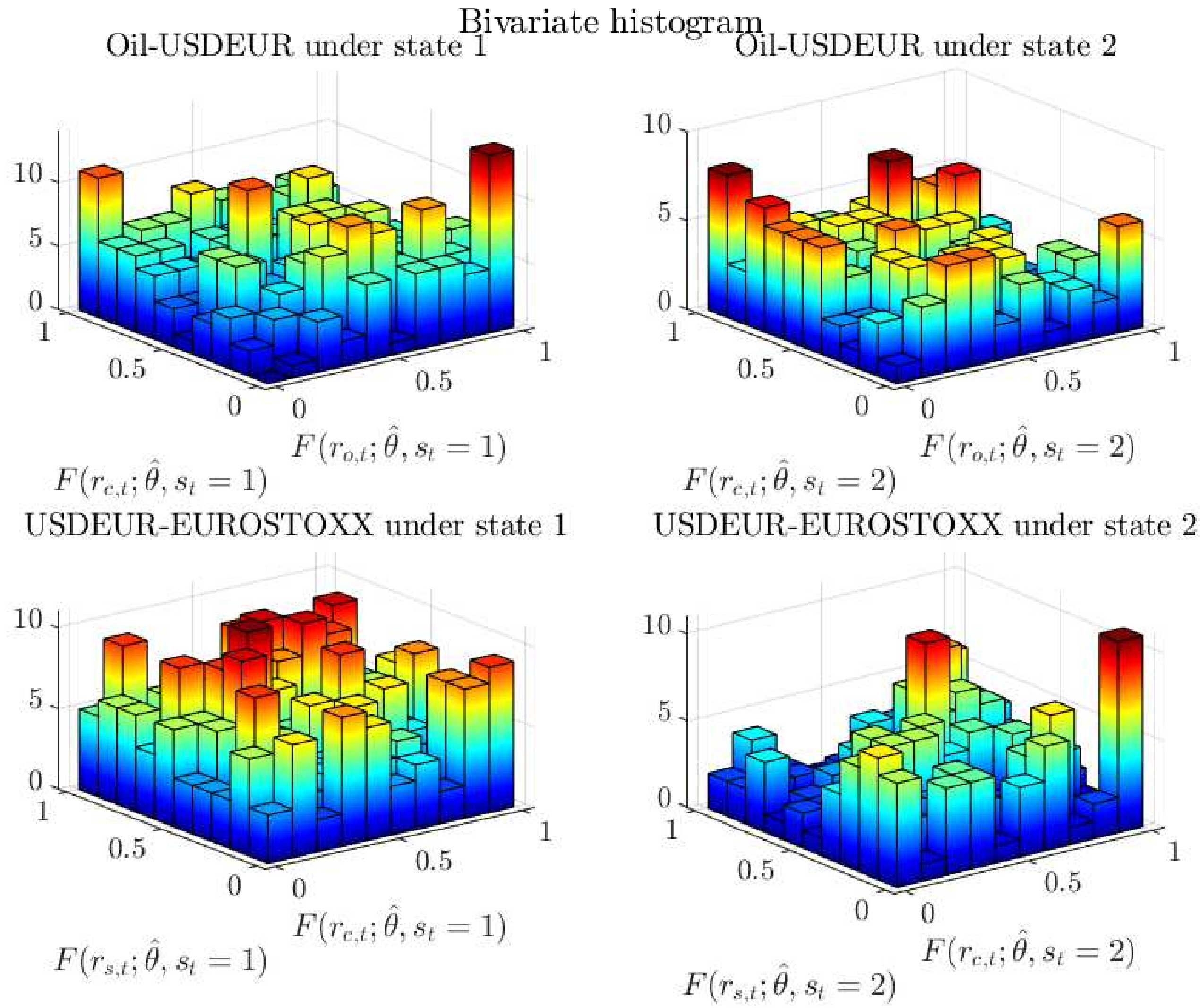
(b) EUROSTOXX returns



(c) USDEUR returns

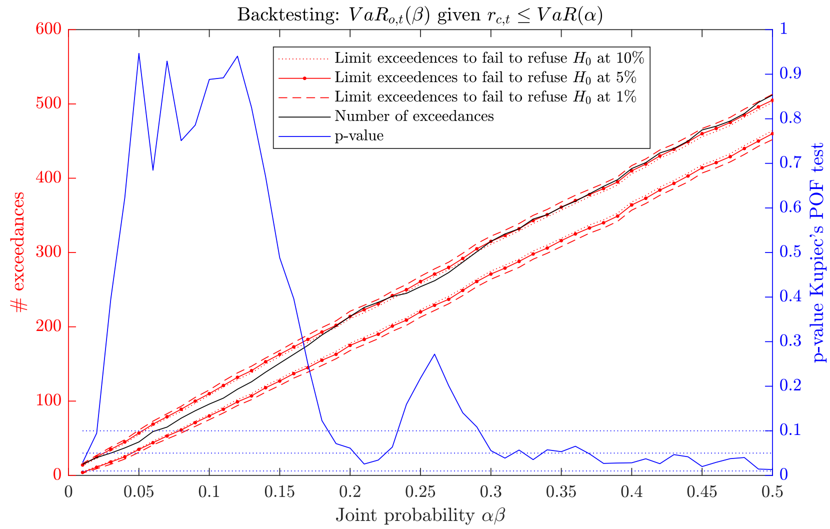
Conditional coverage backtesting test proposed by proposed by Christoffersen (1998) are used for testing the number of exceedances of a VaR with a $\alpha\%$ significance level (x-axis).
Right axis shows the p-value of the Christoffersen (1998)'s test while left axis presents the number of exceedances. Left axis presents the number of observation. Red solid line present the number of observations without exceedances at t and $t-1$. Red dashed line shows the number of pairwise observations where we have an exceedance at t but not at t while the black dotted line shows the opposite case. Red dotted line shows the number of pairwise observations with two consecutive exceedances.

Figure 7: Bivariate Histogram

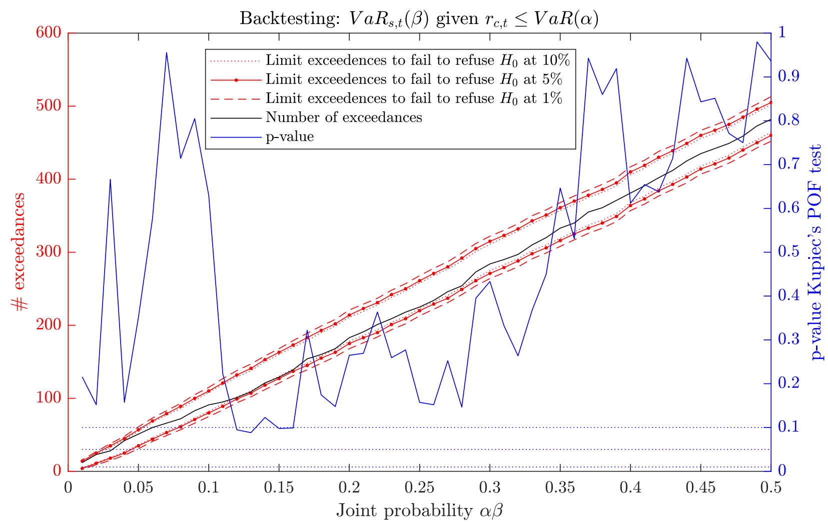


This figure shows the bivariate histogram of the probability integral transforms of oil returns in US dollars and USDEUR returns (top figures) or USDEUR returns and EUROSTOXX returns (bottom figures). We suppose that observation at time t beyond to a regime j if the smoothed probability of being at t in state j is higher than 90%. These figures give us an idea about the type of relationship that we could expect from each set of two variables within each regime.

