Estimation of Stochastic Volatility Models with Jumps in Returns for Stock Market Indices

Ana González[a,*], Alfonso Novales[b] and Gonzalo Rubio[c]

[a]*Fundamentos del Análisis Económico II, Universidad del País Vasco, Bilbao, Spain;*
[b]*Economía Cuantitativa, Universidad Complutense, Madrid, Spain;*
[c]*Economía y Empresa, Universidad CEU Cardenal Herrera, Elche, Spain*

This version: May 30, 2011

Using the Efficient Method of Moments we estimate a continuous time diffusion for the stochastic volatility of some international stock market indices that allows for possible jumps in returns. These jumps are needed for a sensible characterization of the dynamics of the distribution of returns, even under stochastic volatility. Although the stochastic volatility model with jumps in returns tends to exaggerate the negative skewness relative to the sample moments, the inclusion of jumps strongly improves the ability of the model to replicate sample kurtosis. This contrasts with the failure of the pure stochastic volatility model in generating high enough kurtosis. Our results extend the limited available evidence from the U.S. market to several European stock market indices.

Keywords: stochastic volatility; jumps; Efficient Method of Moments EMM; SNP; Hermite polynomials; asymmetry; kurtosis

JEL Classification: C13; C14; C15; C32

1 Introduction

Financial economists achieved unprecedented success over the last thirty years using simple diffusion models to approximate the stochastic process for returns on financial assets. The so-called *volatility smiles and smirks* computed using the volatility implied by the Black-Scholes model reveal, however, that a simple geometric Brownian motion process misses some important features of the data. This limitation is very relevant, since empirical evidence suggests that practical financial decision making based on the continuous time setting will be satisfactory only if it builds upon reasonable specifications of the underlying asset price processes. In other words, the actual distribution of the underlying asset implied by the data must be consistent with the distribution assumed by the theoretical model.

[*]Corresponding author. Email: ana.gonzalezu@ehu.es

High frequency return data displays excess kurtosis (fat tailed distributions), skewness, and volatility clustering. Capturing these essential characteristics with a tractable parsimonious parametric model is difficult, but it is widely accepted that incorporating stochastic volatility or jumps into continuous time diffusion processes can help explain these main statistical characteristics of observed financial returns. Unfortunately, existing results for U.S. data have so far been inconclusive or contradictory, and most studies fail to produce a satisfactory fit to the underlying asset return dynamics.

Andersen, Benzoni and Lund (2002), Chernov et al. (2003) and Eraker, Johannes and Polson (2003), among others, estimate models with stochastic volatility, jumps in prices, and in the latter two papers, jumps in volatility. All of them find strong evidence for stochastic volatility and jumps in prices, but they disagree over the presence and importance of jumps in volatility, and over the convenience to allow for state-dependent arrival of jumps. The available evidence for U.S. data consistently find that allowing for jumps in returns helps matching the observed distribution of returns with relatively smooth volatility. If the process does not allow for jumps, then replication of sample kurtosis requires a higher volatility of the stochastic variance process, to compensate for the absence of jumps. In addition to its kurtosis, the jump-diffusion process allows for two sources of skewness: a nonzero (usually negative) mean jump and the negative correlation between the shocks in returns and volatility. Both features help the model to match the negative skewness observed in sample moments. However, it should be pointed out that stochastic volatility is also important. If we do not allow for stochastic volatility, the estimated frequency of jumps is extraordinarily large, to compensate for the misspecification in the variance process.

There is also disagreement regarding the correct specification of the underlying security process among studies that use option prices instead of time series of returns. Here, the specification is evaluated by fitting option prices, as opposed to fitting the conditional distribution of the return time series. Bakshi, Cao and Chen (1997) find that both stochastic volatility and jumps in prices are important for pricing options although stochastic volatility seems to be enough to design efficient hedging positions in options. On the other hand, Bates (2000), Eraker (2004) and Pan (2002) conclude that jumps in returns are economically small, even with negligible benefits for the pricing of options. A problem with these studies is not only the use of a very different sample period, but also the different estimation methods. More recently, Broadie, Chernov and Johannes (2007) avoid these critiques and, using a long period of option prices, argue that the evidence favours an economically and statistically significant jump risk premia in returns, showing how important are these premia for S&P 500 futures option prices.

The goal of this paper is to identify a diffusion stochastic volatility model with possible jumps in returns that might be successful in approximating the S&P 500 return dynamics as well as some European indices: DAX 30, IBEX 35 and CAC 40. Such model should constitute an adequate basis for continuous time asset pricing applications and therefore, we analyze whether the estimates can reproduce important aspects of the distribution of returns like third and fourth order moments. It is surprising the lack of evidence available regarding the behaviour of continuous-time models for European return indices, which is obviously important for asset allocation or for pricing derivatives. This paper fills that gap.

Until recently, a major obstacle for testing continuous-time models of equity returns was the lack of feasible techniques for estimating and drawing inference on general continuous time models using discrete observations. The main difficulty is that closed form expressions for the discrete transition density generally are not available, especially in the presence of unobserved and serially correlated state variables, as it is the case in stochastic volatility models. One way to respond to this challenge is the Simulated Method of Moments (SMM hereafter) of Duffie and Singleton (1993) that matches sample moments with simulated moments computed from a long time series obtained from the assumed data generating mechanism, also known as the structural model. Together with the Markov Chain Monte Carlo Bayesian estimator (MCMC), the SMM method is increasingly used because of their tractability and potential econometric efficiency, especially in situations with latent variables or under complex specifications of the jump component.

We adopt a variant of the SMM known as the Efficient Method of Moments (EMM hereafter), proposed by Bansal et al. (1993, 1995) and developed by Gallant and Tauchen (1996). EMM is a simulation based moment matching procedure with certain advantages. The moments to be matched are the scores of an auxiliary model called the score generator. As shown by Tauchen (1997) and Gallant and Long (1997), if the score generator is able to approximate the probability distribution of return data reasonably well, then estimates of the parameters of the structural model are as efficient as maximum likelihood estimates.

We find that adding jumps in returns to the stochastic volatility diffusion is needed to explain the statistical characteristics of the return time series. We reject the pure diffusion model with stochastic volatility for all four stock market indices. However, the overall fit of the model improves significantly when we add jumps in returns. In fact, we are not able to reject the jump-diffusion model with stochastic volatility for the S&P 500, the IBEX 35 and the DAX 30 at 5% significance, rejecting it for the CAC 40, which has a p-value of 0.023. Hence, we would not reject the model for either index at 1% significance. Additionally, under the pure stochastic volatility model, the implied skewness is notoriously higher in absolute value than the sample third moment, except for the U.S. market where sample and model skewness are close to each other. On the other hand, the kurtosis generated by the estimated model falls in all cases well below the sample kurtosis. To allow for volatility of variance does not seem to be enough to capture the high levels of kurtosis observed in the sample, which may explain the systematic rejection of this specification for the four indices. Regarding the model with jumps in equity returns, the negative skewness produced by the model is again higher in absolute value than the sample skewness. However, jumps in equity returns help replicating extraordinary well the observed kurtosis in data in the case of European markets, which may explain the overall satisfactory fit of the jumps in returns specification. We may therefore conclude that the stochastic volatility model with jumps in returns replicates much more precisely sample kurtosis than the negative skewness observed in the data.

