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A generalization of Prentice's law for lenses with arbitrary refracting surfaces

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Summary

A generalization of the Prentice's law is presented in this paper. The idea consists of removing some (but not all) of the approximations that comprise the paraxial approach. In that way, we obtain a new formulation that permits us to compute the prismatic power of a lens made up of arbitrary refracting surfaces, and to improve the precision obtained by Prentice's law when applied to monofocal lenses. The resulting formalism is simple and manageable and its derivation leads us to a precise definition of the local dioptric power matrix, introduced in a previous paper, as well as a better understanding of the same.

Introduction

The knowledge of the prismatic effect produced by an ophthalmic lens has great importance in the optometric practice. For monofocal spherical lenses, the Prentice's law establishes a linear relationship between the modulus of the prismatic power and the distance to the optical centre of the lens (Jalie, 1988). This relationship has been extended to both astigmatic and bifocal lenses by means of the dioptric power matrix (Long, 1976; Keating, 1980; Harris, 1988, 1989a,b, 1992, 1993 and 1996). In these cases, a linear relationship is obtained between the components of the prismatic power and the coordinates of the lens point (referred to the lens optical centre) where we want to calculate the prismatic power, using the dioptric power matrix.

Although it is possible to calculate easily the prismatic power components of spherical, cylindrical, spherotorical and even bifocal lenses, it is not possible to do the same for progressive addition lenses (PAL). This is due to the complicated shape of the progressive surface, necessary to obtain different powers for different sight directions.

On this subject we have presented a method for the graphical characterization of a PAL by means of the so called by us *local dioptric power matrix* (Alonso *et al.*, 1997). A generalized expression for the Prentice's law applied to ophthalmic lenses made with arbitrary shaped refracting surfaces is presented in this paper. This expression allows the optical characterization of these lenses by means of the local dioptric power matrix, which is related with the equations describing the lens surfaces. Besides this, our formalism is a more accurate approximation of the exact ray tracing problem than the classical paraxial approach. Thus it gives us a better estimation of prismatic powers for any type of ophthalmic lens.

With the aid of the generalized Prentice's law obtained, we have studied the prismatic effect presented by aspherical lenses formed by conicoidal refracting surfaces. Finally, we have tested the accuracy of the generalized Prentice's law.

Generalized expression for the Prentice's law

When ray tracing is computed within the frame of the classical paraxial approximation, the dependance of the involved angles vs the height of incidence is assumed to be linear. The prismatic power is defined as the difference between the angles that the incident and refracted rays form with the optical axis, so the prismatic effect also depends linearly on the height of

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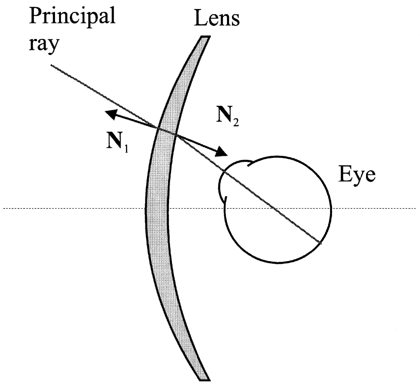

Figure 1. Refraction of a principal ray through the lens–eye system. Because of the meniscus shaped form of the lens, the angles formed by the principal ray the surface normals are small regardless the incidence height.

incidence (that is, we get the Prentice's law). In this framework, there is no way of introducing the shape of the surfaces, because only the curvature radius at the surface vertex are considered, so it makes no sense to talk about the geometry of the surface far from its centre.

If we consider the refraction through an ophthalmic lens, two factors make this classical paraxial scheme inappropriate.

1. The surfaces can be highly aspherical, as happens in a progressive addition lens. In this case, the Prentice's law is useless in order to obtain even an estimation of the prismatic effect.
2. The incidence point is far from the optical axis for oblique sight directions. In such a case, the classical paraxial approach gives inaccurate values of the prismatic power for lenses made up of spherical surfaces.