Figure 8: Kupiec's POF test



(a) Oil returns

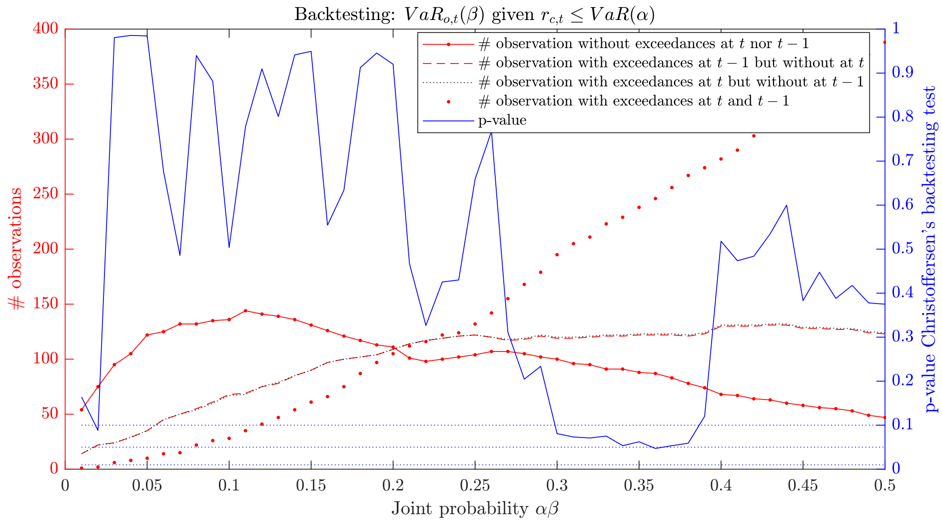


(b) EUROSTOXX returns

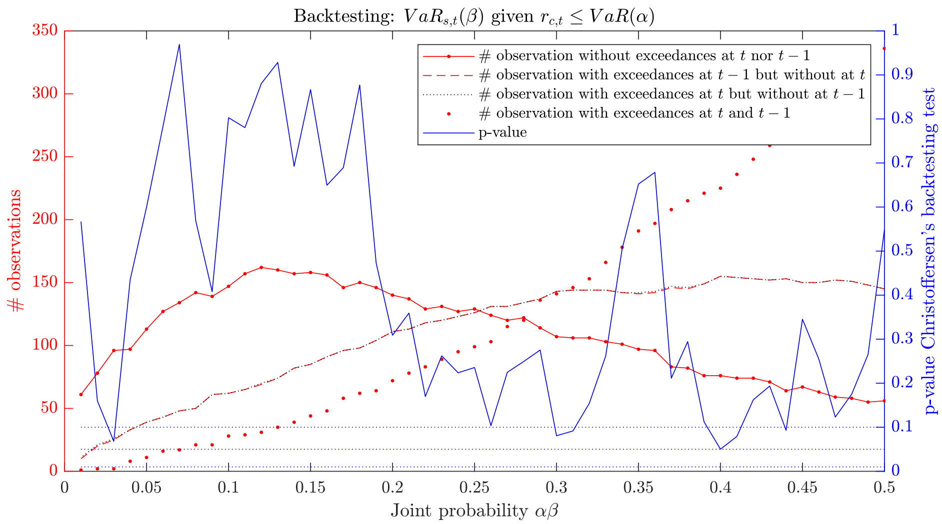
These figures present the unconditional coverage backtesting test proposed by Kupiec (1995) to check the number of exceedances of a $CoVaR(\alpha, \beta)$ with a β% significance level given than exchange rate returns are below $VaR(\alpha)$. This figures sets $\alpha = \beta$ while x-axis shows the joint probability, i.e. $\alpha\beta$.

Right axis shows the p-value of the Kupiec (1995)'s test while left axis presents the number of exceedances. Confidence intervals for the null hypothesis are presented in the red lines for the 1%,5% and 10% significance level. Black line presents the current number of exceedances.

Figure 9: Christoffersen test



(a) Oil returns

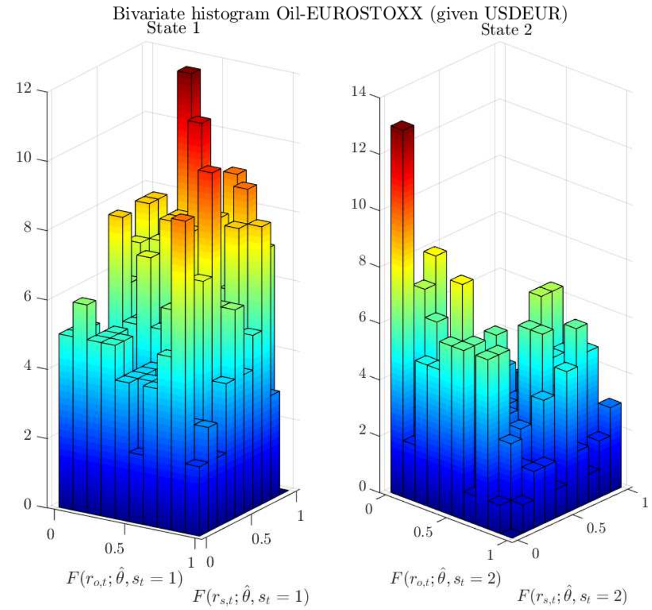


(b) EUROSTOXX returns

Conditional coverage backtesting test proposed by proposed by Christoffersen (1998) are used for testing the number of exceedances of a $CoVaR(\alpha, \beta)$ with a $\beta\%$ significance level given than exchange rate returns are below $VaR(\alpha)$. This figures sets $\alpha = \beta$ while x-axis shows the joint probability, i.e. $\alpha\beta$.

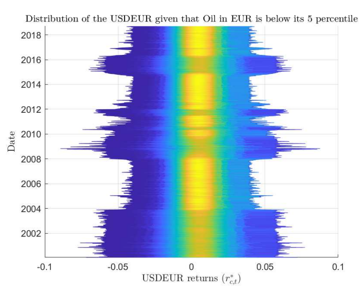
Right axis shows the p-value of the Christoffersen (1998)'s test while left axis presents the number of exceedances. Left axis presents the number of observation. Red solid line present the number of observations without exceedances at t and $t-1$. Red dashed line shows the number of pairwise observations where we have an exceedance at t but not at t while the black dotted line shows the opposite case. Red dotted line shows the number of pairwise observations with two consecutive exceedances.

Figure 10: Bivariate histogram conditioned to the exchange rate returns

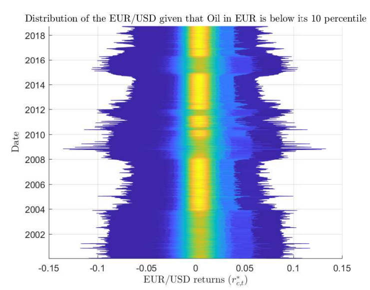


This figure shows the bivariate histogram of the probability integral transforms of oil returns in US dollars and EUROSTOXX returns given the realization of USDEUR returns, i.e. conditional histogram. We suppose that the realization at time t beyond to a regime j if the smoothed probability of being at t in state j is higher than 90%. These figures give us an idea about the type of relationship that we could expect from each set of two variables within each regime, once the dependence through the exchange link is taken into account.

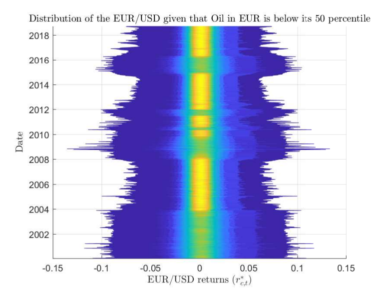
Figure 11: Distribution of USDEUR returns under different scenarios for Oil in EUR.