The paper is organized as follows. Section two presents the continuous time models for stock returns. Section three describes the data employed in the paper, while section four documents the details of the EMM methodology. Empirical results are given in section five. Finally, section six contains the concluding remarks.

2 Model Specification

2.1 Stochastic Volatility (SV) Model

A natural extension of the diffusion models widely applied in the asset pricing literature incorporates stochastic volatility to accommodate the clusters of volatility usually observed in stock market returns.[1] This feature can explain broad general characteristics of actual return data, such as leptokurtosis and persistent volatility, and it is potentially

---

[1] Stochastic volatility models are initially suggested by Clark (1973), Tauchen and Pitts (1983), and Taylor (1986).

useful in pricing derivatives. Hull and White (1987), Melino and Turnbull (1990), and Wiggins (1987) generalize the traditional geometric Brownian motion specification underlying the Black-Scholes expression by allowing for stochastic volatility to price equity and currency options. In a key contribution to literature, Heston (1993) allows for correlation between the Brownian motions in the mean and the variance equations, obtaining closed form expressions for option valuation using the Fourier inverse transform of the conditional characteristic function. In particular, Heston (1993) allows for a volatility risk premium that is proportional to the square root of the stochastic variance. This is the specification we employ in this research.

Let S_t be the price at t of a stock market index, with $s_t = \ln S_t$. The square root stochastic volatility model (SV) is given by,

$$ds_t = \left(\mu - \frac{V_t}{2}\right)dt + \sqrt{V_t}\, dW_{1,t} \tag{1}$$

where the variance V follows a diffusion process with mean reversion in levels:

$$dV_t = (\alpha - \beta V_t)dt + \eta\sqrt{V_t}\, dW_{2,t} \tag{2}$$

with W_1, W_2 being correlated standard Brownian motions, $corr\left(dW_{1,t}, dW_{2,t}\right) = \rho dt$.

Stochastic volatility induces an excess of kurtosis through the values of α, β and η. The parameter β measures the speed at which the process reverts to the long-term variance (α/β), and it captures the persistence in variance. If the variance is highly volatile, i.e., if η is large, the probability of observing large shocks in returns will increase, and the tails of the distribution will be thicker. The asymmetry usually observed in returns can be captured through a negative correlation between shocks in variance and in returns, $\rho < 0$. That way, volatility will increase when prices go down, thereby increasing the likelihood of large negative returns.

We obtain the first-order Euler discretization of our structural continuous-time diffusion process,

$$s_t = s_{t-\Delta} + \left(\mu - \frac{V_{t-\Delta}}{2}\right)\Delta + \sqrt{V_{t-\Delta}}\sqrt{\Delta} z_{1,t-\Delta}\,, \tag{3}$$

together with either:

$$V_t = V_{t-\Delta} + \left(\alpha - \beta V_{t-\Delta}\right)\Delta + \eta\sqrt{V_{t-\Delta}}\sqrt{\Delta}\, z_{2,t-\Delta} \tag{4}$$

where M is the number of subperiods considered each day, $\Delta = \frac{1}{M}$, and $\left(z_{1,t}, z_{2,t}\right)$ are $N(0,1)$ with correlation ρ,

$$\begin{pmatrix} dW_{1,t} \\ dW_{2,t} \end{pmatrix} \approx \begin{pmatrix} z_{1,t} \\ z_{2,t} \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right] \tag{5}$$

This discrete-time representation will provide us with simulated time series for returns that will be used to estimate the parameter vector with a better match to the score vector of an auxiliary model fitted to the data,

2.2 Stochastic Volatility Model with Jumps (SVJ)

It has recently become evident that success in fitting the dynamics of conditional volatility does not guarantee a good fit of the high conditional kurtosis in returns that is observed in many financial assets.[2]

We therefore add a jump component to the previous specification,

$$ds_t = \left(\mu - \lambda_t \bar{k} - \frac{V_t}{2}\right)dt + \sqrt{V_t}\, dW_{1,t} + \ln\left(1 + k_t\right)dq_t \tag{6}$$

where the variance process V follows a mean-reverting diffusion as (2). We denote by q_t a Poisson process, uncorrelated with W_1 and W_2, with a jump intensity λ_t, so that $\Pr\left(dq_t = 1\right) = \lambda_t dt$, and we assume a constant jump intensity, $\lambda_t = \lambda_0$. The size of a jump at time *t*, if it occurs, is denoted by k_t.

[2] See Singleton (2006) for a review of the literature.

We assume that the size of the jump process follows a Normal distribution: $\ln\left(1+k_t\right) \approx N\left(\ln(1+\bar{k}) - 0.5\delta^2, \delta^2\right)$. Finally, $\bar{k}$ is the average size of a jump, so that the average growth rate due to jumps is $\lambda_t \bar{k}$. The correction $\lambda_t \bar{k} dt$ in the drift compensates the non-zero mean of the jump component. A negative (positive) value of $\bar{k}$ implies negative (positive) asymmetry. Hence, the stochastic volatility model with jumps has two sources of asymmetry, the average size of jumps, $\bar{k}$, and a possible negative correlation between the two Brownian motions. Nevertheless, as a first approximation, and following Andersen, Benzoni and Lund (2002), we impose the restriction $\bar{k} = 0$ in the estimation, since this is generally a poorly identified parameter. Even though we lose the contribution of $\bar{k}$ to a negative skewness, imposing $\bar{k} = 0$ leads to a mean jump size of $-0.5\delta^2$, implying more high negative jumps than positive ones, again contributing to negative asymmetry.

Hence, the model we simulate is:

$$
\begin{aligned}
s_t &= s_{t-\Delta} + \left(\mu - \frac{V_{t-\Delta}}{2}\right)\Delta + \sqrt{V_{t-\Delta}}\sqrt{\Delta} z_{1,t} + J_t \\
V_t &= V_{t-\Delta} + \left(\alpha - \beta V_{t-\Delta}\right)\Delta + \eta\sqrt{V_{t-\Delta}}\sqrt{\Delta} z_{2,t} \quad , \\
z_{1,t} &= \xi_{1,t} \\
z_{2,t} &= \rho \xi_{2,t} + \sqrt{1-\rho^2}\, \xi_{2,t}
\end{aligned}
\tag{7}
$$

where $\xi_{1,t}, \xi_{2,t}, i.i.d. \sim N(0,1), corr(\xi_{1,t}, \xi_{2,t}) = 0$. We will start by simulating time series data for ξ_{1t}, ξ_{2t}, and for $\ln\left(1+k_t\right)$ using the Normal distribution we described above. From them, we easily get time series realizations for z_{1t}, z_{2t}. We start the simulations from an initial price: $S_0 = 100$, and stochastic volatility V_t equal to its unconditional mean, $V_0 = \alpha / \beta$. Once we have observations for logged prices and volatility for the 10 subperiods considered each day, we compute log returns each subperiod $r_t = s_t - s_{t-\Delta}$, and add to them the jump component, whenever it is different from zero.