To overcome these problems, we propose a modification of the classical paraxial approach. We will consider the refraction at a point not necessarily located near the optical axis, however we will suppose that the angles formed between the incident and refracted rays with the normal to the surface at the incident point, θ_i and θ_r are small, and first order approximation can be applied. Note that this approximation is adequate in the system ophthalmic lens-eye, as shown in *Figure 1*. This is because ophthalmic lenses usually have positive radii of curvature, and the principal ray goes through the centre of rotation of the eye. A possible change of the surface geometry at the point of incidence can be taken into account by using the principal curvatures as well as surface orientation at this point.

Let us consider an ophthalmic lens composed by two refracting surfaces Σ_1 and Σ_2. These surfaces may be described by means of a Monge's chart (Kreyszig, 1991; Lipschutz, 1970), that is, the coordinates of a point belonging to the surface are given by the expression

$$\left[x, y, z_i(x, y)\right], \qquad {}_{i=1,2} \tag{1}$$

where x, y and z are the co-ordinates along a reference system whose Z axis coincides with the optical axis of the lens (See *Figure 2*). Let us consider a light ray incident at a point P located at Σ_1 and let $\mathbf{k}_1^i$ be the unitary vector in the direction of the incident ray. In the same way, let $\mathbf{k}_1^r$ be the unitary vector parallel to the direction of the refracted ray. The relationship between these vectors is given by the vectorial form of the Snell law (Hecht, 1974)

$$n\mathbf{k}_1^r - \mathbf{k}_1^i = (n\cos\theta_r - \cos\theta_i)\mathbf{N}_1, \tag{2}$$

where n is the refraction index of the lens (we are supposing the refraction index of the incident media as 1) and $\mathbf{N}_1$ is the unitary normal vector at the point P of Σ_1.

As it is well known (Kreyszig, 1991; Lipschutz, 1970), the unitary vector normal to a surface described by Equation (1) is

$$\mathbf{N} = \frac{1}{\sqrt{1 + (\partial_x z_i)^2 + (\partial_y z_i)^2}}(-\partial_x z_i, -\partial_y z_i, 1), \quad {}_{i=1,2} \tag{3}$$

where the symbols ∂_x and ∂_y are a short notation for the partial derivatives $\partial/\partial x$ and $\partial/\partial y$. This expression can be approximated by

$$\mathbf{N} \simeq (-\partial_x z_i, -\partial_y z_i, 1), \qquad {}_{i=1,2} \tag{4}$$

as far as we consider that the lens surfaces are not excessively curved. According with our previous hypotheses the vectors $\mathbf{k}_1^i$ and $\mathbf{k}_1^r$ must be near to the direction of the normal vector $\mathbf{N}_1$, so we can write

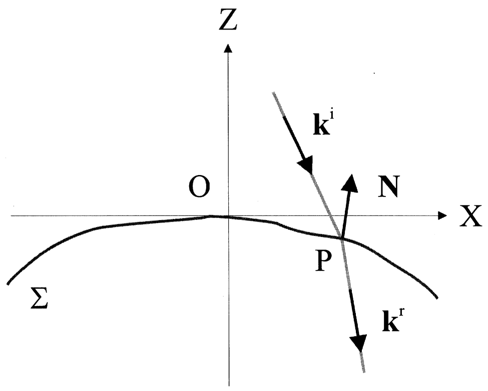

Figure 2. Scheme employed to illustrate the refraction of a ray through an arbitrary surface.

their components as

$$\mathbf{k}_1^i \simeq \left(k_{1x}^i, k_{1y}^i, 1\right), \tag{5a}$$

$$\mathbf{k}_1^r \simeq \left(k_{1x}^r, k_{1y}^r, 1\right). \tag{5b}$$

Substituting Equation (5a), (5b) and (4) in Equation (2), we arrive at the following approximate expressions for their components of $\mathbf{k}_1^r$

$$nk_{1x}^r = k_{1x}^i + (1-n)\,\partial_x z_1, \tag{6a}$$

$$nk_{1y}^r = k_{1y}^i + (1-n)\,\partial_y z_1. \tag{6b}$$

We will make the same assumptions for the refraction at the second lens surface, so we have for the components of $\mathbf{k}_2^r$