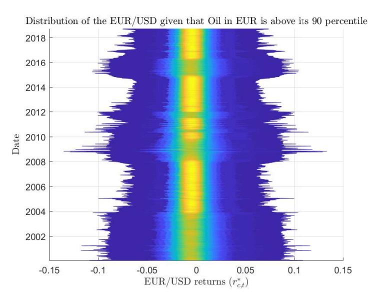
(a) Oil in EUR below 5-th percentile



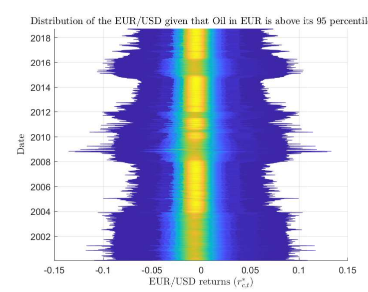
(b) Oil in EUR below 10-th percentile



(c) Oil in EUR below 50-th percentile



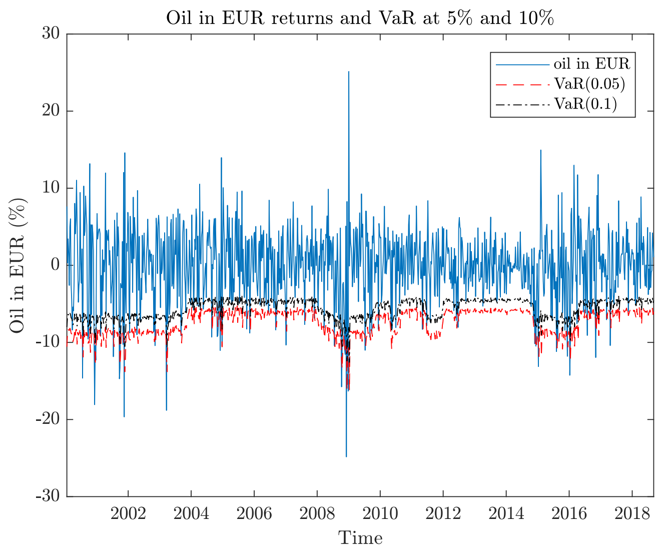
(d) Oil in EUR above 90-th percentile



(e) Oil in EUR above 95-th percentile

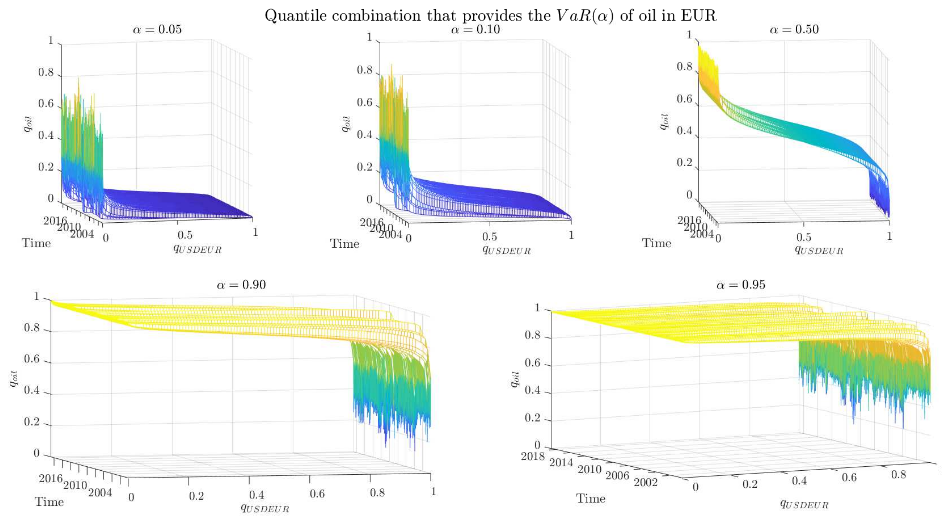
This figure shows the distribution of the exchange rate returns under different scenarios for oil in euros. The distribution of exchange rate returns exhibits skewness features depending on the scenario of oil in Euros. The lighter colour indicates a higher probability for those values. The conditional distribution of the exchange rate is obtained as $f(r_c|r_{oe} < VaR_{oe}(\alpha)) = C_{o|c}(F_o(VaR_{oe} - r_c)|F_c(r_c))f(r_c)\frac{1}{\alpha}$ where the subscript t is ignored for notational convenience.

Figure 12: Oil returns denominated in euros and its 5-th and 10-th percentiles

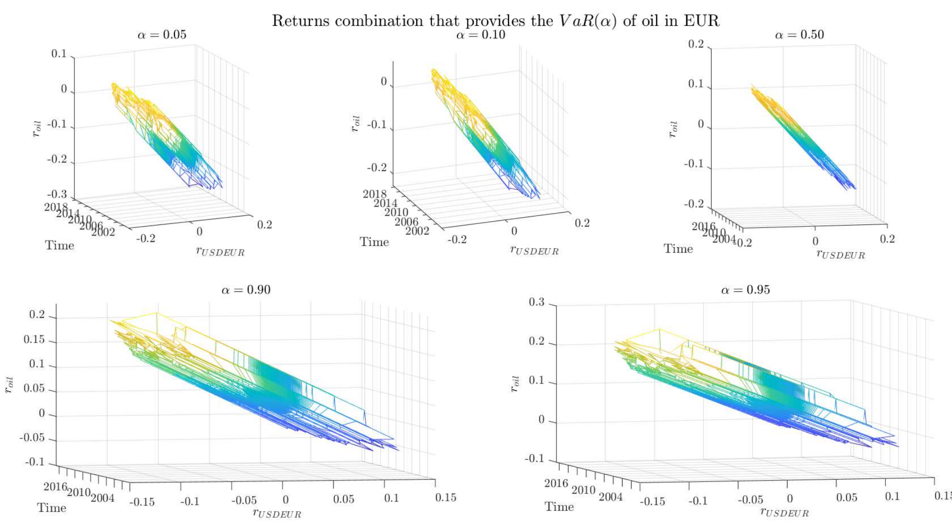


Oil returns denominated in euros and its 5-th and 10-th percentile obtained from the convolution function from Equation (5).

Figure 13: Combination of oil in US dollars and USDEUR such that the sum is the $VaR(\alpha)$ of the oil denominated in euros



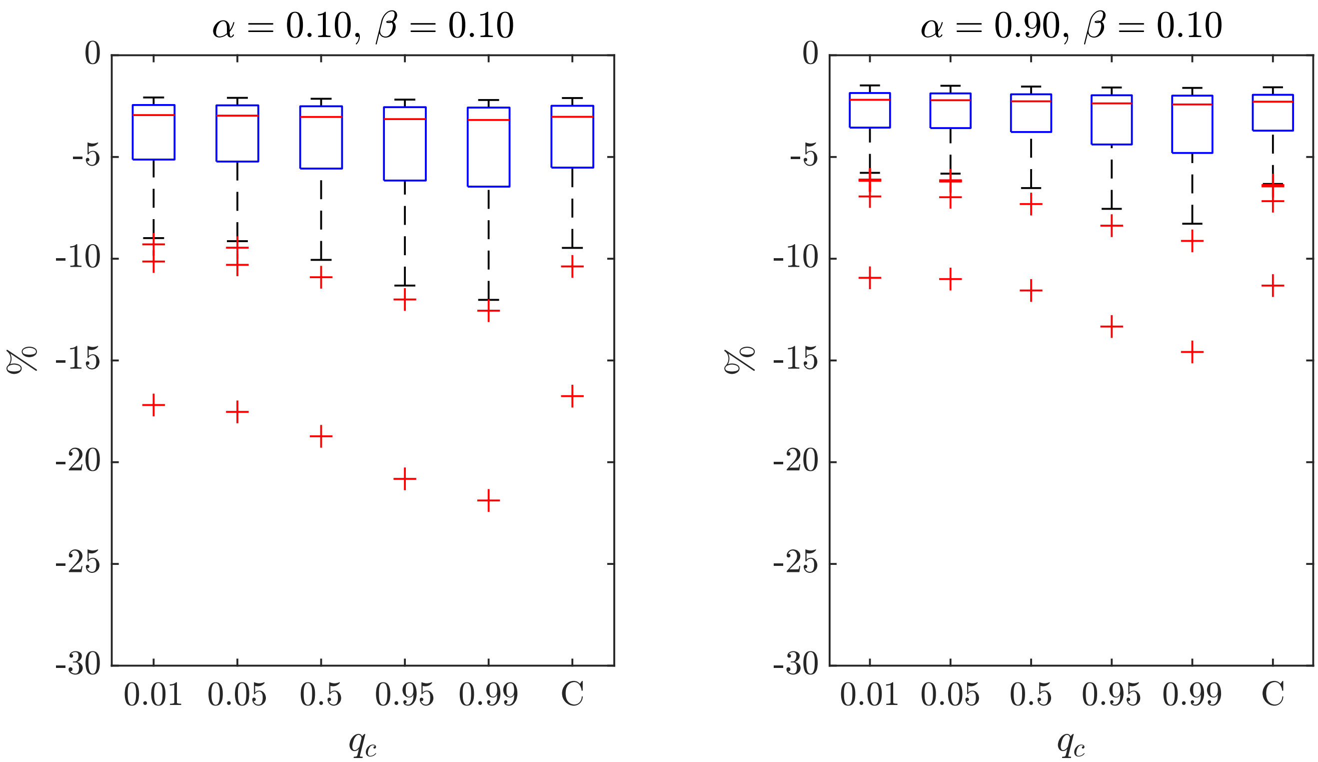
(a) Quantile combination to get $VaR(\alpha)$ of oil returns in euros



