Is it worth *refining* linear approximations to non-linear rational expectations models?*

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Abstract

We characterize the balanced growth path of the basic neoclassical growth economy using standard numerical solution methods which solve a linear or log-linear approximation to the economic model, as well as methods which preserve the nonlinearity in the original model. We also apply the same methods adding indivisible labor to the basic model, and to a monetary version of that economy, subject to a cash-in-advance constraint. In a unified framework, we show that log-linear approximations should generally be preferred to linear approximations. We also provide evidence that preserving the original nonlinear structure of the model when computing the numerical solution generally yields minor gains in accuracy. Methods that use either a linear or a log-linear approximation to the model can produce solutions as accurate as the parameterized expectations method. However, in extreme parametric cases, the solution may be rather sensible to small numerical errors, and even a log-linear approximation may then be inappropriate. Methods using the nonlinear structure of the original model can then perform significantly better.

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1 Introduction

The interaction between economic theory and computational research is a central aspect of modern economics and the suggestion in Lucas (1980) of constructing *fully articulated artificial economies* has led to using rational-expectations dynamic stochastic modelling in almost all fields of economics [see Kydland and Prescott (1996) or Cooley and Prescott (1995) for illustrative reviews]. This generally implies solving a system of stochastic difference equations involving conditional expectations of highly nonlinear functions, or making use of dynamic programming tools when dealing with problems with a recursive structure. The aim is to find the equilibrium solution for all the variables in the economy as well as to characterize the structure of the decision rules that relate state to decision variables. However, the non-linear stochastic structure embedded in these systems makes generally impossible to obtain analytical solutions, which has stimulated the design of a variety of numerical solution methods[1]. Unfortunately, a researcher often does not know how to choose among them, because there is not much systematic evidence concerning the properties of each particular approach.

Focusing on the basic version of the neoclassical growth model, Taylor and Uhlig (1990) considered fourteen different solution methods. Their analysis was quite rich in terms of the variety of methods compared and the comparison measures used, the general conclusion being that differences among methods turned out to be quite substantial for certain aspects of the model. Nonetheless, their study lacked some homogeneity and robustness given the way it was conducted: for each method they used a single solution realization, together with the estimated decision rules. In addition, the probability distribution of the technology shock, the single source of dynamics of the artificial economy, was not the same for all the methods considered.

Other papers analyzing the same model are Christiano (1990), who compared a linear quadratic and a log-linear quadratic method with the solution generated by a discrete-grid value-function iteration procedure, closer to the "true" solution, and Christiano and Fisher (2000), who compared a set of weighted residuals and finite element methods, again with the same type of discrete-grid solution. Barañano, Iza and Vazquez (2002) compared the performance of the solution to an endogenous growth model obtained from the Parameterized Expectations approach with the one obtained from its log-linear approximation. When proposing their accuracy test, den Haan and Marcet (1994) compared the Parameterized Expectations approach with linear quadratic methods when solving the one sector neoclassical growth model as well as the cash-in-advance monetary model of Cooley and Hansen (1989), in which the decentralized solution is not Pareto optimal. Again in a non-optimal environment, Dotsey and Mao (1992) compared different linear and log-linear approximations in a modified version of the basic growth model with taxes on production following a five-state Markov chain and no technology shock, using as a criterion for comparison a discrete state space solution to the Euler equations of the model. İmrohoroğlu (1994) proposed a forward

[1]It is not an objective of this paper to describe the state of the art in this area. For general surveys of existing solution methods see Marimon and Scott (1999), the Winter 1990 issue of the *Journal of Business and Economic Statistic*, Cooley and Prescott (1995), Danthine and Donaldson (1995) or Judd (1998).

solution method and used the den Haan-Marcet (1994) test to compare its performance with the solutions obtained by backsolving [Sims (1990)] as well as from a standard linear quadratic approximation to the model.

In spite of being quite extensive, the picture that emerges from the literature is mixed and scattered. Regarding the basic neoclassical growth model, linear and log-linear quadratic approximation methods are very similar and perform well, except for the den Haan-Marcet test, where linear models usually fail. In non-optimal settings things change. Weighted residuals-finite element methods seem to behave very similarly, although the Parameterized Expectations approach turns out to be the nonlinear solution algorithm most often used, since it seems to be quite convenient when there is a large number of state variables.

In our view two questions arising from this literature have not been sufficiently discussed. First, linear approximation methods are very popular because they are relatively simple to implement, but there is a perceived loss of accuracy due to the approximation, as compared to more elaborate methods. A second question refers to the extended use of the basic neoclassical growth model as a background for comparing solution methods when, most often, they are applied to more complex structures. Hence, a performance analysis of the different methods when departing from the more basic growth model is needed.

To tackle the first issue, we consider two *refinements to linear approximation methods*: *i*) using logged, rather than level variables, to compute an approximation from which to produce the numerical solution, and *ii)* using a linear approximation to derive specific aspects, like stability conditions or decision rules, while using the original nonlinear structure to compute the numerical solution to the model. We want to discuss first, whether each of these refinements to linear approximations increases the accuracy of the numerical solution and, second, the extent to which a refined linear solution performs similarly to nonlinear solutions[2]. Hence, as alternative solution approaches we consider: *i*) the standard linear-quadratic approximation in levels of the variables [Hansen (1985), Díaz-Giménez (1999)] using the original non-linear structure of the problem plus the obtained linear decision rule/s to compute the solution, *ii*) the undetermined coefficients approach applied to the log-linear approximation to the model as proposed by Uhlig (1999), computing the solution from the log-linearized system in state space form, *iii*) a Blanchard and Kahn (1980) and Sims (2002) approach, applied either in levels or in logs of the variables, as described in Novales *et al.* (1999), which uses the original non-linear model together with stability conditions estimated for the linearized/log-linearized system.

These methods are all very similar in spirit, searching for the stable manifold of a linear or log-linear approximation to the original non-linear problem, and imposing stability by selecting the *saddle path equilibrium*. They differ in that they use either the log-linear approximation or the original nonlinear structure to compute the numerical solution. In all cases, either stability conditions or decision rules derived from the linearized or from the log-linearized version of the system are added to the model to compute the solution. We have also solved the models with a nonlinear approximation method, Parameterized Expectations,

[2] A third alternative would be to use second order, as opposed to first order approximation techniques [Judd (1998), Sims (2001), Collard and Juillard (2001), Schmitt-Grohé and Uribe (2002)], which we leave for future research.

which approximates each conditional expectation in the model by a flexible polynomial function, the numerical solution changing with the order of the polynomial. The trade-off then arises between the higher accuracy provided by higher order polynomials, versus the loss of precision in estimation due to collinearity among the parameters. Theoretically, at least, one can approximate arbitrarily well the true solution while maintaining all the non-linear structure in the original problem.

Regarding the models considered, we start by analyzing the standard baseline one-sector stochastic growth model, subject to an autoregressive shock to technology leading the dynamics of the economy. Then, we increase the complexity of the model including indivisible labor as in the real business cycle model of Hansen (1985). In a final step, we add money to the previous model via a cash-in-advance constraint on the consumption commodity, as in Cooley and Hansen (1989). This is a non-Pareto optimal setting with an additional exogenous stochastic process, money growth. With this sequence of models, we try to cover a wide range of standard applications.

Finally, we depart from previous work in using a continuous probability distribution function for the technology shock as well as for the money growth shock in the third model, which turns out to be important when characterizing the statistical properties of a given economy.

We do not attempt to rank different methods or to conclude which one is best, which explains why we do not use a computationally expensive, very accurate algorithm, against which to compare the alternative solution methods considered. As a by-product, we evaluate two widely used proposals in the literature to solve non-linear rational expectations models, Uhlig (1999) and Sims (2002), and provide a user guide to choose among an important set of methods described in Marimón and Scott (1999). To validate a solution we examine a wide set of criteria in a unified and consistent framework. We place a special emphasis on rationality, since fulfilling rationality should be the first requirement for any solution to a rational expectations model. Monte Carlo simulation allows us to extensively check the rationality properties of residuals from stochastic Euler equations: zero mean, lack of serial correlation, zero correlation with variables in the information set, as incorporated in den Haan-Marcet tests. A last test refers to differences with the Parameterized Expectations solution, which can be made to approximate arbitrarily well the "exact" solution.

The main properties of the estimated decision rules implied by each method are also characterized through simulation. For the three model economies considered, the methods proposed in Sims (2002)-Novales *et al.* (1999) and Uhlig (1999) produce solutions which are indistinguishable from those obtained from the Parameterized Expectations approach in all dimensions when the log-linear approximation is used for either computing stability conditions [Sims (2002), Novales et al. (1999)] or for computing the full numerical solution [Uhlig (1999)]. Whether the log-linear approximation or the original nonlinear model are used to compute the solution, is relevant just for extreme parametric cases. However, when a linear, rather than log-linear, approximation in the variables is used, methods that use the original nonlinear structure of the model to compute the solution perform significantly better than those that use the linear approximation to compute the solution.

The rest of the paper is organized as follows. Section 2 presents the versions of the neoclassical growth model we consider. Section 3 briefly describes the four solution methods we use, while Section 4 sets the basis for the evaluation. In Section 5 we show the results and in Section 6 some concluding remarks. The paper is closed with an Appendix where the decision rules for all methods are shown and some guidance on solving the models is given. A Technical Appendix containing a detailed discussion of the implementation of each method to the three models is available from the authors upon request.

2 Description of models

We focus on several standard versions of the neoclassical, exogenous growth model. The sequence begins with a version of the basic one-sector stochastic growth model. Private agents are assumed to choose capital and consumption sequences to maximize

$$\max_{\{k_t,c_t\}_{t=1}^{\infty}} E_0 \sum_{t=1}^{\infty} \beta^{t-1} \left[\frac{c_t^{1-\eta}-1}{1-\eta} \right] \tag{1}$$

subject to technological and resource constraints,

$$\begin{aligned} y_t &= c_t + x_t \\ y_t &= z_t k_{t-1}^{\alpha} \\ k_t &= (1-\delta)k_{t-1} + x_t \\ \log(z_t) &= (1-\rho)\log(z_{ss}) + \rho \log(z_{t-1}) + \epsilon_t \\ \epsilon_t &\sim i.i.d.\, N(0,\sigma_\epsilon^2),\ k_t \geq 0,\ c_t \geq 0 \end{aligned}$$

given k_0 and z_0, where c_t is consumption at time t, k_{t-1} the beginning of period t capital stock, x_t investment, y_t output, and z_t an exogenous technology shock to output. $0 < \beta < 1$ is the subjective discount factor, $\eta > 0$ is the coefficient of relative risk aversion, $0 < \alpha < 1$ the capital share in production, $0 < \delta < 1$ the depreciation rate and $0 < \rho < 1$ controls for the persistence of the shock. Along the paper the ss subscript affecting a given variable denotes its deterministic steady state value. The optimality conditions of this problem are

$$c_t^{-\eta} = \beta E_t \left[c_{t+1}^{-\eta} R_{t+1} \right] \tag{2}$$

together with the previous constraints, where $R_{t+1} = \alpha z_{t+1} k_t^{\alpha-1} + 1 - \delta$. To perform rationality tests, we are concerned with the properties of the prediction error/s. The one-step ahead rational expectation error associated with (2) is,

$$\xi_{t+1} = \left[c_{t+1}^{-\eta} R_{t+1} \right] - E_t \left[c_{t+1}^{-\eta} R_{t+1} \right] \tag{3}$$

with a theoretical white noise structure: $E_t(\xi_{t+1}) = 0$ so that it bears no correlation with any variable contained in the information set available at time t. These are implications of rationality, and we are interested in testing for preservation of these properties as a central issue when evaluating solution methods. Using the time series for consumption and capital that we obtain with each solution method, we will generate time series for the approximate prediction error, ξ_t, as in (3), to test whether it violates rationality.