$$k_{2x}^r = nk_{2x}^i - (1-n)\,\partial_x z_2, \tag{7a}$$

$$k_{2y}^r = nk_{2y}^i - (1-n)\,\partial_y z_2. \tag{7b}$$

With the aid of Equations (6a), (6b), (7a) and (7b), we can compute the paraxial trajectory of a ray through the lens, taking into account that $k^r{}_{1x} = k^i{}_{2x}$ and $k^r{}_{1y} = k^i_{2y}$. Furthermore, within the thin lens approximation, we can substitute (6a) in Equation (7a), and (6b) in Equation (7b), arriving at the following expressions:

$$k_{2x}^r = k_{1x}^i + (1-n)\,(\partial_x z_1 - \partial_x z_2), \tag{8a}$$

$$k_{2y}^r = k_{1y}^i + (1-n)\,(\partial_y z_1 - \partial_y z_2), \tag{8b}$$

The components of the prismatic power, P_x and P_y are given by

$$P_x = k_{2x}^r - k_{1x}^i, \tag{9a}$$

$$P_y = k_{2y}^r - k_{1y}^i, \tag{9b}$$

so we arrive to the following generalized expression of the Prentice's law, for lenses with arbitrary shaped refracting surfaces

$$P_x = (1-n)\,(\partial_x z_1 - \partial_x z_2), \tag{10a}$$
$$P_y = (1-n)\,(\partial_y z_1 - \partial_y z_2). \tag{10b}$$

Local dioptric power matrix

We can rewrite Equations (10a) and (10b) in a more useful matrix form if we use the following Taylor expansion of the surface equations in the neighborhood of a point Q on the lens surface with co-ordinates (x_0, y_0), up to second order

$$\begin{aligned} z_i(x,y) \simeq{}& z_i(x_0,y_0) + \partial_x z_i|_0 (x-x_0) + \partial_y z_i|_0 (y-y_0) + \ldots \\ &+ \frac{1}{2}\partial_{xx}^2 z_i|_0 (x-x_0)^2 + \partial_{xy}^2 z_i|_0 (x-x_0)(y-y_0) + \ldots \\ &+ \frac{1}{2}\partial_{yy}^2 z_i|_0 (y-y_0)^2 \qquad {\scriptstyle i=1,2} \end{aligned} \tag{11}$$

where $\partial_x z_i|_0 = \partial_x z_i(x_0, y_0)$. This means that we are approximating the lens surfaces in the neighborhood of Q by their tangent paraboloids at Q. Substituting these expansions in Equation (10a) and (10b), we arrive to the following expressions for the prismatic powers in the neighborhood of Q

$$\begin{aligned} P_x = (1-n) \\ \left[\partial_x\,\delta z|_0 + \partial_{xx}^2\,\delta z|_0 (x-x_0) + \partial_{xy}^2\,\delta z|_0 (y-y_0)\right], \end{aligned} \tag{12a}$$

$$\begin{aligned} P_y = (1-n) \\ \left[\partial_y\,\delta z|_0 + \partial_{xy}^2\,\delta z|_0 (x-x_0) + \partial_{yy}^2\,\delta z|_0 (y-y_0)\right], \end{aligned} \tag{12b}$$

where $\partial_x \delta z|_0 = \partial_x z_1\ (x_0, y_0) - \partial_x z_2\ (x_0, y_0)$, and so on. Let us consider now a point S related with Q, whose co-ordinates (x_l, y_l) are given by the relationships

$$0 = \left[\partial_x\,\delta z|_0 + \partial_{xx}^2\,\delta z|_0 (x_l-x_0) + \partial_{xy}^2\,\delta z|_0 (y_l-y_0)\right], \tag{13a}$$

$$0 = \left[\partial_y\,\delta z|_0 + \partial_{xy}^2\,\delta z|_0 (x_l-x_0) + \partial_{yy}^2\,\delta z|_0 (y_l-y_0)\right]. \tag{13b}$$

In other words, we are considering the co-ordinates of the optical centre of the astigmatic lens formed by the tangent paraboloids to the surfaces Σ_1 and Σ_2 at the point Q. The point S, that we call *local optical centre* of the lens at the point Q, was introduced by us in previous paper (Alonso *et al.*, 1997) from a conceptual point of view. From Equations (13a) and (13b), we obtain

$$\partial_x\,\delta z|_0 = -\,\partial_{xx}^2\,\delta z|_0 (x_l-x_0) - \partial_{xy}^2\,\delta z|_0 (y_l-y_0)\big], \tag{14a}$$

$$\partial_y\,\delta z|_0 = -\,\partial_{xy}^2\,\delta z|_0 (x_l-x_0) - \partial_{yy}^2\,\delta z|_0 (y_l-y_0)\big]. \tag{14b}$$