(b) Returns combination to get $VaR(\alpha)$ of oil returns in euros

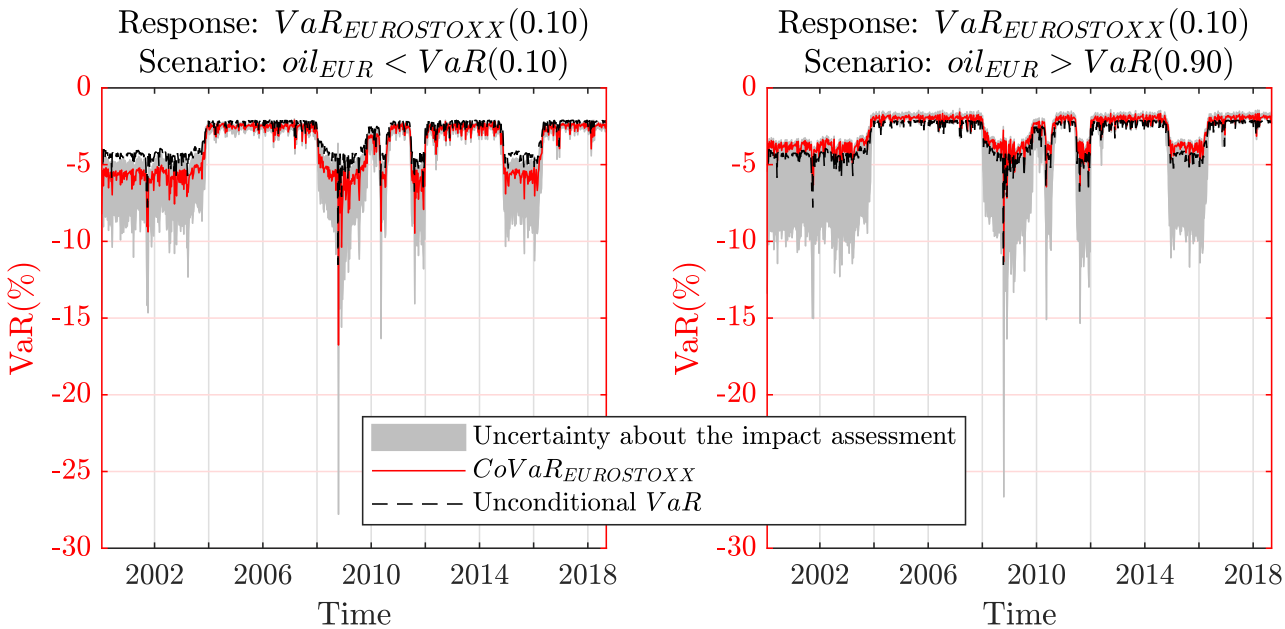
This figure shows the combination of quantiles (top) / returns (bottom) of oil in US dollars and USDEUR that provides the $VaR(\alpha)$ of the oil returns in euros. Note that the bottom figure is a straight line, because oil return in US dollar is a linear function of the $VaR(\alpha)$ of the oil returns in euros and the exchange rate return. The changes over time are due to the changes in the $VaR(\alpha)$ of the oil in euros. Note that when we are dealing with quantiles (top chart) the relationship is not linear.

Figure 14: Boxplot of the CoVaR distribution of the EUROSTOXX over the full sample



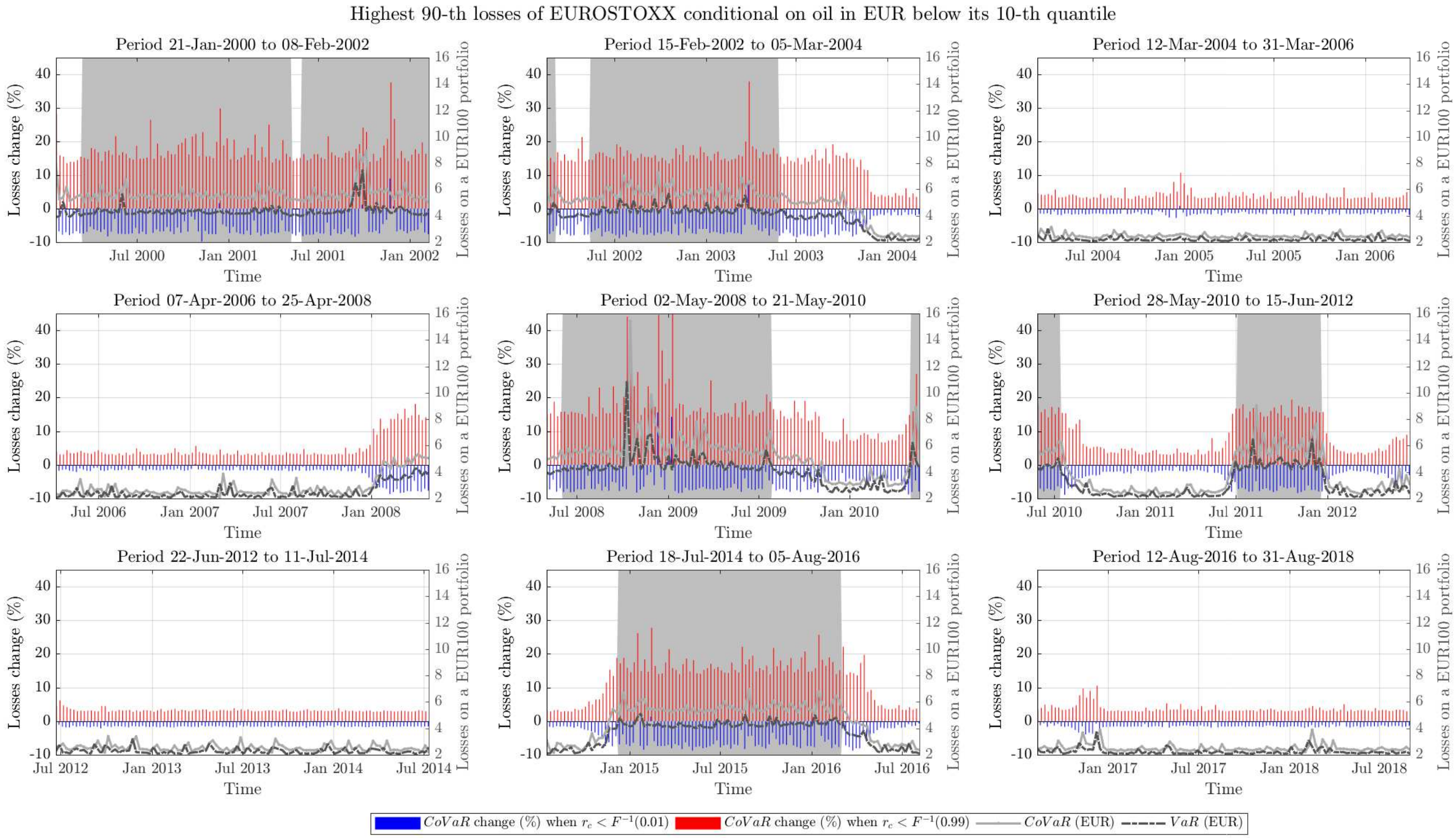
This figure shows the distribution of the CoVaR over the sample 2000-2018. Left chart presents a scenario where the oil returns in euros is below its $10-th$ percentile ($\alpha = 0.1$) while right chart shows a scenario where oil returns in euros is above its $90-th$ percentile ($\alpha = 0.9$). X-axis compares the same scenario depending on the quantile of the exchange rate (q_c). Left figure shows a bearish scenario for oil returns in euros and USDEUR is below its q_c100-th percentile, while right graph presents a bullish scenario for oil returns in euros where the USDEUR is above its its q_c100-th percentile. Label C in the x-axis refers to the convolution of oil returns and the exchange rate, i.e. without doing any assumption regarding the stress in the exchange rate.

Figure 15: Value at Risk of the EUROSTOXX under different oil-related scenarios



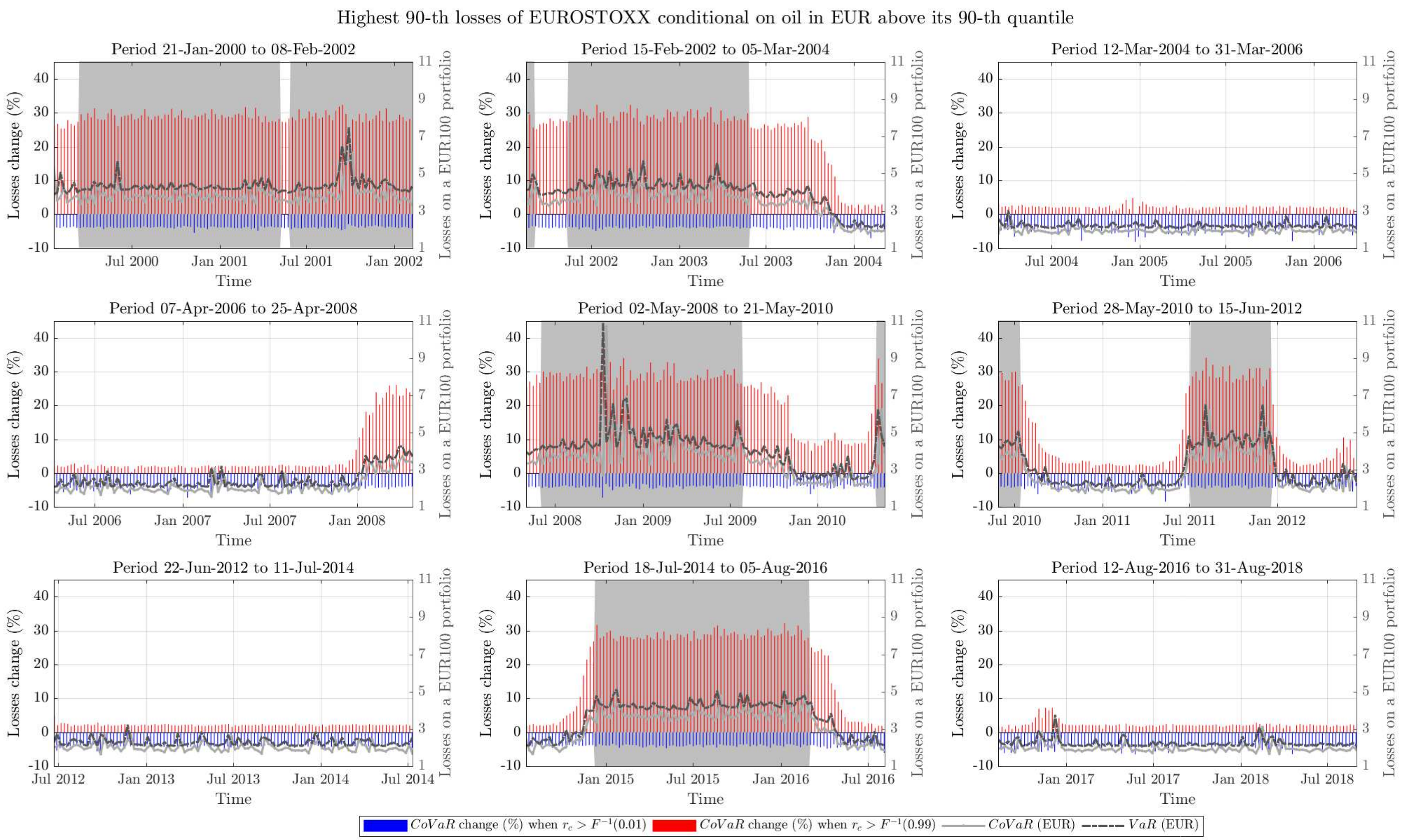
This figure shows the Value at Risk of EUROSTOXX (black dashed line), the Value at Risk of the EUROSTOXX under a distress scenario for oil in euros (red solid line) and its range of potential values depending on the source of risk that triggers the distress scenario for oil prices in euros (grey area). Left figure shows a bearish scenario for oil in euros, i.e. below its 10-th percentile, while the right chart indicates a bullish scenario, i.e. oil returns in euros above its 90-th percentile. The response of the EUROSTOXX VaR might be different depending on the source of the shock, i.e. arising from the exchange rate or from the oil trade. Grey areas show how the response of EUROSTOXX could vary under the same scenario for oil in euros depending on the source of the scenario. This allows us to build a range of *uncertainty* regarding the impact of the scenario.