The second model is proposed in Hansen (1985). It is slightly more non-linear than the previous one in that it includes a non-convexity, indivisible labor. Here the representative household faces the problem,

$$\max_{\{k_t, c_t, N_t\}_{t=1}^{\infty}} E_0 \sum_{t=1}^{\infty} \beta^{t-1} \left[\frac{c_t^{1-\eta} - 1}{1-\eta} - A_N N_t \right] \tag{4}$$

subject to

$$\begin{aligned} y_t &= c_t + x_t \\ y_t &= z_t k_{t-1}^{\alpha} N_t^{1-\alpha} \\ k_t &= (1-\delta) k_{t-1} + x_t \\ \log(z_t) &= (1-\rho)\log(z_{ss}) + \rho \log(z_{t-1}) + \epsilon_t \\ \epsilon_t &\sim i.i.d.\, N(0, \sigma_\epsilon^2),\ k_t \geq 0,\ c_t \geq 0 \end{aligned}$$

given k_0 and z_0. N_t denotes labor and A_N is a parameter that measures the relative weight of labor in the utility function. The remaining parameters are as in the previous model. Again (2) is the single equation involving expectations terms, from the first order condition for capital and consumption, where now $R_{t+1} = \alpha z_{t+1} k_t^{\alpha-1} N_t^{1-\alpha} + 1 - \delta$, and the rational expectations error is defined as in (3). In addition to (2) and the constraints there is now another optimality condition from maximizing with respect to labor which, using the first order condition for consumption, can be written,

$$A_N = (1-\alpha) c_t^{-\eta} z_t k_{t-1}^{\alpha} N_t^{-\alpha} \tag{5}$$

The last economy considered, Cooley and Hansen (1989), is a version of Hansen (1985), with money introduced via a cash-in-advance constraint in consumption. The competitive equilibrium is non-Pareto-optimal in this case, and the second welfare theorem does not apply. The representative firm solves a standard profit maximization problem, while households seek to maximize their time preferences subject to their holdings of money balances and a set of standard budget constraints. There are two sources of uncertainty in this economy: the autoregressive shock to technology, z_t, and an autoregressive logged money growth rate, $\log(g_{t+1}) = (1-\rho_g)\log(g_{ss}) + \rho_g \log(g_t) + \epsilon_{g_{t+1}}$. In equilibrium, we have two first order conditions involving expectations terms,

$$\lambda_t = \beta E_t \left[\lambda_{t+1} R_{t+1} \right] \tag{6}$$

$$\lambda_t c_t = \beta E_t \frac{1}{g_{t+1}} \tag{7}$$

where $R_{t+1} = \alpha z_{t+1} k_t^{\alpha-1} N_{t+1}^{1-\alpha} + 1 - \delta$ and λ_t is the Lagrange multiplier associated with the household's budget constraint. The first equation is the optimization condition for capital, with an expectation error

$$\xi_{t+1} = \left[\lambda_{t+1} R_{t+1} \right] - E_t \left[\lambda_{t+1} R_{t+1} \right], \tag{8}$$

The second expectation arises from the first order conditions for real money balances and consumption, and the budget constraint. Assuming normality of the innovation ϵ_{g_t}, this expectation has an easy to derive analytical form, linear in the logs of the variables.

3 Solution Methods

We evaluate two sets of methods. On the one hand, we use three "almost" linear methods[3] preserving different degrees of the non-linear structure in the original problem that are easy to implement and computationally fast: i) the standard linear-quadratic approximation in levels of the variables (LQA henceforth), ii) the approach proposed in Uhlig (1999) (UHL) and iii) the method proposed by Sims (2002)-Blanchard and Kahn (1980) as described in Novales *et al.* (1999) either in levels or in logs of the variables (SIM / SIL, respectively). The first one is a Value-Function-based method while the other two are Euler-equation-based methods. It is important to notice that we evaluate the different methods as they are usually implemented in practice. It is because of specific details of their implementation that makes them different. More fundamentally, they all search for the same stable subspace, and can be adapted to become essentially indistinguishable from each other. On the other hand, we also use a nonlinear type method, Parameterized Expectations (PEA), an Euler-equation-based method.

We do not provide in the paper computing times because they depend on the programming language and specific code used. However, the PEA method was clearly the most time consuming.

3.1 "Almost" Linear Methods

LQA uses the non-linear structure of the model, adding linear decision rules for consumption, investment or labor. SIM, also implemented in levels of the variables, only adds linear stability conditions to the original, non-linear model. These conditions guarantee that the numerical solution to the non-linear system of equations is stable. For each of the three models in the paper, just a single stability condition is needed. A comparison between these two solutions will allow us to discuss whether the higher complexity produced by preserving more non-linear structure in the SIM method pays in terms of increased accuracy. We also apply the SIM method to a log-linear approximation to the model around steady-state, which we will denote by SIL. This produces a stability condition linear in logged variables, instead of one such condition linear in the variables. Comparing SIM with SIL we can test whether performing the approximation in logs implies any accuracy gain. Finally, since UHL solves the log-linearized system while SIL uses the original nonlinear model plus the stability condition obtained from the log-linear approximation, we can again evaluate the benefits of preserving non-linearity.

Relative to the discussion in the Introduction to this paper, SIL is the most refined of the "almost" linear methods, and LQA, as implemented here, the less refined.

3.1.1 Standard Linear Quadratic Approximation (LQA)

The LQA approach consists in approximating a non-linear problem by one with a linear-quadratic structure, for which the solution is always known [for a detailed description see for

[3]We call them "almost" linear, in the Marimon and Scott (1999) terminology, because they combine the stable manifold of the linear/log-linear approximation with the original non-linear problem.

example Hansen and Prescott (1995)]. This method deals with solving a dynamic programming problem[4] of the form:

$$V^{n+1}(\mathbf{z}_t, s_t) = \max_{d_t} \{r(\mathbf{z}_t, s_t, d_t) + \beta E\left[V^n(\mathbf{z}_{t+1}, s_{t+1}|\mathbf{z}_t)\right]\} \tag{9}$$

subject to

$$\begin{bmatrix} \mathbf{z}_{t+1} \\ s_{t+1} \end{bmatrix} = A\varepsilon_{t+1} + B(\mathbf{z}_t, s_t, d_t)$$

where $V^n(\mathbf{z}_t, s_t)$ is the n^{th}-iteration on the optimal value function, β the discount factor, $\mathbf{z}_t$ a vector of exogenous state variables, s_t a vector of endogenous state variables, d_t a vector of decision variables, $r(\mathbf{z}_t, s_t, d_t)$ the return function for the problem, ε_t a vector of exogenous i.i.d. stochastic processes, and the constraints describe the evolution of the state variables. We will maintain this notation across methods. What LQA does is to compute a linear quadratic approximation to the original economy (9) around steady-state, and then search for the solution to this approximate linear quadratic economy.

The solution to the linear-quadratic problem produces a linear function that maps states into decisions, $d_t = H[1, \mathbf{z}_t, s_t]^T$, with H being a matrix with as many rows as decision variables in d_t. To generate artificial time series we use the original non-linear problem (production function, resource constraint, law of motion of capital) plus the linear decision rule/s. This is the procedure we followed to solve the basic stochastic growth model and the Hansen (1985) model. In the first model, the outcome of the algorithm is a linear decision rule for investment as a function of technology and lagged capital. For the Hansen (1985) economy we obtain linear decision rules for investment and labor as functions of technology and lagged capital. For the cash-in-advance model, important changes are needed, due to the distortion introduced by the cash-in-advance constraint. In addition to taking a quadratic approximation to the return function, it is necessary to assume that the perceived law of motion for the inverse of real money balances is linear in the state variables. These changes are described in detail in Kydland (1989) and Cooley and Hansen (1989). To solve this monetary model, we simply take the decision rules provided by Cooley and Hansen (1989) and restrict ourselves to parametric cases considered in that paper, to make our work comparable to the analysis in den Haan and Marcet (1994), who use the same parameters.

3.1.2 Undetermined Coefficients (UHL)

This method consists of log-linearizing the equations characterizing the equilibrium and solving for the recursive laws of motion with the method of undetermined coefficients. We use the approach in Uhlig (1999). Let the recursive equilibrium law of motion of the economy be those matrices Ξ_1, Ξ_2, Ξ_3 and Ξ_4 that make stable the system

$$\begin{bmatrix} s_t \\ v_t \end{bmatrix} = \begin{bmatrix} \Xi_1 & \Xi_2 \\ \Xi_3 & \Xi_4 \end{bmatrix} \begin{bmatrix} s_{t-1} \\ \mathbf{z}_t \end{bmatrix} \tag{10}$$

where, again, s_t is a vector with the endogenous states, $\mathbf{z}_t$ contains the exogenous states and v_t is a vector of other endogenous variables of the system. To find estimates for matrices

[4]In many applications, this is a social-planning problem for the economy.

Ξ_1, Ξ_2, Ξ_3 and Ξ_4 it is necessary to equate the coefficients of (10) to the corresponding ones in the system formed by the log-linearized version of the equations characterizing the equilibrium. To do so, the well-known method of undetermined coefficients is applied, choosing among possible parameter values those that make (10) stable.

One can easily generate time series of size T for all the elements of s_t and v_t using the state-space representation (10) and the law of motion for $\mathbf{z}_t$, given s_0 and $\mathbf{z}_0$.

3.1.3 Eigenvalue/Eigenvector Decompositions (SIM, SIL)

This approach rests heavily on Blanchard and Kahn (1980) and, specially, on Sims (2002), and it is explained in detail and applied to different nonlinear systems in Novales *et al.* (1999). A related contribution is Klein (1998). Its specific characteristic is that each conditional expectation is considered as an additional variable to solve for (say W_t), being defined as the realized value of the function inside the expectation, plus a forecast error[5]. Stability conditions associated with the linear approximation to the model are added to the original non-linear problem.

Let the linearized (SIM) version of the set of equations around steady-state (or log-linearized in the case of the SIL method) be:

$$\Gamma_0 u_{t+1} = \Gamma_1 u_t + \Psi \varepsilon_{t+1} + \Pi \zeta_{t+1} \tag{11}$$

where u_t is a subset of the vector $\{s_t, v_t, \mathbf{z}_t, W_t\}$, ε_t contains the innovations in the laws of motion of the exogenous states, and ζ_t is the vector of expectations errors. Let matrix $\Gamma_0^{-1}\Gamma_1$ have a Jordan decomposition $P\Lambda P^{-1}$, where Λ is a diagonal matrix containing the eigenvalues of $\Gamma_0^{-1}\Gamma_1$, P^{-1} is the matrix which has as rows the left eigenvectors, and let P^s be the rows of P^{-1} associated with an unstable eigenvalue. A stationary solution to model (11) requires the time paths of the variables to lie on the stable manifold of the solution space, which can be achieved by imposing every period the condition,

$$P^s u_t = 0, \ \forall t \tag{12}$$

This condition can be written to relate the conditional expectation, W_t, to the other variables in u_t in a linear (SIM case) or an exponential way (SIL case). To simulate the approximate economy, take the original non-linear problem (Euler equations, production function, resource constraint, law of motion of capital) and solve for the expectation through the stability condition. Combining the original non-linear structure with the stability condition implies solving a non-linear system of equations in each step of the simulation process, and so the solution method tends to be computationally more demanding than other methods based on linear approximations.