Finally, by substituting Equations (14a) and (14b) in (12a) and (12b) respectively, it is possible to obtain the following matricial expression for the prismatic power at the point Q

$$\begin{pmatrix} P_x \\ P_y \end{pmatrix} = -(n-1)\begin{bmatrix} \partial_{xx}^2\,\delta z|_0 & \partial_{xy}^2\,\delta z|_0 \\ \partial_{xy}^2\,\delta z|_0 & \partial_{yy}^2\,\delta z|_0 \end{bmatrix}\begin{pmatrix} x_0 - x_l \\ y_0 - y_l \end{pmatrix}. \tag{15}$$

We will call *local dioptric power matrix*, to the matricial function

$$\mathbf{F}(x, y) = (n-1)\begin{bmatrix} \partial^2_{xx}\,\delta z(x, y) & \partial^2_{xy}\,\delta z(x, y) \\ \partial^2_{xy}\,\delta z(x, y) & \partial^2_{yy}\,\delta z(x, y) \end{bmatrix}, \quad (16)$$

which allows the characterization of ophthalmic lenses made up of surfaces of arbitrary shape as stated in Alonso *et al.* (1997). From this definition and the expressions (10a) and (10b), we obtain the relation between the elements of the local dioptric power matrix $f_{xx}(x, y)$, $f_{xy}(x, y)$ and $f_{yy}(x, y)$, and the prismatic power components. These are

$$f_{xx}(x, y) = -\,\partial_x P_x, \quad (17a)$$

$$f_{xy}(x, y) = -\,\partial_y P_x = -\,\partial_x P_y, \quad (17b)$$

$$f_{yy}(x, y) = -\,\partial_y P_y. \quad (17c)$$

Equation (15) may be written in this alternative form

$$\begin{pmatrix} P_x \\ P_y \end{pmatrix} = \begin{pmatrix} P^L_x \\ P^L_y \end{pmatrix} - \mathbf{F}(x_0, y_0)\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \quad (18)$$

where P^L_x, P^L_y are the components of a point dependent prismatic effect which is due to surface orientation and is not directly related to power. We call this *local ground prism* at the point Q (Alonso *et al.*, 1997), and it is given by

$$P^L_x = (1-n)\left[\partial_x\,\delta z|_0 - \partial^2_{xx}\,\delta z|_0\,x_0 - \partial^2_{xy}\,\delta z|_0\,y_0\right], \quad (19a)$$

$$P^L_y = (1-n)\left[\partial_y\,\delta z|_0 - \partial^2_{xy}\,\delta z|_0\,x_0 - \partial^2_{yy}\,\delta z|_0\,y_0\right]. \quad (19b)$$

According to the previous description, there are two factors that account for the total deviation of a ray refracted at any point on the lens surfaces. The first one is related to the local curvatures at the point of incidence, which are directly related to refracting power. This factor can be calculated by means of the local dioptric power matrix, in a similar way as we do with the matricial Prentice's law. The second factor is related to the orientation of the normal vectors to the surfaces at the point of incidence, and depends not only on the second derivatives of the surfaces at that point but also on the first order derivatives. For that reason, we can think of it as a local ground prism. In fact, Equation (18) is a generalization of the implicit expression for the prismatic power presented by a thin lens with allowance for prism given by Harris and Abelman (1991).

From Equations (15) and (18), it is possible to obtain the following relationship for the co-ordinates of the local optical centre

$$\begin{pmatrix} x_l \\ y_l \end{pmatrix} = \mathbf{F}^{-1}(x_0, y_0)\begin{pmatrix} P^L_x \\ P^L_y \end{pmatrix}. \quad (20)$$

This relation can be applied only if the determinant of the local dioptric power matrix is not zero. Otherwise, the inverse $\mathbf{F}^{-1}(x_0, y_0)$ does not exist and we have to employ the Moore–Penrose formalism extensively used by Harris (1992, 1993) with the dioptric power matrix.