Figure 16: Losses on a 100EUR portfolio when the distress scenario (VaR or $CoVaR$) materializes and the percentage change of $CoVaR$ losses depending on the source of the scenario. Bearish scenario for oil in euros.



This figure shows the losses of a 100EUR portfolio when the distress scenario defined by the risk measures materializes (right axis). The oil price denominated in euros presents a downward movement under the CoVaR scenario. Left axis shows the percentage change of the $CoVaR$ losses depending on the source of the shock. Grey areas indicate periods where the smoothed probabilities of being at a high-volatility state are higher than 90%.

Figure 17: Losses on a 100EUR portfolio when the distress scenario (VaR or CoVaR) materializes and the percentage change of CoVaR losses depending on the source of the scenario. Bullish scenario for oil in euros.



This figure shows the losses of a 100EUR portfolio when the distress scenario measured by the risk measures materializes (right axis). The oil price denominated in euros presents a upward movement under the CoVaR scenario. Left axis shows the percentage change of the $CoVaR$ losses depending on the source of the shock. Grey areas indicate periods where the smoothed probabilities of being at a high-volatility state are higher than 90%.

B Tables

Table 1: Main tail dependence features for each copula

Family	τ_L	τ_U
Gaussian	$-$ (if $\rho = 1$ then 1)	$-$ (if $\rho = 1$ then 1)
Student t	$2t_{\eta+1}\left(-\sqrt{\frac{(\eta+1)(1-\rho)}{1+\rho}}\right)$	$2t_{\eta+1}\left(-\sqrt{\frac{(\eta+1)(1-\rho)}{1+\rho}}\right)$
Clayton	$2^{-1/\theta}$	$-$
Gumbel	$-$	$2 - 2^{1/\theta}$

Note:

$-$ represents no tail dependence.

Source: (Ao et al., 2017, p. 22), Jiang (2012), Joe and Hu (1996), Fischer (2003) and (Joe, 1997, p. 193–204).

Let u_1 and u_2 denote two uniform-distributed variables across (0,1).

The lower tail dependence, τ_L, is defined as $\tau_L = lim_{q\to 0}P(u_2 < q|u_1 < q)$.

The upper tail dependence, τ_U is defined as $\tau_U = lim_{q\to 1}P(u_2 > q|u_1 > q)$.

Table 2: Descriptive statistic for the variables

	Full sample			Pre-crisis period			Post-crisis period		
	A	B	C	A	B	C	A	B	C
μ	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
σ	0,05	0,01	0,03	0,05	0,01	0,03	0,05	0,01	0,03
skewness	-0,50	0,08	-0,96	-0,60	-0,01	-0,23	-0,41	0,17	-1,38
kurtosis	5,19	4,10	9,94	3,95	3,03	4,81	6,77	5,04	12,30
q=95%	0,07	0,02	0,04	0,08	0,02	0,04	0,07	0,02	0,04
q=5%	-0,09	-0,02	-0,05	-0,09	-0,02	-0,05	-0,09	-0,02	-0,05
ρ_{USDEUR}	-0.1933	-	-0.0513	-0.1271	-	0.1649	-0.2554	-	-0.2188
ρ_o	-	-0.1933	0.2153	-	-0.1271	-0.0379	-	-0.2554	0.4326
ARCH test	0,0000	0,0007	0,0000	0,0000	0,6604	0,0000	0,0000	0,0031	0,0251
LBQ test	0,4992	0,7454	0,0090	0,4223	0,5941	0,8540	0,1537	0,2942	0,0125

A: Oil in USD, B: USDEUR exchange rate, C: EUROSTOXX. All the series are shown in returns.

LBQ test refers to the p-value of the Ljung-Box Q-test for autocorrelation performed with 20 lags.

ARCH test refers to the p-value of the Engle's ARCH Test for heteroscedasticity performed with 1 lag.

The 15 September 2008 is chosen as breakpoint to define a crisis date.

ρ_{USDEUR} and ρ_o shows the Pearson's linear correlation coefficient of the variables against the $USDEUR$ and the Oil in USD respectively.

Table 3: Gaussian and Student t models

	Gaussian model			Student model		
	A	B	C	A	B	C
ϕ_0	0.00 **	-0.00	0.00 **	0.00 *	-0.00	0.00 ***
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
ϕ_1	0.06 **	0.04	-0.05	0.06 **	0.04 *	-0.06 *
	(0.04)	(0.03)	(0.04)	(0.04)	(0.03)	(0.04)
$\kappa_{s_t=2}$	2.27 ***	2.25 ***	3.74 ***	2.21 ***	2.21 ***	3.64 ***
	(0.32)	(0.26)	(0.32)	(0.26)	(0.29)	(0.08)
α_0	0.00 ***	0.00 ***	0.00 ***	0.00 ***	0.00 ***	0.00 ***
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
α_1	0.12 ***	0.08 **	0.15 ***	0.12 ***	0.09 **	0.16 ***
	(0.04)	(0.04)	(0.05)	(0.04)	(0.04)	(0.05)

	Gaussian			Student	
State 1	$\rho_{A,B}$	-0.23 ***	State 1	$\rho_{A,B}$	-0.22 ***
		(0.05)			(0.05)
	$\rho_{B,C}$	-0.09 *		$\eta_{A,B}$	13.35 ***
		(0.05)			(0.24)
State 2	$\rho_{A,B}$	-0.17 ***		$\rho_{B,C}$	-0.08 *
		(0.05)			(0.05)
	$\rho_{B,C}$	-0.03		$\eta_{B,C}$	24.80 ***
		(0.05)			(0.24)
	p_{11}	0.99 ***	State 2	$\rho_{A,B}$	-0.17 ***
		(0.00)			(0.05)
	p_{22}	0.98 ***		$\eta_{A,B}$	100.00 ***
		(0.01)			(1.17)
	LL	6668.53		$\rho_{B,C}$	0.07
					(0.07)
				$\eta_{B,C}$	7.35 ***
					(0.52)
				p_{11}	0.99 ***
					(0.00)
				p_{22}	0.98 ***
	RL	5,51			(0.01)
	RL p-value	0,0263		LL	6674.04

The table reports the estimates and the standard deviation (in parenthesis) for the parameters of the marginal model in Equations (6) and (9) and for the parameters of the Gaussian and Student t copula.

LL is the log-Likelihood value. RL is the logarithm of the likelihood ratio between the Student (unrestricted model) and the Gaussian (restricted model). RL p-value is the probability a results at least as extreme as the one obtained under the null hypothesis. The likelihood ratio is distributed under the null hypothesis as

$$-2(log(Likel_R) - log(Likel_{UR})) \sim X_{k_{UR}-k_R}$$

A: Oil in USD, B: USDEUR exchange rate, C: EUROSTOXX.