3.2 Parameterized expectations (PEA)

This approach consists in parameterizing the conditional expectation in the stochastic Euler equation. The conditional expectation is specified as a function of the state of the system,

[5]As an example, to solve the basic growth model, define $W_t = E_t[c_{t+1}^{-\eta} R_{t+1}]$. Then, to implement this method substitute equation (2) for $c_t^{-\eta} = \beta W_t$, and rewrite (3) as $\xi_t = [c_t^{-\eta} R_t] - W_{t-1}$.

and the parameters of that function are estimated before solving the model. For a detailed explanation see den Haan and Marcet (1990) or Marcet and Lorenzoni (1999). We refine our PEA approximation until the prediction error from the stochastic Euler equation passes the den Haan and Marcet (1994) test. The steps to follow are:

1. Substitute each conditional expectation, W_t, by a parameterized polynomial function $\psi(q; s_t, \mathtt{z}_t)$, where q is a vector of parameters. Define the residual $\hat{W}_t - \psi_t$, where $\hat{W}_t$ is the realized value of W_t. In principle ψ_t should approximate the conditional expectation arbitrarily well by increasing the order of the polynomial.
2. Choose an initial value for q. Use the first order conditions and constraints of the problem (with the conditional expectation substituted by $\psi(q; s_t(q), \mathtt{z}_t)$) to generate time series paths for the variables of the economy.
3. Define $S : \Re^m \to \Re^m$, where m is the dimension of q, and

$$S(q) = argmin_q \, E_t \left[\hat{W}_t(q) - \psi_t(q; s_t(q), \mathtt{z}_t) \right]^2 .$$

4. Iterate until $q = S(q)$. This guarantees that if agents use ψ_t as their expectation function, then q is the best parameter vector they could use, in the sense that it minimizes the mean squared error to the true expectation. To find each q^{i+1} starting from a previous q^i, take the residual sum of squares from a nonlinear regression of $\hat{W}_t(q^i)$ on $\psi_t(q^i; s_t(q^i), \mathtt{z}_t)$ as an approximation to $S(q^i)$ and update q according to the rule $q^{i+1} = q^i + \lambda_q S(q^i)$, where λ_q controls the degree of updating in each iteration[6].

4 The evaluation exercise

In this section we describe the parametric cases considered in each of the three models, as well as the tools used in the comparative evaluation of the different solution methods.

In the first two models we analyze the robustness of the results to changes in the relative risk aversion parameter and the variance of the technology shock, suggested in the literature as being the most influential parameters. An increase in risk aversion implies more concavity in the utility function and a more non-linear problem. The technology shock is the main source of dynamics, so a larger variance will produce bigger deviations around steady-state for all the variables, which should be expected to deteriorate the performance of methods

[6]PEA is substantially more complex than alternative linear methods, due to some practical difficulties. One relates to selecting initial values for the q vector: this generally requires hard computational work, and if one starts to search for the fixed point in q from arbitrary initial conditions, convergence is hard to achieve. Instead of using standard homothopy techniques to determine initial conditions, as suggested in den Haan and Marcet (1990), we estimated them from a log-linear solution method [see Pérez (2001)]. This proved to be faster and computationally efficient, since stationarity and ergodicity of the time paths obtained under the initial parameterization is guaranteed. In addition, it is very important for the solution to be accurate to select an adequate order for the polynomial, which requires going repeatedly over the steps outlined above. This mixed approach of using the numerical solution obtained from a log-linear approximation to compute robust initial conditions for the parameterized expectations method seems to be a promising approach when solving more complex models.

that use linear approximations around steady-state. For sensitivity analysis, we consider three values of σ_ϵ: 0.01, which is close to a usual choice in the literature (0.00721), 0.02 and 0.06. Concerning risk aversion, we moved between a lower bound of 0.5 and a highest value of 3.0. The remaining parameter values are standard: $\beta = 0.99, \rho = 0.95, \alpha = 0.36$, and $\delta = 0.025$, and remained constant in all the experiments. For the Hansen model $A_N = 2.86$. Hence, we have nine parametric cases - see table 1.

Insert table 1

In the Cooley-Hansen economy we focus on the variance of the technology perturbation, as well as on the steady-state money growth rate, analyzing the same cases as in Cooley and Hansen (1989). Parameter values are now $\beta = 0.99, \alpha = 0.36, \delta = 0.025, A_N = 2.86$. To control for persistence of the exogenous shocks, we chose as coefficients of the first-order autoregressive processes for technology and money growth: $\rho_z = 0.95$ and $\rho_g = 0.48$, and as standard deviation for the innovation in the money growth process: $\sigma_{\epsilon_g} = 0.009$. We then changed the money growth rate and the variance of the technology shock, to consider six parametric cases [see table 1].

We solved each model for each parametric case with all the methods. For the sake of robustness, we computed 250 simulations of length $T = 150$, and 250 simulations of length $T = 3000$. Size 150 is representative of a standard quarterly sample length, while a size of 3000 is a more reliable sample length for statistical purposes.

In the basic growth model, results for cases 8 and 9 when $T = 3000$ when solving with SIM are not shown, due to some negative value of k_t arising for every draw of z_t. For the Hansen model, in the high variance cases $7, 8$ and 9 with $T = 3000$, it was not possible to find a solution with the LQA and SIM methods for the same reason. For the Cooley-Hansen model, the same problem occurred with the SIM method in the high variance cases 5 and 6, with $T = 3000$. When $T = 150$ this method generated negative values for the capital stock for about 30% of the realizations of the shocks in those parametric cases, and we repeated the simulation process until we had 250 valid simulations. We had to go through the same exercise when solving this model with LQA, because it generated negative values of the capital stock for about 70% of realizations of the shocks under the mentioned parameterizations. In contrast to SIM, it was always feasible to achieve a solution using SIL.

For each simulation we calculated two sets of measures, described in the following subsections. The first set has to do with the numerical accuracy of the solution, which we discuss by testing whether the stochastic Euler equation residual ξ_t, defined by (3) for the first two models and (8) for the Cooley-Hansen economy, satisfies the properties implied by rationality. The second set of measures deals with the statistics usually examined in empirical studies to assess the model's responses to meaningful economic questions. It is crucial to analyze whether the answer to these questions depends on the solution method being implemented.

4.1 Expectations error properties

4.1.1 Correlations of ξ_t with the available information set: the den Haan and Marcet accuracy test

The idea of the test proposed in den Haan and Marcet (1994) is to check whether there exists any function of variables dated t or earlier that helps predict ξ_{t+1}. That would be a strong deviation from rationality. To implement the test, the steps to follow are: i) obtain a large number of observations by simulating the model for a long realization of the exogenous processes; ii) run a regression of ξ_{t+1} over I_t, a list of instruments selected from the set of variables in the time t information set; iii) define $\hat{a} = (\sum I_t^T I_t)^{-1}(\sum I_t^T \xi_{t+1})$ and form the statistic:

$$M = \hat{a}^T(\sum I_t^T I_t)(\sum I_t^T I_t \xi_{t+1}^2)^{-1}(\sum I_t^T I_t)\hat{a} \sim \chi^2_{m_1 m_2},$$

where m_2 is the number of instruments chosen and m_1 is the number of Euler equation errors, which is equal to one in our three models. The statistic M provides a test for the rational expectations hypothesis: $E_t(\xi_{t+1}) = 0$. It is worth noting that the alternative hypothesis is that the error is not a martingale; so if the value of the statistic belongs to the *upper* critical region of the $\chi^2_{m_1,m_2}$ distribution, there is evidence against the accuracy of the solution.

The number of observations used can be interpreted as a measure of how stringent the criterion is: that the solution passes the test even for a very large number of data points should be taken as evidence that the solution is very accurate. We have chosen as set of instruments $I_t = [1, k_t, k_{t-1}, \ k_{t-2}, \log(z_t), \log(z_{t-1}), \ \log(z_{t-2})]$, so that the test statistic has a χ^2_7 distribution. This is the same set of instruments used by den Haan and Marcet (1994) for the basic model and a standard deviation for the technology shock of 0.02 or 0.06. Even though they could only use a constant as instrument in the low variance case, $\sigma_\epsilon = 0.01$, we were able to use the full set of instruments I_t in all our parametric cases. The better behavior of our PEA solution seems to arise from using as initial conditions for vector q in the expectations polynomial the numerical estimates obtained from the log-linear version of the model. We also used I_t as instruments when testing accuracy of the solutions to the Hansen (1985) and Cooley and Hansen (1989) models.

4.1.2 Time series dependence properties of ξ_t

We also checked for autocorrelation in the conditional expectation residual, ξ_t obtained from each model. We first fitted an AR(1) process with constant to the generated residual ξ_t,

$$\xi_t = \mu + \rho\xi_{t-1} + \epsilon_{\xi_t}, \tag{13}$$

and tested the two null hypothesis $H_0 : \mu = 0$ (zero-mean) and $H_0 : \rho = 0$ (no serial correlation) using conventional t-tests. Under rationality, the conditional expectation residual should have no significant mean and no autocorrelation, since it is a one-period-ahead, rational expectations prediction error. The resulting information on these two issues is complementary to that provided by the den Haan-Marcet test.

4.2 Other characteristics of the implied solutions

4.2.1 Decision Rules

For each solution method and model, we tabulated the values of the decision variables at alternative points in the space of state variables. After building a grid of values for the state variables, we used the decision rules to obtain the implied values for the decision variables. The LQA decision rules arise, as already mentioned, from the linear function $d_t = H[1, \mathbf{z}_t, s_t]^T$. For UHL they are obtained from the log-linear relation $s_t = \Xi_1 s_{t-1} + \Xi_2 \mathbf{z}_t$ in (10), while SIM/SIL's decision rules correspond to the stability conditions $P^s u_t = 0$. Concerning PEA, a system of equations of the kind $F(d_t, \psi_t(q; s_t, \mathbf{z}_t)) = 0$ is used. The reader can see for each model and method the exact definition of the vectors d_t, s_t and $\mathbf{z}_t$, as well as those of H, Ξ_1, Ξ_2, P^s and $\psi_t(q; s_t, \mathbf{z}_t)$ in the Appendix.

Concerning the capital stock, for each of the three models we selected 25 equally spaced values in a ten percent interval around k_{ss}. In relation to the technology shock, for the basic growth model we got again 25 equally spaced values, between 0.4 and 1.6. For Hansen's model the range of variation for the technology shock was narrower, between 0.8 and 1.2, due to numerical problems with the LQA and SIM decision rules. As for the Cooley-Hansen model, we performed two similar exercises: on the one hand, we fixed z_t at its steady state value of 1.0, and selected 25 equally spaced observations for g in a ten percent interval around g_{ss}. On the other hand, we fixed g_t at its steady state, and chose 25 equally spaced data in a 20% interval around z_{ss}.

4.2.2 Sample cross correlations, standard deviations and means

We compute the autocorrelation function for output, $\rho(y_t, y_{t-j})$, in each simulation. For a given variable X_t, we also obtain its cross-correlation with output, $\rho(y_t, X_{t+j})$, $j \gtrless 0$, standard deviation, σ_X, and sample mean, $\bar{X}$. This way, we produce random samples of size 250 for each statistic.

Since most papers report average values across simulations for some of these statistics, we check whether they differ among solution methods[7]. Dispersion in the sample of N values of a given statistic obtained from a solution method is usually very small for reasonable values of N. This is the main reason why sample means may turn out to be significantly different for different methods, since no method produces a systematic bias in any variable. In other cases, a method may have some difficulty in fully capturing the serial correlation in a variable or the correlation between two variables, this test again showing statistically significant differences between average values of the relevant statistics across the set of N

[7] Let us denote by γ_i^k a particular statistic obtained from the $i-$th simulation, $1 \leq i \leq N$, with method k . Let μ_{γ^k} denote the population mean for γ_i^k and $a_{\gamma^k}, s_{\gamma^k}$ the sample mean and standard deviation calculated from the sample of N simulations. To test $H_0 : \mu_{\gamma^{k_1}} = \mu_{\gamma^{k_2}}$ for any two different methods k_1, k_2, we can use the large sample approximation,

$$\| \frac{a_{\gamma^{k_1}} - a_{\gamma^{k_2}}}{\sqrt{\frac{s^2_{\gamma^{k_1}} + s^2_{\gamma^{k_2}}}{N}}} \| \sim N(0,1)$$

simulations. Even though we performed the calculations for a wide set of variables, we only show the results for those variables we deem more representative. In the basic growth model we only look at consumption, $X_t = [c_t]$. As regards Hansen's model we considered employment, given the emphasis placed on the labor market, $X_t = [N_t]$. Finally, for the Cooley-Hansen model, we present statistics for labor and inflation, $X_t = [N_t, \pi_t]$ [8].