Verification of the Prentice's law generalized form

In order to evaluate the generalized expression of the Prentice's law, Equations (10a) and (10b), we will derive the prismatic effect produced by a monofocal ophthalmic lens composed of conicoidal refracting surfaces (which includes as a particular case the monofocal spherical lenses). According to Jalie (1988) the sagittae of the conicoidal refracting surfaces are given by

$$z_i(x, y) = \frac{R_i - \sqrt{R_i^2 - p_i r^2}}{p_i}, \qquad i=1,2 \quad (21)$$

where R_i is the curvature radius of the ith surface, $r = (x^2+y^2)^{1/2}$ is the radial co-ordinate and p_i is the aspherical coefficient which specifies the type of conicoidal surface (for example $p_i=0$ corresponds with a paraboloid, $p_i=1$ corresponds with a sphere, and so on). In order to simplify the equations, we will use the notation $\rho_i=\sqrt{R_i^2 - p_i r^2}$ for each surface. Substituting (21) in Equations (10a) and (10b), we arrive at the result

$$P_x(x, y) = -(n-1)\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)x, \quad (22)$$

$$P_y(x, y) = -(n-1)\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)y. \quad (23)$$

The term $(n-1)(1/\rho_1 - 1/\rho_2)$ that appears in the prismatic effect along the x and y directions, resembles the classical expression for the power of a thin lens with curvatures radius ρ_1 and ρ_2, although in this case these parameters depend on the co-ordinates x and y.

The elements of the local dioptric power matrix are easily obtained by applying Equation (16),

$$f_{xx} = (n-1)\left[\frac{1}{\rho_1} - \frac{1}{\rho_2} + x^2\left(\frac{p_1}{\rho_1^3} - \frac{p_2}{\rho_2^3}\right)\right], \quad (24)$$

$$f_{yy} = (n-1)\left[\frac{1}{\rho_1} - \frac{1}{\rho_2} + y^2\left(\frac{p_1}{\rho_1^3} - \frac{p_2}{\rho_2^3}\right)\right], \quad (25)$$

$$f_{xy} = (n-1)xy\left(\frac{p_1}{\rho_1^3} - \frac{p_2}{\rho_2^3}\right). \quad (26)$$

The components of the ground prism, obtained from expressions (19a), (19b) are given by

$$P_x^L = (n-1)\left(\frac{p_1}{\rho_1^3} - \frac{p_2}{\rho_2^3}\right)x^2(x+y), \quad (27)$$

$$P_y^L = (n-1)\left(\frac{p_1}{\rho_1^3} - \frac{p_2}{\rho_2^3}\right)y^2(x+y), \quad (28)$$

and finally the co-ordinates of the local optical centre are

$$x_L = \frac{\left(\frac{p_1}{\rho_1^3} - \frac{p_2}{\rho_2^3}\right)r^2 x}{\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) + \left(\frac{p_1}{\rho_1^3} - \frac{p_2}{\rho_2^3}\right)r^2}, \quad (29)$$

$$y_L = \frac{\left(\frac{p_1}{\rho_1^3} - \frac{p_2}{\rho_2^3}\right)r^2 y}{\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) + \left(\frac{p_1}{\rho_1^3} - \frac{p_2}{\rho_2^3}\right)r^2}. \quad (30)$$

The prismatic power presents rotational symmetry, as well as the sagittae in expression (21). The orientation of the prismatic power is radial, and the modulus of the prismatic power is given by

$$P(r) = -(n-1)\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)r. \quad (31)$$

In *Figure 3* we present the modulus of the prismatic power for a –7 D aspheric lens vs the radial co-ordinate, for several values of the aspheric coefficient of the second lens surface (being $p_1 = 1$). The generalized Prentice's law predicts a nonlinear behavior of the prismatic power except when the aspherical coefficients p_1 and p_2 are both zero, in which case $\rho_1 = R_1$ and $\rho_2 = R_2$.

