Valuation of commodity derivatives when spot prices revert to a cyclical mean

April 11, 2014

Abstract

This paper introduces a new continuous-time model based on the logarithm of the commodity spot price assuming that the log-price converges to a cyclic level. In short, this model incorporates that assumption by modelling the mean reversion level through a Fourier series. Under this assumption, we compute closed-form expressions for the values of several commodity derivatives. Finally, we analyze the empirical in-sample performance of our model versus two alternative competitors, namely, those proposed in Schwartz (1997) and Lucia and Schwartz (2002). Our findings show that this model outperforms both benchmarks, providing a simple and powerful tool for portfolio management, risk management and derivative pricing.

Keywords: Fourier series, Pricing, Energy Market, Seasonal

JEL classification: C51, G12, G13.

1 Introduction

Characterizing the stochastic behaviour of commodity prices constitutes an issue of special relevance for practitioners in financial markets and it has been deeply analysed in many academic papers throughout the years. That is hardly surprising, since some commodity markets are very liquid and they move every day a huge number of financial trades. Furthermore, many financial contingent claims such as futures, options and options on futures use some commodity as the underlying asset. Given the seasonal behaviour exhibited by most commodities, this paper introduces a new continuous-time model based on an Ornstein-Uhlenbeck process for the logarithm of the commodity spot price, with a reversion to a time dependent long-run level, whose time variation is characterized by a Fourier series. The underlying idea behind this assumption is that the pricing process is driven by market forces and dominated by a strong seasonal component. Intuitively, some commodity prices are pulled back to a lower mean reversion level in periods of high supply or low demand, while this reversion level tends to be higher whenever the supply is low or the demand is high. In other cases, a given commodity may be perceived as a refuge against bad economic times, and the cyclical behaviour in its price may reflect in part the evolution of the business cycle in some major economy. Under this framework, we compute closed-form expressions for the prices of futures, European options and European options on futures.[1]

In the academic literature we can find a significant number of papers addressing empirically and theoretically the commodity valuation problem. A pioneer contribution can be found in Schwartz (1997), who compares three mean-reverting models for the stochastic behaviour of a commodity price. The first model is a simple one-factor model based on the logarithm of the commodity spot price, constituting the starting point of our posited model. The second is the two-factor model proposed in Gibson and Schwartz (1990), where the second factor accounts for the convenience yield of the commodity. Finally, the third model is an extension of the Gibson and Schwartz (1990) model that incorporates the stochastic behaviour of interest rates as in Vasicek (1977). An interesting twist of the two-factor model is presented in Schwartz and Smith (2000), where the log-spot price is described as the sum of two state variables referred to as the short-term deviation in prices and the equilibrium price level, respectively. In more detail, short-run deviations are assumed to revert toward zero and the equilibrium level is assumed to follow a Brownian motion process.

Addressing the possible seasonal behaviour of the commodity price, a simple and clever contribution can be found in Lucia and Schwartz (2002). In this paper the authors use the Scandinavian electricity market to compare a number of models based on the spot price and the logarithm of the spot price, where the seasonal component is arbitrarily incorporated in an ad-hoc fashion to the process for the log-spot price and modelled by a deterministic trigonometric function with annual

[1] In this paper we make no distinction between futures and forward agreements.

frequency. An interesting extension of the one-factor log-spot price model presented in Lucia and Schwartz (2002) can be found in Cartea and Figueroa (2005), where the stochastic process follows a zero level mean-reverting jump-diffusion process for the underlying log-spot price and the exponential of the trigonometric function is replaced by a Fourier series of order five. For a thorough description of some commodity models see, for instance, Pilipović (1998).

Energy and power markets present a perfect framework to analyse the suitability of this kind of models with a seasonal component. By its own nature, any source of energy is difficult to store or transport. For instance, the low density of natural gas makes highly impractical its storability and transportation, and it has a deep impact on its price, specially in those periods of high demand or production shortages. Additionally, there is a bunch of seasonal variables driving the commodity price, such as business activity, weather conditions, market regulations, etc. There is a rich academic literature focused on energy markets and the corresponding pricing issues. Some interesting contributions on this area can be found in Clewlow and Strickland (2000), Eydeland and Wolyniec (2003), Geman (2005), Burger, Graeber, and Schindlmayr (2007), Forsythe (2007), Weron (2007), and Carmona and Coulon (2012), among many others.

In this paper, we focus our analysis on natural gas as a source of energy, taking Henry Hub as the pricing point for natural gas futures contracts. We compare the fitting ability of our model to market data against two alternative benchmarks. In particular, we use the one-factor models proposed in Schwartz (1997) and Lucia and Schwartz (2002) for the logarithm of the commodity spot price. Since the seasonal component varies among commodities and it could even be different for the same commodity at different maturities, it will be crucial to identify those underlying periods driving the market forces. We use spectral analysis to identify such frequencies, and in particular, to identify the fundamental frequency of the pricing process.

This paper is organized as follows. Section 2 presents the benchmark models and their main features. Section 3 derives the posited model and the futures pricing formula. Section 4 provides closed-form expressions for prices of different derivatives. Section 5 presents the empirical analysis. Finally, Section 6 summarizes the main findings and provides some concluding remarks.

2 Benchmark models

This Section introduces the benchmark models presented by Schwartz (1997) and Lucia and Schwartz (2002), Model 1 and Model 2, respectively.

2.1 Model 1

This model assumes that the commodity spot price S_t follows a stochastic process given by,

$$dS_t = \kappa\left(\mu - \ln(S_t)\right) S_t dt + \sigma S_t dW_t$$

where κ, μ, and σ are constant parameters, and W_t is a standard Wiener process.

Moreover, defining $X_t = \ln(S_t)$, assuming a constant market price of risk λ, and applying Ito's Lemma, the log price can be represented by the following risk-neutral process

$$dX_t = \kappa\left(\widetilde{\alpha} - X_t\right) dt + \sigma d\widetilde{W_t}$$

where

$$\widetilde{\alpha} = \mu - \frac{\sigma^2}{2\kappa} - \frac{\lambda\sigma}{\kappa}$$

where $\widetilde{\alpha}$, κ and σ are constant parameters and $\widetilde{W_t} = W_t + \lambda t$ is a standard Wiener process under the risk-neutral measure $\widetilde{P}$. In addition, under this measure, the solution to equation (2.1) is given as

$$X_s = e^{-\kappa(s-t)} X_t + \left(1 - e^{-\kappa(s-t)}\right)\widetilde{\alpha} + \sigma \int_t^s e^{-\kappa(s-u)} d\widetilde{W_u}$$

which is normally distributed with mean and variance at time T as follows

$$\begin{aligned}
\widetilde{E}[X_T|F_t] &= e^{-\kappa(T-t)} X_t + \left(1 - e^{-\kappa(T-t)}\right)\widetilde{\alpha} \\
\widetilde{V}[X_T|F_t] &= \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa(T-t)}\right)
\end{aligned}$$

Since the spot price of the commodity at time T is log-normally distributed, the forward price of the commodity is given as

$$\begin{aligned}
F(S_t, t, T) &= \widetilde{E}\left[S_T|F_t\right] = \exp\left\{\widetilde{E}\left[X_T|F_t\right] + \frac{1}{2}\widetilde{V}\left[X_T|F_t\right]\right\} \\
&= \exp\left\{e^{-\kappa(T-t)}\ln(S_t) + \left(1 - e^{-\kappa(T-t)}\right)\widetilde{\alpha} + \frac{\sigma^2}{4\kappa}\left(1 - e^{-2\kappa(T-t)}\right)\right\}
\end{aligned}$$

Alternatively,

$$\ln(F(S_t, t, T)) = e^{-\kappa(T-t)}\ln(S_t) + \left(1 - e^{-\kappa(T-t)}\right)\widetilde{\alpha} + \frac{\sigma^2}{4\kappa}\left(1 - e^{-2\kappa(T-t)}\right) \tag{1}$$

2.2 Model 2

Lucia and Schwartz (2002) propose an alternative one-factor model for the log-spot price that incorporates an interesting feature capturing the seasonal nature of some commodity prices. Specifically, model 2 incorporates a deterministic function of time, $f(t)$, of the form

$$\begin{aligned}\ln S_t &= f(t) + Y_t \\ f(t) &= \alpha + \vartheta D_t + \gamma \cos\left((t+\varphi)\cdot \frac{2\pi}{365}\right)\end{aligned}$$

where α, ϑ, γ, and φ are constant parameters, $D_t = 1$ if date t is holiday or weekend, $D_t = 0$ otherwise, and Y_t is a zero level mean-reverting stochastic process given as

$$dY_t = -\kappa Y_t dt + \sigma dW$$

where κ and σ are positive and constant parameters. Under the risk-neutral measure, that is, defining a constant market price of risk $\Lambda(t) = \lambda$, the risk-neutral process is given as

$$dY_t = \kappa(\alpha^* - Y_t)dt + \sigma d\widetilde{W}$$

where $\alpha^* = -\lambda\sigma/\kappa$ is a constant parameter.

In addition, defining $X_t = \ln(S_t)$ and applying some basic algebra we find that the solution for X_s under the risk-neutral measure is given as

$$X_s = f(s) + Y_t e^{-\kappa(s-t)} + \left(1 - e^{-\kappa(s-t)}\right)\alpha^* + \sigma \int_t^s e^{-\kappa(s-u)} d\widetilde{W}$$

Again, since the spot price of the commodity at time T is log-normally distributed, the forward price of the commodity is given as

$$\begin{aligned}F(S_t,t,T) &= \widetilde{E}\left[S_T|F_t\right] = \exp\left\{\widetilde{E}\left[X_T|F_t\right] + \frac{1}{2}\widetilde{V}\left[X_T|F_t\right]\right\} \\ &= \exp\left\{f(T) + e^{-\kappa(T-t)}(\ln(S_t) - f(t)) + \left(1 - e^{-\kappa(T-t)}\right)\alpha^* + \frac{\sigma^2}{4\kappa}\left(1 - e^{-2\kappa(T-t)}\right)\right\}\end{aligned} \tag{2}$$

with $\alpha^* = -\lambda\sigma/\kappa$

3 A New Model for the Commodity Price

In this section we introduce a new valuation model for commodity prices, obtaining the corresponding expression for pricing futures contracts.

3.1 The New Model

Let S_t denote the commodity spot price available at time t. Then, the evolution of the commodity spot price, S_t, is given by the stochastic differential equation

$$dS_t = \kappa\left(f(t) - \ln(S_t)\right)S_t dt + \sigma S_t dW_t \tag{3}$$

where $\kappa,\ \sigma \in \mathbb{R}^+$ and W_t is a standard Wiener process. The main assumption made in this model is that the mean reversion level, $f(t)$, follows a time-dependent periodic function characterized by the real part a Fourier series,[2]

$$f(t) = \sum_{n=0}^{\infty} Re\left[A_n e^{inwt}\right]$$

Note that, $\forall n \mid A_n \in \mathbb{C}$, so that there is a phase factor contained in A_n. In more detail, consider $A_n = A_{x,n} + iA_{y,n}$ where $A_{x,n}, A_{y,n} \in \mathbb{R}$. Hence, $A_{x,n}$ and $A_{y,n}$ denote the amplitude and phase of each term in the Fourier expansion, respectively. Note that this model nests model 1 presented in Schwartz (1997) by taking $A_n = 0,\ \forall n \in \mathbb{N} - \{0\}$.

Moreover, defining $X_t = \ln(S_t)$, assuming a constant market price of risk, that is $\Lambda(S_t, t) = \lambda$, and applying Ito's Lemma, the log price can be represented by the following risk-neutral process

$$dX_t = \mu_t dt + \sigma d\widetilde{W}_t \tag{4}$$

where

$$\begin{aligned} \mu_t &= \kappa\left(\widetilde{\alpha} + g(t) - X_t\right) && (5) \\ \widetilde{\alpha} &= A_0 - \frac{\sigma^2}{2\kappa} - \frac{\lambda\sigma}{\kappa} && (6) \\ g(t) &= \sum_{n=1}^{\infty} Re\left[A_n e^{inwt}\right] && (7) \end{aligned}$$

where $A_0 \in \mathbb{R}$ and $\widetilde{W}_t = W_t + \lambda t$ is a standard Wiener process under the risk-neutral measure $\widetilde{P}$.

The following Proposition establishes the solution of the stochastic differential equation (4).

Proposition 1 *The solution of the risk-neutral process followed by the logarithm of the commodity spot price is given as*

$$X_s = e^{-\kappa(s-t)}X_t + \left(1 - e^{-\kappa(s-t)}\right)\widetilde{\alpha} + \sum_{n=1}^{\infty} Re\left[\frac{\kappa A_n}{\kappa + inw}\left(e^{inws} - e^{-\kappa(s-t)+inwt}\right)\right] + \sigma\int_t^s e^{-\kappa(s-u)}d\widetilde{W}_u$$

■

Figure 1 presents the evolution of the spot price time series for four different set of parameters. In the first graph we only consider the drift process, that is $\sigma = 0$. The flexibility of the model is reflected in the fact that any scenario can be replicated by increasing the number of terms in the Fourier expansion. The second graph represents the aggregate of the drift and the diffusion processes, presenting a simulated spot price walk under each parameter scenario. For illustrative

[2] It is only the real part of the Fourier series that makes economic sense.

purposes, Figures 2 and 3 show how the spot price responds to different values of the parameters $\widetilde{\alpha}$, κ, $A_{n,x}$, $A_{n,y}$, and ω with $n=1$, $\sigma=0$.

From Proposition 1, it is clear that the conditional distribution of the logarithm of the commodity spot price at time T follows a normal distribution where the mean and variance under the risk-neutral probability measure $\widetilde{P}$ are given as

$$\widetilde{E}\left[X_T|F_t\right] = e^{-\kappa(s-t)}X_t + \left(1-e^{-\kappa(s-t)}\right)\widetilde{\alpha} + \sum_{n=1}^{\infty} Re\left[\frac{\kappa A_n}{\kappa+inw}\left(e^{inws} - e^{-\kappa(s-t)+inwt}\right)\right] \tag{8}$$

$$\begin{aligned}\widetilde{V}\left[X_T|F_t\right] &= \widetilde{V}\left[\sigma\int_t^T e^{-\kappa(T-u)}d\widetilde{W}_u\right] = \left(\sigma\int_t^T e^{-\kappa(T-u)}d\widetilde{W}_u\right)^2 = \sigma^2\int_t^T e^{-2\kappa(T-u)}du \\ &= \frac{\sigma^2}{2\kappa}\left(1-e^{-2\kappa(T-t)}\right)\end{aligned} \tag{9}$$

where we have applied the isometry property for stochastic integrals in the variance.