5 Results

To compute empirical distributions for each statistic, we repeated the following steps for each of the 250 simulations run with each model, parameter vector, and sample size: *i)* generate a realization of the exogenous shock $\{\mathsf{z}_t\}_{t=1}^{T}$, *ii)* use it to implement each method (LQA, UHL, SIM, SIL, PEA) to generate time series for all the variables, *iii)* compute the set of statistics. We show here a sample of results, selected according to their relevance for the aim of the paper. The whole set of results can be obtained from the authors upon request, and is partially available in the working paper version of this article, Novales and Pérez (2002).

5.1 Basic Neoclassical Growth Model

5.1.1 Expectations error properties

Tables 2 and 3, and figures 1 and 2, summarize the main results for the basic growth model using the five solution approaches.

Insert table 2, table 3, figure 1 and figure 2

In figure 1, we show the results of the den Haan-Marcet test for the linear/log-linear approximation-based methods: LQA, SIM, SIL and UHL. The performance of the LQA and SIM solutions deteriorates for a large standard deviation of the technology shock, for any sample size, rejecting the null hypothesis of zero correlation between the expectations error and variables in the information set much more often than in 5% of the simulations. This result is intuitive, since a larger deviation from steady-state makes local approximations in levels to be less accurate. When T=150, SIM tends to behave slightly better than LQA, although both solutions fail to pass the test when T=3000, in the sense that the percentage of rejections of the null hypothesis is well above 5%. As already mentioned, the SIM solution could not be obtained for T=3000 and σ_ϵ=0.06. The SIL and UHL solutions, based on the log-linear approximation to the model, are fairly accurate for the nine parametric cases analyzed and both sample sizes, passing the den Haan-Marcet test in about 95% of the realizations. This is the most salient feature in figure 1: when working with logged variables,

[8] We also implemented non-parametric Kolmogoroff-Smirnov tests, to see whether the empirical *distribution* of a given statistic was the same across the different solution methods. The results pointed in the same direction than those obtained with the previous test, and are not reported. Similarly, we used the set of first order conditions and decision rules to generate the response functions of the main variables to a one standard deviation impulse in the shocks. As differences across methods were again negligible, we do not provide the results to save space.

as in the SIL and UHL methods, an increase in the variance of the technology shock does not deteriorate the statistical properties of the solution, possibly because of the homoskedasticity effect induced by the log-transformation. As regards the effect of the relative risk aversion parameter (still in figure 1), the performance of the SIM and LQA solutions in terms of the den Haan-Marcet test deteriorates for low values of η, i.e. for high values of the elasticity of intertemporal substitution of consumption, while the SIL and UHL solutions are again barely affected[9].

Table 2 shows the results of testing for a significant mean as well as for significant autocorrelation structure fitting an estimated AR(1) model for the expectations error. There is no evidence of a significant mean in any parametric case and sample size, but there is evidence of a significant autoregressive coefficient under some parameterizations for the LQA and SIM solutions. Serial correlation arises more often than suggested by the 5% significance level for simulations with high elasticity of intertemporal substitution of consumption and high innovation variance. Rejection becomes much more frequent when T=3000, most likely because higher precision in estimation increases the power of the test. Autocorrelation in ξ_t for high values of the elasticity of substitution may explain the more important failure of the den Haan-Marcet test in those cases. The representative agent then does little smoothing, adjusting consumption to income fluctuations, and the LQA and SIM methods fail to fully capture the higher consumption volatility in these cases. These methods seem to impose more inertia in the expectations mechanism than there actually is in such cases, thereby inducing some spurious autocorrelation in the expectation error.

A similar effect is produced by an increase in the volatility of the exogenous shock. That will again produce a more volatile decision variable, and methods that impose more inertia in the expectations mechanism will tend to exhibit deviations from rationality. So, it is not surprising that rejections of the den Haan-Marcet test as well as evidence of autocorrelation are more important for high elasticity of intertemporal substitution, as well as for a high variance of the exogenous shock.

These tests refer to possible deviations of rationality. The LQA and SIM solutions tend to produce expectations errors that display evidence of autoregressive structure, and show significant correlation with variables in the information set available when forming the conditional expectation. These characteristics, related to each other, are very damaging for an interpretation of the time series obtained from the described implementation of these methods as rational expectations solutions. On the other hand, there is essentially no evidence on violation of rationality for the SIL, UHL and PEA solutions, which show no significant evidence of autocorrelation in ξ_t.

5.1.2 Other measures

To evaluate the decision rules according to whether decision variables are increasing or decreasing in the state variables we present figure 2 for the LQA, SIM, SIL and UHL solutions.

[9]Regarding PEA solutions, the statistic associated to the den Haan-Marcet test was precisely the criterion used to accept a particular parameterization for the polynomial approximation to the expectations equation in each case. So, it is not surprising that the PEA solution passes the test at roughly the chosen 5% significance level in all cases.

We only show parametric cases 2, 5 and 8 (η =1.5), being the qualitative results identical in the other cases. In them, the stock of capital takes 25 equally spaced values in a ten percent interval around the deterministic steady-state k_{ss},while z takes 25 values around its steady state value of 1.0, from 0.4 to 1.6. The SIM, SIL and UHL decision rules are monotonically increasing over the selected values of the state variables, for all parametric cases (as it is the case with PEA, not shown in the figure). The LQA decision rule for consumption is non-monotonic in technology, although it is always increasing in capital. According to this decision rule, for any given level of capital, consumption falls when the value of the technology shock moves from zero to 0.90, ten percent below its deterministic steady state value of 1.0, increasing above the 0.90 threshold. This lack of monotonicity is unlikely to reflect an optimal consumption behavior. If it did, it would be a feature not captured by any other solution approach, which seems unlikely. Solving the basic growth model under full depreciation and a discrete three state first order Markov chain for technology, Christiano (1990) reports the same lack of monotonicity, for a high standard deviation of the technology shock, $\sigma_\epsilon = 0.1$, the anomaly not arising in his work for a low standard deviation, $\sigma_\epsilon = 0.01$.

No significant differences were observed among sample means, standard deviations and cross correlations generated with different solution methods in cases 1 to 6. Only in the high variance cases, $\sigma_\epsilon = 0.06$, we can appreciate some deterioration in methods that rely on linear approximations in levels around steady-state, LQA and SIM, in that the statistics they produce are significantly different from those of other methods. From the results of the previous tests, we believe that these two methods are to blame for the differences. Table 3 presents the outcome for case 9: the contemporaneous and lagged correlation of consumption with output, as well as the first two output autocorrelations do not statistically differ across methods. However, the mean of the consumption series generated by the LQA and UHL approximations significantly differs from those generated by the SIL and PEA methods when $T = 3000$. When $T = 150$, the standard deviation of consumption generated with SIM is different from those obtained with SIL, UHL and PEA. When $T = 3000$ the SIM solution could not be computed, but the standard deviation of consumption from the LQA solution is significantly different from those of the UHL, SIL and PEA methods.

To summarize: the performance of the UHL, SIL and PEA solutions is almost identical in all the analyzed dimensions. Linear approximations in levels (LQA, SIM) are less accurate when looking at properties of the prediction error, showing significant deviations from rationality. They also tend to perform slightly worse for high technology shock variances in terms of the mean and variance of decision variables, for which they occasionally produce values significantly different in average from those obtained with the other solution methods. We also observe a non-monotonic behavior in the linear LQA decision rule for consumption that does not appear with any other solution method.

5.2 Hansen (1985) Model

5.2.1 Expectations error properties

Qualitative results emerging from the battery of tests are similar to those obtained for the basic growth model. For the sake of saving space we only show the results for the den Haan and Marcet test (figure 3) and a table containing statistics to assess the equality among the means, standard deviations and cross correlations (table 4).

Insert table 4 and figure 3

Figure 3 summarizes the results of the test for the linear approximation-based methods: the SIM method seems now to be more sensitive than LQA to a higher technology shock variance. Both solutions, and specially the former, deteriorate in terms of the den Haan-Marcet test for both sample sizes when the variance of the shock increases. As in the more basic model, the SIL and UHL solutions are fairly accurate for the nine parametric cases analyzed and both sample sizes, passing the den Haan-Marcet test in approximately 95% of the simulations. This consistent behavior seems to arise from performing the approximation in logged-variables. As regards the effect of the relative risk aversion parameter, the SIM and LQA solutions again behave worse for low values of η, reaching a very high percentage of rejections of the null hypothesis of lack of correlation between expectations errors and variables in the information set. The performance of SIL and UHL is uniformly good for all values of η.

As in the basic growth model, the results of the tests on the estimated AR(1) model for the expectations error showed no evidence of a significant mean in the expectations error. Statistically significant autoregressive coefficients for the expectation error that emerges from the LQA and SIM solutions tend to be again associated to a high elasticity of intertemporal elasticity of substitution and to a high variance of the technology shock. Jointly with the rejections to the den Haan-Marcet test, this result raises again serious questions regarding the interpretation of the obtained time series as being the rational expectations solution to the model. Reasons for this failure are again those described in the basic growth model.

5.2.2 Other measures

SIM, SIL, UHL and PEA decision rules showed consumption increasing with both state variables, capital and technology, their values being essentially identical. Again, the LQA decision rule was non-monotonic in technology, although it is increasingly monotonic in capital. The non-monotonicity effect is less important than in the basic growth model. It shows, for any level of capital, consumption falling when technology moves from zero to 0.90, ten percent below its deterministic steady state value of 1.0, and increasing from that level on.

No significant differences were appreciated among the means, standard deviations and cross correlations generated with different methods in cases 1 to 5. But, even with not very large volatility, SIM contemporaneous and lagged correlations of labor with output for $T = 3000$ in case 6 are significantly different from those obtained with the PEA method at the 95% level, and from those obtained with LQA, UHL and SIL at the 90% level. We present table 4 as an example of results for high technology shock variance cases: SIM correlations between output and labor are significantly different from those obtained from alternative solutions at the 95% level when $T = 150$ (remember we could not solve with SIM for $T = 3000$). The mean of labor from the UHL solution differs from that obtained from

SIL and PEA when $T = 3000$.

Summing up, the performance of UHL, SIL and PEA solutions to the "indivisible labor" model is, again, almost indistinguishable in all the dimensions analyzed, except for discrepancies in the mean value of labor in extreme parametric cases among UHL on the one hand, and SIL and PEA on the other. In those cases, using the original nonlinear structure of the model seems to be very relevant. Concerning the den Haan-Marcet accuracy test, LQA and SIM behave badly, showing correlation between the expectations error and variables which were known when the conditional expectation was made. They also tend to present significant autocorrelation in the expectations error for high variance cases and low elasticity of intertemporal substitution of consumption. As in the basic growth model, these failures are related to each other. Again, a strict interpretation of these as being rational expectations solutions is questionable. The non-monotonic performance of the linear LQA decision rule for consumption relative to technology appears again, although it is now weaker.

5.3 Cooley and Hansen (1989) Model

5.3.1 Expectations error properties

The most salient results for the Cooley-Hansen model are shown in table 5, and figure 4.

Insert table 5 and figure 4

It is important to recall that the implementation of the LQA method to solve this model is different from that used for the two previous economies, in which the competitive solution was Pareto efficient. Therefore, comments regarding the LQA solution should not be read as a smooth transition from those made when applied to the two non-monetary models.

Regarding the den Haan-Marcet test, the results obtained when solving the previous models also hold for the monetary model. Figure 4 now shows the percentage of rejections as a function of the steady state rate of money growth and the variance of the technology shock, so they are not comparable to those in the previous models. SIL and UHL solutions passed the test with an approximate significance level of 5%, and did not deteriorate with an increased variance for the technology shock. The effect of an increased rate of growth of money on the den Haan-Marcet test for these two solutions is also negligible.