The jump component is obtained as:

$$J_t = I(U(0,1) < \lambda_0 \Delta).\ln\left(1+k_t\right) \quad (8)$$

where I denotes an indicator function that takes a value of 1 when the condition in brackets holds, and it is equal to zero otherwise. Jumps are added to each return when they happen to materialize. In principle, it is possible that more than one jump occurs in a single day, although the probability of such event is very small. Finally, we aggregate log returns over each market day.

3 Data

We have daily from January 3, 1988 to December 30, 2010, with 5799 sample observations data for S&P 500, 5915 observations for DAX 30, 5764 for IBEX 35 and 5806 observations for CAC 40. Table 1 shows the sample mean, standard deviation, asymmetry, kurtosis, and minimum and maximum values for the daily return series (log returns in percentage form), as well as the augmented Dickey-Fuller statistic (ADF) for the stock market index and for the returns on the index. The ADF statistic suggests that returns are stationary, and they are shown in Figure 1. Index returns display important kurtosis and negative asymmetry, so the data generating process must be able to produce these same statistical characteristics in simulated returns.

4 EMM Estimation Methodology

We start by computing the quasi-maximum likelihood (QML) estimate of the parameters in the conditional density of index returns, which is approximated by a semi-nonparametric (SNP) density function.[3] The approximation considers an auxiliary

[3] Gallant and Long (1997) show that among discrete time models, SNP densities, proposed by Gallant and Tauchen (1989), provide such an approximation. The estimated SNP density is also a consistent estimator

model made up by a constant plus a MA(1) innovation for the conditional mean, and a GARCH(1,1) representation for the conditional variance for the residuals from the previous estimation. As in Andersen, Benzoni and Lund (2002), we start by prefiltering the return data series, $\tilde{s}_t$, using a simple MA(1) model, $\tilde{s}_t = \theta_0 + \varepsilon_t - \theta_1\varepsilon_{t-1}$, and rescaling the residuals, so that $\tilde{r}_t = E(\tilde{s}_t) + stdev(\tilde{s}_t)\frac{\hat{\varepsilon}_t - E(\hat{\varepsilon}_t)}{stdev(\hat{\varepsilon}_t)}$, to match the sample mean and variance in the original data set. This residual series $\tilde{r}_t$ is then treated as the observed return process, to which we fit a GARCH(1,1) model. We also include a number of Hermite polynomials in the SNP model to make up the non-parametric term in the approximation to the density function in order to adapt to all non-Gaussian features of the process.

The election of model is not arbitrary. There is almost no serial correlation structure in daily returns, and it can be appropriately captured by an MA term. An alternative would be to consider a long-memory process, given the evidence that has recently been provided in the literature in that respect. In our simple treatment of returns, we start by estimating the MA term to fit a GARCH structure for the conditional variance of the series of residuals from that estimation. Proceeding this way, we can focus just on the variance dynamics.

The SNP density f_K takes the form

$$f_K(r_t \mid x_t;\xi) = \left(\nu + (1-\nu)\frac{[P_K(z_t, x_t)]^2}{\int_R [P_K(u, x_t)]^2 \varphi(u)du} \right)\frac{\varphi(z_t)}{\sqrt{h_t}} \qquad (9)$$

where $r_t, x_t = (r_{t-1},...,r_{t-L}), t = 1,...,\infty$ are the random variables corresponding to the index return process and lags of those returns, $\varphi(\cdot)$ denotes the standard normal density, ν is a

(Gallant and Nychka 1987), efficient (Fenton and Gallant 1996a; Gallant and Long 1997) and with desirable qualitative features (Fenton and Gallant 1996b).

small constant (0.01)[4], $z_t = \frac{r_t - \upsilon_t}{\sqrt{h_t}}$ is the standardized process of daily returns, which is supposed to be i.i.d., and υ_t and h_t are the conditional mean and variance of the auxiliary model, which are given by

$$\begin{aligned} \upsilon_t &= 0 \\ h_t &= \gamma_0 + \gamma_1 r_{t-1}^2 + \vartheta_1 h_{t-1} \sim GARCH(1,1) \end{aligned} \tag{10}$$

and the polynomial:

$$P_K(z,x) = \sum_{i=0}^{K_z} a_i(x) z^i = \sum_{i=0}^{K_z} \left(\sum_{j=0}^{K_x} a_{ij} x^j \right) z^i, \quad a_{00} = 1 \tag{11}$$

is a K_z-th order polynomial in z, whose coefficients are represented by a polynomial of degree K_x in x. The condition a_{00} = 1 is imposed for identification purposes (in order to obtain a unique representation). Specifically, we employ polynomials that are of Hermite form and we fix K_x equal to zero, which induces a time-homogeneous non-Gaussian error structure, letting $K_z > 0$. So that the polynomial becomes

$$P_K(z,x) = \sum_{i=0}^{K_z} a_i \overline{He_i}(z), \quad a_0 = 1 \tag{12}$$

and $\overline{He_i}(z)$ is the orthogonal Hermite polynomial of degree *i*.

With this normalization, the f_K density is interpreted as an expansion whose leading term is the Normal density $\varphi(\cdot)$, while higher order terms adapt to minor deviations from the Normal. In fact, the main task of the nonparametric polynomial expansion in the conditional density is to capture any excess kurtosis in the return process and any asymmetry which has not already been accommodated by the leading term.

---

[4] This constant ν is used to avoid numerical problems during EMM estimation, guaranteeing $P_K(z_t, x_t)$ not to be zero.

The parameters $\xi = \left(\gamma_0, \gamma_1, \vartheta_1, a_1, a_2, \ldots, a_{K_z}\right)$ of the auxiliary model are estimated by QML by solving the problem:

$$\tilde{\xi} = \arg\max_{\xi} \frac{1}{n} \sum_{t=1}^{n} \ln\left[f_K\left(\tilde{r}_t \mid \tilde{x}_t ; \xi\right)\right] \tag{13}$$

where $\tilde{r}_t, \tilde{x}_t = \left(\tilde{r}_{t-1}, \ldots, \tilde{r}_{t-L}\right), t = 1,\ldots,n$ are the observed data, and n denotes the sample size.