Since $X_t = \ln(S_t)$, the forward price of a commodity maturing at time T is a straightforward application of the properties of the log-normal distribution under the risk-neutral measure. Hence, the following proposition arises

Proposition 2 *Assuming a constant interest rate, the forward price of a commodity maturing at time T is given by*

$$\begin{aligned}F(S_t,t,T) &= \widetilde{E}\left[S_T|F_t\right] = \exp\left\{\widetilde{E}\left[X_T|F_t\right] + \frac{1}{2}\widetilde{V}\left[X_T|F_t\right]\right\} \\ &= \exp\left\{e^{-\kappa(T-t)}\ln(S_t) + \left(1-e^{-\kappa(T-t)}\right)\widetilde{\alpha} + \frac{\sigma^2}{4\kappa}\left(1-e^{-2\kappa(T-t)}\right)\right. \\ &\quad \left. + \sum_{n=1}^{\infty} Re\left[\frac{\kappa A_n}{\kappa+inw}\left(e^{inws} - e^{-\kappa(s-t)+inwt}\right)\right]\right\}\end{aligned}$$

Alternatively,

$$\begin{aligned}\ln(F(S_t,t,T)) &= e^{-\kappa(T-t)}\ln(S_t) + \left(1-e^{-\kappa(T-t)}\right)\widetilde{\alpha} + \frac{\sigma^2}{4\kappa}\left(1-e^{-2\kappa(T-t)}\right) \\ &\quad + \sum_{n=1}^{\infty} Re\left[\frac{\kappa A_n}{\kappa+inw}\left(e^{inws} - e^{-\kappa(s-t)+inwt}\right)\right]\end{aligned} \tag{10}$$

■

4 Option Pricing

This section focuses on option pricing under our proposed model, and provides closed-form expressions for the prices of European options whose underlying asset is a certain commodity or a forward on this commodity.

- **European option on the commodity**

  Consider a call option maturing at time T with strike K, written on a commodity. Let $c_t(S_t; t; T; K)$ denote the price at time t of this call option. Then, the terminal condition to this call option is given by

$$c_T(S_T; T; T; K) = \max\{F(S_T; T; T) - K; 0\}$$

  Hence, under the risk-neutral measure $\widetilde{P}$, the price at time t of this option will be given by

$$c_t(S_t; t; T; K) = \widetilde{E}\left[e^{-r(T-t)}(F(S_t; t; T) - K)^+ | F_t\right]$$

  The call option price is given by the following Proposition.

  Proposition 3 *The price at time t of a European call option with maturity T written on a commodity is given by*

$$\begin{aligned} c_t(S_t; t; T; K) &= \widetilde{E}\left[e^{-r(T-t)}(S_T - K)^+ | F_t\right] \\ &= e^{-r(T-t)} \int_{-\infty}^{\infty} (S_T - K)^+ \rho(\mu, \Sigma) dX_T \\ &= e^{-r(T-t)} \left[e^{\mu + \frac{1}{2}\Sigma^2} \Phi(d_1) - K\Phi(d_2)\right] \end{aligned}$$

  where $\rho(\mu, \Sigma)$ defines the normal density function and

$$\begin{aligned} \mu &= \widetilde{E}[X_T | F_t] \\ \Sigma &= \widetilde{V}[X_T | F_t] \\ d_1 &= \frac{\mu + \Sigma^2 - \ln(K)}{\Sigma} \\ d_2 &= d_1 - \Sigma \end{aligned}$$

  with $\widetilde{E}[X_T | F_t]$ and $\widetilde{V}[X_T | F_t]$ given by equation (8) and (9), respectively.

- **European option on the commodity forward**

  Consider a European forward call option that matures at time T with strike K. If this option is exercised, the call-holder pays K and receives a forward maturing at time s on a commodity. Let $c_t(S_t; t; T; s; K)$ denote the price at time t of this option. The terminal condition of this option is given as

$$c_T(S_T; T; T; s; K) = max\{F(S_T; T; s) - K, 0\}$$

Under the risk-neutral measure $\widetilde{P}$, the price at time t of this option is given as

$$c_t(S_t; t; T; s; K) = \widetilde{E}\left[e^{-r(T-t)}(F(S_T; T; s) - K)^+ | F_t\right]$$

Hence, the following proposition arises.

Proposition 4 *The price at time t of a European forward call option with maturity T on a forward contract expiring at time s written on a commodity is given by*

$$\begin{aligned}
c_t(S_t; t; T; s; K) &= \widetilde{E}\left[e^{-r(T-t)}(F(S_T, T, s) - K)^+ | F_t\right] \\
&= e^{-r(T-t)} \int_{-\infty}^{\infty} (F(S_T, T, s) - K)^+ \rho(\mu, \Sigma) dX_T \\
&= e^{-r(T-t)} \left[\exp\left\{\Omega + \mu e^{-\kappa(s-T)} + \frac{1}{2}\Sigma^2 e^{-2\kappa(s-T)}\right\} \Phi(d_1) - K\Phi(d_2)\right]
\end{aligned}$$

where $\rho(\mu, \Sigma)$ denotes the normal density function and

$$\begin{aligned}
\mu &= \widetilde{E}\left[X_T | F_t\right] \\
\Sigma^2 &= \widetilde{V}\left[X_T | F_t\right] \\
\Omega &= \left(1 - e^{-\kappa(s-T)}\right)\widetilde{\alpha} + \left(1 - e^{-2\kappa(s-T)}\right)\frac{\sigma^2}{4\kappa} + \sum_{n=1}^{\infty} Re\left[\frac{\kappa A_n}{\kappa + inw}\left(e^{inws} - e^{-\kappa(s-T)+inwT}\right)\right] \\
\nu &= (\ln(K) - \Omega)\, e^{\kappa(s-T)} \\
d_1 &= \frac{\mu + \Sigma^2 e^{-\kappa(s-T)} - \nu}{\Sigma} \\
d_2 &= \frac{\mu - \nu}{\Sigma}
\end{aligned}$$

with $\widetilde{E}[X_T|F_t]$ and $\widetilde{V}[X_T|F_t]$ given by equation (8) and (9), respectively.

5 Empirical Analysis

5.1 Data

The data set used for the empirical study consist of daily observations of futures contracts written on natural gas. In more detail, we take Henry Hub as the pricing point for natural gas futures contracts, which is traded on the New York Mercantile Exchange (NYMEX). We have complete data for the spot prices and the twelve contracts closest to maturity from 02/02/1998 to 07/03/2011. In this analysis we take into consideration the Ng-5, Ng-8 and Ng-12, where Ng-5 is the fifth contract closest to maturity, and so on.

Since the models for the forward price have the form given as in the previous Sections, we estimate the structural parameters by minimizing the fitting error of each model as:

$$Y_t = \sum_{i=1}^{6} \beta_i z_{it} + u_t$$

where u_t can be interpreted either as a measurement or as an approximation error in the pricing formula. For every model we follow a non-weighted least-squares approach to obtain the parameter estimates.

Model 1

We hope to identify the values of the structural parameters: $\theta = (\widetilde{\alpha}, \kappa, \sigma)$.

$$\begin{aligned}
Y_t &= \ln(F(S_t, t, T)) - e^{-\kappa(T-t)} \ln(S_t) \\
z_{1t} &= 1 - e^{-\kappa(T-t)} \\
z_{2t} &= \left(1 - e^{-2\kappa(T-t)}\right) / 4\kappa \\
\beta_1 &= \widetilde{\alpha};\ \beta_2 = \sigma^2;\ \beta_3 = \beta_4 = \beta_5 = \beta_6 = 0
\end{aligned}$$

Model 2

Neglecting ϑ, considering trading days and with some basic algebra we reorganized model 2. In this case, we hope to identify the values of the structural parameters: $\theta = (\widetilde{\alpha}, \kappa, \sigma, \gamma, \varphi)$.

$$\begin{aligned}
Y_t &= \ln(F(S_t, t, T)) - e^{-\kappa(T-t)} \ln(S_t) \\
z_{1t} &= 1 - e^{-\kappa(T-t)} \\
z_{2t} &= \left(1 - e^{-2\kappa(T-t)}\right) / 4\kappa \\
\beta_1 &= \widetilde{\alpha} = \alpha + \alpha^*;\ \beta_2 = \sigma^2;\ \beta_3 = \gamma;\ \beta_4 = 0 \\
f_1(t, T) &= \cos\left((T + \varphi) \cdot 2\pi\right) - e^{-\kappa(T-t)} \cos\left((t + \varphi) \cdot 2\pi\right)
\end{aligned}$$

Model 3

This model assumes only one term in the Fourier expansion, hence we hope to identify the values of the structural parameters $\theta = \left(\widetilde{\alpha}, \kappa, \sigma, A_{(x,n_1)}, A_{(y,n_1)}, \omega\right)$.

$$\begin{aligned}
Y_t &= \ln(F(S_t, t, T)) - e^{-\kappa(T-t)} \ln(S_t) \\
z_{1t} &= 1 - e^{-\kappa(T-t)} \\
z_{2t} &= \left(1 - e^{-2\kappa(T-t)}\right) / 4\kappa \\
\beta_1 &= \widetilde{\alpha};\ \beta_2 = \sigma^2
\end{aligned}$$

and $Re\left[(A_{(x,n_1)} + iA_{(y,n_1)}) \dfrac{\kappa}{\kappa + inw} \left(e^{inws} - e^{-\kappa(s-t)+inwt}\right)\right]$ derives in $\beta_3 z_{3t} + \beta_4 z_{4t}$.

Model 4

In this case we assume two terms in the Fourier expansion, which leads us to identify the values of the structural parameters $\theta = \left(\widetilde{\alpha}, \kappa, \sigma, A_{(x,n_1)}, A_{(y,n_1)}, A_{(x,n_2)}, A_{(y,n_2)}, \omega\right)$.

$$\begin{aligned}
Y_t &= \ln(F(S_t,t,T)) - e^{-\kappa(T-t)}\ln(S_t) \\
z_{1t} &= 1 - e^{-\kappa(T-t)} \\
z_{2t} &= \left(1 - e^{-2\kappa(T-t)}\right)/4\kappa \\
\beta_1 &= \widetilde{\alpha};\ \beta_2 = \sigma^2
\end{aligned}$$

where $\sum_{n=n_1,n_2} Re\left[(A_{(x,n)} + iA_{(y,n)})\frac{\kappa}{\kappa + inw}\left(e^{inws} - e^{-\kappa(s-t)+inwt}\right)\right]$ derives in $\sum_{i=3}^{6} \beta_i z_{it}$.

5.2 In-Sample Analysis

The key assumption in this paper is that there is a seasonal pattern in futures prices, besides being non-stationary. Table 1 presents the Augmented Dickey-Fuller (ADF) test for the time series for the log-spot price as well as for the log-price of each of the futures contracts considered, and for the first differences of each of these series. Clearly, the unit root hypothesis cannot be rejected in any case, meaning that both, the spot and futures price series are non-stationary. On the other hand, the presence of a unit root is rejected when considering the first difference of each price. Figures 4 to 7 present the autocorrelation and partial autocorrelation function for each time series.

Table 2 shows the estimation results for the long-run relationship between the futures and the spot price:

$$\ln F_t(\tau) = \alpha + \beta \ln S_t + a_t$$

where τ corresponds to each futures tenor, Ng-5, Ng-8, and Ng-12, respectively. Table 3 presents the Augmented Dickey-Fuller test for each residual time series and its first difference. In every case we reject the existence of a unit root. Figures 8 to 10 present the residual and its first difference time series corresponding to the estimation of Ng-5, Ng-8, and Ng-12. Figures 11 to 13 present the autocorrelation and partial autocorrelation function for each residual time series and its first difference time series. Hence, the logarithm of each futures price is cointegrated with the logarithm of the spot price, as it is usually the case in most liquid futures markets.

The expected seasonal pattern suggests that it is reasonable to study the spectral density of futures prices, bearing in mind that for any such time series the spectral density should be expected to have a maximum at the zero frequency, due to the presence of a unit root. Figures 14 to 16 present the logarithm of the price of natural gas for Ng-5, Ng-8 and Ng-12 futures contracts and the associate spectral density, where $f(Hz) = \frac{\omega}{2\pi}$. As expected, these figures confirm the existence of a maximum in the spectral density at the zero frequency. However, an additional interesting result arises in these spectra: aside from the zero frequency, the three time series of futures prices reach a maximum in their spectral density at rather low frequency, which should be interpreted as the fundamental frequency. This frequency indicates that there is an underlying long-run period driving the behaviour of futures prices, of about 15 to 16 years.

This result is quite interesting for our purposes. However, since the three models we consider are driven by the commodity spot price, we are specifically interested in the spectral component which is not explained by the spot price. The reason is that we are interested in the seasonal period that is specific to futures prices and hence, on the seasonal component that it is not inherited through the dependence of futures prices from the spot price. This is a not trivial endeavour because, according to our model, the relationship between spot and futures prices is not straightforward. In fact, it depends on the speed of mean reversion parameter κ, which should be estimated for each model. In addition, we should remember that our proposed model has been developed under the assumption that the mean reversion level follows over time an evolution characterized by a Fourier series. Alternatively, the model posited by Lucia and Schwartz (2002) assumes a zero level mean-reverting process and arbitrarily adds a trigonometric function with an annual frequency. Therefore, relaxing the annual frequency assumption will not collapse the Lucia and Schwartz model into our model.

The estimated spectra are precisely very important to conduct the specification of our model for estimation purposes. We need to truncate the infinite Fourier series, and it is important to have some idea about how many terms may be needed to fit the futures price data, and which frequencies should be incorporated into the chosen terms of the Fourier expansion. At this point, we already know that to appropriately capture the dynamics in natural gas futures prices it is necessary to include in the model the estimated fundamental frequency.