$\rho_{A,B}$ is the correlation between Oil in USD and USDEUR returns. $\rho_{B,C}$ is the correlation between USDEUR exchange rate and EUROSTOXX returns.

$***/**/*$ indicates statistical significance at 1/5/10%

Table 4: AICC values to choose the best model fit

A	B	C	D	E
-13294.01	-13296.70	-13298.53	-13292.29	-13293.90
F	G	H	I	J
-13287.44	**-13298.66**	-13293.60	-13293.43	-13288.06

Notes: *AICC* denotes Akaike Information Criterion corrected for small sample bias.

$AICC = 2k\frac{T}{T-k-1} - 2\log(\hat{L})$ where T is the sample size, k is the number of estimated parameters and $\hat{L}$ is the Log-likelihood value. Minimum *AICC* value (in bold letters) indicates the best copula fit.

A: *Oil-USDEUR*- State 1: Gaussian, State 2: Gaussian
USDEUR-EUROSTOXX- State 1: Gaussian, State 2: Gaussian.

B: *Oil-USDEUR*- State 1: Student, State 2: Student
USDEUR-EUROSTOXX- State 1: Student, State 2: Student.

C: *Oil-USDEUR*- State 1: Student, State 2: Gaussian
USDEUR-EUROSTOXX-State 1: Gaussian, State 2: 90° Clayton.

D: *Oil-USDEUR*-State 1: 90° Clayton, State 2: Gaussian
USDEUR-EUROSTOXX-State 1: Gaussian, State 2: 90° Clayton.

E: *Oil-USDEUR*-State 1: Student, State 2: 90° Gumbel.
USDEUR-EUROSTOXX-State 1: Gaussian, State 2: 90° Clayton.

F: *Oil-USDEUR*-State 1: 90° Clayton, State 2: 90° Gumbel.
USDEUR-EUROSTOXX-State 1: Gaussian, State 2: 90° Clayton.

G: *Oil-USDEUR*- State 1: Student, State 2: Gaussian.
USDEUR-EUROSTOXX-State 1: Independence, State 2: 90° Clayton.

H: *Oil-USDEUR*-State 1: 90° Clayton, State 2: Gaussian.
USDEUR-EUROSTOXX-State 1: Independence, State 2: 90° Clayton.

I: *Oil-USDEUR*-State 1: Student, State 2: 90° Gumbel.
USDEUR-EUROSTOXX-State 1: Independence, State 2: 90° Clayton.

J: *Oil-USDEUR*-State 1: 90° Clayton, State 2: 90° Gumbel.
USDEUR-EUROSTOXX-State 1: Independence, State 2: 90° Clayton.

Table 5: Model with a truncated vine structure

	A	B	C
ϕ_0	0.00 *	-0.00	0.00 **
	(0.00)	(0.00)	(0.00)
ϕ_1	0.07 **	0.04	-0.05 *
	(0.04)	(0.03)	(0.04)
$\kappa_{s_t=2}$	2.17 ***	2.13 ***	3.78 ***
	(0.31)	(0.33)	(1.54)
α_0	0.00 ***	0.00 ***	0.00 ***
	(0.00)	(0.00)	(0.00)
α_1	0.13 ***	0.08 **	0.15 **
	(0.04)	(0.04)	(0.07)

State 1		State 2	
$\rho_{A,B}$	-0.20 ***	$\rho_{A,B}$	-0.18 ***
	(0.06)		(0.05)
$\eta_{A,B}$	12.22 ***	$\theta_{B,C}$	0.07 **
	(1.40)		(0.04)
p_{11}	0.99 ***	p_{22}	0.98 ***
	(0.00)		(0.01)
LL	-6670.82		

The table reports the estimates and the standard deviation (in parenthesis) for the parameters of the marginal model in Equations (6) and (9) and for the parameters of the best copula choice according to the $AICC$ value reported by Table 4.
LL is the log-Likelihood value.
A: Oil in USD, B: USDEUR exchange rate, C: EUROSTOXX. $\rho_{1,2}$ and $\eta_{A,B}$ is the correlation and number of degrees of freedom between Oil in USD and USDEUR returns. $\theta_{B,C}$ is the estimate of the 90° Rotated Clayton under state 2.
Vine structure: *Oil-USDEUR*- State 1: Student, State 2: Gaussian. *USDEUR-EUROSTOXX*-State 1: Independence, State 2: 90° Clayton.

Table 6: Conditional independence test result

	$s_t = 1,\, s_{t-1} = 1$	$s_t = 1,\, s_{t-1} = 2$	$s_t = 2,\, s_{t-1} = 1$	$s_t = 2,\, s_{t-1} = 2$
$\hat{\tau}$	0.1062	0.1080	0.1103	0.1130
a	4.9582	5.0422	5.1518	5.2778
p-value	0.0000	0.0000	0.0000	0.0000

The p-values of the the independence test is built as $p - value = 2(1 - \Phi(a))$ where Φ is the Gaussian c.d.f. and $a = \sqrt{\frac{9T(T-1)}{2(2T+5)}}|\hat{\tau}|$ where T is the sample size, and $\hat{\tau}$ is the empirical Kendall's τ of the conditional distribution of oil and EUROSTOXX returns given a certain quantile of the returns of USDEUR exchange rate (see Brechmann and Schepsmeier (2013)). The conditional distribution is obtained given the best copula fit according to the AICC criterion from Table 4. The conditional independence is rejected for the four regimes.

Table 7: AICC to choose the best model fit for the stage 2 within the vine structure (T_2)

Model		
A	B	C
-13316,45	-13316,39	-13313,64

Notes: $AICC$ denotes Akaike Information Criterion corrected for small sample bias.

$AICC = 2k\frac{T}{T-k-1} - 2\log(\hat{L})$ where T is the sample size, k is the number of estimated parameters and $\hat{L}$ is the Log-likelihood value. Minimum $AICC$ value (in bold letters) indicates the best copula fit.

A: *Oil-USDEUR*- State 1: Student, State 2: Gaussian
USDEUR-EUROSTOXX- State 1: Independence, State 2: 90° Clayton .
Oil-EUROSTOXX|USDEUR- State 1: Gaussian, State 2: Clayton.

B: *Oil-USDEUR*- State 1: Student, State 2: Gaussian
USDEUR-EUROSTOXX- State 1: Independence, State 2: 90° Clayton .
Oil-EUROSTOXX|USDEUR- State 1: Independence, State 2: Clayton.

C: *Oil-USDEUR*- State 1: Student, State 2: Gaussian
USDEUR-EUROSTOXX- State 1: Independence, State 2: 90° Clayton .
Oil-EUROSTOXX|USDEUR- State 1: 90° Gumbel, State 2: Clayton.

Table 8: Model with a complete vine structure

	A	B	C
ϕ_0	0.00 **	-0.00	0.00 ***
	(0.00)	(0.00)	(0.00)
ϕ_1	0.06 *	0.04	-0.05 *
	(0.04)	(0.03)	(0.04)
$\kappa_{s_t=2}$	2.16 ***	2.17 ***	3.77 ***
	(0.24)	(0.23)	(0.43)
α_0	0.00 ***	0.00 ***	0.00 ***
	(0.00)	(0.00)	(0.00)
α_1	0.12 ***	0.09 **	0.15 ***
	(0.04)	(0.04)	(0.06)

State 1		State 2	
T_1 (A,B)			
$\rho_{A,B}$	-0.20 ***	$\rho_{A,B}$	-0.18 ***
	(0.05)		(0.05)
$\nu_{A,B}$	12.54 ***		
	(0.41)		
T_1 (B,C)			
		$\theta_{B,C}$	0.06 **
			(0.03)
T_2 (A,C—B)			
$\rho_{A,C}$	0.08 *	$\theta_{A,C}$	0.14 ***
	(0.05)		(0.05)
p_{11}	0.99 ***	p_{22}	0.98 ***
	(0.01)		(0.01)
		LL	-6671.80

The table reports the estimates and the standard deviation (in parenthesis) for the parameters of the marginal model in Equations (6) and (9) and for the parameters of the best copula choice according to the $AICC$ value reported by Table 7.

LL is the log-Likelihood value.

A: Oil in USD, B: USDEUR exchange rate, C: EUROSTOXX. $\rho_{A,B}$ and $\eta_{A,B}$ is the correlation and number of degrees of freedom between oil in USD and USDEUR returns. $\theta_{B,C}$ is the estimate of the 90° Rotated Clayton under state 2 between USDEUR and EUROSTOXX. $\rho_{A,C}$ is the correlation between oil in USD and EUROSTOXX under state 1 once the dependence between those variables and USDEUR has been considered. $\rho_{A,C}$ is the estimate of the Clayton copula between oil in USD and EUROSTOXX under state 2 once the dependence between those variables and USDEUR has been considered.