The second step of the estimation procedure consists of estimating the parameters of the structural model. We search for a parameter vector that allows the assumed diffusion to capture the main statistical characteristics in the data. This possibility is measured through the expected value of the QML gradient, $m\left(\psi, \tilde{\xi}\right) = \int \frac{\partial \ln f_K\left(r \middle| x; \tilde{\xi}\right)}{\partial \xi} dP\left(r \middle| x; \psi\right)$. For this, we use the sample moment,

$$m_N\left(\psi, \tilde{\xi}\right) = \frac{1}{N} \sum_{t=1}^{N} \frac{\partial \ln f_K\left(\hat{r}_t\left(\psi\right) \middle| \hat{x}_t\left(\psi\right); \tilde{\xi}\right)}{\partial \xi} \tag{14}$$

Here, a GMM estimation technique is used, minimizing a quadratic form made up with the mathematical expectation of the elements of the gradient of the likelihood function, and an appropriate weighting matrix. Minimization of the quadratic form needs to be implemented by simulation, since it is not feasible to compute the analytical expression for the gradient of the likelihood under the structural model. Let $\left\{\hat{r}_t\left(\psi\right), \hat{x}_t\left(\psi\right)\right\}_{t=1}^{N}$ denote a sample simulated from the structural model using the parameter vector ψ. The EMM estimator of ψ is then defined by

$$\hat{\psi} = \arg\min_{\psi}\left[m_N\left(\psi, \tilde{\xi}\right)' \tilde{I}^{-1} m_N\left(\psi, \tilde{\xi}\right)\right] \tag{15}$$

where, as explained above, $m_N\left(\psi, \tilde{\xi}\right)$ is the expectation of the score function of the auxiliary model, evaluated by Monte Carlo integration at the quasi-maximum likelihood estimate of the parameter vector $\tilde{\xi}$ in the auxiliary model, and the weighting matrix $\tilde{I}^{-1}$ is a consistent estimate of the asymptotic covariance matrix of the density f_K, which is estimated from the outer product of the gradient:

$$\tilde{I} = \frac{1}{n}\sum_{t=1}^{n}\left[\frac{\partial \ln f_K\left(\tilde{r}_t \middle| \tilde{x}_t ; \tilde{\xi}\right)}{\partial \xi}\right]\left[\frac{\partial \ln f_K\left(\tilde{r}_t \middle| \tilde{x}_t ; \tilde{\xi}\right)}{\partial \xi}\right]' \tag{16}$$

Therefore, this approach is similar in some aspects to the *Simulated Method of Moments* (SMM) of Duffie and Singleton (1993). The expectation of the score function for the auxiliary model provides the moment conditions for the Simulated Method of Moment estimation of the continuous time structural model.

We use $M = 10, \Delta = 1/10$, $N_1 = N_2 =$ 1000 and $N =$ 10000.[5] N must be large enough so that the Monte Carlo simulation error in the gradient of the log likelihood can be considered to be negligible. The problem is that we would literally need *millions* of observations so that the error is insignificant as discussed by Andersen and Lund, 1997. We also use the variance reduction technique of antithetic variables suggested by Geweke, 1996, which is quite effective as shown, among others, by Andersen and Lund, (1997). The idea is to average two estimations of the integral $m\left(\psi, \tilde{\xi}\right) = \int \frac{\partial \ln f_K\left(r \middle| x; \tilde{\xi}\right)}{\partial \xi} dP\left(r \middle| x; \psi\right)$ which are supposed to be negatively correlated. We first compute the gradient of the likelihood using random variables $\left(z_{1,t}, z_{2,t}\right)$, $m_N^{(z_{1t}, z_{2t})}\left(\psi, \tilde{\xi}\right)$. The second estimation, $m_N^{(-z_{1t}, -z_{2t})}\left(\psi, \tilde{\xi}\right)$, is computed using the same random numbers with the opposite sign: $\left(-z_{1,t}, -z_{2,t}\right)$. Finally,

[5] The Euler approximation with $M =$ 1 is frequently used to estimate parameters in stochastic differential equations from discrete observed data. To estimate by simulation, a value $M >$ 1 is needed to reduce the discretization bias (Kloeden and Platen, 1992). An open question would be to examine the behavior of estimates as the number of subperiods per day, *M*, increases.

$m_N\left(\psi,\tilde{\xi}\right)=\frac{1}{2}\left[m_N^{(z_{1t},z_{2t})}\left(\psi,\tilde{\xi}\right)+m_N^{(-z_{1t},-z_{2t})}\left(\psi,\tilde{\xi}\right)\right]$. The use of antithetic realizations also helps reducing the discretization bias.

Hence, in simulating the return sequence $\{\hat{r}_t(\psi)\}_{t=1}^N$, two antithetic samples of $(N+N_2)\times M+N_1=(10000+1000)\times 10+1000$ log-returns are generated using an Euler approximation (of order one) from the continuous time model at time intervals of $1/10$ of a day. We discard the first N_1 simulated values of s_t to eliminate the effect of the initial conditions, so that we can think of the time series for s_t as coming from a stationary distribution. Then, a sequence of 11000 daily returns is obtained by summing the elements of the simulated sample in groups of 10. Again to eliminate the potential bias that might be produced by the random number generator, we discard the first N_2 observations of those returns, to obtain a stationary path of the score $m_N\left(\psi,\tilde{\xi}\right)$. In estimation, we maintain fixed the realization of the two fundamental $N(0,1)$ innovations. The realizations for $(z_{1,t},z_{2,t})$ will nevertheless change, because the parameter vector is changing in each iteration of the algorithm.

In our choice of auxiliary model, there are 3 parameters in the parametric component of the auxiliary model and 6 coefficients in the Hermite polynomials, for a total of 9 parameters. On the other hand the structural model has 5 parameters if we do not include jumps in returns, and 7 parameters if jumps are considered. Hence, the identification condition $\dim(\psi)\leq\dim(\xi)$ holds, and we can proceed to implement the global specification test.

Once we get the estimate of the parameters in the auxiliary model, the minimized value of the objective function follows a chi-distribution:

$$\chi^2 = n \cdot m_N\left(\hat{\psi}, \tilde{\xi}\right)' \tilde{I}^{-1} m_N\left(\hat{\psi}, \tilde{\xi}\right) = n \cdot f\left(\hat{\psi}, \tilde{\xi}\right) \tag{17}$$

where $f\left(\hat{\psi}, \tilde{\xi}\right)$ is the numerical value of the objective function at the final estimate and we can implement a global specification test by comparing the statistic above with the appropriate percentile of a Chi-square distribution with $\dim(\xi) - \dim(\psi)$ degrees of freedom, which is 4 or 2, depending on whether we consider the basic stochastic volatility model or the specification with jumps in returns. We can also compute t-ratios for the individual elements of the score, by dividing their estimates by their standard errors,

$$\hat{t} = \frac{m_N\left(\hat{\psi}, \tilde{\xi}\right)}{\sqrt{diag(S)}} \tag{18}$$

where $S = \frac{1}{n}\left(\tilde{I} - M_\psi \left(M_\psi' \tilde{I}^{-1} M_\psi\right)^{-1} M_\psi'\right)$ and $M_\psi = \frac{\partial m_N\left(\hat{\psi}, \tilde{\xi}\right)}{\partial \psi}$, that must be computed by numerical differentiation. Individual significance tests for these components can throw some light on the appropriateness of the auxiliary ability of the model to capture the main statistical features of the structural model, or some features of the data that the model cannot approximate.