To detect the frequencies that are relevant to explain the dynamics in futures prices, we create a grid of frequencies and fit our model to the observed time series for each value of ω in the grid. These estimations will provide us with a measure of the fitting errors for each frequency, thereby exposing the cyclical component not captured by the spot price. Figure 17 shows the residual sum of squares of estimating our model for a fixed value of ω in $(0, 2\pi)$. The results are quite conclusive: there is a well defined minimum fitting error at a point very close to the fundamental frequency obtained in estimated spectrum, indicating an underlying long run period of 15 to 16 years. This analysis reveals another interesting feature: the second relevant term in the Fourier series for the Ng-5 and Ng-8 is the annual frequency, which is, of course, a multiple of the fundamental frequency $(n \cdot \omega = 2\pi n \cdot f(Hz))$. However, the importance of the annual frequency decreases with maturity, completely disappearing beyond the futures expiring in one year, Ng-12. This may be reasonable: since the Ng-12 expiration date is exactly one wavelength of the annual frequency, then it makes sense that the annual frequency has a negligible effect on the Ng-12 time series. Since the analysis considers the relation between the spot and futures prices, it is fair to say that this frequency is indicating a cyclical behaviour in the futures price which is not captured by the spot price.

To obtain the spectrum of the component of futures prices which is not explained by spot prices, we use the estimated parameters associated with this fundamental frequency to compute the function $\hat{s}(F_t, S_t)$, given as

$$\hat{s}(F_t, S_t) = \ln(F(S_t, t, T)) - e^{-\hat{\kappa}(T-t)} \ln(S_t) - \hat{\tilde{\alpha}} \left(1 - e^{-\hat{\kappa}(T-t)}\right) - \frac{\hat{\sigma}^2}{4\hat{\kappa}} \left(1 - e^{-2\hat{\kappa}(T-t)}\right)$$

The spectrum of this function will expose the underlying component that is not fully explained by the spot price or by any trend. Figures 18 to 20 present the spectrum of the $\hat{s}$-function for each futures price, suggesting that at most three terms in the Fourier expansion should be enough to attain a good fit of the $\hat{s}$-function. As expected, the frequencies identified in each spectrum match exactly the frequencies detected in the graphs of the residual sum of squares for fixed values of ω, confirming that we certainly have spotted the frequencies we need to obtain accurate estimates.

Tables 4 to 6 present the estimated parameters and the corresponding standard deviation for each chosen futures price and for the whole sample, as well as goodness of fit measures for each model. For the whole period we present the minimized value of the function $\sum_{i,t} \min \mathrm{SR}(\hat{\theta}_{i,t})$ and $\sum_{i,t} |\hat{u}_{i,t}|$, the sum of the absolute value of pricing errors for the whole period, to represent how well each model fits the observed futures prices. In addition, Figures 21 to 23 present the Ng-5, Ng-8 and Ng-12 adjustment error time series for Models 2, 3, and 4. To keep the graphs as clear as possible we have intentionally excluded Model 1. On February 25, 2003 every model shows a particularly poor fit. That day, United States, United Kingdom and Spain presented to the UN Security Council a resolution stating that Iraq “has failed to take the final opportunity” to disarm. Rumors of an imminent war plunged stock markets all over the world, while many commodities prices raised till historical maximums. Henry Hub spot price has closed at 19.38$, when average spot price oscillated at 5$.

Regarding the goodness of fit the results are conclusive. Compared with the benchmark models, both representations of our model dramatically improve the in-sample fit of every observed futures time series. Model 3, the model with just one term in the Fourier expansion, reduces the aggregate sum of squares of Model 2 by 28%, 54% and 79%, for Ng-5, Ng-8 and Ng-12, respectively. Comparing Model 3 with Model 1, the improvement is of 48%, 61% and 79% for Ng-5, Ng-8 and Ng-12, respectively. It is encouraging to know that we do not need to go farther away in the Fourier expansion to achieve a good fitting, even though increasing the number of terms in the Fourier expansion would eventually allow for fitting arbitrarily well the observed time series. On this regard, it is interesting to point out that the annual frequency proposed by the Schwartz and Lucia model has little impact by itself. In fact, for the futures contract expiring in one year, Model 2 provides no further improvement from the model with no seasonal component. In fact, there is an annual frequency in the process driving the futures price, but that annual fluctuation is mostly explained by the spot price. On the other hand, the long run frequency, between 15 and 16 years, explains the seasonality in futures prices that is not captured by the spot price. This frequency might well be related to the business cycle.

Although the main improvement comes with the incorporation of the fundamental frequency, adding a second term in the model still provides further improvements. Comparing Model 4 against Model 3, i.e., the models with two and one term in the Fourier expansion, the relative improvement is given as 45%, 40% and 27% for Ng-5, Ng-8 and Ng-12, respectively. For contracts Ng-5 and Ng-8, the second term incorporates the annual frequency, while the second term for the Ng-12 futures contract suggests a period of 4 years.

Figures 24 to 26 show the spectral density of fitting errors obtained with Models 3 and 4 for each futures contract. As expected, the fundamental frequency has been completely removed from the spectrum. Model 3 fitting error of futures series Ng-5 and Ng-8 is dominated by the annual frequency, and it is completely eliminated from the model 4 fitting error spectra. Model 4 fitting error spectra reveals no dominating frequency, although we can spot some frequencies standing from the noise which could be incorporated in further term of the Fourier expansion. On this respect, adding a third term in the Fourier expansion provides a relative improvement over the two-terms model of 11.5%, 20.5% and 20% for the estimation of Ng-5, Ng-8, and Ng-12, respectively.

6 Conclusions

This paper has introduced a continuous-time model for the logarithm of the commodity spot price, assuming that it reverts to a mean level that follows a cyclical behaviour over time that is characterized by a Fourier series. Under this assumption, our model nests the original one-factor model presented in Schwartz (1997), while allowing for a more flexible evolution of the commodity spot price and preserving the analytical tractability of the Schwartz model. Under this framework, we have obtained analytical expressions for the prices of futures, European option on the commodity and European options on commodity futures.

Considering Natural gas as the underlying asset of the futures contract, we have also analysed the empirical performance of two versions of our model against two different one-factor benchmarks, those proposed in Schwartz (1997) and Lucia and Schwartz (2002). In order to identify the fundamental frequency and the underlying period driving the futures contract price, we have conducted a spectral analysis of three futures with different tenors, in particular Ng-5, Ng-8 and Ng-12. The spectrum revealed that there is a short frequency driving the futures price behaviour of about 15 to 16 years. Although the annual frequency has some relevance in Ng-5 and Ng-8, its importance tends to decrease with maturity. Considering the effect of the fundamental frequency, even in its simplest representation based on a single term of the Fourier expansion, our model outperforms both benchmark models, providing a better and more reliable in-sample fitting of the commodity futures price. Adding a second term in the Fourier expansion provides an improvement relative to the one term representation, although the improvement tends to be lower for longer maturities. On that count, it is worth pointing out that increasing the number of terms in the Fourier expansion would eventually allow for fitting the observed time series arbitrarily well. These results are very relevant, suggesting that our proposed Fourier model provides a simple and powerful tool for portfolio management, risk management and derivative pricing on commodities.

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7 Appendix of Tables

Table 1: Augmented Dickey-Fuller test

	ADF (Level)		ADF (First Difference)	
	Lags	t-stat (p-value)	Lags	t-stat (p-value)
Spot	26	-2.2492(0.189)	25	-13.0611(3.833e-29)
Ng-5	25	-2.0527(0.264)	24	-10.5281(6.566e-21)
Ng-8	23	-1.9750(0.298)	22	-9.7038(2.933e-18)
Ng-12	19	-1.6753(0.444)	25	-12.3007(1.096e-26)

Note: Augmented Dickey-Fuller test for the log spot and futures price, and the first differences of each of these series.

Table 2: Estimation results

	Ng-5	Ng-8	Ng-12
α	0.2311(0.0089)	0.2703(0.0101)	0.3046(0.0109)
β	0.9095(0.0056)	0.8945(0.0063)	0.8721(0.0068)
Log-likelihood function	1422.346	1024.496	775.8086
R^2	0.890935	0.860997	0.834877

Note: Estimation results for the long-run relationship between the futures and the spot price given by process 5.2. Standard errors in parentheses

Table 3: Augmented Dickey-Fuller test

	ADF (Level)		ADF (First Difference)	
	Lags	t-stat (p-value)	Lags	t-stat (p-value)
a_t(Ng-5)	30	-4.6453(1.025e-4)	29	-11.2560(2.803e-23)
a_t(Ng-8)	30	-3.8854(2.156e-3)	26	-13.6307(5.906e-31)
a_t(Ng-12)	27	-3.1480(2.323e-2)	26	-14.3477(3.446e-33)

Note: Augmented Dickey-Fuller test for each residual time series and its first difference.

Table 4: Parameters estimates. In-Sample Estimation Ng-5

Parameter	Model 1	Model 2	Model 3	Model 4
$\hat{\beta}_1$	-14.7756(7.0527)	2.4228(0.0337)	0.4320(0.3487)	1.4190(0.2193)
$\hat{\beta}_2$	9.2165(10.2398)	0.1134(0.0271)	3.8248(1.0256)	0.8830(0.5870)
$\hat{\kappa}$	0.2539(0.1846)	0.2309(0.0079)	1.2085(0.0210)	1.1148(0.0129)
$\hat{\beta}_3$	-	0.0661(0.0085)	-	-
$\hat{\varphi}$	-	0.0607(0.0032)	-	-
$\hat{A}_{(x,n_1=1)}$	-	-	-0.2224(0.0279)	-0.1561(0.0107)
$\hat{A}_{(y,n_1=1)}$	-	-	0.4984(0.0228)	0.5529(0.0087)
$\hat{A}_{(x,n_2=15)}$	-	-	-	-0.3117(0.0109)
$\hat{A}_{(x,n_2=15)}$	-	-	-	-0.3839(0.0092)
$\hat{\omega_0}$	-	2π	0.4152(0.0032)	0.4175(0.0003)
$\sum_{i,t} \min \mathrm{SR}(\hat{\theta}_{i,t})$	80.0594	57.9226	41.7740	22.9802
$\sum_{i,t} \lvert\hat{u}_{i,t}\rvert$	346.6403	309.4022	291.4021	205.9710

Note: $\sum_{i,t} \min \mathrm{SR}(\hat{\theta}_{i,t})$ represents the least squares pricing error, $\sum_{i,t} |\hat{u}_{i,t}|$ shows the pricing errors in absolute value.

Table 5: Parameters estimates. In-Sample Estimation Ng-8

Parameter	Model 1	Model 2	Model 3	Model 4
$\hat{\beta}_1$	-7.1126(6.5074)	2.5991(0.1140)	0.9698(0.4022)	1.6985(0.0039)
$\hat{\beta}_2$	3.6508(2.7470)	0.0000(0.0000)	1.7696(0.9716)	0.0000(0.0000)
$\hat{\kappa}$	0.1787(0.0219)	0.1721(0.0209)	0.9308(0.0152)	0.9223(0.0132)
$\hat{\beta}_3$	-	-0.0557(0.0284)	-	-
$\hat{\varphi}$	-	0.5826(0.0827)	-	-
$\hat{A}_{(x,n_1=1)}$	-	-	-0.0928(0.0313)	-0.1979(0.0116)
$\hat{A}_{(y,n_1=1)}$	-	-	0.6017(0.0250)	0.5228(0.0101)
$\hat{A}_{(x,n_2=16)}$	-	-	-	-0.4101(0.0124)
$\hat{A}_{(x,n_2=16)}$	-	-	-	-0.3657(0.0130)
$\hat{\omega}_0$	-	2π	0.4020(0.0027)	0.3901(0.0003)
$\sum_{i,t} \min \mathrm{SR}(\hat{\theta}_{i,t})$	102.1405	86.6176	39.8256	23.9565
$\sum_{i,t} \lvert\hat{u}_{i,t}\rvert$	406.3963	390.8210	285.4515	205.1990

Note: $\sum_{i,t} \min \mathrm{SR}(\hat{\theta}_{i,t})$ represents the least squares pricing error, $\sum_{i,t} |\hat{u}_{i,t}|$ shows the pricing errors in absolute value.

Table 6: Parameters estimates. In-Sample Estimation Ng-12

Parameter	Model 1	Model 2	Model 3	Model 4
$\hat{\beta}_1$	2.3811(0.8887)	1.8400(0.0839)	1.3963(0.5728)	1.1713(0.2376)
$\hat{\beta}_2$	0.0000(0.0000)	0.1640(0.0179)	0.5359(1.2134)	1.1805(0.5932)
$\hat{\kappa}$	0.1431(0.4613)	0.1464(0.0051)	0.7846(0.0120)	0.8904(0.0085)
$\hat{\beta}_3$	-	0.0858(0.0101)	-	-
$\hat{\varphi}$	-	0.3240(0.0179)	-	-
$\hat{A}_{(x,n_1=1)}$	-	-	-0.1175(0.0299)	-0.1913(0.0224)
$\hat{A}_{(y,n_1=1)}$	-	-	0.5820(0.0253)	0.5116(0.0178)
$\hat{A}_{(x,n_2=3)}$	-	-	-	-0.4965(0.0206)
$\hat{A}_{(x,n_2=3)}$	-	-	-	-0.4196(0.0206)
$\hat{\omega_0}$	-	2π	0.3758(0.0031)	0.3723(0.0021)
$\sum_{i,t} \min \mathrm{SR}(\hat{\theta}_{i,t})$	118.9080	117.6984	24.4762	17.7831
$\sum_{i,t} \lvert\hat{u}_{i,t}\rvert$	490.5892	480.9532	217.4176	188.5439

Note:$\sum_{i,t} \min \mathrm{SR}(\hat{\theta}_{i,t})$ represents the least squares pricing error, $\sum_{i,t} |\hat{u}_{i,t}|$ shows the pricing errors in absolute value.