On the other hand, when $T = 3000$, LQA and SIM solutions deteriorate for a higher variance of the technology shock, as in the previous models. Moving from $\sigma_\epsilon = 0.01$ to $\sigma_\epsilon = 0.02$ in the LQA solution, the percentage of rejections to the den Haan-Marcet test jumps from 21% to 64%, and from 37% to 96% in the SIM method. Also, for a given variance of the technology shock, the greater the growth rate of money, the worse the performance of the LQA and SIM solutions. This is intuitive since as g_t is log-normal, an increase in g_{ss} implies an increase not only in the mean of g_t, but also in its variance σ_{ϵ_g}. When $T = 150$, the LQA and SIM solutions do not fail to pass the test so often as when $T = 3000$, due to the lack of power of the test for low values of T.

Regarding the AR(1) structure estimation, we did not detect any evidence of a significant mean in the expectations error, although there was some indication of serial correlation, specially in the higher variance cases, for the LQA and SIM solutions. For the LQA solution, that evidence was very clear for T=3000.

5.3.2 Other measures

As regards decision rules, the LQA approach to solving non Pareto optimal problems proposed by Kydland (1989) and Cooley and Hansen (1989) did not present the non-monotonicity problem we obtained for standard social planner problems. Consumption was increasing in both, technology and capital, and decreasing in money growth. The SIM, SIL, UHL and PEA solutions also have these properties. What is more, the grids are in this case quite similar among all four "almost" linear solution approaches [see Novales and Pérez (2002)].

Finally, table 5 shows the statistics to test for differences in the mean of sample averages, standard deviations and cross correlations generated with the different solution methods for the case with a higher variance for the technology shock and a higher money growth: σ_ϵ =0.06 and g_{ss} =1.15. We did not appreciate any significant difference in these tests for cases 1 to 4, while the picture for case 5 was very similar to that for case 6. When T =150, the statistics to compare LQA with the SIM, SIL, UHL and PEA solutions exceed the 5% or the 15% [critical value 1.0364] significance level when applied to the second autocorrelation of output as well as to the contemporaneous and lagged correlations of output with labor and inflation. For that sample size, and at the 5% or 10% level, SIM also tends to differ from SIL, UHL and PEA concerning the mean of labor, the contemporaneous and lagged correlation of labor with output, and the contemporaneous correlation of inflation with output. The same applies when T = 3000: the LQA solution seems to be significantly different from those obtained with SIL, UHL and PEA [remember that it was not possible to implement SIM in this case]. A last observation in table 5 is that the mean of labor that arises from UHL solution when T = 3000 is significantly different from those obtained with SIL and PEA, a phenomenon similar to that observed in Hansen's model for extreme parameter values. As in the previous model, using the original nonlinear structure of the economy to compute the numerical solution seems to be important in these extreme cases.

Hence, the performance of UHL, SIL and PEA solutions to the Cooley-Hansen cash-in-advance economy is again almost identical in all the analyzed dimensions, except for average labor in the high variance cases. As in the previous non-monetary models, the LQA and SIM solutions violate rationality, since they perform badly in terms of the den Haan-Marcet test and tend to show some significant autoregression coefficients for the expectations error. The non-monotonicity of the linear LQA decision rule for consumption in the non-monetary models disappears in the version of the method designed to cope with non Pareto optimal settings that we have applied here.

6 Concluding remarks

We have characterized numerical solutions to three different versions of the neoclassical exogenous growth economy using standard solution methods for nonlinear rational expectations models. The methods considered are very similar in spirit, all of them searching for the stable manifold of the analytical representation of the economy, although important differences arise from the way they are usually implemented. Our comparison includes numerical solutions obtained from linear and log-linear approximations to the economic model, as well as solutions obtained from the original nonlinear structure of the model economy. The latter includes the Parameterized Expectations methods, with whom we compare numerical solutions obtained from all other methods. It is a merit of this paper to present an homogeneous evaluation and comparison of the properties of the different solution methods, using a common realization of the shock/s in the economy.

For the economies considered, the main difficulties with analyzed methods are: *i)* strange non-monotonicity properties tend to appear with the standard linear quadratic approximation which do not arise for alternative solution methods, *ii)* approximations in levels, like the standard linear quadratic approximation approach, tend to produce deviations from rationality in the expectations error, a failure that should be considered central, since we are supposedly computing the rational expectations solution to the dynamic optimization model. There is some evidence suggesting that linear approximations fail to fully capture the dynamics embedded in the conditional expectations, imposing more inertia in the conditional expectations than there actually is in the model. That leaves some spurious autocorrelation in the expectation error that leads to failure of the rationality tests. We have also shown that, as pointed out in Taylor and Uhlig (1990), numerical solutions that differ in their behavior with respect to the den Haan-Marcet rationality test may also show significant differences in terms of other statistics: means, standard deviations or the sign of the relationships involved in the decision rules.

On the positive side, we have found solutions computed using a log-linear approximation to the model around steady-state to be generally undistinguishible from solutions obtained using the original nonlinear model, for the three economies considered. Solving the log-linear approximation to the model as in Uhlig (1999) or following Sims (2002) and Novales *et al.* (1999) to compute stability conditions from that log-linear approximation are generally as accurate as Parameterized Expectations in all the analyzed dimensions (rationality of expectation errors, first moments of decision variables, cross correlations with output, induced decision rules). In particular, these methods do not present any evidence on violation of rationality. It is only under extreme parameterizations that using the original nonlinear structure of the model as in Sims (2002) and Novales *et al.* (1999) or in the Parameterized Expectations approach to compute the numerical solution differs significantly from using the log-linear approximation.

When a version of Sims (2002) is implemented through a linear, rather than a log-linear approximation, performance deteriorates and we fall under criticism *ii)* in the previous paragraph. The fact that using log-linear approximations should always be preferred is one of the messages of this paper. In fact, even though we have just considered the linear-quadratic approach as it is usually implemented, a log-linear version of that method would produce essentially the same numerical solution than solving the log-linear approximation to the model, as in Uhlig (1999), or following Sims (2002) to solve the original nonlinear model together with the stability condition obtained from the log-linear approximation.

When working in logs, an increase in the variance of exogenous shock/s does not deteriorate the solution, possibly due to the homoskedasticity effect induced by the log-transformation.

Working in logs seems to be very advisable when solving non-linear rational expectations models, specially in view of the rationality properties of the solutions obtained in both cases. In fact, with independence of the solution approach, writing the model in logs of the variables seems to be more important than preserving the nonlinear structure of the original model.

The performance of the alternative numerical solutions relative to the parameterized expectations approach did not worsen when departing from the basic growth model. This may be due to the fact that endogenous dynamics are weak in all the model economies considered, and the shape of the time series generated by different methods inherit the pattern of the common exogenous shocks. The result may not stand in more complex economic structures.

These results contrast with Dotsey and Mao (1992), where log-linear methods did not dominate linear methods, and where *more refined* linear or log-linear methods did not dominate *less refined* ones. Although the model they consider departs from the neoclassical growth model in a different way than those we have analyzed, we presume that the different results may arise from their use of a five-state Markov chain for the only source of exogenous dynamics, a process for tax rates. Discussing solutions to the basic neoclassical growth model in section 5.1.2, we have also seen how the UHL and SIL log-linearizations seem to perform better than the log-linearization of the same model in Christiano (1990) where the log-linear decision rule presents some lack of monotonicity for high variance technology shocks. These comparisons suggest that, contrary to some conventional wisdom, the choice of using discrete versus continuous probability distributions for exogenous shocks is fully relevant.

Several interesting questions have been left aside in this paper and are important in the context of solution methods evaluation: considering models with more state variables, economies in transition to steady state after having experienced some perturbation, economies with a richer endogenous dynamics, or economies with heterogeneous agents. These are some of the interesting extensions of this work.

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A Appendix

A.1 Basic Neoclassical Growth Model

For the LQA solution, we have $s_t = [k_{t-1}]$, $\mathbf{z}_t = [\log(z_t)]$ and $d_t = [x_t]$. For all the parametric cases considered, the coefficients in the decision rule $d_t = H[1, \mathbf{z}_t, s_t]^T$ are,

H		
CASES 1,4,7	CASES 2,5,8	CASES 3,6,9
[1.9190, 3.2243, -0.0255]	[1.0512, 2.7668, -0.0027]	[0.7015, 2.7244, 0.0065]

changing only for different degrees of risk aversion. From the resource constraint and the production function, we can write consumption as a function of last period capital and the contemporaneous technology shock, $c_t = z_t k_{t-1}^{\alpha} - H[1, \log(z_t), k_{t-1}]^T$.

To solve with the UHL method, we choose: $s_t = [\tilde{k}_t]$, $v_t = [\tilde{c}_t,\ \tilde{R}_t,\ \tilde{y}_t]^T$, $\mathbf{z}_t = [\tilde{z}_t]$, where, along this Appendix, ˜ denotes log-deviations from steady state. Then, for the analyzed cases, the matrices in (10) become,

CASE	Ξ_1	Ξ_2	Ξ_3^T	Ξ_4^T
1,4,7	0.9495	0.0849	[0.8361, 0.1742, -0.0222]	[0.0348, 0.3600, 1.000]
2,5,8	0.9723	0.0728	[0.5210, 0.3403, -0.0222]	[0.0348, 0.3600, 1.000]
3,6,9	0.9815	0.0717	[0.3940, 0.3557, -0.0222]	[0.0348, 0.3600, 1.000]

As regards SIM method, we have: $u_t = [c_t - c_{ss},\ k_t - k_{ss},\ W_t - W_{ss},\ \log(z_t)]^T$, $\varepsilon_t = [\epsilon_t]$ and $\zeta_t = [\xi_t]$, while for the SIL method: $u_t = [\tilde{c}_t,\ \tilde{k}_t,\ \tilde{W}_t, \tilde{z}_t]^T$, $\varepsilon_t = [\epsilon_t]$ and $\zeta_t = [\xi_t]$. Numerical estimates for the stability condition P^s are in each case,

CASE	P^s - SIM method	P^s - SIL method
1,4,7	[0.0000, 0.0071, 1.0000, 0.0303]	[0.0000, 0.4403, 1.0000, 0.0497]
2,5,8	[0.0000, 0.0047, 1.0000, 0.0999]	[0.0000, 0.8037, 1.0000, 0.4519]
3,6,9	[0.0000, 0.0015, 1.0000, 0.0474]	[0.0000, 1.2043, 1.0000, 0.9807]

Concerning the PEA solution to this model, in all the considered parametric cases, a second order polynomial approximation proved to be useful:

$$\begin{aligned} \psi_t(q; k_{t-1}, z_t) &= q_1 \exp\left(q_2 \log(k_{t-1}) + q_3 \log(z_t) + q_4 (\log(k_{t-1}))^2\right) \\ &\times \exp\left(q_5 \log(k_{t-1}) \log(z_t) + q_6 (\log(z_t))^2\right). \end{aligned}$$

The fixed point for vector q was calculated in each case using a sample size of 25000 observations and a four-digit accuracy stopping criterion. This applies to all considered models. We set λ_q equal to one except for the cases when $\eta = 0.5$, that we chose $\lambda_q = 0.5$. Estimated parameter values were,

CASE	q_1	q_2	q_3	q_4	q_5	q_6
1	2.3473	-0.3253	-0.2258	-0.0126	0.0382	-0.0055
2	1.6293	-0.3156	-2.2440	-0.0642	0.4766	-0.4221
3	0.1162	-0.7187	-4.8308	-0.2635	1.0314	-0.5244
4	2.7395	-0.4170	-0.1762	-0.0008	0.0245	-0.0214
5	0.7466	0.1009	-1.0839	-0.1195	0.1561	-0.0971
6	1.6741	-0.7658	-3.5681	-0.0567	0.6828	-0.1533
7	2.4171	-0.3407	-0.2021	-0.0106	0.0315	-0.0207
8	3.1017	-0.6436	-0.9073	-0.0220	0.1080	-0.0861
9	2.8286	-1.0233	-2.3828	-0.0218	0.3569	-0.2553