5 Empirical Results

For each stock market index, we start by estimating the symmetric GARCH(1,1) auxiliary model, with parameter estimates and the corresponding standard errors shown in Table 2. We present estimates for the pure GARCH(1,1) model as well as for the SNP specification. As expected, volatility displays high persistence in the four indices, and the long term GARCH volatility is close to the sample standard deviation, reflecting the fact that the model specification allows for almost no predictability of daily returns.[6] By and large, estimated parameters in the SNP density are statistically significant.

Once we have numerical estimates for the auxiliary model, we can proceed to estimating the parameters in the two structural models, SV and SVJ. Table 3 displays results in daily percent terms for each index and each structural model. Panel A shows parameter values and the minimized value of the objective function, together with the corresponding Chi-square statistic, while Panel B shows values for the t-ratios for the score vector, together with their p-values. Panel C compare sample moments to those obtained from the simulated time series from the estimated structural model.

Most parameter estimates are statistically significant for the models fitted to DAX 30 and CAC 40 returns, while the opposite is the case for S&P 500 and IBEX 35. By comparing estimated standard deviations for the former and the latter indices, we can see that it is a problem of loss precision, i.e., high standard deviations in estimating the models for S&P 500 and IBEX 35. It is particularly encouraging that the estimates of the two parameters characterizing the structure of jumps, δ and λ_0, are significant in most cases. And the same is true for the parameter η that characterizes the volatility of the latent variance process. The main identification problem has to do with the correlation parameter, which is consistently estimated as negative, but with very low precision, as indicated by the large standard deviation. Even relatively large changes in the value of ρ would not affect the objective function substantially. The negative sign of the correlation parameter ρ, allows for capturing the observed asymmetry in the return process.

The specification with no jumps in returns is rejected for the four indices at standard significance levels. As shown in Panel C, the SV model does not do a good job

[6] The γ_0 and γ_1 parameters of the variance equation add up to more than one in the SNP estimates, but the unconditional variance in that model is no longer determined by the value of these two parameters.

in replicating the sample asymmetry and kurtosis statistics. The kurtosis is in the four indices not too far above 3.

After incorporating jumps in returns, the objective function reduces considerably for all indices. The reduction is of 45% for the S&P 500, 34% for DAX 30, 72% for IBEX 35 and 31% for CAC 40. As a consequence, the Chi-square statistic drops well below its value in the SV model. Now, at the 1% significance level, the model is not rejected for any of the four indices, and at 5% significance, it would be rejected just for CAC 40.

Of particular interest is the jump component. The estimation of λ_0 is significantly lower for S&P 500 than for the European indices. Estimated values imply an average of about 3 jumps per year for S&P 500 against 5, 10 and 6 jumps per year for DAX 30, IBEX 35 and CAC 40, respectively. The estimated average jump size, $-0.5\hat{\delta}$, is –1.34% for S&P 500, –1.43% for DAX 30, –1.40% for IBEX 35, and –1.48% for CAC 40. Therefore, jumps are less frequent in USA (lower estimated λ_0), despite the larger sample kurtosis reported for the U.S. data.

Incorporating jumps greatly improves the ability of the model to reproduce the levels of kurtosis observed in actual European return data. Simulated kurtosis in pseudo-daily returns increases from 3.7 to 5.6 for the S&P 500 when including jumps in returns, from 4.5 to 8.6 for DAX 30, from 4.6 to 9.6 for IBEX 35, and from 4.7 to 8.0 for CAC 40. On the other hand, the skewness of actual data is poorly explained by both specifications.

A systematic result is that the range of returns implied by estimated models is shifted to the left, relative to actual data, as indicated by the minimum and maximum returns in the simulated time series for the four indices. That is, both the minimum and the maximum returns are higher in absolute value than those in the data.[7] This comes about because of having jumps in returns as a mechanism to produce thick tails. We can attain the same level of kurtosis as in the data, together with negative skewness, because of the predominance of negative jumps in the simulated time series of returns. The level of volatility falls short in the simulated series relative to actual data, while the level of negative skewness is higher in simulated returns than in actual returns. The three observations on sample moments are consistent with each other. Even though jumps in returns under stochastic volatility help to explain the high levels of kurtosis observed in actual return data, some other relevant characteristics of the data remain unexplained. This suggests that some additional model features might be needed. An already tested candidate with U.S. data is jumps in volatility. This is tested with mixed results by Broadie, Chernov and Johannes (2007) and Eraker, Johannes and Polson (2003). A second, and probably more useful extension given difficulty in replicating the negative skewness in the sample, is to allow for state-dependent correlation between the innovations in the return and volatility equations. If, for example, the negative variance risk premium reported in the literature is indeed a premium on correlation as suggested by Driessen, Maenhout and Vilkov (2009), then we might want to allow for the correlation between the two innovations to depend upon the variance risk premium.

Finally, it should be pointed out that the estimation algorithm seems to work well for all the indices, as reflected in the fact that the p-values for the t-ratios of the components of the score vector are close to zero in the model with jumps in returns, with no statistical significance that could suggest some pattern of misspecification in any direction, in spite of the limitations we have pointed out throughout the paper.

[7] With the only exception of the minimum return for the S&P 500 index.

We also estimated the structural model adding to the objective function penalty terms capturing the inability of the model to reproduce higher order moments of sample returns. Specifically, we added to the objective function in (15), three terms defined as 10^{-4} times the squared difference between the sample and simulated variance, skewness and kurtosis of index returns. We can then obtain numerical values for the parameters in the structural model that fit well variance, skewness and kurtosis, but the numerical value of the quadratic form $m_N\left(\psi,\tilde{\xi}\right)' \tilde{I}^{-1} m_N\left(\psi,\tilde{\xi}\right)$ deteriorates drastically, suggesting that the SNP density incorporates characteristics of the density of returns that cannot be reasonably fitted when using 'brute force' to fit the three higher order moments.

6 Conclusions

It is widely accepted that incorporating stochastic volatility or jumps to continuous time diffusion processes can help explaining the main statistical characteristics of observed stock market index returns. Unfortunately, existing results for U.S. data are contradictory and fail to satisfactorily approximate the dynamics of the underlying return process. We attempt to identify a model that adequately fits the dynamics of returns over the January 1988 to December 2010 period, and extend the analysis to European indices: DAX 30, IBEX 35 and CAC 40.

We incorporate a Poisson process with constant intensity to a stochastic volatility diffusion process for returns, and perform EMM estimation, where a simple GARCH(1,1) auxiliary model is taken into account. We start by showing that the standard stochastic volatility is unable to explain the higher order moments of the sample distribution of stock market index returns. After that, we find that adding jumps in returns to the stochastic volatility diffusion can help explaining some of the statistical characteristics of return data series. Specifically, with such a model, we are able to replicate the degree of kurtosis observed in the European stock market indices considered. Adding jumps in returns drastically improves the fit, and the model is no longer rejected at the 1% significance level for any of the four indices. However, the model overestimates the degree of asymmetry and underestimates volatility, relative to sample moments. Hence, additional features are needed to improve the fit. Allowing for a state dependent correlation between the two Brownian motions in the model is our candidate to advance along this line in future research.