8 Appendix of Figures

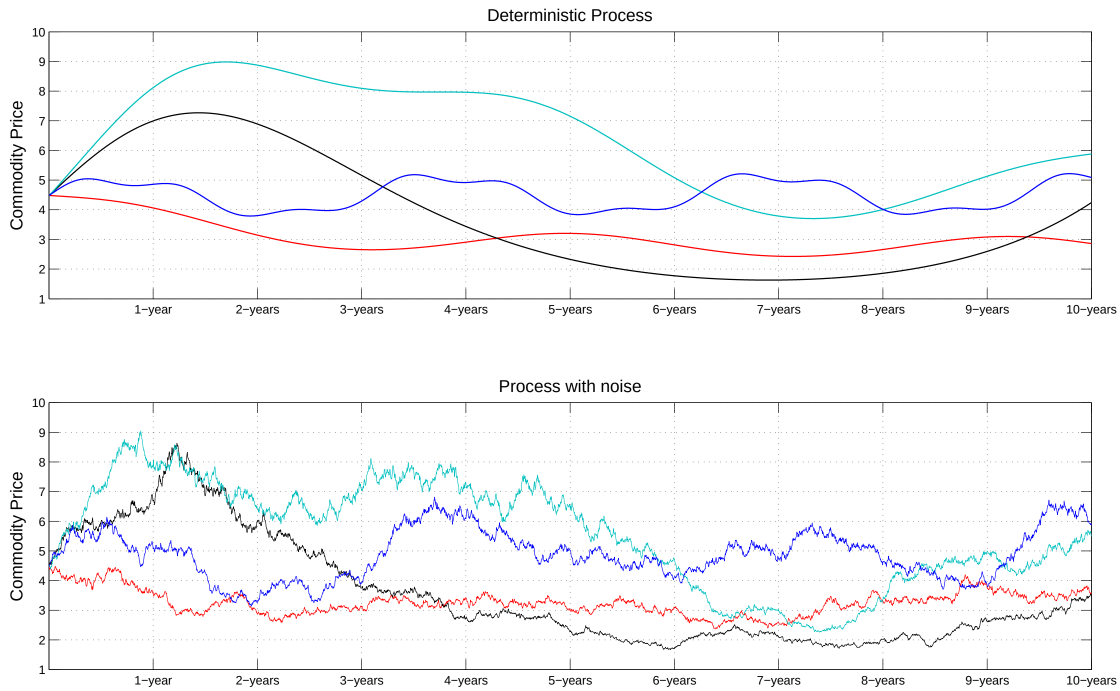

Figure 1: Spot price time series simulation for an arbitrary set of parameters. The first graph represents the drift process, that is setting $\sigma = 0$ in equation (3). The second graph represents the whole process with $\sigma = 0.2$ in equation (3). We consider the following parameters:

Red line: $\widetilde{\alpha} = 1,\ \kappa = 0.5,\ A_{n=1,x} = 0.4,\ A_{n=1,y} = 0,\ A_{n=3,x} = 0,\ A_{n=3,y} = 0,\ \omega = 1.5.$

Black line: $\widetilde{\alpha} = 2,\ \kappa = 0.5,\ A_{n=1,x} = 1,\ A_{n=1,y} = \frac{\pi}{2},\ A_{n=3,x} = 0,\ A_{n=3,y} = 0,\ \omega = 0.4.$

Lightblue line: $\widetilde{\alpha} = 2,\ \kappa = 0.5,\ A_{n=1,x} = 0.8,\ A_{n=1,y} = 0,\ A_{n=3,x} = 0.4,\ A_{n=3,y} = 0,\ \omega = 0.5.$

Blue line: $\widetilde{\alpha} = 1.5,\ \kappa = 0.5,\ A_{n=1,x} = 0.6,\ A_{n=1,y} = 0,\ A_{n=3,x} = 0.5,\ A_{n=3,y} = 0,\ \omega = 2.$

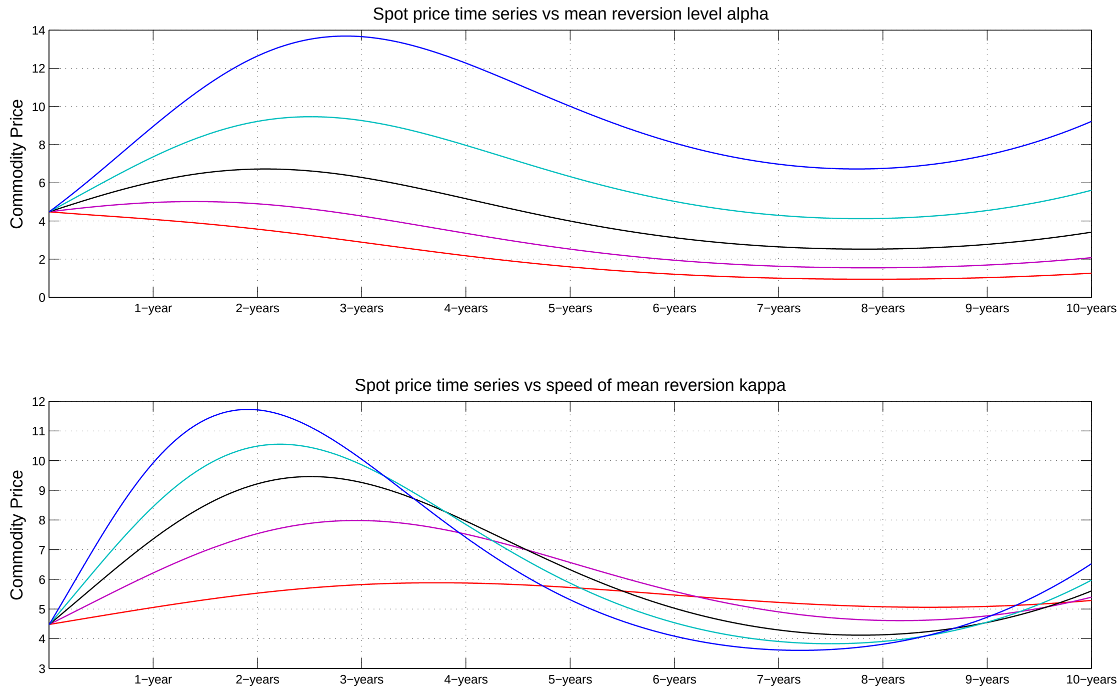

Figure 2: Spot price time series simulation for an arbitrary set of parameters and no diffusion process, $\sigma = 0$ in equation (3). For both graphs: $A_{n=1,x} = 0.8,\ A_{n=1,y} = 0,\ n = 1,\ \omega = 0.5$.

The first graph represents the spot price time series for $\kappa = 0.5$ and different values of $\widetilde{\alpha}$:

Red line: $\widetilde{\alpha} = 0.5$, *Violet line:* $\widetilde{\alpha} = 1$, *Black line:* $\widetilde{\alpha} = 1.5$, *Lightblue line:* $\widetilde{\alpha} = 2$, *Blue line:* $\widetilde{\alpha} = 2.5$.

The second graph represents the spot price time series for $\widetilde{\alpha} = 2$ and different values of κ:

Red line: $\kappa = 0.1$, *Violet line:* $\kappa = 0.3$, *Black line:* $\kappa = 0.5$, *Lightblue line:* $\kappa = 0.7$, *Blue line:* $\kappa = 1$.

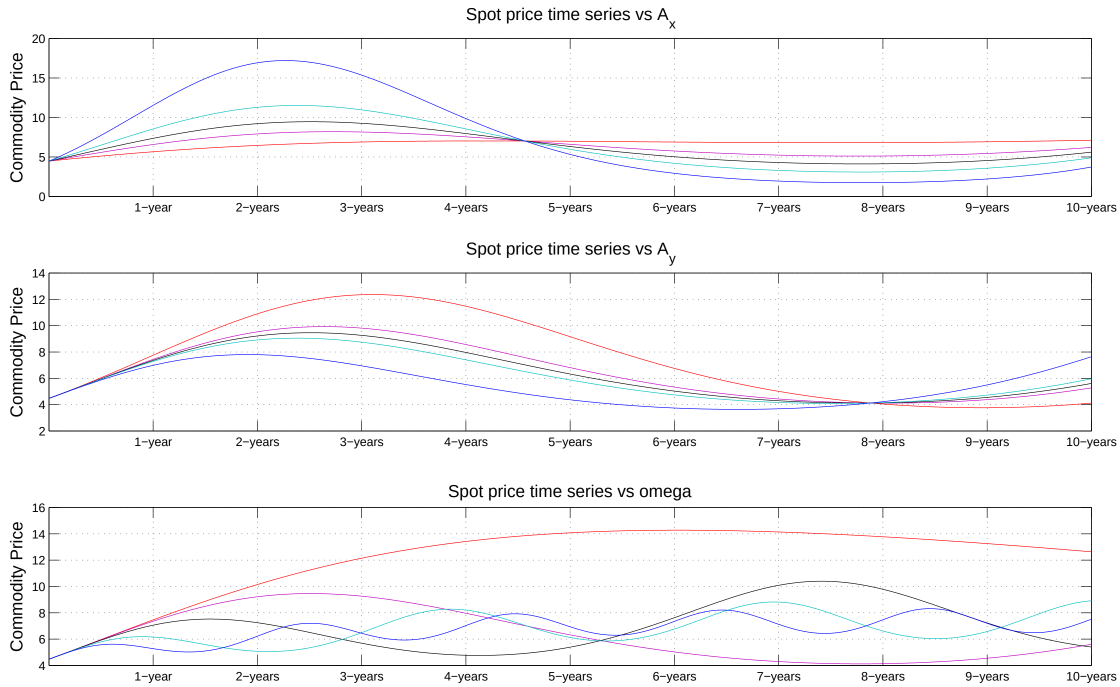

Figure 3: Spot price time series simulation for an arbitrary set of parameters and no diffusion process, $\sigma = 0$ in equation (3). For the three graphs: $\widetilde{\alpha} = 2,\ \kappa = 0.5,\ n = 1$.

The first graph represents the spot price time series for $A_{n=1,y} = 0$, $\omega = 0.5$ and different values of $A_{n=1,x}$:

Red line: $A_{n=1,x} = 0.1$, *Violet line:* $A_{n=1,x} = 0.5$, *Black line:* $A_{n=1,x} = 0.8$, *Lightblue line:* $A_{n=1,x} = 1.2$, *Blue line:* $A_{n=1,x} = 2$.

The second graph represents the spot price time series for $A_{n=1,x} = 0.8$, $\omega = 0.5$ and different values of $A_{n=1,y}$:

Red line: $A_{n=1,y} = -0.5$, *Violet line:* $A_{n=1,y} = -0.1$, *Black line:* $A_{n=1,y} = 0$, *Lightblue line:* $A_{n=1,y} = 0.1$, *Blue line:* $A_{n=1,y} = 0.5$.

The third graph represents the spot price time series for $A_{n=1,x} = 0.8$, $A_{n=1,y} = 0$ and different values of ω:

Red line: $\omega = 0.1$, *Violet line:* $\omega = 0.5$, *Black line:* $\omega = 1$, *Lightblue line:* $\omega = 2$, *Blue line:* $\omega = \pi$.

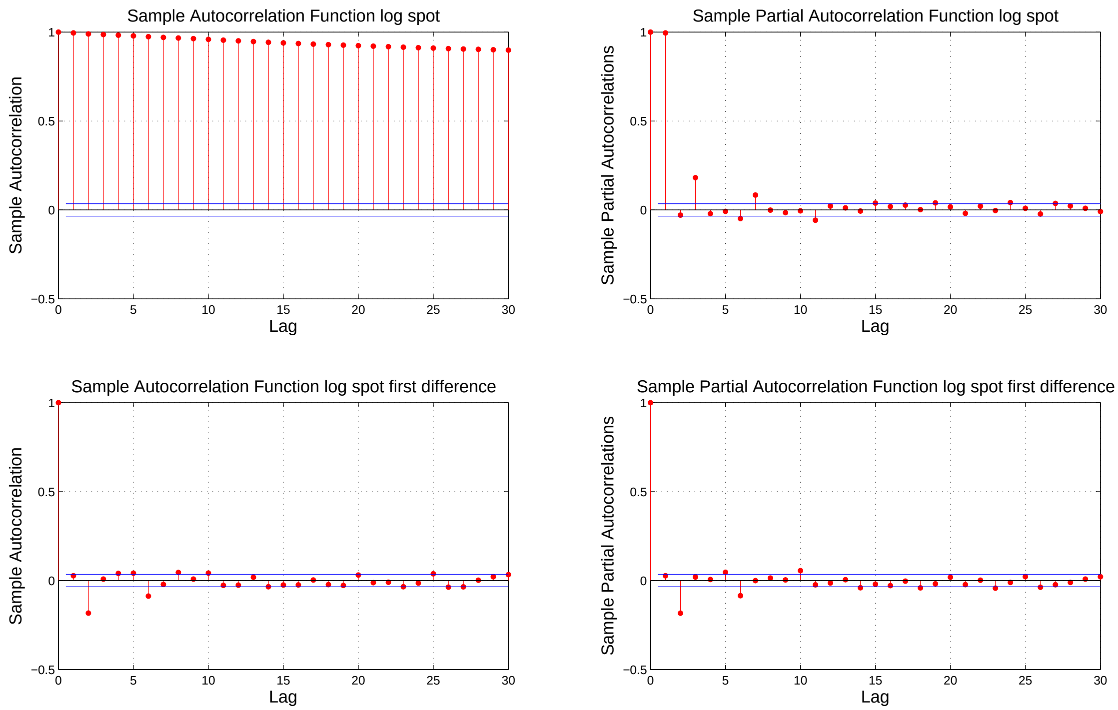

Figure 4: Autocorrelation and Partial autocorrelation function for the logarithm of the spot price series and its first difference series.