Vine structure: *Oil-USDEUR*- State 1: Student, State 2: Gaussian. *USDEUR-EUROSTOXX*-State 1: Independence, State 2: 90° Clayton. *Oil-EUROSTOXX*|*USDEUR*-State 1: Gaussian, State 2: Clayton.

C Algorithm for the simulation process

Algorithm 1 Simulation of dependence under a Vine in dimension N=3 over a time period τ and a copula structure that follows a two-state Switching Markov.

procedure SIM-DEPENDENCE($\theta, P(s_{t-1} = 1|I_T), P(s_t = 1|I_T), p_{11}, p_{22}$)
  for $\omega \leftarrow 1, \ldots, W$ **do**
    if $rand < P(s_{t-1} = 1|I_T)$ **then**
      $state_{1,\omega} = 1$
    else
      $state_{2,\omega} = 2$
    end if
    if $rand < P(s_t = 1|I_T)$ **then**
      $state_{2,\omega} = 1$
    else
      $state_{2,\omega} = 2$
    end if
    for $t \leftarrow 1, \ldots, \tau$ **do**
      if $state_{t+1,\omega} = 1$ **then**
        if $rand < p_{11}$ **then**
          $state_{t+2,\omega} = 1$
        else
          $state_{t+2,\omega} = 2$
        end if
      else
        if $rand < p_{22}$ **then**
          $state_{t+2,\omega} = 2$
        else
          $state_{t+2,\omega} = 1$
        end if
      end if
      $u_{t,\omega,1} = rand$
      $u_{t,\omega,2} = C^{-1}_{2|1}\left(rand|u_{t,\omega,1}; \theta_{state_{t+2,\omega}}\right)$
      for $n \leftarrow 3, \ldots N$ **do**
        $u_{t,\omega,n} = rand$
        for $k \leftarrow 1, \ldots, n-1$ **do**
          $u_{t,\omega,n} = C^{-1}_{n|k}\left(u_{t,\omega,n}|u_{t,\omega,k}; \theta_{state_{t+2,\omega}}\right)$
        end for k
      end for n
    end for t
  end for ω
  Return u and $state$
end procedure

θ_s are the set of parameters for the copula structure under regime s. $P(s_{t-1} = 1|I_T)$ and $P(s_t = 1|I_T)$ are the smoothed probabilities of being in state 1 at $t-1$ and t.
p_{11} and p_{22} are the diagonal values from the transition matrix (see Equation (17)).
$rand$ refers to an uniform-distributed random realization.
The $OUTPUT$ u is a uniform-distributed matrix that has the joint dependence presented in the model.
The $OUTPUT$ $state$ is a matrix that indicates in which regime is the model at each time within each simulation.

Algorithm 2 Simulation from a AR(1)-SWARCH(2,1) over a time period τ and Gaussian distribution assumption for the innovation process.

procedure SIM-PATH$(u, state, \phi_0, \phi_1, \alpha_0, \alpha_1, \kappa_2, r_{T-1:T})$
  for $n \leftarrow 1, \ldots, N$ **do**
    for $w \leftarrow 1, \ldots, W$ **do**
      for $t \leftarrow 1, \ldots, \tau$ **do**
        if $t = 1$ **then**
          $\varepsilon = r_{n,T} - \phi_{n,0} - \phi_{n,1} r_{n,T-1}$
        end if
        if $state_{t,\omega} = 1$ **then**
          $h_{t,\omega,n} = \alpha_{n,0} + \alpha_{n,1}\varepsilon^2$
        else
          $h_{t,\omega,n} = \alpha_{n,0} + \alpha_{n,1}\frac{\varepsilon^2}{\kappa_{n,2}}$
        end if
        if $state_{t+1,\omega} = 1$ **then**
          $\sigma_{t,\omega,n} = \sqrt{h_{t,\omega,n}}$
        else
          $\sigma_{t,\omega,n} = \sqrt{\kappa_{n,2} h_{t,\omega,n}}$
        end if
        $\varepsilon = \Phi^{-1}(u_{t,\omega,n})\sigma_{t,\omega,n}$
        if $t = 1$ **then**
          $r_{t,\omega,n} = \phi_{n,0} + \phi_{n,1} r_{n,T} + \varepsilon$
        else
          $r_{t,\omega,n} = \phi_{n,0} + \phi_{n,1} r_{t-1,\omega,n} + \varepsilon$
        end if
      end for t
    end for w
  end for n
  Return r
end procedure

u is a N-dimension matrix $(TxWxN)$ obtained from Algorithm 1.
ϕ_0 and ϕ_1 are vectors of parameters of length N that drive the dynamic in Equation (6).
α_0, α_1, κ_2 are vectors of parameters of length N that drive the dynamic in Equation (9).
The $OUTPUT$ r is a N-dimension matrix $(\tau xWxN)$ of W simulated paths of length τ for the N returns.

D Copula set for modelling joint distribution

Gaussian and Student copula are elliptical copulas, i.e., the bivariate joint density under these copulas has elliptic isodensities.
Gumbel and Clayton are Archimedean copulas, which implies that can be expressed as a function of the generate function ϕ and its inverse ϕ^{-1}, i.e. $C(u_1, u_2, \theta) = \phi^{-1}[\phi(u_1; \theta) + \phi(u_2; \theta); \theta]$ where θ is the copula parameter.
To enhance the features of copulas that only allow for positive dependence, they are rotated to capture negative tail dependence. The next table shows the tail dependence for the 90° rotated copulas. The 90° rotated copulas are built modifying slightly the standard copula, i.e.

$$C_{90}(u_1, u_2) = u_2 - C(1 - u_1, u_2)$$

Table 9: Tail dependence for the 90° rotated copulas

	$\tau_{L\|U}$	$\tau_{U\|L}$
$90°R$ Clayton	$2^{-1/\theta}$	-
$90°R$ Gumbel	-	$2-2^{1/\theta}$

θ is the parameter from the original copula. Further information about the rotated copula can be found in Brechmann and Schepsmeier (2013), Cech (2006), Georges et al. (2001) and Luo (2010).

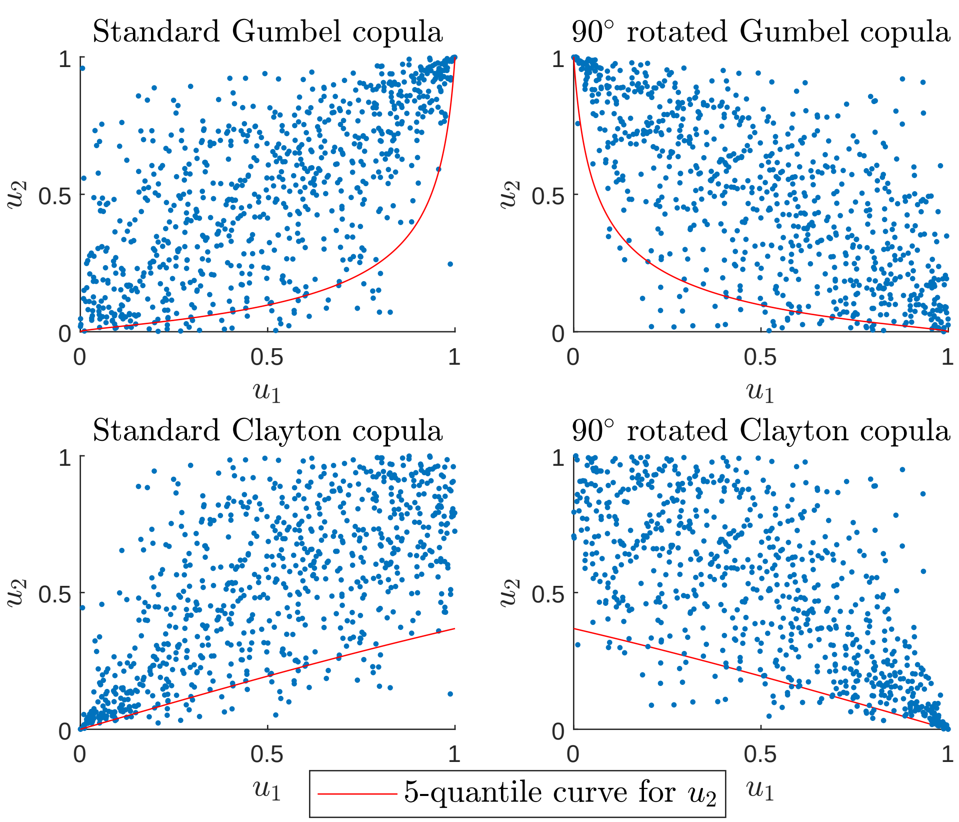
Let u_1 and u_2 denote two variables uniformly distributed across (0,1).

The negative lower tail dependence, $\tau_{L|U}$, is defined as $\tau_{L|U} = lim_{q\to 0}P(u_2 < q|u_1 > 1-q)$.