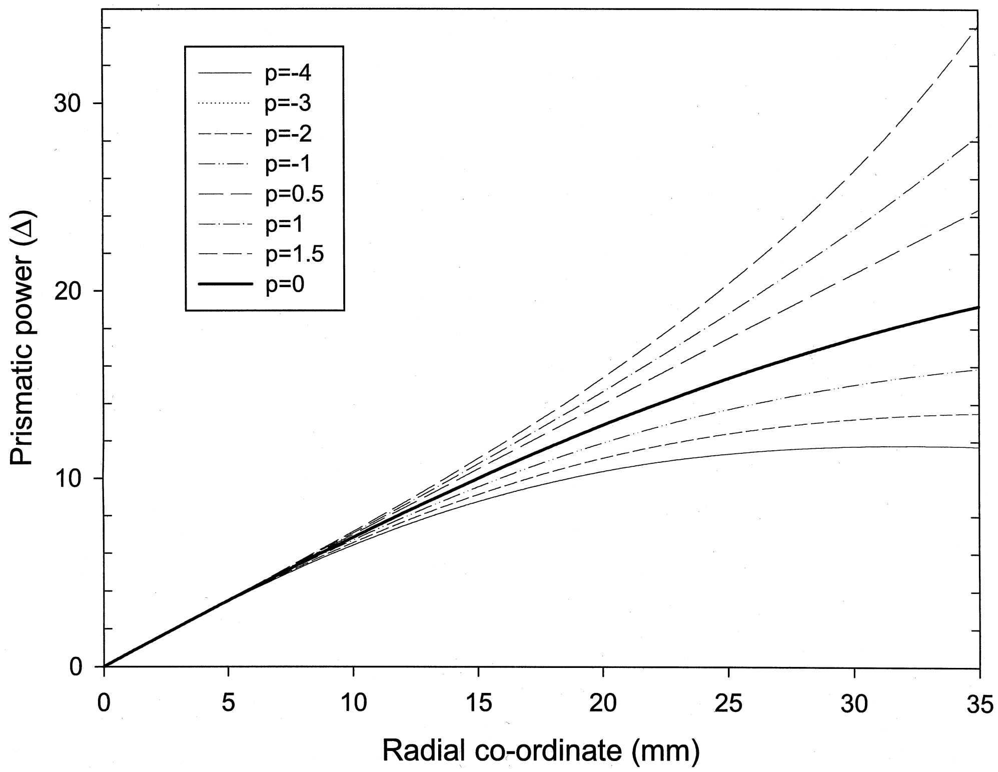

Figure 3. Plot of the prismatic power produced by a –7 D aspherical vs the radial co-ordinate for several values of the second lens surface. The lens is considered thin and the curvature radius of the surfaces are R_1=41.84 mm, R_2=95.09 mm.