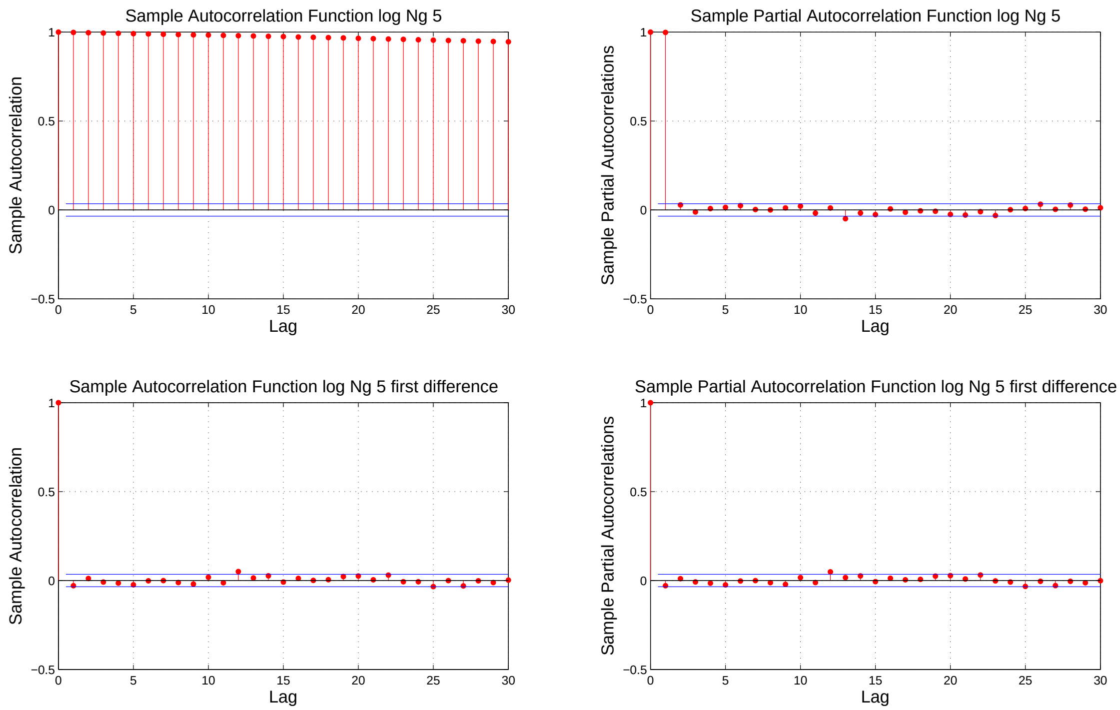

Figure 5: Autocorrelation and Partial autocorrelation function for the logarithm of the Ng-5 futures price series and its first difference series.

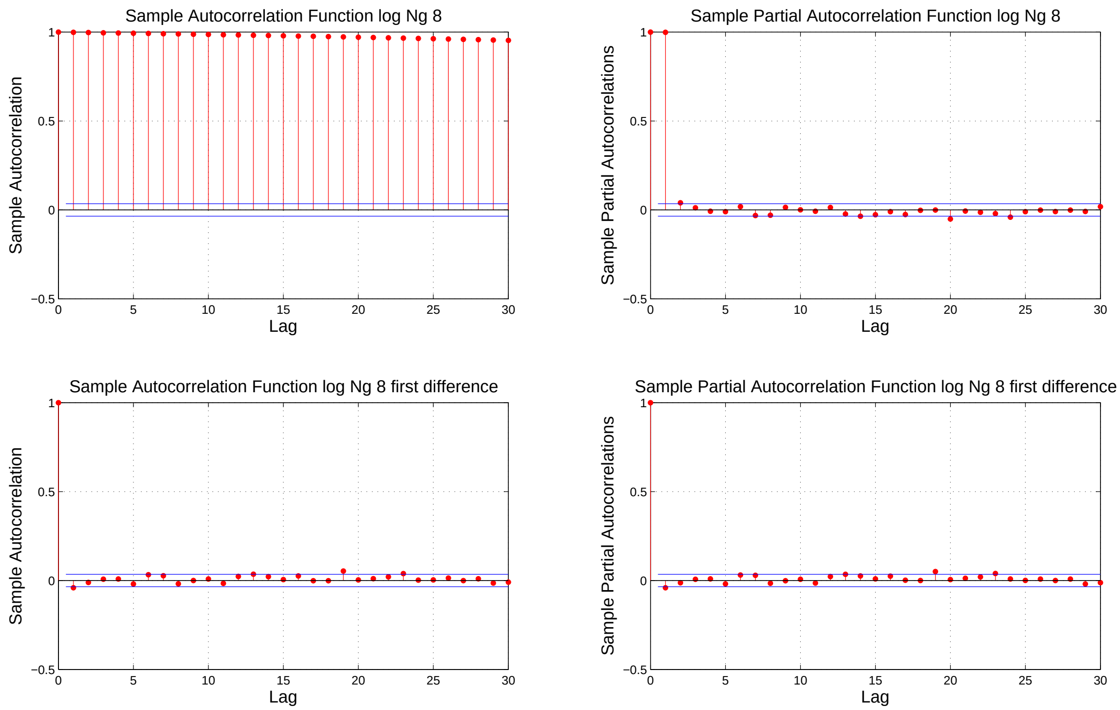

Figure 6: Autocorrelation and Partial autocorrelation function for the logarithm of the Ng-8 futures price series and its first difference series.

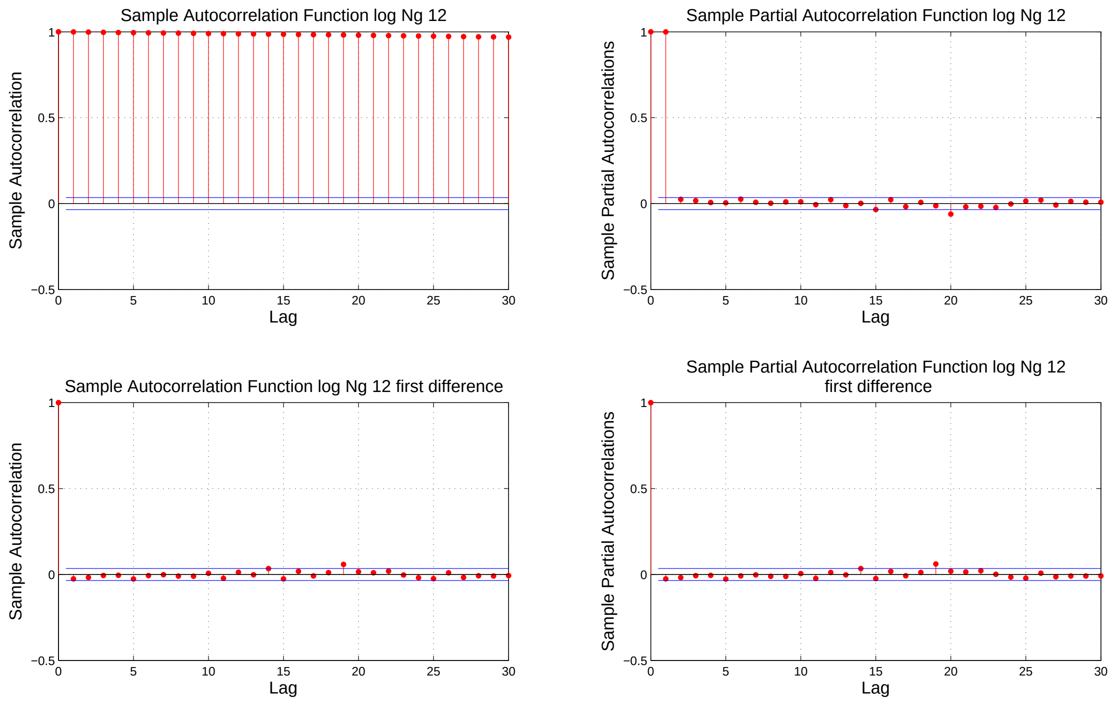

Figure 7: Autocorrelation and Partial autocorrelation function for the logarithm of the Ng-12 futures price series and its first difference series.

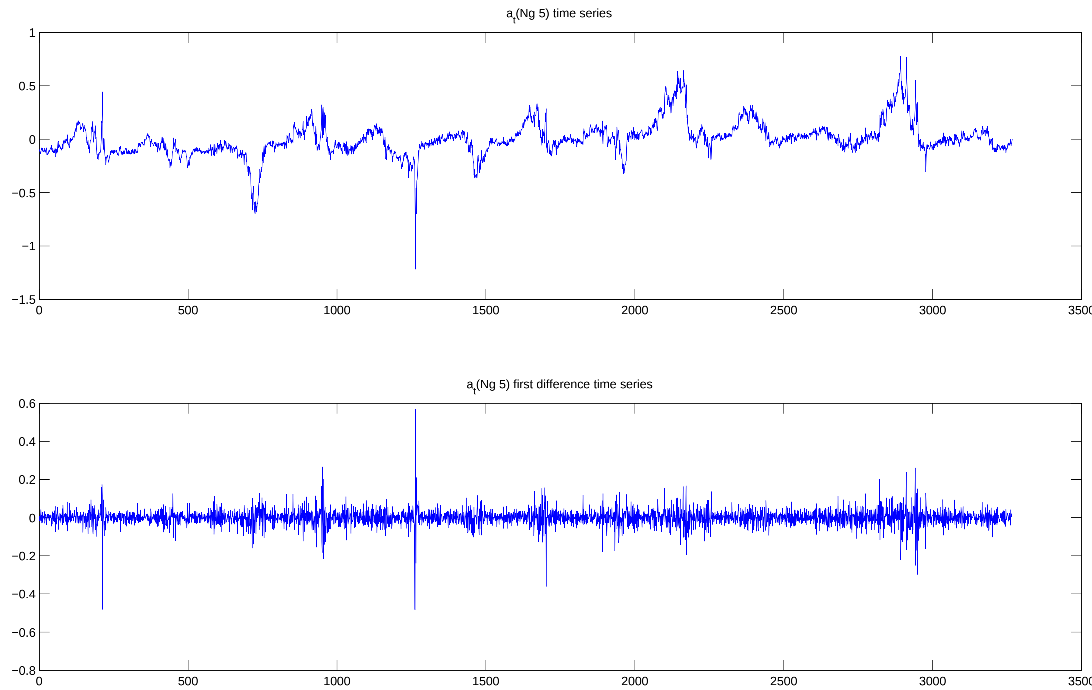

Figure 8: a_t(Ng-5) time series and its first difference series.

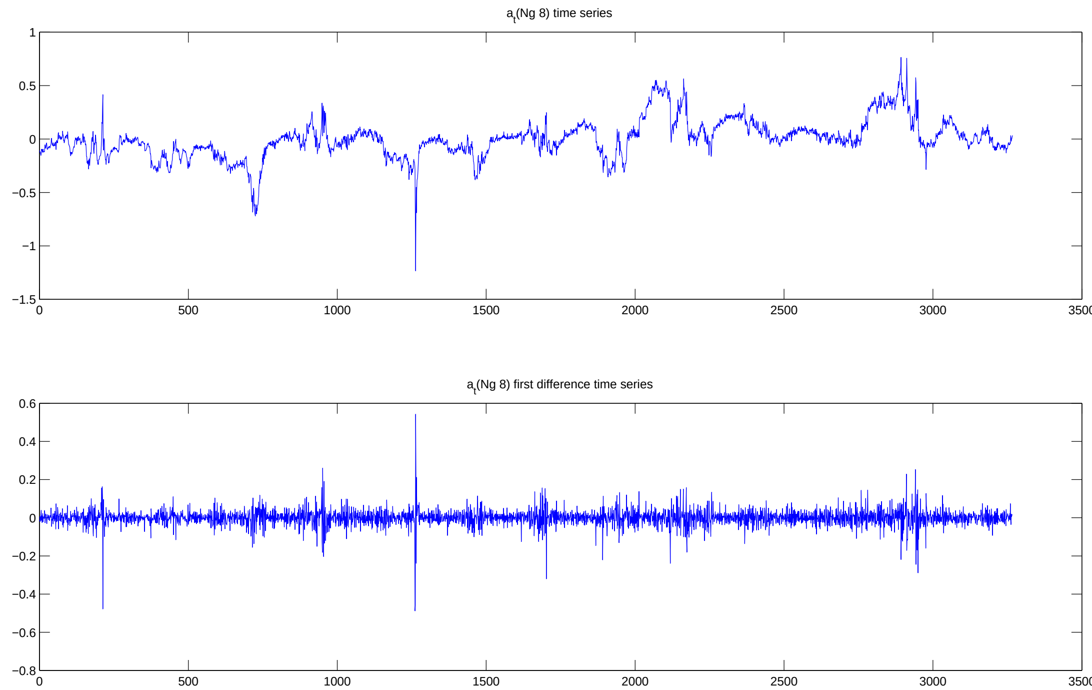

Figure 9: a_t(Ng-8) time series and its first difference series.

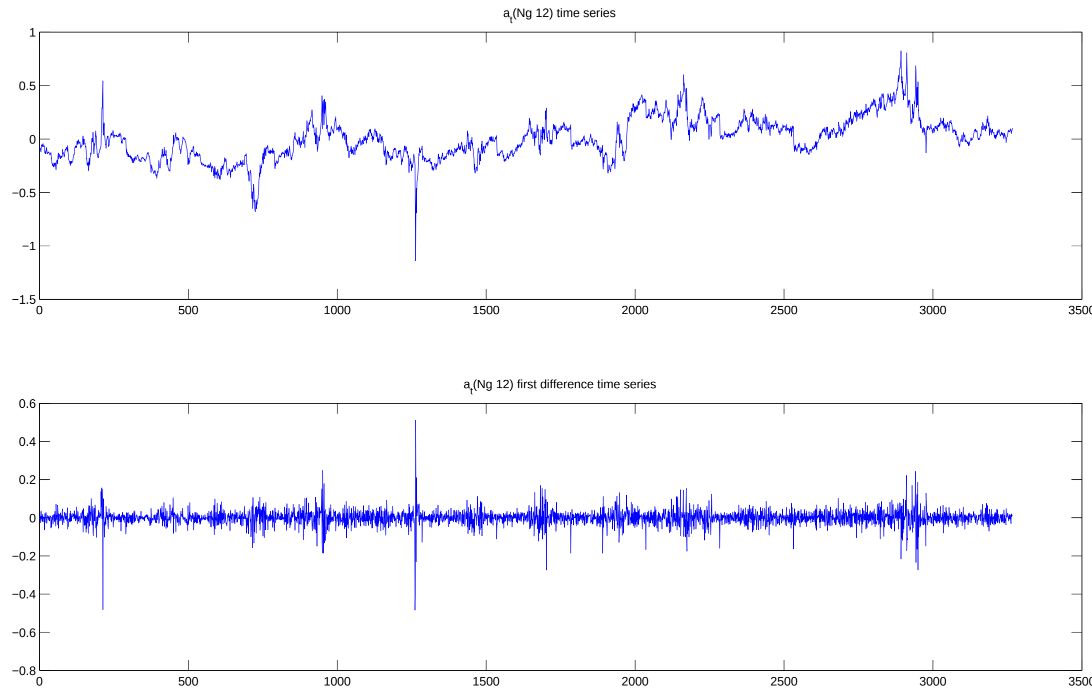

Figure 10: a_t(Ng-12) time series and its first difference series.