Acknowledgements

The authors thank participants in the XVI Foro de Finanzas at ESADE Business School in Barcelona. Ana González acknowledges financial support from the Spanish Ministry for Science and Innovation through grants SEJ2006-06309 and BES-2007-15631. Alfonso Novales thanks financial support from MEC ECO2009-10398 and Prometeo2008/106. Gonzalo Rubio thanks financial support from Prometeo2008/106 and MEC ECO2008-03058.

References

Andersen, T. and J. Lund. 1997. Estimating continuous-time stochastic volatility models of the short term interest rate. *Journal of Econometrics* 77: 343-377.

Andersen, T., L. Benzoni and J. Lund. 2002. An empirical investigation of continuous-time equity return models. *Journal of Finance* 57: 1239-1284.

Bakshi, G., C. Cao and Z. Chen. 1997. Empirical performance of alternative option pricing models. *Journal of Finance* 52: 2003-2049.

Bansal, R., A. Gallant, R. Hussey and G. Tauchen. 1993. Computational aspects of nonparametric simulation estimation. In *Computational techniques for econometrics and economic analysis*, ed. D.A. Belsley, 3-22. Kluwer Academic Publishers, Boston.

Bansal, R., A. Gallant, R. Hussey and G. Tauchen. 1995. Nonparametric estimation of structural models for high-frequency currency market data. *Journal of Econometrics* 66: 251-287.

Bates, D. 2000. Post-'87 crash fears in S&P 500 futures option market. *Journal of Econometrics* 94: 181-238.

Broadie, M., M. Chernov and M. Johannes. 2007. Model specification and risk premia: evidence from futures options. *Journal of Finance* 62: 1453-1490.

Chernov, M., A. Gallant, E. Ghysels and G. Tauchen. 2003. Alternative models for stock price dynamics. *Journal of Econometrics* 116: 225-257.

Clark, P. 1973. A subordinated stochastic process model with finite variance for speculative prices. *Econometrica* 41: 135-156.

Driessen, J., P. Maenhout and G. Vilkov. 2009. The price of correlation risk: evidence form equity options. *Journal of Finance* 64: 1377-1406.

Duffie, D. and K. Singleton. 1993. Simulated moments estimation of markov models of asset prices. *Econometrica* 61: 929-952.

Eraker, B. 2004. Do stock prices and volatility jump? Reconciling evidence from spot and option prices. *Journal of Finance* 59: 1367-1404.

Eraker, B., M. Johannes and N. Polson. 2003. The impact of jumps in volatility and returns. *Journal of Finance* 58: 1269-1300.

Fenton, V. and A. Gallant. 1996a. Convergence rates of SNP density estimators. *Econometrica* 64: 719-727.

Fenton, V. and A. Gallant. 1996b. Qualitative and asymptotic performance of SNP density estimators. *Journal of Econometrics* 74: 77-118.

Gallant, A. and J. Long 1997. Estimating stochastic differential equations efficiently by minimum chi-square. *Biometrika* 84: 125-141.

Gallant, A. and D. Nychka. 1987. Semi-nonparametric maximum likelihood estimation. *Econometrica* 55: 363-390.

Gallant, A. and G. Tauchen. 1989. Seminonparametric estimation of conditionally constrained heterogeneous processes: asset pricing applications. *Econometrica* 57: 1091-1120.

Gallant, A. and G. Tauchen. 1996. Which moments to match?. *Econometric Theory* 12, 657-681.

Geweke, J. 1996. Monte Carlo simulation and numerical integration. In *Handbook of Computational Economics*, ed. Amman, H., Kendrick, D. and Rust, J., Handbooks in Economics, North-Holland.

Heston, S. 1993. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies* 6: 327-343.

Hull, J. and A. White. 1987. The pricing of options on assets with stochastic volatilities. *Journal of Finance* 42: 281-300.

Kloeden, P. and E. Platen. 1992. *Numerical solution of stochastic differential equations,* Springer-Verlag, Berlin.

Melino, A. and S. Turnbull. 1990. Pricing foreign currency options with stochastic volatility. *Journal of Econometrics* 45: 239-265.

Pan, J. 2002. The jump-risk premia implicit in options: evidence from an integrated time-series study. *Journal of Financial Economics* 63: 3-50.

Singleton, K. 2006. *Empirical Dynamic Asset Pricing,* Princeton University Press, Princeton.

Tauchen, G. 1997. New minimum chi-square methods in empirical finance. In *Advances in Economics and Econometrics: Theory and Applications*, ed. Kreps, D., Wallis, K., Seventh World Congress, Cambridge University Press, Cambridge.

Tauchen, G. and M. Pitts. 1983. The price variability-volume relationship on speculative markets. *Econometrica* 51: 485-505.

Taylor, S. 1986. *Modelling financial time series*. Wiley, New York, NY.

Wiggins, J. 1987. Option values under stochastic volatility: theory and empirical estimates. *Journal of Financial Economics* 19: 351-372.

Table 1. S&P 500, DAX 30, IBEX 35 and CAC 40 daily Rates of Return.

In all figures and tables, returns are expressed on a daily basis, in percentage form, from January 4, 1988 to December 30, 2010. *Panel A:* Sample descriptive statistics for daily rates of return on the S&P 500, DAX 30, IBEX 35 and CAC 40. *Panel B:* Augmented Dickey Fuller (ADF) test for the presence of a unit root. The test is based on the regression:

$$\Delta s_t = \omega + \varsigma t + \theta s_{t-1} + \sum_{j=1}^{12} \tau_j \Delta s_{t-j} + \varepsilon_t$$

***Panel A:* Sample Descriptive Statistics**

	S&P 500	**DAX 30**	**IBEX 35**	**CAC 40**
Mean	0.0275	0.0246	0.0292	0.0233
Standard Deviation	1.1535	1.4385	1.3353	1.3880
Asymmetry	-0.2638	-0.2433	-0.1635	-0.0346
Kurtosis	12.0352	9.4817	8.1412	7.9109
Minimum	-9.4695	-13.7074	-9.5859	-9.4715
Maximum	10.9572	10.7974	10.1176	10.5946

***Panel B:* Augmented Dickey Fuller Test**

ADF (p-value)	**S&P 500**	**DAX 30**	**IBEX 35**	**CAC 40**
Stock market index	-1.35 (0.61)	-1.63 (0.46)	-1.03 (0.74)	-1.79 (0.38)
Returns	-58.99 (0.00)	-77.51 (0.00)	-75.10 (0.00)	-47.64 (0.00)

For our sample size, critical values at 5% and 1% significance levels are -2.86 and -3.43, respectively.

Table 2. SNP Model Estimates.