The negative upper tail dependence, $\tau_{U|L}$ is defined as $\tau_{U|L} = lim_{q\to 1}P(u_2 > q|u_1 < 1-q)$.

Figure 18 shows an example of how change the distribution and the tail joint behaviour when the 90° rotated copula is employed. See Zhang (2008) for further details about negative tail dependence.

Figure 18: Rotated copulas employed to capture negative tail dependence



This figure shows 800 simulations from the same seed but under different copula assumptions. Rotating 90 degrees allows us to capture negative upper tail dependence (90° rotated Gumbel), negative lower tail dependence (90° rotated Clayton). The red line indicates the threshold below which the 5% of the u_2 are found given the values taken by u_1. Gumbel and Clayton copula has a copula parameter $\theta = 2$.

Gaussian copula. This copula has a parameter ρ that gathers linear correlation. When $\rho = 1$ the tail dependence is 1, otherwise this copula does not present tail dependence. There is not a closed form expression due to the fact that Gaussian copula is an implicit copula. Meyer (2013) takes a in-depth look

at this copula.
The copula probability density function is

$$c(u_1, u_2; \rho) \quad = \quad \frac{1}{\sqrt{1-\rho^2}} \exp\left\{ -\frac{\rho^2 \Phi^{-1}(u_1)^2 - 2\rho\Phi^{-1}(u_1)\Phi^{-1}(u_2) + \rho^2\Phi^{-1}(u_2)^2}{2(1-\rho^2)} \right\},$$

where Φ^{-1} stands for the Gaussian inverse cumulative distribution function.
The conditional copula $C_{2|1}(u_2|u_1; \rho)$ is

$$\Phi\left(\frac{\Phi^{-1}(u_2) - \rho\Phi^{-1}(u_1)}{\sqrt{1-\rho^2}} \right).$$

Student copula. This copula allows for positive and negative symmetric tail dependence. The parameter ρ measures correlation and the parameter η, the number of degrees of freedom, controls the probability mass assigned to extreme joint co-movements of risk factors changes.[11] When $\eta \to \infty$ corresponds to the Gaussian copula.[12] Student copula has not a closed form because it is aN implicit copula.
The copula probability density function is

$$\begin{aligned} c(u_1, u_2; \eta, \rho) \quad = \quad & K \frac{1}{\sqrt{1-\rho^2}} \\ & \left[1 + \frac{T_\eta^{-1}(u_1)^2 - 2\rho T_\eta^{-1}(u_1) T_\eta^{-1}(u_2) + T_\eta^{-1}(u_2)^2}{\eta(1-\rho^2)} \right]^{-\frac{\eta+2}{2}} \\ & \left[(1 + \eta^{-1} T_\eta^{-1}(u_1)^2)(1 + \eta^{-1} T_\eta^{-1}(u_2)^2) \right]^{\frac{\eta+1}{2}}, \end{aligned}$$

where $K = \Gamma(\frac{\eta}{2})\Gamma(\frac{\eta+1}{2})^{-2}\Gamma(\frac{\eta+2}{2})$.
The conditional copula $C_{2|1}(u_2|u_1; \rho, \eta)$ is

$$T_{\eta+1}\left(\sqrt{\frac{\eta+1}{\eta + (T_\eta^{-1}(u_1))^2}} \frac{T_\eta^{-1}(u_2) - \rho T_\eta^{-1}(u_1)}{\sqrt{1-\rho^2}} \right)$$

where T_η is the cdf of a t-Student with the numbers of degrees of freedom equal to η and T_η^{-1} represents its inverse [13]

Clayton copula. This copula allows positive dependence and asymmetric lower tail dependence. The Clayton copula has a dependence parameter $\theta \in (0, +\infty)$. When $\theta \to 0$ implies independence and when $\theta \to \infty$ implies perfect dependence.
The Clayton copula is

$$C(u_1, u_2; \theta) = \left(u_1^{-\theta} + u_2^{-\theta} - 1 \right)^{-1/\theta},$$

and the copula probability density function is

$$c(u_1, u_2; \theta) = (\theta + 1) \left(u_1^{-\theta} + u_2^{-\theta} - 1 \right)^{-2-\frac{1}{\theta}} (u_1 u_2)^{-\theta-1}.$$

The conditional copula $C_{2|1}(u_2|u_1; \theta)$ is

$$\left(u_1^{-\theta} + u_2^{-\theta} - 1 \right)^{-\frac{1+\theta}{\theta}} u_1^{-\theta-1}$$

[11] For more information about the properties of the t-Student copula see Demarta and McNeil (2005)

[12] The Gaussian copula underestimates the probability of joint extreme co-movements in high volatility and correlation scenarios (see Aussenegg and Cech (2011))

[13] See for instance Cech (2006)

Gumbel copula. This copula allows for positive dependence and asymmetric upper tail dependence. The Gumbel copula has a dependence parameter $\theta \in [1, +\infty)$. When $\theta = 1$ implies independence and when $\theta \to \infty$ implies perfect dependence.
The Gumbel copula is

$$C(u_1, u_2; \theta) = \exp\left(-\left\{(-\log u_1)^\theta + (-\log u_2)^\theta\right\}^{1/\theta}\right),$$

and the copula probability density function is

$$\begin{aligned} c(u_1, u_2; \theta) &= (A + \theta - 1)\, A^{1-2\theta} \exp(-A) \\ &\quad (u_1 u_2)^{-1}(-\log u_1)^{\theta-1}(-\log u_2)^{\theta-1}, \end{aligned}$$

where $A = \left[(-\log u_1)^\theta + (-\log u_2)^\theta\right]^{\frac{1}{\theta}}$.
The conditional copula $C_{2|1}(u_2|u_1; \theta)$ is

$$\exp\left(-\left\{(-\log u_1)^\theta + (-\log u_2)^\theta\right\}^{1/\theta}\right)\left\{(-\log u_1)^\theta + (-\log u_2)^\theta\right\}^{1/\theta-1}(-\log u_1)^{\theta-1}\frac{1}{u_1}$$

E Considering the role of the exchange rate in a bullish scenario for oil returns in euros

Following Ojea Ferreiro (2019), I define the bullish $CoVaR_{s|oe}(\alpha, \beta)$ as the $\beta 100\%$ lowest stock returns given that oil returns in euros are above its α quantile, i.e.

$$\begin{aligned} P\left(r_s < CoVaR_{s|oe} | r_{oe} > VaR_{oe}(\alpha)\right) &= \frac{P\left(r_m < CoVaR_{s|oe}, r_{oe} > VaR_{oe}(\alpha)\right)}{P(r_{oe} > VaR_{oe}(\alpha))} \\ &= \beta, \end{aligned}$$

where $P(r_{oe} > VaR_{oe}(\alpha)) = 1 - \alpha$.
Following the same reasoning that in Subsection 2.3 for a given lower bound q_c for the quantile of the exchange rate returns we get

$$r^*_{oe}(\alpha) \leq F_c^{-1}(q_c) + r_o,$$

where $r^*_{oe} = VaR_{oe}(\alpha)$. Consequently, oil returns denominated in US dollars should be greater

$$r_o \geq r^*_{oe} - F_c^{-1}(q_c)$$

which in terms of quantiles would be

$$\begin{aligned} P\left(r_o \geq r^*_{oe} - F_c^{-1}(q_c)\right) &= 1 - F_o(r^*_{oe} - F_c^{-1}(q_c)) \\ &= 1 - q_o. \end{aligned} \tag{27}$$

Hence, the bullish $CoVaR(\alpha, \beta)$ when the exchange rate returns are above its q_c100-th quantile would be obtained implicitly from

$$\begin{aligned} P\left(r_s < CoVaR_{s|oe} | r_{oe} > VaR_{oe}(\alpha), r_c > VaR_c(q_c)\right) &= \frac{P\left(r_s < CoVaR_{s|oe,c}, r_{oe} > VaR_{oe}(\alpha), r_c > VaR_c(q_c)\right)}{P(r_{oe} > VaR_{oe}(\alpha), r_c > VaR_c(q_c))} \\ &= \beta. \end{aligned}$$

Taking into account the chosen vine copula structure, where the first link between the variables arises from a common exposure to the exchange rate while the direct relationship between oil and stock returns is modelled once this connection through the exchange rate has been considered, we get the following expression

$$\frac{\int_{q_c}^{1} C_{s|c}(F_s(CoVaR_{s|oe,c})|u) - C_{s,o|c}\left(C_{s|c}(F_s(CoVaR_{s|oe,c})|u), C_{o|c}(q_o|u)\right) du}{1 - q_o - q_c + C_{o,c}(q_o, q_c)} = \beta, \tag{28}$$

where the probabilities of being above the threshold are obtained considering the rotation of copulas.[14]

[14] See Ojea Ferreiro (Ojea Ferreiro) as a reference on this topic.