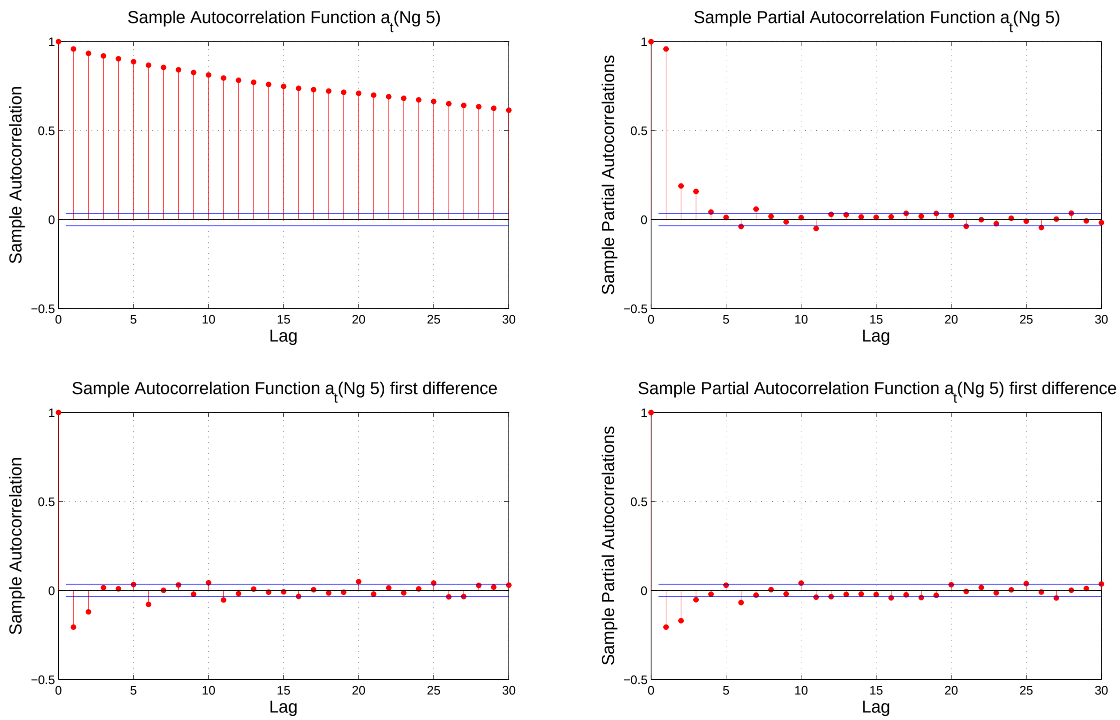

Figure 11: Autocorrelation and Partial autocorrelation function for the a_t(Ng-5) time series and its first difference series.

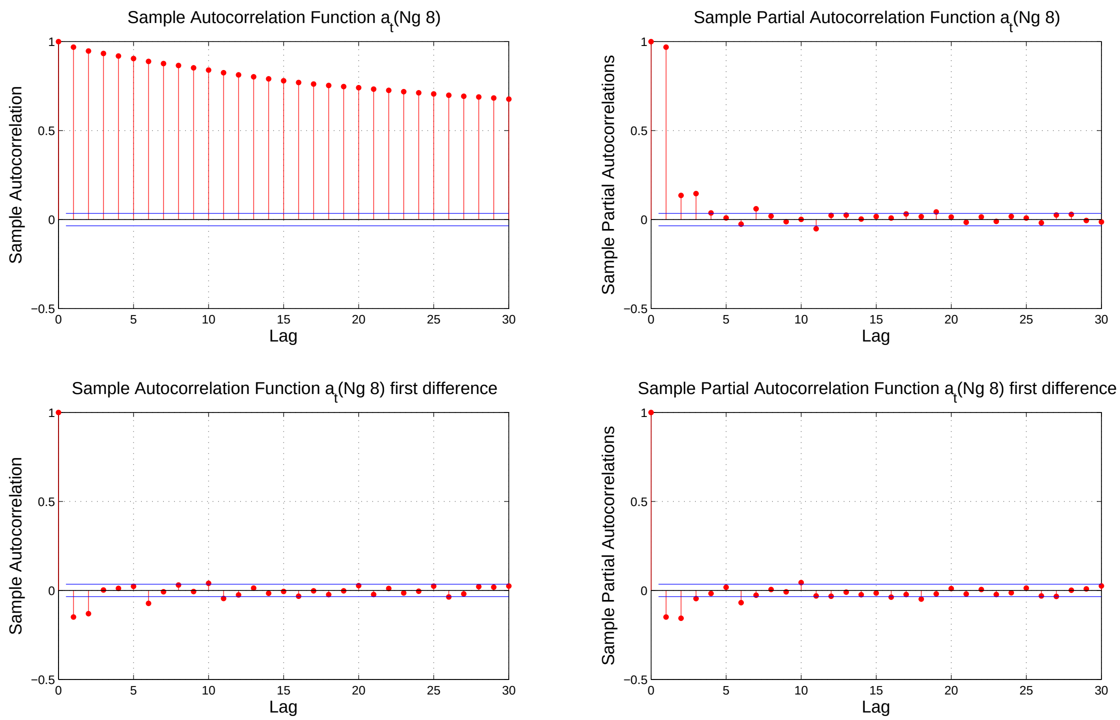

Figure 12: Autocorrelation and Partial autocorrelation function for the a_t(Ng-8) time series and its first difference series.

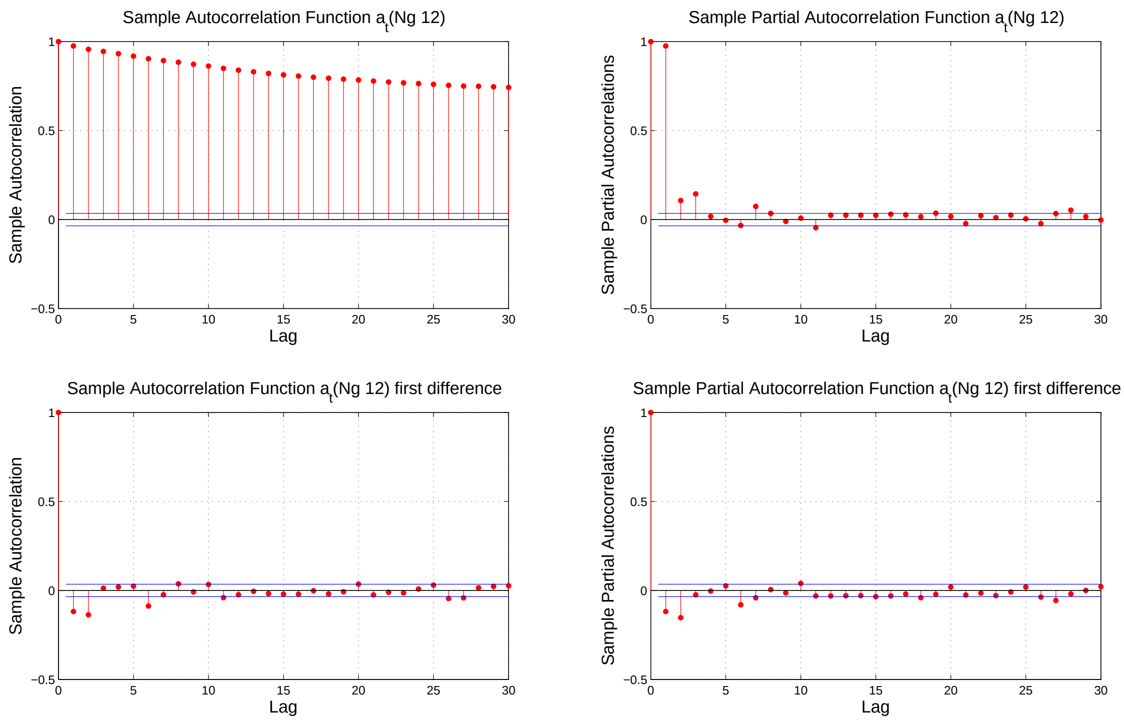

Figure 13: Autocorrelation and Partial autocorrelation function for the a_t(Ng-12) time series and its first difference series.

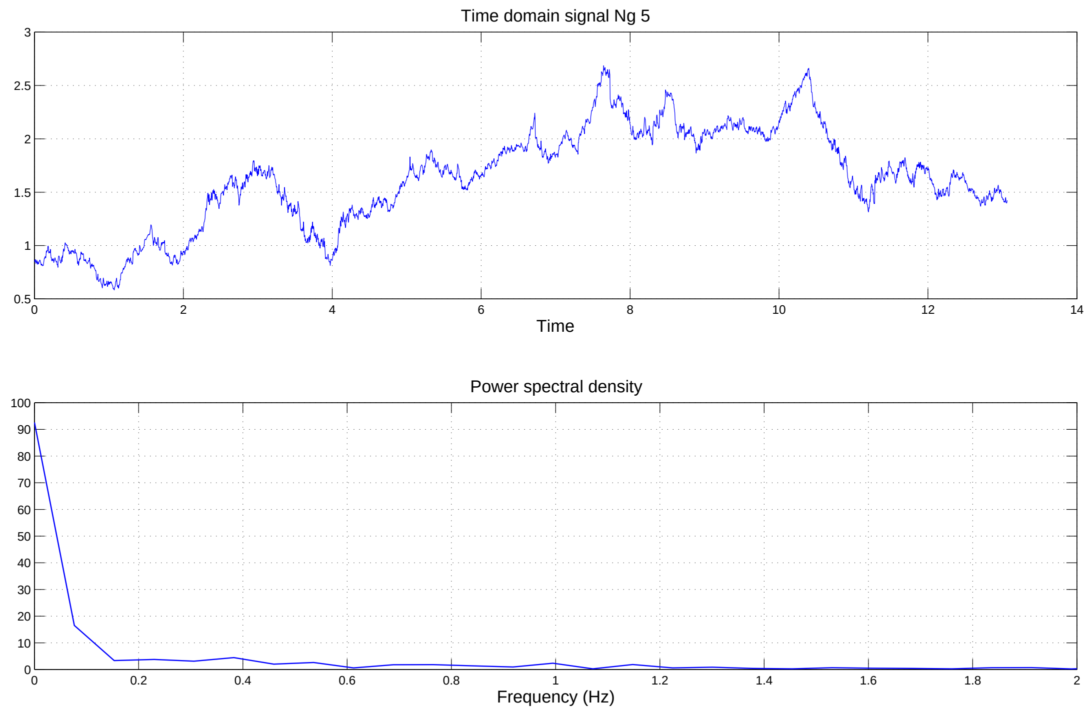

Figure 14: Log(Ng-5) Spectral Density.

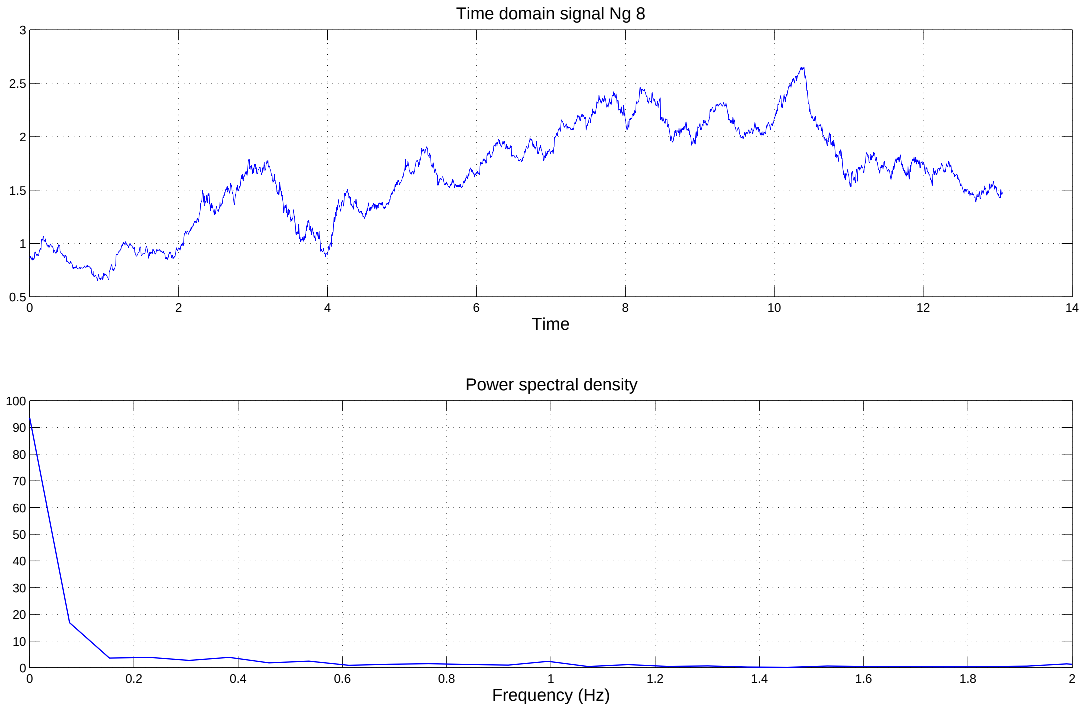

Figure 15: Log(Ng-8) Spectral Density.