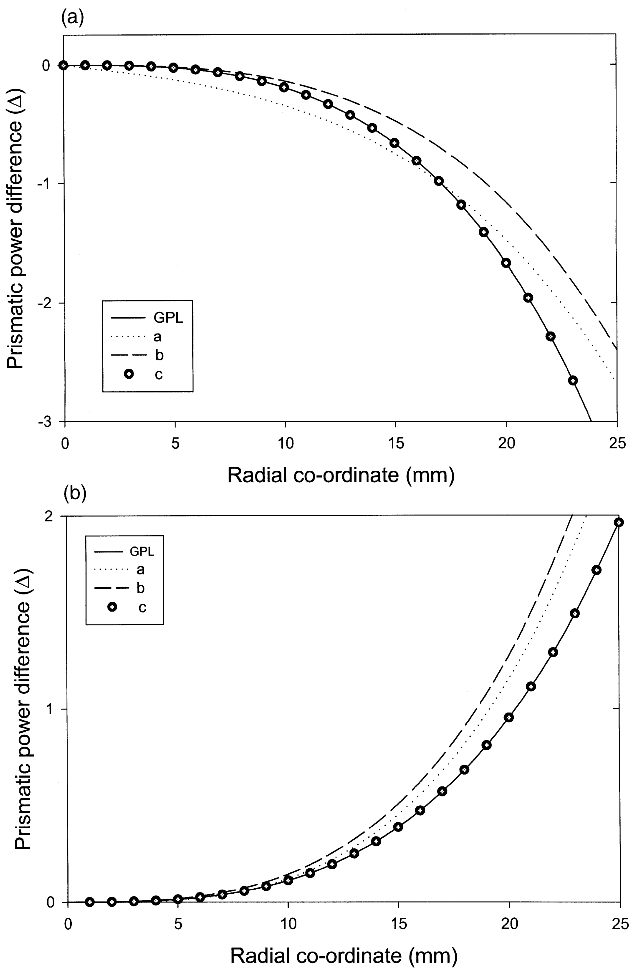

Figure 4. Prismatic power difference vs radial co-ordinate. The solid curve has been obtained using the generalized Prentice's law, whereas the curves named *a*, *b* and *c* correspond with the numerical ray tracing calculations as explained in the text. (a) +6 D spherical lens (R_1=51.024 mm, R_2=123.058 mm, t_c=7.62 mm). (b) –6 D spherical lens (R_1=209.200 mm, R_2=61.529 mm, t_c=1 mm).

In order to test the generalized form of the Prentice's law, we have computed the prismatic power along a radial direction for two spherical lenses by means of numerical ray tracing. The ray tracing has been done with three levels of approximation:

(a) Exact trigonometric ray tracing.
(b) Trigonometrical ray tracing within the thin lens approximation, that is, the height of incidence on the second lens surface is assumed to be the same as on the first surface.
(c) Ray tracing within the thin lens approximation. In this approach we substitute the *sine* functions of the Snell's law by their arguments, but we keep an accurate calculation of the angles of incidence by means of the local computation of the normal vectors to the surfaces at the points of incidence.

The comparison between the ray tracing techniques and the generalized Prentice's law for two spherical lenses with powers –6 D and +6 D is shown in *Figure 4*. In order to obtain a better visualization of the prismatic power behavior, we have excluded its linear dependance on the radial coordinate, that is, we have substracted the linear prismatic power predicted by the Prentice's law from the values calculated from ray tracing and the generalized Prentice's law.

At 20 mm from the optical centre, the Prentice's law gives errors of 11% and 9% for the +6 D and –6 D lenses, respectively. Comparatively, the generalized Prentice's law gives errors of 1.5% and 1.6% for the same lenses. We can notice also in *Figure 4* that the values calculated with the generalized Prentice's law coincide with those obtained with approximation *c*. This fact evidences the nature of the approximations made in our formalism.

Conclusions

The matricial formulation of Prentice's law is a useful rule to calculate the prismatic power of any type of monofocal, bifocal or even trifocal lenses. The matrix appearing in this formulation, the dioptric power matrix, has been thoroughly studied in the literature and, within the scope of the paraxial approximation, it turns out to be a fundamental description of vergences, refracting errors and lenses. The paraxial description of a diopter (or a wavefront) only considers its power (or curvatures) at the axis, and the exact shape of the surface out of axis cannot be taken into account. Then, because of the paraxial nature of the dioptric power matrix, there is no sense in considering power variations, so the techniques related to the matrix are no longer applicable for progressive addition lenses.

In this work we have addressed this problem. We have reconsidered the paraxial assumptions in such a way that power variations far from the axis can be taken into account. Of course, although exact ray tracing can be easily carried out by means of a computer, we have searched for a similar formalism to that of the dioptric power matrix, because of its usefulness and compactness. We have obtained a function matrix that can be considered a direct generalization of the dioptric power matrix, and it allows us, along with the concept of local optical centre, to write a generalization of the Prentice's law which is valid for lenses with arbitrary geometry. The local optical centre is defined as the optical centre of the monofocal lens formed by the tangent paraboloids to the lens surfaces at the point of incidence. We have introduced another way of expressing this concept, by means of the local ground prism, and we provide simple expressions to compute all of them.

Another interesting application for the presented formalism is the possibility of a more precise computation of prismatic effects in monofocal lenses. In effect, even in monofocal lenses, the ray deviations predicted by the paraxial approximation differs appreciably from the correct values for medium and high powered lenses and for points a few millimeters far from the axis. Even more, Prentice's law does not distinguish between spherical or aspherical designs as far as their curvatures at the axis are identical. The presented formalism does allow for this distinction, and gives more accurate results for the prismatic effect in any type of ophthalmic lens. Finally, we have presented a verification of the accuracy of the method for ± 6 D spherical lenses.

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