A.2 Hansen (1985) Model

For the LQA solution, we have $s_t = [k_{t-1}]$, $\mathbf{z}_t = [\log(z_t)]$ and $d_t = [x_t, N_t]^T$. For the different parameter vectors considered, the decision rules $d_t = H[1, \mathbf{z}_t, s_t]^T$ are,

H		
CASES 1,4,7	CASES 2,5,8	CASES 3,6,9
0.7368, 2.6129, -0.0332	0.7368, 1.7499, -0.0332	0.7368, 1.5342, -0.0332
0.3801, 0.7383, -0.0037	0.5459, 0.3718, -0.0168	0.6127, 0.2242, -0.0221

For the UHL method: $s_t = [\tilde{k}_t]$, $v_t = [\tilde{c}_t,\ \tilde{y}_t,\ \tilde{N}_t,\ \tilde{R}_t,\ \tilde{x}_t]^T$, $\mathbf{z}_t = [\tilde{z}_t]$. Then, we have,

CASE	Ξ_1	Ξ_2	Ξ_3^T	Ξ_4^T
1,4,7	0.9418	0.2063	[0.8210, 0.2702, -0.1403, -0.0254, -1.3273]	[0.4052, 2.4176, 2.2150, 0.0840, 8.2537]
2,5,8	0.9418	0.1382	[0.3930, -0.0481, -0.6376, -0.0364, -1.3273]	[0.3989, 1.7139, 1.1155, 0.0596, 5.5276]
3,6,9	0.9418	0.1212	[0.2206, -0.1763, -0.8380, -0.0409, -1.3273]	[0.2526, 1.4304, 0.6725, 0.0497, 4.8461]

Concerning SIM, we have: $u_t = [c_t - c_{ss},\ N_t - N_{ss},\ k_t - k_{ss},\ W_t - W_{ss},\ \log(z_t)]^T$, $\varepsilon_t = [\epsilon_t]$ and $\zeta_t = [\xi_t]$. For the SIL method: $u_t = [\tilde{c}_t,\ \tilde{N}_t,\ \tilde{k}_t,\ \tilde{W}_t,\ \tilde{z}_t]^T$, $\varepsilon_t = [\epsilon_t]$ and $\zeta_t = [\xi_t]$. Then,

CASE	P^s-SIM method	P^s-SIL method
1,4,7	[0.0000, 0.0000, 0.0363, 1.0000, 0.1188]	[0.0000, 0.0000, 0.4359, 1.0000, 0.1127]
2,5,8	[0.0000, 0.0000, 0.0568, 1.0000, 0.5878]	[0.0000, 0.0000, 0.6260, 1.0000, 0.5119]
3,6,9	[0.0000, 0.0000, 0.0724, 1.0000, 0.8781]	[0.0000, 0.0000, 0.7026, 1.0000,0.6728]

For PEA in all the considered parametric cases a second order polynomial approximation proved again to be useful. We set λ_q equal to one except for the cases when $\eta = 0.5$ that we set $\lambda_q = 0.5$. Estimated coefficients in the parameterized expectation were,

CASE	q_1	q_2	q_3	q_4	q_5	q_6
1	2.9009	-0.3869	-0.3265	-0.0047	0.0490	-0.0611
2	2.1570	0.0846	-0.7523	-0.1310	0.0612	-0.1099
3	2.2404	0.2335	-1.7301	-0.1757	0.3847	-0.2542
4	2.8956	-0.3866	-0.3594	-0.0046	0.0620	-0.0978
5	3.9471	-0.3810	-1.1585	-0.0415	0.2274	-0.1638
6	3.9666	-0.2217	-1.1962	-0.0848	0.1683	-0.0769
7	2.8364	-0.3693	-0.3593	-0.0082	0.0614	-0.1045
8	3.8988	-0.3776	-1.0778	-0.0405	0.1882	-0.1368
9	3.5446	-0.1183	-1.2966	-0.1065	0.2031	-0.0884

A.3 Cooley and Hansen (1989) Model

The equilibrium conditions for the Cooley and Hansen's problem are (6), (7) together with (5), the resource constraint and a last condition associated with the cash-in-advance constraint: $\hat{p}_t = \frac{1}{c_t}$, where $\hat{p}_t$ denotes the inverse of real money balances.

To solve this model using the LQA approach we simply took the H matrix reported by Cooley and Hansen in their paper. Now $[\hat{p}_t, N_t]^T = H\ [1, \log(z_t), \log(g_t), k_{t-1}]^T$, where

H				
CASES	1.88633	-0.58175	0.55474	-0.05898
1,3,5	0.64419	1.73073	0.30219	-0.03318
CASES	2.07319	-0.66585	0.63537	-0.07726
2,4,6	0.52716	1.51216	0.26423	-0.03318

Concerning the undetermined coefficients method, UHL, we have: $s_t = [\tilde{k}_t]$, $v_t = [\tilde{c}_t,\ \tilde{y}_t,\ \tilde{N}_t,\ \tilde{x}_t,\ \tilde{\hat{p}}_t,\ \tilde{\lambda}_t,\ \tilde{R}_t]^T$, $\mathbf{z}_t = [\tilde{z}_t,\ \tilde{g}_t]$. Then, for all the analyzed cases, we have $\Xi_1 = [0.9418]$, $\Xi_2 = [0.1552,\ 0.0271]$, $\Xi_3 = [0.5316, 0.0550, -0.4766,\ -1.3273, -0.5316,\ -0.5316, -0.0328]^T$, and

$$\Xi_4 = \begin{bmatrix} 0.4703 & 1.9417 & 1.4715 & 6.2091 & -0.4703 & -0.4703 & 0.0675 \\ -0.4488 & -0.5555 & -0.0867 & 1.0850 & 0.4488 & -0.0312 & -0.0019 \end{bmatrix}^T.$$

To implement the SIM solution: $u_t = [c_t - c_{ss},\ N_t - N_{ss},\ k_t - k_{ss},\ W_t - W_{ss},\ \log(z_t),\ \log(g_t) - \log(g_{ss})]^T$, $\varepsilon_t = [\epsilon_{z_t},\ \epsilon_{g_t}]$ and $\zeta_t = [\xi_t]$. For the SIL method: $u_t = [\tilde{c}_t,\ ,\ \tilde{N}_t,\ \tilde{k}_t,\ \tilde{W}_t,\ \tilde{z}_t,\ \tilde{g}_t]^T$, $\varepsilon_t = [\epsilon_{z_t},\ \epsilon_{g_t}]$ and $\zeta_t = [\xi_t]$. The single stability condition in each case is

CASE	P^s - SIM method	P^s - SIL method
1,3,5	[0.0000, 0.0000, 0.0617, 1.0000, 0.4663, 0.0194]	[0.0000, 0.0000, 0.5644, 1.0000, 0.3827, 0.0159]
2,4,6	[0.0000, 0.0000, 0.0699, 1.0000, 0.4663, 0.0194]	[0.0000, 0.0000, 0.5644, 1.0000, 0.3827, 0.0159]

Finally, as regards the PEA solution for this model, to have an appropriate approximation we needed to use a third order polynomial in all the considered cases,

$$\begin{aligned} \psi_t(q; k_{t-1}, z_t) &= q_1 \exp\left(q_2 \log(k_{t-1}) + q_3 \log(z_t) + q_4 \log(g_t) + q_5 (\log(k_{t-1})^2\right) \\ &\times \exp\left(q_6 \log(k_{t-1}) \log(z_t) + q_7 (\log(z_t))^2 + q_8 (\log(z_t))^3\right) \end{aligned}$$

From (6) we have $\lambda_t = \beta \psi_t(q; k_{t-1}, z_t)$, and then consumption can be obtained from (7). The fixed point for q was calculated in each case as in the previous models. We set λ_q equal to one in all the cases. Estimated coefficients for the different parameter vectors were,

CASE	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8
1	3.0710	-0.2498	-0.8460	-0.0362	-0.0569	0.1543	-0.1351	0.2371
2	3.6111	-0.4189	-0.7175	-0.0288	-0.0257	0.1111	-0.3122	-0.9004
3	3.8793	-0.4407	-0.9017	-0.0297	-0.0179	0.1790	-0.1316	0.1953
4	3.0250	-0.2686	-0.8818	-0.0223	-0.0576	0.1831	-0.1849	-0.6025
5	3.9664	-0.4567	-0.7432	-0.0582	-0.0154	0.1125	-0.1128	-0.0125
6	3.5363	-0.4038	-0.7614	-0.0191	-0.0291	0.1286	-0.1064	-0.0294

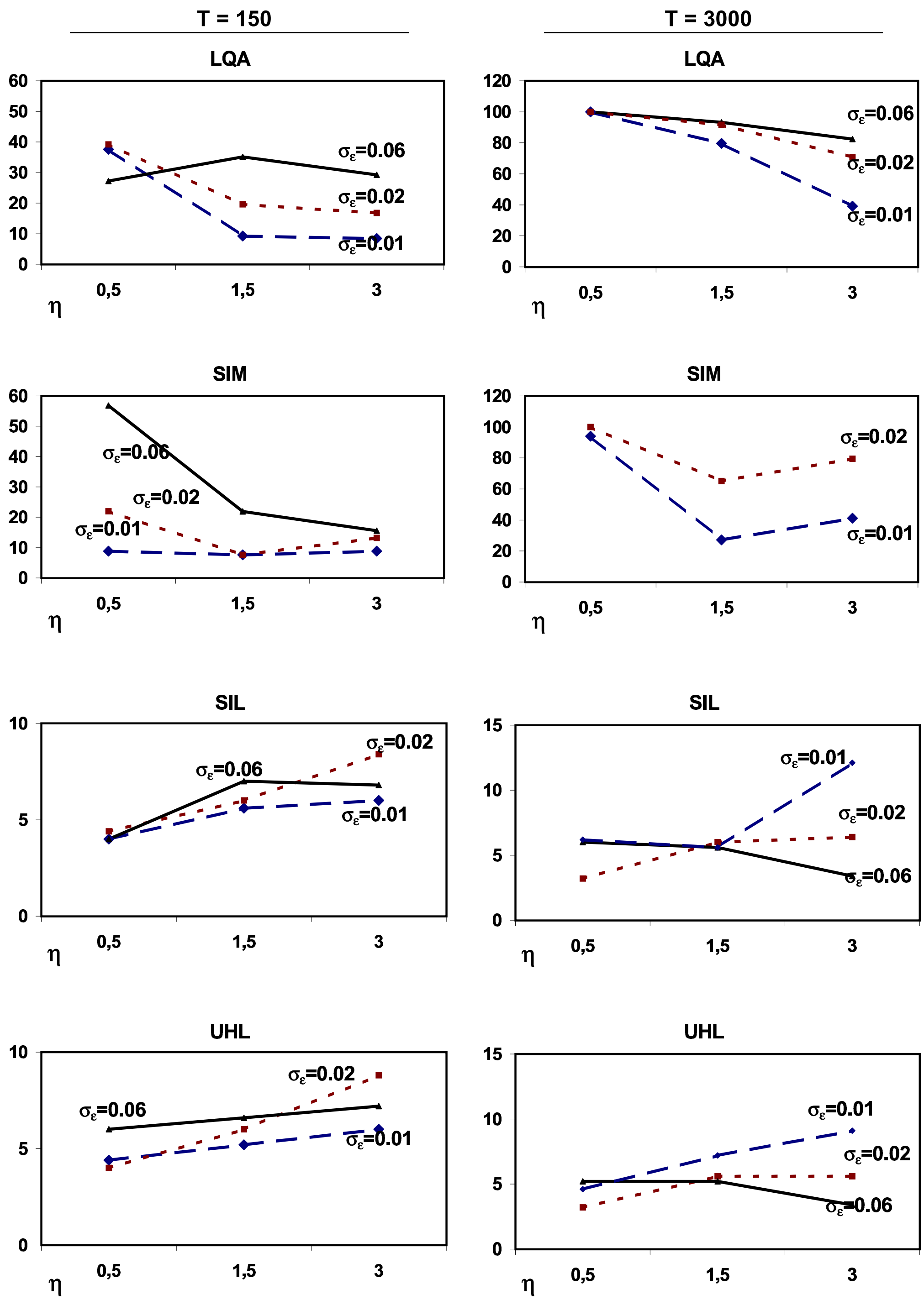

Figure 1: den Haan and Marcet (1994) test: Basic Neoclassical Growth Model. "Almost" linear methods. Percentage of realizations of the statistic in the 5% rejection region for the null hypothesis: $H_0 : E_t(\xi_{t+1}) = 0$. Instruments used: I_t = [constant, k_t, k_{t-1}, k_{t-2}, $\log(z_t)$,$\log(z_{t-1})$, $\log(z_{t-2})$].