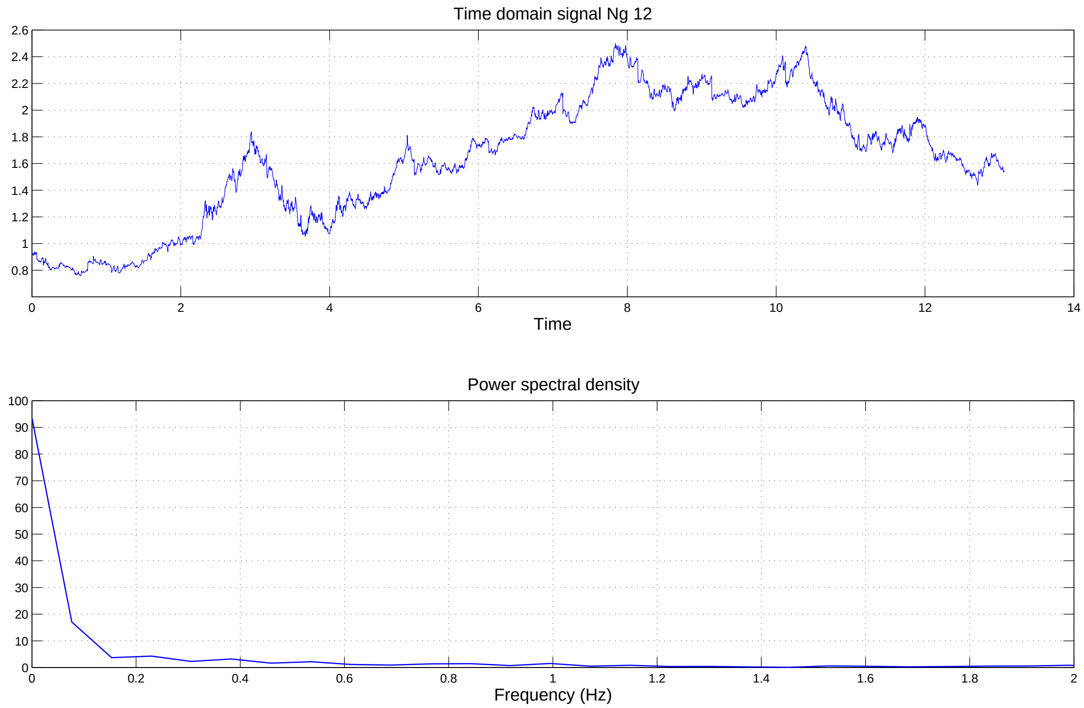

Figure 16: Log(Ng-12) Spectral Density.

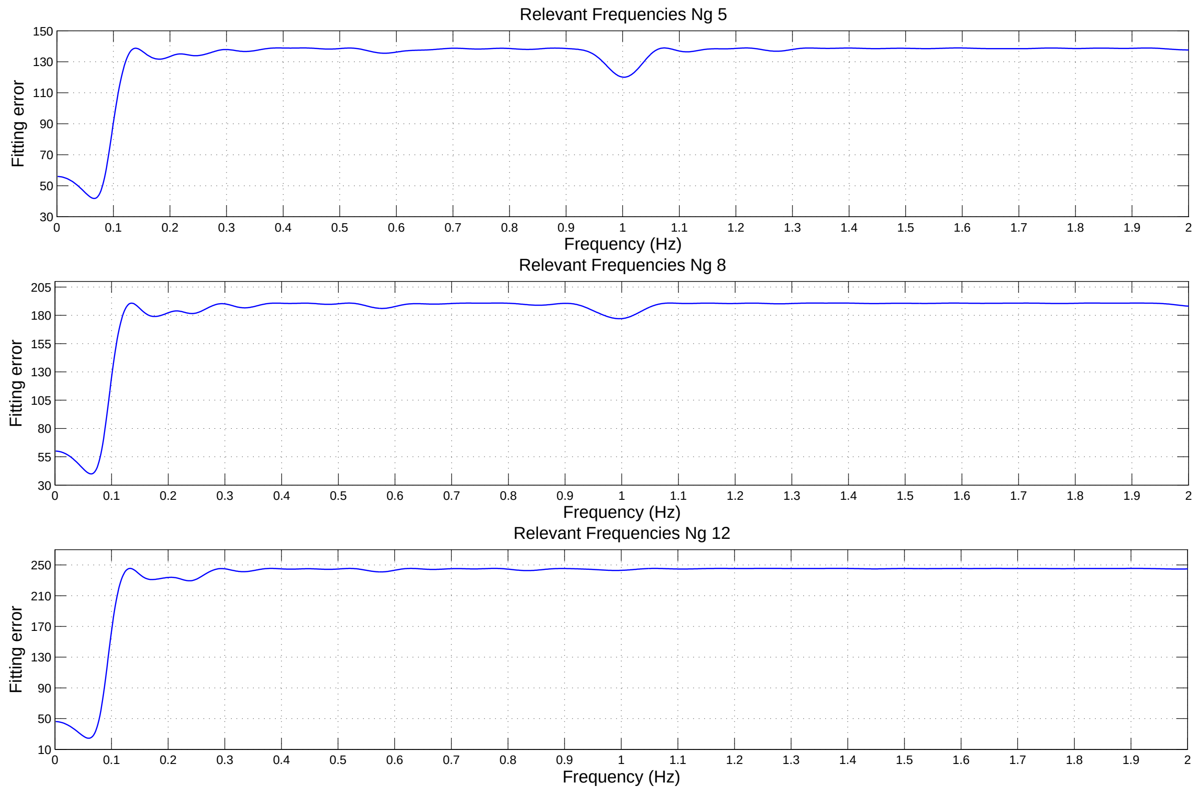

Figure 17: Residual sum of squares for log futures prices estimating model 3 for a fixed frequency, indicated in the horizontal axis.

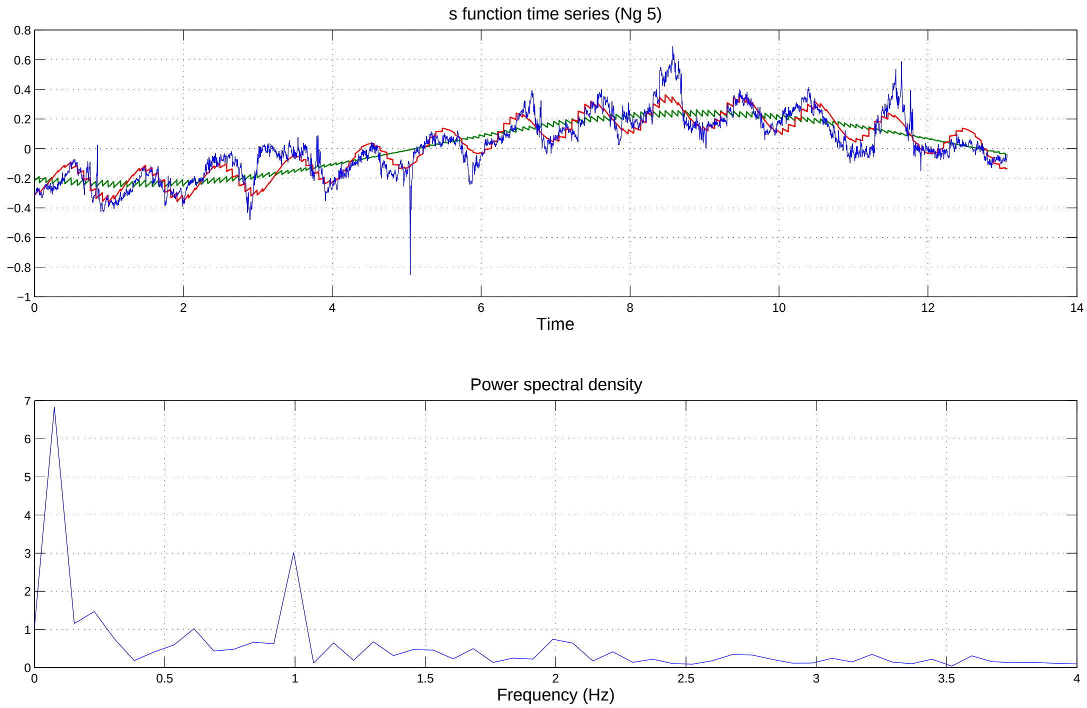

Figure 18: Spectral density for the $\hat{s}$-function corresponding to Ng-5 futures contract. The green line in the first graph shows how one term in Fourier expansion fits the s-function. Similarly, the red line in the first graph shows how two term in Fourier expansion fits the s-function.

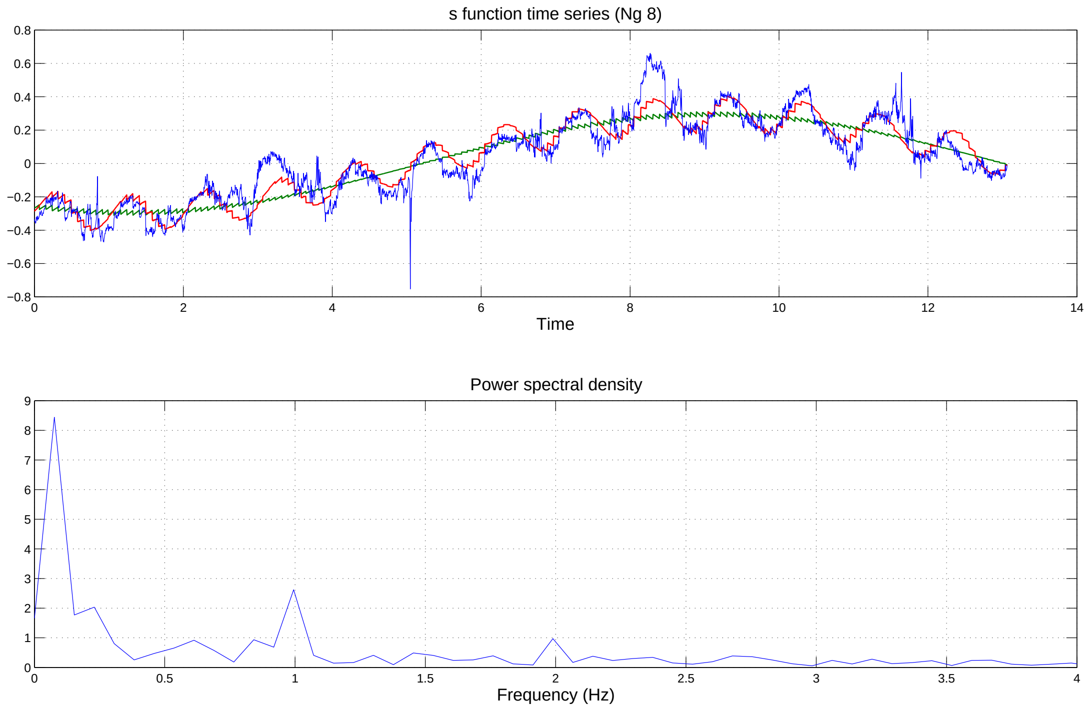

Figure 19: Spectral density for the $\hat{s}$-function corresponding to Ng-8 futures contract. The green line in the first graph shows how one term in Fourier expansion fits the s-function. Similarly, the red line in the first graph shows how two term in Fourier expansion fits the s-function.

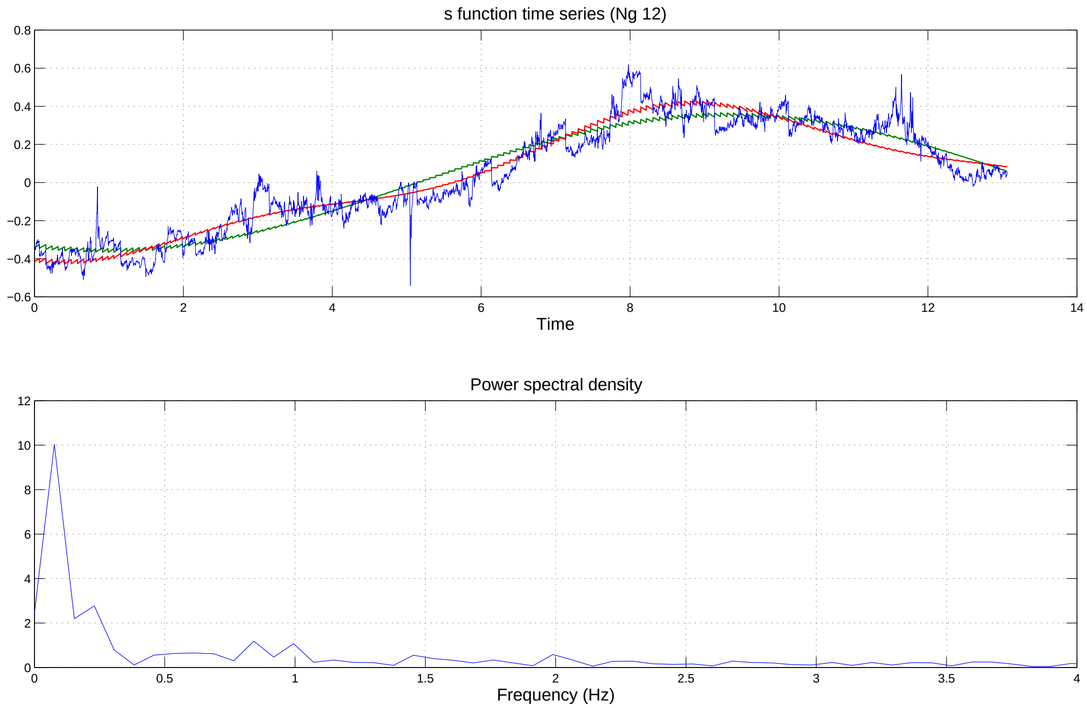

Figure 20: Spectral density for the $\hat{s}$-function corresponding to Ng-12 futures contract. The green line in the first graph shows how one term in Fourier expansion fits the s-function. Similarly, the red line in the first graph shows how two term in Fourier expansion fits the s-function.

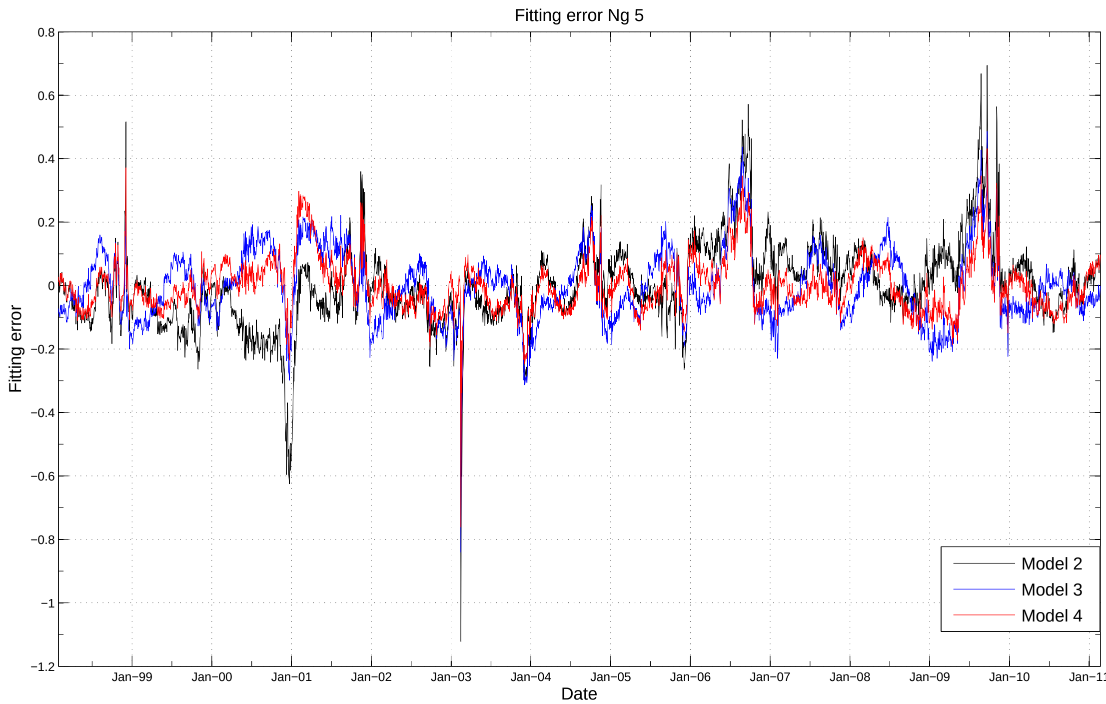

Figure 21: Fitting errors from Models 2, 3 and 4 for Ng-5 futures prices.

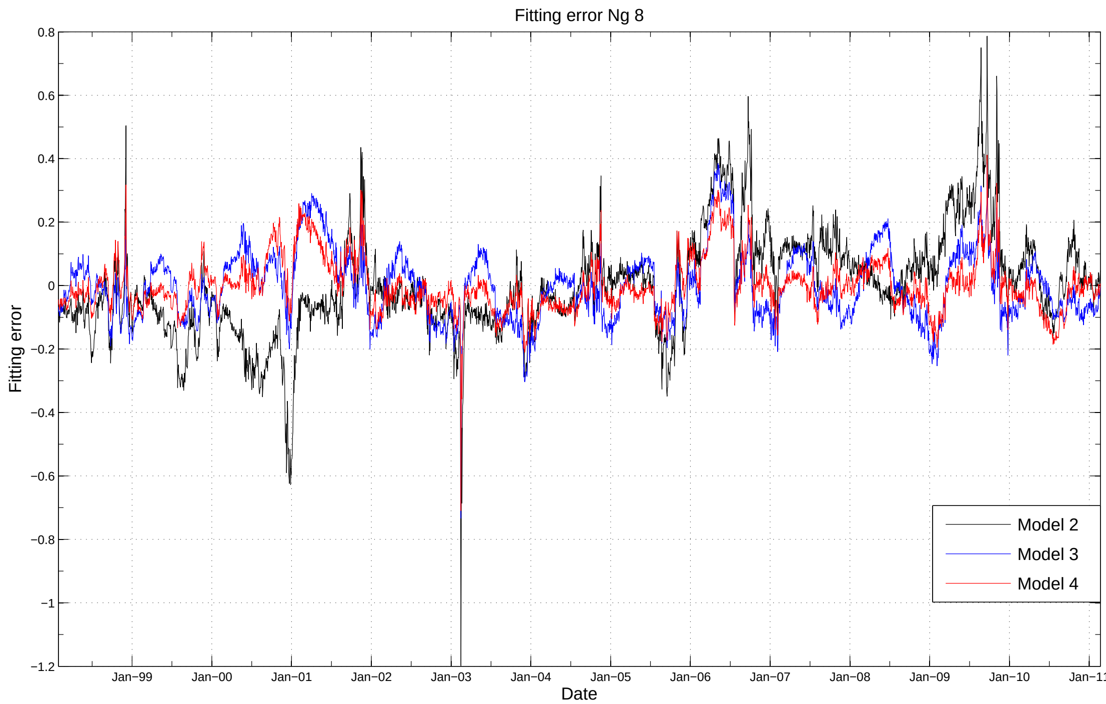

Figure 22: Fitting errors from Models 2, 3 and 4 for Ng-8 futures prices.

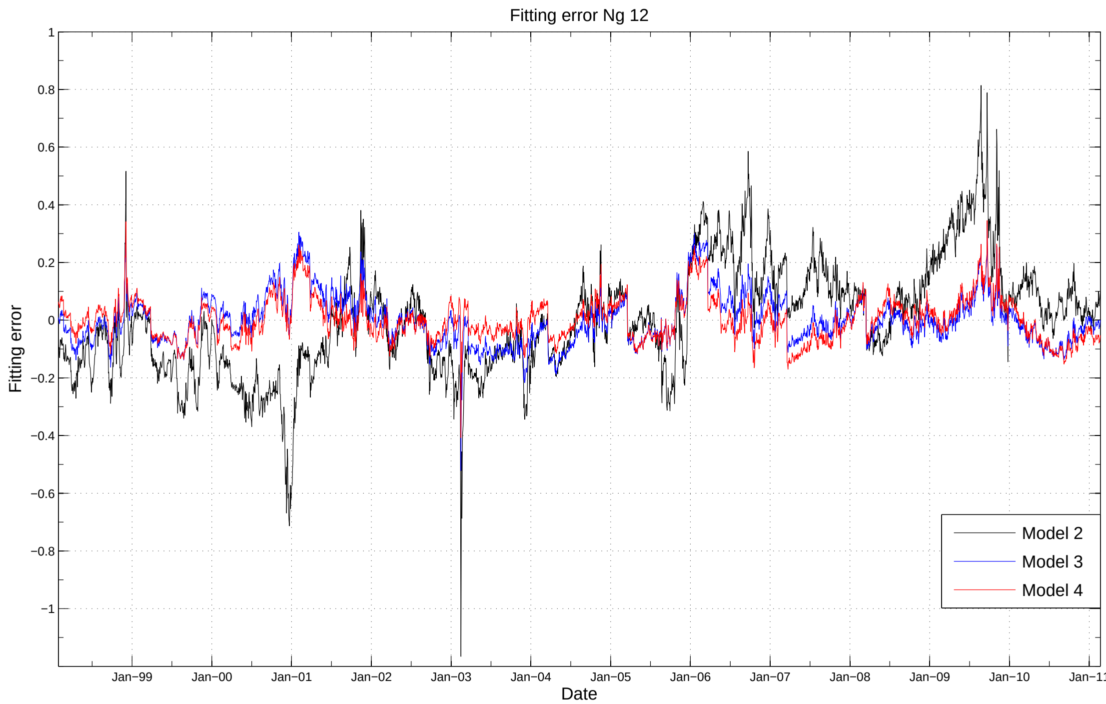

Figure 23: Fitting errors from Models 2, 3 and 4 for Ng-12 futures prices.

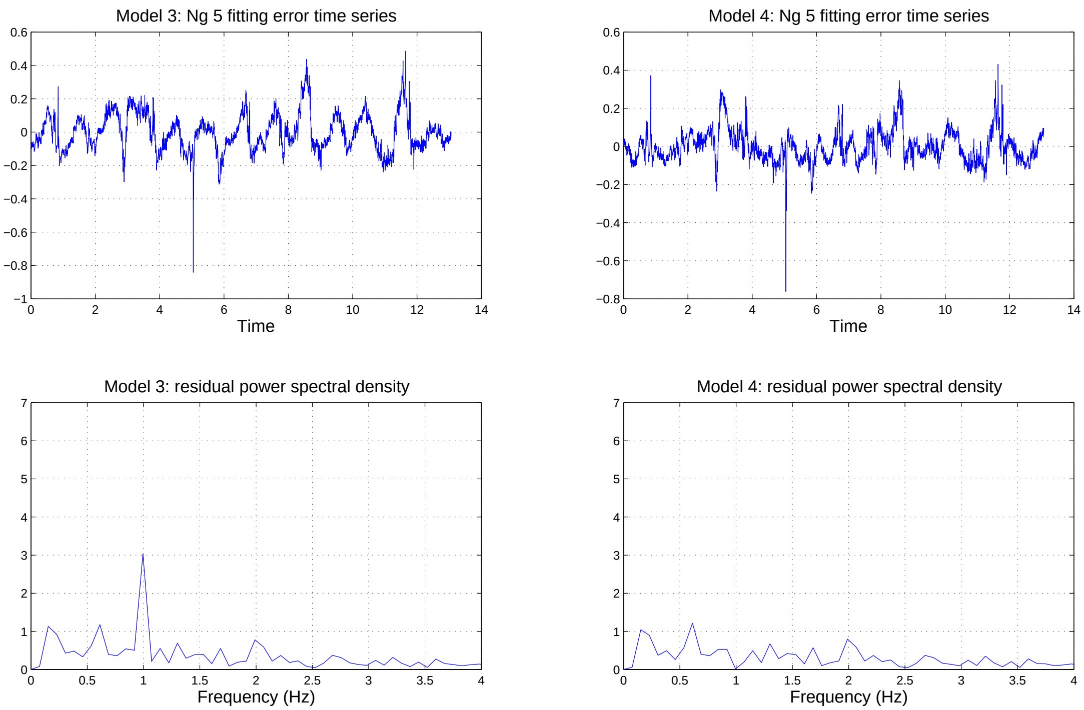

Figure 24: Spectral density for fitting errors for the Ng-5 futures prices from models 3 and 4.

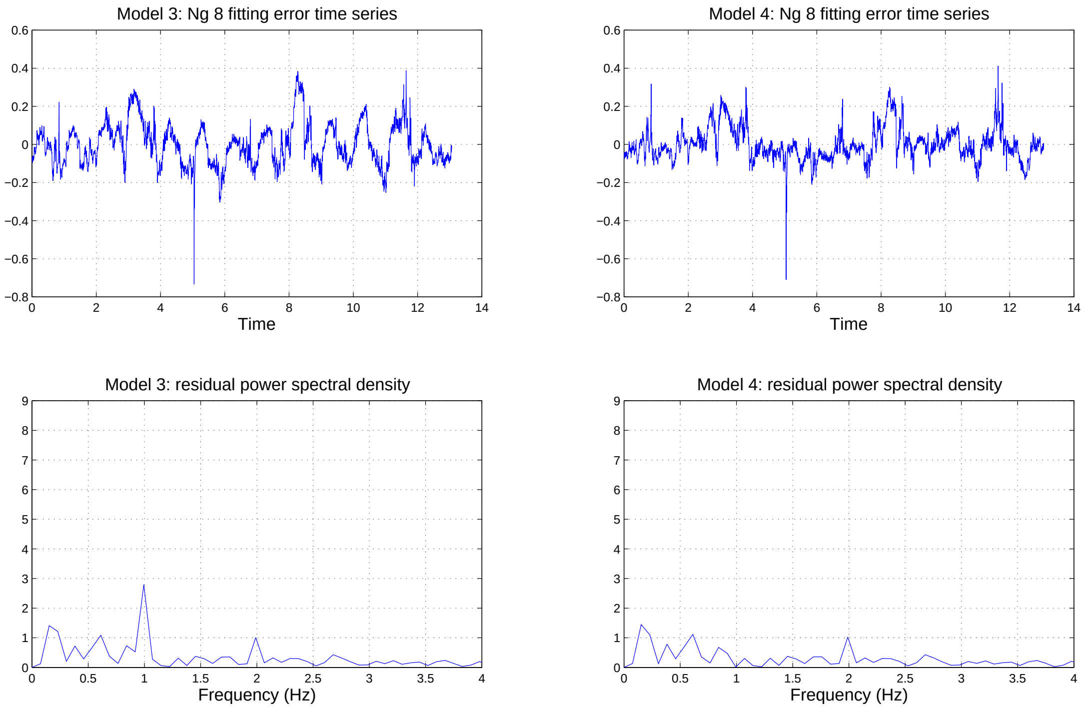

Figure 25: Spectral density for fitting errors for the Ng-8 futures prices from models 3 and 4.

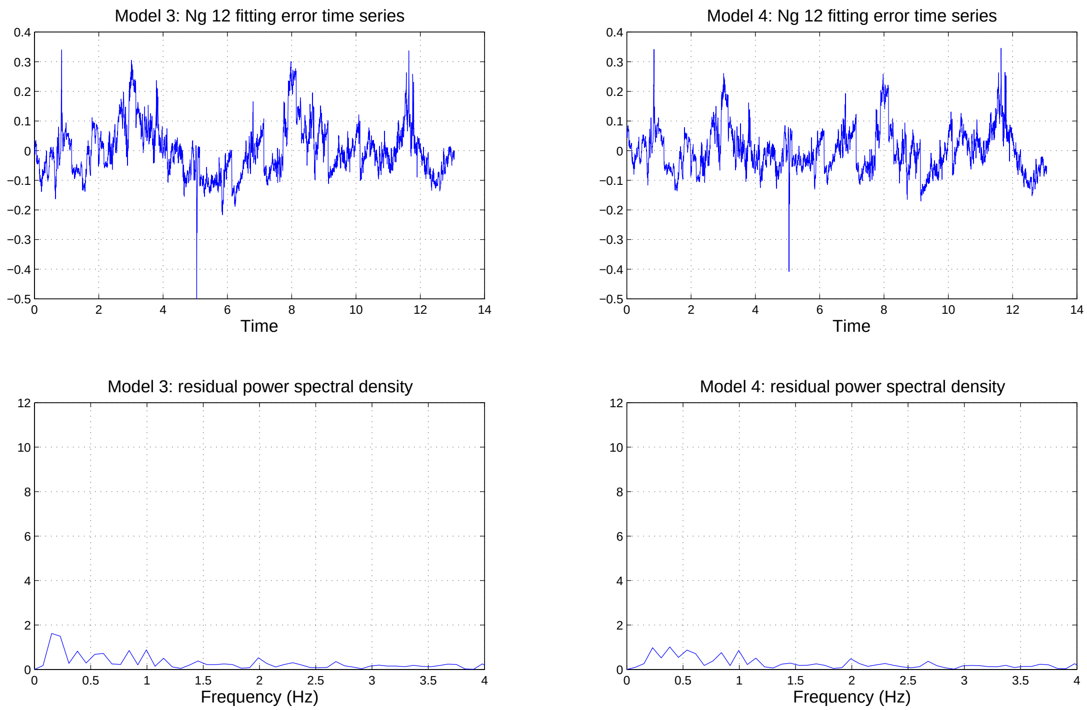

Figure 26: Spectral density for fitting errors for the Ng-12 futures prices from models 3 and 4.