The reported results are expressed in percentage form. They are obtained from daily returns, filtered using a MA(1). The SNP model is: $f_K(r_t \mid x_t;\xi) = \left(v + (1-v)\frac{[P_K(z_t,x_t)]^2}{\int_R [P_K(u,x_t)]^2 \varphi(u)du} \right)\frac{\varphi(z_t)}{\sqrt{h_t}}$, where v = 0.01, $\varphi(\cdot)$ is the standard normal density, $z_t = \frac{r_t - \upsilon_t}{\sqrt{h_t}}$, $\begin{array}{l} \upsilon_t = 0 \\ h_t = \gamma_0 + \gamma_1 r_{t-1}^2 + \vartheta_1 h_{t-1} \sim GARCH(1,1) \end{array}$ and $P_K(z,x) = \sum_{i=0}^{K_z} a_i \overline{He_i}(z), \quad a_0 = 1$.

Standard errors are given in parenthesis, except for the long-term variance $\frac{\gamma_0}{1-\gamma_1-\vartheta_1}$, where we show in brackets the long-term GARCH standard deviation.

GARCH(1,1) Model Estimates				
Parameter	**S&P 500**	**DAX 30**	**IBEX 35**	**CAC 40**
γ_0	0.0079 (0.0010)	0.0389 (0.0029)	0.0279 (0.0024)	0.0307 (0.0032)
γ_1	0.0552 (0.0034)	0.0989 (0.0039)	0.0871 (0.0053)	0.0904 (0.0059)
ϑ_1	0.9380 (0.0039)	0.8827 (0.0056)	0.8959 (0.0063)	0.8936 (0.0066)
$\frac{\gamma_0}{1-\gamma_1-\vartheta_1}$	1.162 (1.077)	2.114 (1.454)	1.641 (1.281)	1.919 (1.385)

SNP Model Estimates				
Parameter	**S&P 500**	**DAX 30**	**IBEX 35**	**CAC 40**
γ_0	0.0127 (0.0018)	0.0544 (0.0077)	0.0429 (0.0058)	0.0577 (0.0091)
γ_1	0.0800 (0.0074)	0.2056 (0.0136)	0.1558 (0.0100)	0.1870 (0.0177)
ϑ_1	0.9555 (0.0038)	0.8908 (0.0082)	0.9150 (0.0057)	0.8895 (0.0093)
a_1	-0.0064 (0.0066)	0.0022 (0.0068)	0.0005 (0.0071)	0.0023 (0.0068)
a_2	-0.2426 (0.0122)	-0.2509 (0.0098)	-0.2487 (0.0124)	-0.2203 (0.0141)
a_3	-0.0220 (0.0071)	-0.0307 (0.0067)	-0.0422 (0.0070)	-0.0202 (0.0067)
a_4	0.1227 (0.0082)	0.1022 (0.0088)	0.1339 (0.0090)	0.0970 (0.0074)
a_5	-0.0036 (0.0078)	0.0138 (0.0073)	0.0012 (0.0046)	0.0255 (0.0062)
a_6	-0.0559 (0.0081)	-0.0893 (0.0085)	-0.0657 (0.0080)	-0.0536 (0.0088)

Table 3. EMM Results of the Stochastic Volatility Model (SV) and Stochastic Volatility Model with Jumps in Returns (SVJ).

Panel A: EMM estimates: Estimates are expressed in percentage form on a daily basis. The rates of return of the S&P 500, DAX 30, IBEX 35 and CAC 40, correspond to the sample period from January 4, 1988 to December 30, 2010. Returns of the stock market indices have 5799, 5915, 5764 and 5806 observations respectively. The estimates refer to the following models:

$$\text{SV: } ds_t = \left(\mu - \frac{V_t}{2}\right)dt + \sqrt{V_t}\,dW_{1,t}\ ,\ dV_t = (\alpha - \beta V_t)dt + \eta\sqrt{V_t}\,dW_{2,t}$$

$$\text{SVJ: } ds_t = \left(\mu - \lambda_t \bar{k} - \frac{V_t}{2}\right)dt + \sqrt{V_t}\,dW_{1,t} + \ln(1+k_t)dq_t\ ,\ dV_t = (\alpha - \beta V_t)dt + \eta\sqrt{V_t}\,dW_{2,t}\ ,$$

$$\ln(1+k(t)) \approx N\left(\ln(1+\bar{k}) - 0.5\delta^2, \delta^2\right),\ \bar{k} = 0\ ,\ \text{cov}(dW_{1,t}, dW_{2,t}) = \rho dt\ ,\ \Pr(dq_t = 1) = \lambda(t)dt\ ,\ \lambda(t) = \lambda_0\ .$$

EMM Estimates for the Structural Model
January 4, 1988-December 30, 2010
(Standard deviations in brackets)

	S&P500		**DAX 30**		**IBEX 35**		**CAC 40**	
Parameters	**SV**	**SVJ**	**SV**	**SVJ**	**SV**	**SVJ**	**SV**	**SVJ**
μ	0.0287 (0.0085)	0.0304 (0.0033)	0.0458 (0.0041)	0.0492 (0.0048)	0.0437 (0.0040)	0.0473 (0.0056)	0.0511 (0.0030)	0.0558 (0.0048)
α	0.1214 (0.0853)	0.0153 (0.0162)	0.0267 (0.0086)	0.0209 (0.0065)	0.0148 (0.0042)	0.0218 (0.0212)	0.0245 (0.0062)	0.0233 (0.0085)
β	0.2134 (0.1510)	0.0242 (0.0257)	0.0285 (0.0112)	0.0201 (0.0060)	0.0170 (0.0057)	0.0233 (0.0265)	0.0224 (0.0070)	0.0191 (0.0068)
η	0.1813 (0.0760)	0.0745 (0.0320)	0.1427 (0.0021)	0.1312 (0.0418)	0.1023 (0.0200)	0.1148 (0.0276)	0.1400 (0.0206)	0.1393 (0.0262)
ρ	-0.0971 (0.2542)	-0.2703 (0.4167)	-0.1167 (0.3716)	-0.0114 (0.7936)	-0.2423 (0.6448)	-0.2056 (0.9473)	-0.1552 (0.3939)	-0.0155 (0.4271)
δ		2.6877 (0.1506)		2.8807 (0.1823)		2.7963 (0.0702)		2.9640 (0.1057)
100 λ_0		0.1186 (0.1421)		0.2036 (0.0812)		0.3865 (0.0907)		0.2260 (0.0807)
100 ***f***	0.1956	0.1080	0.2341	0.1551	0.2333	0.0677	0.2837	0.1945
n	5799		5915		5764		5806	
χ^2 ***(p-value)***	11.34 (0.003)	6.27 (0.180)	13.85 (0.001)	9.18 (0.056)	13.45 (0.001)	3.90 (0.419)	16.47 (0.000)	11.29 (0.023)