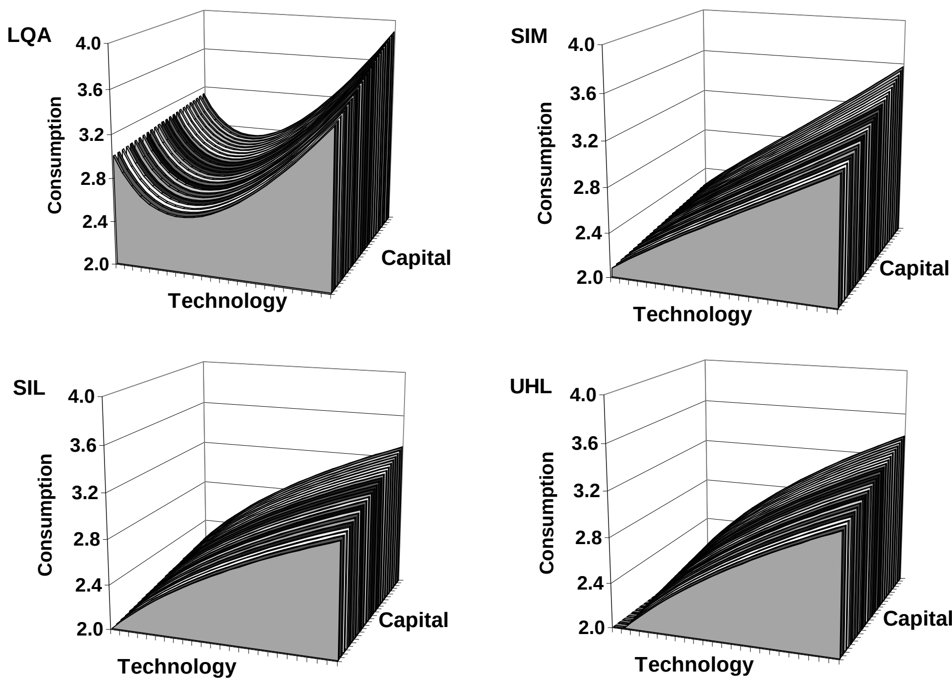

Figure 2: Decision rule for consumption. Basic Neoclassical Growth Model, cases 2, 5 and 8. "Almost" linear methods.

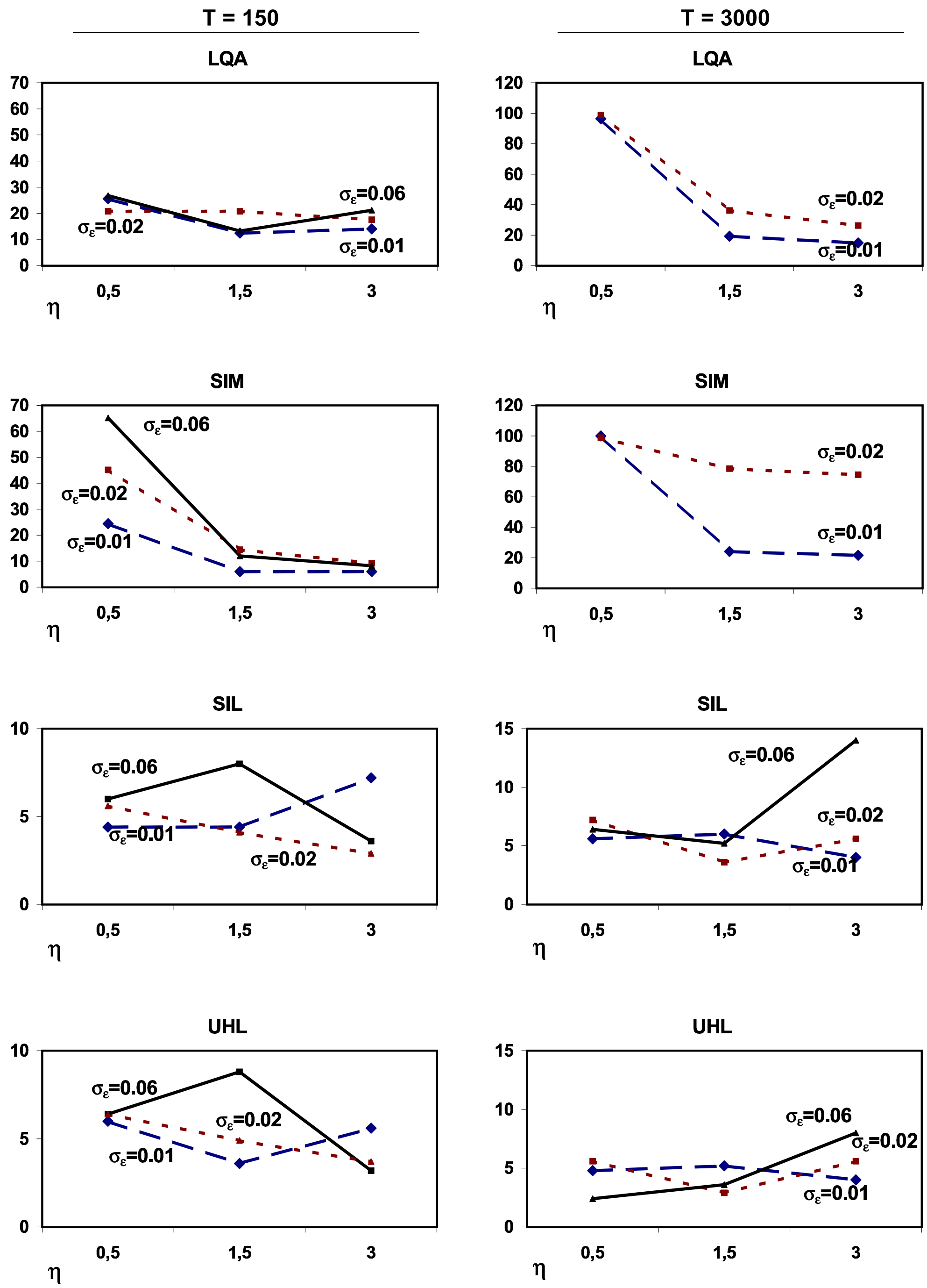

Figure 3: den Haan and Marcet (1994) test: Hansen (1985) Model. "Almost" linear methods. Percentage of realizations of the statistic in the 5% rejection region for the null hypothesis: H_0 : $E_t(\xi_{t+1}) = 0$. Instruments used: I_t = [constant, k_t, k_{t-1}, k_{t-2}, $\log(z_t)$,$\log(z_{t-1})$, $\log(z_{t-2})$].

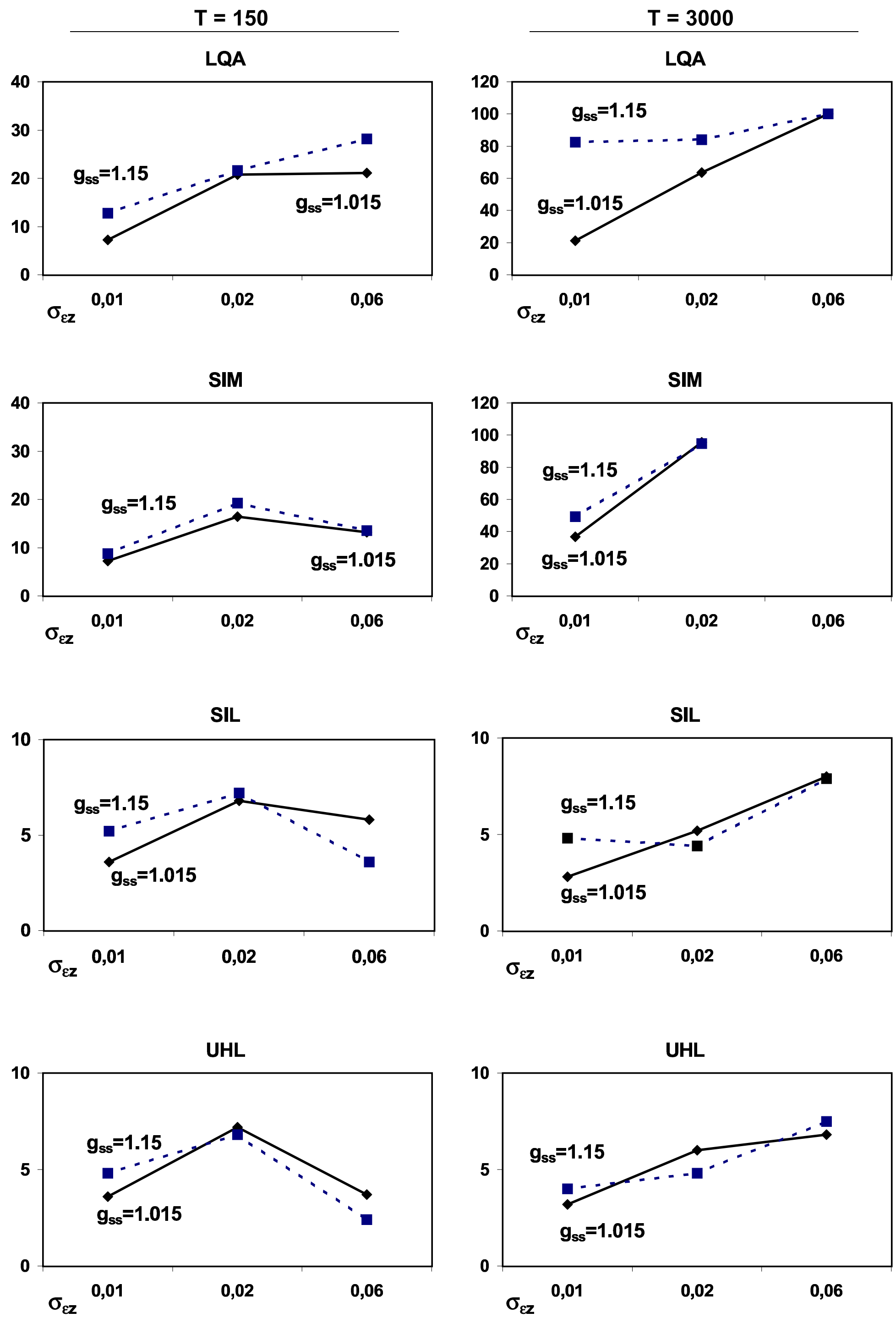

Figure 4: den Haan and Marcet (1994) test: Cooley and Hansen (1989) Model. "Almost" linear methods. Percentage of realizations of the statistic in the 5% rejection region for the null hypothesis: $H_0 : E_t(\xi_{t+1}) = 0$. Instruments used: I_t = [constant, k_t, k_{t-1}, k_{t-2}, $\log(z_t)$,$\log(z_{t-1})$, $\log(z_{t-2})$].

Table 1: Parametric cases considered for the evaluation exercise

CASE	1	2	3	4	5	6	7	8	9
Basic Neoclassical Growth model and Hansen (1985) model									
σ_ϵ	0.01	0.01	0.01	0.02	0.02	0.02	0.06	0.06	0.06
η	0.5	1.5	3.0	0.5	1.5	3.0	0.5	1.5	3.0
Cooley and Hansen (1989) model									
g_{ss}	1.015	1.15	1.015	1.15	1.015	1.15			
σ_{ϵ_z}	0.01	0.01	0.02	0.02	0.06	0.06			

Table 2: Basic Neoclassical Growth Model. Estimates of AR(1) for the expectations error. Percentage of realizations of the t-statistic in the 5% rejection region for the null hypothesis: $H_0 : \mu = 0$ (upper row), and $H_0 : \rho = 0$ (lower row).

T=150		LQA	SIM	SIL	UHL	PEA
Case 1	μ	0.0	0.0	0.0	0.0	0.0
	ρ	8.8	5.2	3.2	3.2	3.2
Case 2	μ	0.0	0.0	0.0	0.0	0.0
	ρ	5.2	4.4	4.8	4.8	4.8
Case 3	μ	0.0	0.0	0.0	0.0	0.0
	ρ	4.8	4.0	4.4	4.4	4.4
Case 4	μ	0.0	0.0	0.0	0.0	0.0
	ρ	12.4	14.8	2.4	2.4	2.4
Case 5	μ	0.0	0.0	0.0	0.0	0.0
	ρ	5.6	4.8	4.8	4.8	4.4
Case 6	μ	0.0	0.0	0.0	0.0	0.0
	ρ	6.0	4.0	2.8	2.0	2.0
Case 7	μ	0.0	0.0	0.0	0.0	0.0
	ρ	10.0	53.6	4.4	5.2	4.8
Case 8	μ	0.0	0.0	0.0	0.0	0.0
	ρ	11.6	8.8	6.8	6.4	7.6
Case 9	μ	0.0	0.0	0.0	0.0	0.0
	ρ	11.2	3.2	4.4	5.2	4.8

T=3000		LQA	SIM	SIL	UHL	PEA
Case 1	μ	0.0	0.0	0.0	0.0	0.0
	ρ	9.6	16.0	3.6	3.6	3.2
Case 2	μ	0.0	0.0	0.0	0.0	0.0
	ρ	8.0	6.4	8.4	8.4	8.0
Case 3	μ	0.0	0.0	0.0	0.0	0.0
	ρ	6.0	5.6	5.6	5.6	5.6
Case 4	μ	0.0	0.0	0.0	0.0	0.0
	ρ	19.2	75.6	4.4	4.4	2.8
Case 5	μ	0.0	0.0	0.0	0.0	0.0
	ρ	13.6	14.8	9.2	9.2	6.4
Case 6	μ	0.0	0.0	0.0	0.0	0.0
	ρ	18.7	24.9	6.2	6.2	5.3
Case 7	μ	0.0	0.0	0.0	0.0	0.0
	ρ	45.6	100	4.2	4.6	4.6
Case 8	μ	0.0	—	0.0	0.0	0.0
	ρ	83.8	—	7.4	7.9	9.6
Case 9	μ	4.4	—	0.0	0.0	0.0
	ρ	76.9	—	15.0	13.1	14.3

Table 3: Basic Neoclassical Growth Model, case 9. Test statistic for differences between cross correlations, means and standard deviations. In each panel, the upper corner corresponds to T=3000, the lower corner to T=150. Critical values at 95% and 90% significance levels are 1.6449 and 1.2816.

$\rho(y_t, y_{t-1})$	LQA	SIM	SIL	UHL	PEA	$\rho(y_t, y_{t-2})$	LQA	SIM	SIL	UHL	PEA
LQA	—	—	0.1357	0.0376	0.2143	LQA	—	—	0.1877	0.0466	0.2939
SIM	0.0971	—	—	—	—	SIM	0.1329	—	—	—	—
SIL	0.0305	0.1272	—	0.0974	0.0794	SIL	0.0396	0.1721	—	0.1403	0.1072
UHL	0.0469	0.1437	0.0163	—	0.1759	UHL	0.0663	0.1990	0.0267	—	0.2463
PEA	0.0286	0.1251	0.0018	0.0181	—	PEA	0.0377	0.1702	0.0018	0.0284	—

mean(c_t)	LQA	SIM	SIL	UHL	PEA	σ_{c_t}	LQA	SIM	SIL	UHL	PEA
LQA	—	—	1.4382	0.0108	2.1787	LQA	—	—	2.3481	2.0376	2.9012
SIM	0.0772	—	—	—	—	SIM	1.0565	—	—	—	—
SIL	0.1065	0.1837	—	1.4624	0.7434	SIL	0.9891	1.9992	—	0.2453	0.5521
UHL	0.5667	0.6409	0.4648	—	2.2100	UHL	0.9398	1.9478	0.0429	—	0.7808
PEA	0.1916	0.2688	0.0852	0.3829	—	PEA	1.0951	2.0984	0.1077	0.1498	—

$\rho(y_t, c_t)$	LQA	SIM	SIL	UHL	PEA	$\rho(y_t, c_{t+1})$	LQA	SIM	SIL	UHL	PEA
LQA	—	—	0.3374	0.1749	0.9169	LQA	—	—	0.1705	0.0125	0.6916
SIM	0.8374	—	—	—	—	SIM	0.5649	—	—	—	—
SIL	0.3526	0.6739	—	0.1925	0.7164	SIL	0.3835	0.2072	—	0.1962	0.5677
UHL	0.3210	0.7206	0.0438	—	0.8910	UHL	0.3752	0.2175	0.0100	—	0.7545
PEA	0.1744	0.8778	0.2281	0.1864	—	PEA	0.2491	0.3615	0.1535	0.1438	—

Table 4: Hansen (1985) Model, case 8. Test statistic for differences between cross correlations, means and standard deviations. In each panel, the upper corner corresponds to T=3000, the lower corner to T=150. Critical values at 95% and 90% significance levels are 1.6449 and 1.2816.

$\rho(y_t, y_{t-1})$	LQA	SIM	SIL	UHL	PEA	$\rho(y_t, y_{t-2})$	LQA	SIM	SIL	UHL	PEA
LQA	—	—	—	—	—	LQA	—	—	—	—	—
SIM	0.1331	—	—	—	—	SIM	0.1534	—	—	—	—
SIL	0.0943	0.2253	—	0.0743	0.1434	SIL	0.1438	0.2940	—	0.0936	0.1832
UHL	0.0562	0.1878	0.0380	—	0.2176	UHL	0.0970	0.2481	0.0467	—	0.2766
PEA	0.1611	0.2910	0.0664	0.1044	—	PEA	0.2274	0.3762	0.0833	0.1300	—

mean(N_t)	LQA	SIM	SIL	UHL	PEA	σ_{N_t}	LQA	SIM	SIL	UHL	PEA
LQA	—	—	—	—	—	LQA	—	—	—	—	—
SIM	1.2136	—	—	—	—	SIM	0.5834	—	—	—	—
SIL	0.3164	0.9159	—	1.6474	0.0786	SIL	0.0978	0.4969	—	0.3882	0.1349
UHL	0.4625	1.6724	0.7857	—	1.5759	UHL	0.0388	0.5486	0.0589	—	0.2527
PEA	0.2139	1.0172	0.1039	0.6831	—	PEA	0.0484	0.6241	0.1457	0.0870	—

$\rho(y_t, N_t)$	LQA	SIM	SIL	UHL	PEA	$\rho(y_t, N_{t+1})$	LQA	SIM	SIL	UHL	PEA
LQA	—	—	—	—	—	LQA	—	—	—	—	—
SIM	3.2537	—	—	—	—	SIM	3.0337	—	—	—	—
SIL	0.2161	3.0502	—	0.2296	0.7624	SIL	0.1705	2.8699	—	0.2420	0.7494
UHL	0.4761	2.8184	0.2586	—	0.9874	UHL	0.4570	2.6265	0.2836	—	0.9867
PEA	0.1757	3.3882	0.3885	0.6460	—	PEA	0.2255	3.2013	0.3915	0.6747	—

Table 5: Cooley and Hansen (1989) Model, case 6. Test statistic for differences between cross correlations, means and standard deviations. In each panel, the upper corner corresponds to T=3000, the lower corner to T=150. Critical values at 95% and 90% significance levels are 1.6449 and 1.2816.

$\rho(y_t, y_{t-1})$	LQA	SIM	SIL	UHL	PEA	$\rho(y_t, y_{t-2})$	LQA	SIM	SIL	UHL	PEA
LQA	—	—	0.3708	0.4047	0.4108	LQA	—	—	0.7159	0.7629	0.7831
SIM	1.2309	—	—	—	—	SIM	1.5889	—	—	—	—
SIL	0.8021	0.4411	—	0.0743	0.0877	SIL	1.0430	0.5613	—	0.0878	0.1254
UHL	0.8446	0.3971	0.0439	—	0.0134	UHL	1.0924	0.5107	0.0507	—	0.0376
PEA	0.8396	0.4027	0.0385	0.0054	—	PEA	1.0928	0.5109	0.0507	0.0000	—

$\text{mean}(N_t)$	LQA	SIM	SIL	UHL	PEA	σ_{N_t}	LQA	SIM	SIL	UHL	PEA
LQA	—	—	1.0810	0.6934	1.1183	LQA	—	—	2.1126	2.0987	2.1147
SIM	1.5367	—	—	—	—	SIM	1.1693	—	—	—	—
SIL	0.7345	1.2395	—	2.1197	0.2086	SIL	0.5762	0.9115	—	0.3825	0.0614
UHL	0.3216	1.8793	0.6334	—	2.3468	UHL	0.5005	1.0072	0.1108	—	0.4417
PEA	0.8139	1.1240	0.1202	0.7564	—	PEA	0.5704	0.9180	0.0083	0.1024	—

$\rho(y_t, N_t)$	LQA	SIM	SIL	UHL	PEA	$\rho(y_t, N_{t+1})$	LQA	SIM	SIL	UHL	PEA
LQA	—	—	2.3463	2.2124	2.2568	LQA	—	—	2.4838	2.3217	2.3763
SIM	4.3355	—	—	—	—	SIM	4.2143	—	—	—	—
SIL	2.0852	2.7667	—	0.2451	0.1773	SIL	2.0822	2.7167	—	0.2882	0.2149
UHL	2.0817	2.7355	0.0181	—	0.0704	UHL	2.1199	2.6337	0.0743	—	0.0785
PEA	2.1431	2.7288	0.0546	0.0359	—	PEA	2.1623	2.6615	0.0801	0.0042	—

$\text{mean}(\pi_t)$	LQA	SIM	SIL	UHL	PEA	σ_{π_t}	LQA	SIM	SIL	UHL	PEA
LQA	—	—	0.1258	0.1201	0.1262	LQA	—	—	0.7278	0.7159	0.7340
SIM	0.1008	—	—	—	—	SIM	0.3894	—	—	—	—
SIL	0.0268	0.1343	—	0.0073	0.0005	SIL	0.8684	0.5184	—	0.0793	0.0428
UHL	0.0205	0.0839	0.0496	—	0.0078	UHL	0.8446	0.4924	0.0262	—	0.1214
PEA	0.0185	0.1255	0.0087	0.0409	—	PEA	0.9050	0.5589	0.0420	0.0681	—

$\rho(y_t, \pi_t)$	LQA	SIM	SIL	UHL	PEA	$\rho(y_t, \pi_{t+1})$	LQA	SIM	SIL	UHL	PEA
LQA	—	—	2.9119	3.0739	3.1353	LQA	—	—	3.6807	3.7865	3.8903
SIM	0.5384	—	—	—	—	SIM	1.1232	—	—	—	—
SIL	1.2243	1.7588	—	0.3015	0.4137	SIL	1.8476	0.7659	—	0.1512	0.2989
UHL	1.5472	2.0784	0.3286	—	0.1118	UHL	2.0158	0.9392	0.1694	—	0.1477
PEA	1.7787	2.3061	0.5673	0.2396	—	PEA	2.2728	1.2274	0.4713	0.3066	—