Panel B: EMM Diagnosis: t-ratios of the elements of the score vector, which are given by $\hat{t} = \frac{m_N(\hat{\psi}, \tilde{\xi})}{\sqrt{diag(S)}}$ where $S = \frac{1}{n}\left(\tilde{I} - M_\psi\left(M_\psi'\tilde{I}^{-1}M_\psi\right)^{-1}M_\psi'\right)$, $M_\psi = \frac{\partial m_N(\hat{\psi}, \tilde{\xi})}{\partial \psi}$. They correspond to the auxiliary model: $f_K(r_t \mid x_t; \xi) = \left(v + (1-v)\frac{[P_K(z_t, x_t)]^2}{\int_R [P_K(u, x_t)]^2 \varphi(u) du}\right)\frac{\varphi(z_t)}{\sqrt{h_t}}$, where v = 0.01, $\varphi(\cdot)$ is the standard normal density, $z_t = \frac{r_t - \upsilon_t}{\sqrt{h_t}}$, $\upsilon_t = 0$, $h_t = \gamma_0 + \gamma_1 r_{t-1}^2 + \vartheta_1 h_{t-1} \sim GARCH(1,1)$ and $P_K(z,x) = \sum_{i=0}^{K_z} a_i \overline{He_i}(z)$, $a_0 = 1$.

***t*-ratios for the elements of the score vector** **(p-values in brackets)** **Sample: January 4, 1988-December 30, 2010**								
	S&P 500		**DAX 30**		**IBEX 35**		**CAC 40**	
Parameter	**SV**	**SVJ**	**SV**	**SVJ**	**SV**	**SVJ**	**SV**	**SVJ**
γ_0	-0.285	0.281	-1.937	-0.236	-1.365	1.689	-3.455	6.307
	(0.79)	(0.81)	(0.25)	(0.84)	(0.24)	(0.23)	(0.03)	(0.98)
γ_1	-0.546	-1.271	-3.035	-2.493	-1.288	-1.712	-4.069	-7.110
	(0.61)	(0.33)	(0.73)	(0.13)	(0.27)	(0.23)	(0.02)	(0.64)
ϑ_1	-0.556	-1.867	-2.695	-1.445	-1.521	-2.455	-3.578	-6.775
	(0.61)	(0.20)	(0.56)	(0.13)	(0.20)	(0.13)	(0.02)	(0.69)
a_1	1.792	-1.540	2.404	-2.153	1.690	-1.696	7.076	-4.711
	(0.15)	(0.26)	(0.02)	(0.16)	(0.17)	(0.23)	(0.00)	(0.15)
a_2	-3.265	-4.686	-4.282	-4.880	-2.997	-57.657	-3.681	-4.456
	(0.03)	(0.04)	(0.23)	(0.04)	(0.04)	(0.00)	(0.02)	(0.45)
a_3	0.321	0.377	0.720	-0.442	0.394	0.285	-3.264	0.496
	(0.76)	(0.74)	(0.47)	(0.70)	(0.71)	(0.80)	(0.03)	(0.83)
a_4	-0.666	-7.125	-2.159	-1.149	-0.877	-1.919	0.594	-3.313
	(0.54)	(0.02)	(0.57)	(0.37)	(0.43)	(0.19)	(0.58)	(0.67)
a_5	0.624	6.434	3.624	2.144	0.888	1.386	1.570	4.347
	(0.57)	(0.02)	(0.40)	(0.17)	(0.42)	(030)	(0.19)	(0.63)
a_6	1.163	-0.456	1.064	3.285	0.241	-0.235	0.446	-0.925
	(0.31)	(0.69)	(0.12)	(0.08)	(0.82)	(0.84)	(0.68)	(0.93)

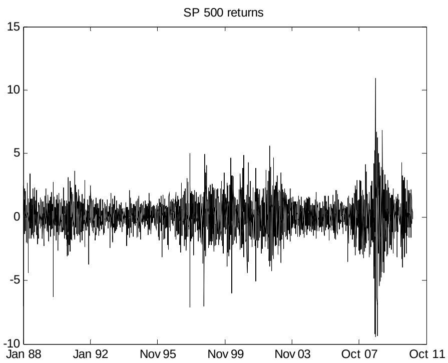
Panel C: Basic Statistics from the sample data and the SV/SVJ simulations obtained under the $\hat{\psi}$ estimates of the structural model:

		Mean	Std. Dev.	Asymmetry	Kurtosis	Minimum	Maximum
S&P 500	**Sample**	0.027	1.153	-0.264	12.035	-9.470	10.960
	SV	0.013	0.758	-0.209	3.460	-3.690	3.100
	SVJ	-0.001	0.816	-0.528	5.570	-7.530	3.570
DAX 30	**Sample**	0.025	1.438	-0.243	9.482	-13.710	10.800
	SV	0.005	1.007	-0.607	4.540	-6.440	4.090
	SVJ	-0.018	1.095	-1.023	8.620	-14.470	4.270
IBEX 35	**Sample**	0.029	1.335	-0.163	8.141	-9.580	10.120
	SV	0.020	0.967	-0.613	4.570	-6.310	4.080
	SVJ	0.012	1-040	-1.070	9.550	-13.990	4.050
CAC 40	**Sample**	0.023	1.388	-0.035	7.911	-9.471	10.595
	SV	-0.016	1.099	-0.690	4.720	-7.410	4.390
	SVJ	-0.042	1.195	-1.021	7.970	-14.980	4.530

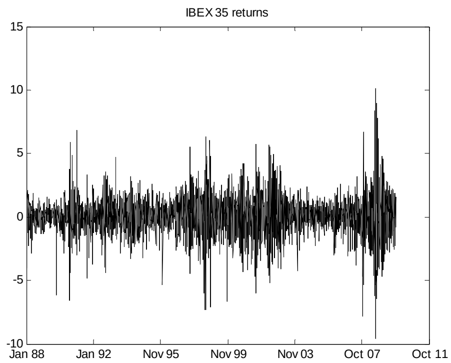
Figure 1. S&P 500, DAX 30, IBEX 35 and CAC 40 Daily Rate of Return.

All data are expressed on a daily basis percentage form, from January 4, 1988 to December 30, 2010. Daily rates of return of the S&P 500 (*Panel A*), DAX 30 (*Panel B*), IBEX 35 (*Panel C*) and CAC 40 (*Panel D*), have 5799, 5915, 5764 and 5806 observations respectively.

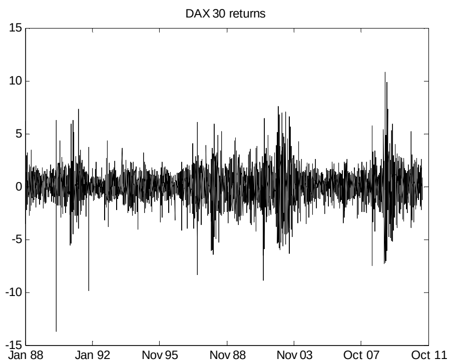
Panel A:



Panel C:



Panel B:



Panel D: