Indeterminacy under non-separability of public consumption and leisure in the utility function

Esther Fernández, Alfonso Novales,
Jesús Ruiz*
Departamento de Economía Cuantitativa
Universidad Complutense de Madrid

September 28, 2001

Abstract

In a one sector growth model with public consumption in the utility function, the competitive equilibrium can be indeterminate for plausible values of the elasticity of intertemporal substitution of consumption, under constant returns to scale and endogenous government expenditures. Non-separability between public consumption and leisure in the utility function is crucial for this result.

JEL: E32, E62, D90.

*Financial support from Spanish Ministry of Education through grant PB98-0831 is gratefully acknowledged. Correspondence: A. Novales, Departamento de Economía Cuantitativa, Universidad Complutense de Madrid, Campus de Somosaguas, 28223 Madrid, Spain. Fax: (34) 91 394 26 13, e-mail: anovales@sis.ucm.es.

1 Introduction

We characterize conditions on preferences under which competitive equilibrium can be indeterminate in a single sector growth model with public consumption in the utility function, constant returns to scale in production and endogenous public expenditures financed through a constant rate income tax.

General equilibrium models that display indeterminacy have been focus of attention in recent years. Indeterminacy implies that there will be multiple paths converging to a given steady state. Hence, indeterminacy guarantees existence of a continuum of sunspot stationary equilibria, *i.e.*, stochastic rational expectations equilibria determined by perturbations unrelated to the uncertainty in economic fundamentals. The interest of sunspot equilibria is that they provide a theoretical justification for 'animal spirits' underlying economic instability.

Characteristics that produce indeterminacy of equilibria in one- or multi-sector real business cycle models or in endogenous growth models have been widely studied. Initially, it was shown that when these models are extended to include either productivity externalities or some market imperfection, indeterminacy can arise if social returns to scale in production are sufficiently high so that the labor demand curve has a slope which is not only positive, but also greater than that of the labor supply curve (see Benhabib and Farmer (1994), and Farmer and Guo (1994), for one-sector models). These models have been widely criticized because to produce multiple equilibria, they require larger returns to scale than observed in actual data [see Aiyagari (1995)].

More recent work has described two situations in which increasing returns needed to produce indeterminacy are lower than initially thought, sometimes allowing for negatively sloped labor demand curves: i) two sector economies with externalities in one sector (Benhabib and Farmer (1996), Perli (1998), Benhabib and Nishimura (1998), Weder (1998), who introduces a durable consumption sector in addition to the investment and non-durable consumption sectors, or Barinci and Chéron (2001) in an environment with financial constraints); ii) one sector models with non-standard characteristics, such as non-separable preferences (Bennett and Farmer (2000)), or capacity utilization affecting production (Wen (1998)). These models are able to produce indeterminacy of equilibria with relatively low increasing returns to scale, but the elasticity of intertemporal substitution needed is often too large, relative to values considered in the real business cycle literature (Weder (1998) and Bennett and Farmer (2000), among others). This will generally lead to rather volatile consumption paths, against the observation in most actual economies. The same criticism applies to existing endogenous growth models producing indeterminacy (see Benhabib and Perli (1994), and Xie (1994)).

We show that if preferences of the representative agent are non-separable in public consumption and leisure, there are conditions under which competitive equilibrium is indeterminate for standard values of the elasticity of intertemporal substitution and constant returns to scale in production. If, in addition, public and private consumption enter as a composite good in the utility function, indeterminacy is compatible with labor supply (defined as in Benhabib and Farmer (1994), ie., holding constant consumption) and demand curves having the standard slopes. As for the most part of the literature on indeterminacy, our model is subject to an externality, public consumption entering into the utility function. This extension can be justified from the point of view that to design fiscal policy, it is important to bear in mind the effect that public consumption may have on private consumption and work-leisure decisions. In this respect, there is some empirical work that provides evidence on complementary or sustitutability between private and public consumption (Ni (1995), among others).

Schmitt-Grohé and Uribe (1997) also characterize conditions for indeterminacy of equilibria under constant returns to scale without including public consumption in the utility function. These authors show that under a constant level of public expenditures and endogenous proportional income taxes that balance the budget every period, indeterminacy arises for a range of values of the tax rate and the elasticity of intertemporal substitution consistent with those in the real business cycle literature. However, indeterminacy in their economy would be precluded under fixed income tax rates and endogenous government expenditures.

Cazzavillan (1996) studies indeterminacy in a simple one sector model with public expenditures entering into the utility and production functions. The presence of both externalities is crucial to get indeterminacy in his economy, which displays endogenous growth.

Hence, the contribution of our work is to show that there can be indeterminacy in a simple neoclassical growth model with only a source of externality: public consumption entering in the utility function non-separably with leisure. Relative to fiscal policy, we assume that i) public consumption is financed through distorting taxes (a proportional tax on labor and capital income), and ii) the government budget balances every period. Our economy can display indeterminacy for fixed income tax rates, plausible values of the intertemporal elasticity of substitution of consumption, constant returns to scale and endogenous government expenditures. Furthermore, a) the region of the parameter space producing indeterminacy is independent of the income tax rate, and b) the elasticity of the constant-consumption labor supply function with respect to the level of public expenditures can have either sign. Finally, our indeterminacy condition can be interpreted in a similar manner to Benett and Farmer (2000): the economy displays indeterminacy if the Frisch labor supply curve (holding constant marginal utility of consumption) crosses the labor demand curve with the "wrong slope".

The model is described in section 2. In section 3 we characterize the transitional dynamics of the model and the conditions for indeterminacy. In section 4 we provide an economic interpretation of the condition that generates indeterminacy. We provide a numerical example in section 5, and the paper closes with some remarks in section 6.

2 The Economy

2.1 The Consumer´s Problem

There is a continuum of identical consumers, all with the same preferences defined on private and public consumption and leisure, according to the utility function:

$$U(c,g,n) = \frac{(c+\phi g)^{1-\rho}}{1-\rho} - g^{1-\rho} \ \ n^{1+\chi} \ , \rho > 0, \ \rho \neq 1 \ , \tag{1}$$

where c is private consumption, g public consumption, and n is the supply of labor[1]. To simplify notation, time dependence of all variables is suppressed. The parameter $\phi \geq 0$ determines the degree of substitution between private and public consumption (as in Ni (1995)), $\chi \geq 0$ defines the elasticity of labor supply (as in Benhabib and Farmer (1994)), and ρ is the inverse of the elasticity of intertemporal substitution of consumption, defined in a broad sense ($C = c + \phi g$). The possibility that $\rho \lesseqgtr 1$ allows for public consumption to either be complement or substitute for leisure. The single period utility function in (1) is homogeneous of degree 1-ρ in c and g. In our model, this condition allows for the existence of a balanced growth path although, for simplicity, we do not consider growth in the economy.[2]

Each consumer receives income on labor and capital, that can be used to consume, save, and pay taxes at a constant rate $\tau \in (0,1)$ on the two sources of income:

$$c + \dot{k} + \delta k = (1-\tau)\,(\omega n + rk) \quad , \tag{2}$$

where ω is the real wage, and r is the return on capital (k). Hence, the representative consumer faces the optimization problem:

$$Max \quad \int_0^\infty e^{-\beta t} U(c,g,n) \ dt \tag{3}$$

subject to (2) and given k_0 , with $\beta > 0$ being the time discount.

First order conditions for this problem are:

$$(c+\phi g)^{-\rho} = \lambda \quad , \tag{4}$$

$$(1+\chi)\, g^{1-\rho} n^{\chi} = \lambda (1-\tau)\omega \quad , \tag{5}$$

$$-\frac{\dot{\lambda}}{\lambda} = (1-\tau) r - (\delta + \beta) \quad , \tag{6}$$

[1] Farmer (1997) considers a similar class of utility functions which includes real balances instead of public consumption.

[2] Farmer (1997) showed that, in the case of a monetary model with real balances, M/P, in the utility function, the condition that allows for the possibility of balanced growth is that $U(c, M/P, n)$ be homogenous in c and M/P.

$$\lim_{t \to \infty} e^{-\beta t} \lambda = 0 \quad , \tag{7}$$

λ being the multiplier associated to the consumer´s budget constraint.

Combining (4) and (5):

$$\omega = \frac{1+\chi}{1-\tau} \left(c + \phi g\right)^{\rho} g^{1-\rho} n^{\chi}, \tag{8}$$

so that the marginal rate of substitution between private consumption and leisure must be equal to the after tax real wage. Combining (4) and (6) and using $C = c + \phi g$:

$$\frac{\dot{C}}{C} = \frac{1}{\rho} \left[(1-\tau)r - (\delta + \beta)\right] \quad , \tag{9}$$

so that consumption growth is every period proportional to the return on capital, net of taxes and depreciation.

2.2 Firms

There is a continuum of identical firms operating in a competitive environment. For simplicity, we normalize their number to one. Aggregate production exhibits constant returns to scale:

$$y = Ak^{\alpha} n^{1-\alpha}, \quad \alpha \in (0,1) \quad , \tag{10}$$

y being aggregate output and k, n the demand for the two production factors.

Under perfectly competitive markets for the two factors, profit maximizing conditions are:

$$r = \alpha \frac{y}{k} \quad , \tag{11}$$

$$\omega = (1-\alpha) \frac{y}{n} \quad . \tag{12}$$

2.3 Government

The government chooses a tax rate τ and balances its budget every period. Hence, the instantaneous government budget constraint is:

$$g = \tau \left(\omega n + rk\right) \quad , \tag{13}$$

where g represents government spending on goods and services that contribute to household´s utility.

2.4 Equilibrium

Given τ and k_0 , a *competitive equilibrium* is a set of trajectories $\{c, n, k, g, \omega, r\}$ such that: i) given the paths for $\{\omega, r\}$, $\{c, n, k\}$ solve consumer´s problem [(2), (7), (8), (9), with $C = c + \phi g$]:, ii) given $\{\omega, r\}$, $\{n, k\}$ solve the firm´s problem [(11) and (12)], iii) government budget constraint (13) holds every period and iv) markets clear, in particular, from (2), (10), (13):

$$Ak^{\alpha}n^{1-\alpha} = c + \left(\dot{k} + \delta k\right) + g \ , \tag{14}$$

where

$$g = \tau A k^{\alpha} n^{1-\alpha} \ . \tag{15}$$

3 Local Dynamics

Let us now discuss the local properties of the equilibrium dynamics of the system. A *steady state* is a vector $(c_{ss}, k_{ss}, n_{ss}, \lambda_{ss}, g_{ss})$ satisfying the equations for competitive equilibrium such that if it is ever reached, the system will stay at that point forever ($\dot{C}_{ss} = \dot{c}_{ss} = \dot{k}_{ss} = \dot{\lambda}_{ss} = 0$):

$$\frac{k_{ss}}{n_{ss}} = \left[\frac{(1-\tau)\, A\alpha}{\delta + \beta}\right]^{\frac{1}{1-\alpha}} , \tag{16}$$

$$\frac{c_{ss}}{k_{ss}} = \frac{\delta\,(1-\alpha) + \beta}{\alpha}, \tag{17}$$

$$\frac{c_{ss}}{g_{ss}} = \frac{\delta\,(1-\alpha) + \beta}{\delta + \beta}\frac{1-\tau}{\tau}, \tag{18}$$

$$n_{ss} = \left[\frac{1-\tau}{\tau}\frac{1-\alpha}{1+\chi}\left(\phi + \frac{\delta\,(1-\alpha)+\beta}{\delta+\beta}\frac{1-\tau}{\tau}\right)^{-\rho}\right]^{\frac{1}{1+\chi}} , \tag{19}$$

$$\lambda_{ss} = \left(c_{ss} + \phi g_{ss}\right)^{-\rho} , \tag{20}$$

where (16) comes from (6), (10) and (11); (17) refers to (14)-(16); (18) is obtained from (15)-(17); (19) comes from (8) together with (10), (12), (15), (16) and (18); and (20) is obtained from (4). All these equations are evaluated at steady-state.

To characterize the local dynamics of the system it is enough to analyze the transition of the state and co-state variables (k, λ). That requires to write the control variables (c, n) as a function of (k, λ).

$$n = \left[\frac{(1-\alpha)(1-\tau) A^{\rho}}{\tau^{(1-\rho)}(1+\chi)} \lambda k^{\alpha\rho} \right]^{\frac{1}{1+\chi-\rho(1-\alpha)}} , \tag{21}$$

$$c = \lambda^{-1/\rho} - \phi \left[\left(\frac{(1-\alpha)(1-\tau)}{1+\chi} \right) \tau^{\frac{\chi+\alpha}{1-\alpha}} A^{\frac{1+\chi}{1-\alpha}} k^{\frac{\alpha(1+\chi)}{(1-\alpha)}} \lambda \right]^{\frac{1-\alpha}{1+\chi-\rho(1-\alpha)}} , \tag{22}$$

where (21) comes from (5), (10) and (12); and (22) is obtained from (4) together with (15) and (21).

The first order approximation to the dynamic system made up by equations (6) and (14) is:

$$\begin{pmatrix} \dot{k} \\ \dot{\lambda} \end{pmatrix} = \Gamma \begin{pmatrix} k - k_{ss} \\ \lambda - \lambda_{ss} \end{pmatrix} ,$$

with transition matrix Γ (see Appendix):

$$\Gamma = \begin{pmatrix} \frac{(\delta+\beta)(1+\chi)}{1+\chi-\rho(1-\alpha)} \left(1 + \phi \frac{\tau}{1-\tau}\right) - \delta & \left(\frac{k_{ss}}{\lambda_{ss}}\right) \left[\frac{(\delta+\beta)(1+\chi)\left(1+\phi\frac{\tau}{1-\tau}\right)}{\alpha\rho(1+\chi-\rho(1-\alpha))} - \frac{\delta}{\rho} \right] \\ \left(\frac{\lambda_{ss}}{k_{ss}}\right)(1-\alpha)(\delta+\beta)\frac{1+\chi-\rho}{1+\chi-\rho(1-\alpha)} & -\frac{(\delta+\beta)(1-\alpha)}{1+\chi-\rho(1-\alpha)} \end{pmatrix} . \tag{23}$$

Since one of the variables in the system is predetermined and the other is free, the system will have a unique, locally determined equilibrium when the steady state is a saddle point, which requires the two characteristic roots of matrix Γ to be of opposite sign. Indeterminacy of equilibria arises only when the two roots of Γ are negative. This means that trace of Γ be negative, and its determinant be positive.

For any given values of the vector $(\alpha, \delta, \tau, \phi, \beta)$, let $\varphi_1(\rho)$, $\varphi_2(\rho)$ denote the linear functions:

$$\begin{aligned} \varphi_1(\rho) &= -1 + (1-\alpha)\rho , \\ \varphi_2(\rho) &= \frac{\delta - (\delta+\beta)\left[\alpha + \phi\frac{\tau}{1-\tau}\right]}{\beta + (\delta+\beta)\phi\frac{\tau}{1-\tau}} - \frac{\delta(1-\alpha)}{\beta + (\delta+\beta)\phi\frac{\tau}{1-\tau}} \rho . \end{aligned}$$

Proposition 1 *Det*$(\Gamma) > 0$ *if and only if* $\chi < \varphi_1(\rho)$.

Proof. See Appendix. ■

Proposition 2 *Tr*$(\Gamma)<0$ *if and only if* $\frac{\chi-\varphi_2(\rho)}{\chi-\varphi_1(\rho)}<0$.

Proof. See Appendix. ■

Let ρ^*, ρ^{**} be the zeroes of $\varphi_1(.)$ and $\varphi_2(.)$. We have:
$\rho^*=\frac{1}{1-\alpha}$, $\rho^{**}=\frac{1}{1-\alpha}\left[1-\frac{\delta+\beta}{\delta}\left(\alpha+\phi\frac{\tau}{1-\tau}\right)\right]$ and $\rho^*>\rho^{**}$ for all $\alpha,\delta,\tau,\phi,\beta$.

Proposition 3 *Let* (χ_0,ρ_0) *be a point in the* (χ,ρ)*-plane with* $0\leq\chi_0<\varphi_1(\rho_0)$. *Then* $\frac{\chi_0-\varphi_2(\rho_0)}{\chi_0-\varphi_1(\rho_0)}<0$.

Proof. Since ρ^* is the single root of $\varphi_1(\rho)=0$, which is an increasing linear function of ρ, $0\leq\chi_0<\varphi_1(\rho_0)\Rightarrow\rho_0>\rho^*>\rho^{**}$ and hence, $\varphi_2(\rho_0)<0$, since ρ^{**} is the single root of $\varphi_2(\rho)=0$, $\varphi_2(.)$ being a decreasing linear function of ρ. This implies that $\chi_0-\varphi_2(\rho_0)>0$, and, hence $\frac{\chi_0-\varphi_2(\rho_0)}{\chi_0-\varphi_1(\rho_0)}<0$. ■

Proposition 4 *The competitive equilibrium is indeterminate in economy (1)-(15), if and only if* $0\leq\chi_0<\varphi_1(\rho_0)$.

Proof. Let us assume the competitive equilibrium is indeterminate. Then, $Det(\Gamma)>0$ and $Tr(\Gamma)<0$. By proposition 1, $\chi_0<\varphi_1(\rho_0)$, while χ_0 is constrained to be non-negative in (1). Conversely, let us assume that $0\leq\chi_0<\varphi_1(\rho_0)$. Propositions 1, 2 and 3 then imply that $Det(\Gamma)>0$ and $Tr(\Gamma)<0$ and hence, the competitive equilibrium is indeterminate. ■

Corollary 5 *A necessary condition for indeterminacy of the competitive equilibrium is* $\rho>\frac{1}{1-\alpha}$.

Proof. This condition is needed for $\varphi_1(\rho)>0$, and only then condition in Proposition 4 can hold. ■

The previous results show that, in this economy:

i) For values of α as usually calibrated in the real business cycle literature (around 0.35), indeterminacy can arise for values of the relative risk aversion parameter $\rho>1.54$, (*i.e.*, intertemporal elasticity of substitution smaller than 0.65), a range of values consistent with existing empirical estimates.

ii) fiscal policy parameters do not affect the conditions guaranteeing indeterminacy of equilibria, at a difference of Schmitt-Grohé and Uribe (1997). In their model, indeterminacy is possible in a range of intermediate steady-state tax rates. In particular, indeterminacy is not possible for tax rates in the decreasing part of the Laffer curve and, as the authors point out, it is crucial for their results that labor income taxes be endogenous. In the economy we consider, indeterminacy can arise for any value of the income tax rate and hence, for any size of the public sector, defined as the ratio of government expenditures to output. The goverment cannot do anything to stabilize the economy.

Guo and Lansing (1998) show indeterminacy in a one-sector real business cycle under increasing returns to scale with a tax policy similar to ours, but without public consumption in the utility function. They consider alternative parameterizations allowing for regressive, proportional and progressive taxes, showing that progressive taxes may act as an economic stabilizer, precluding indeterminacy of equilibria. These same fiscal rules could be used in our framework to discuss whether they could be used to stabilize the economy, in the same sense of these authors.

Non-separability of public expenditures and leisure in the utility function is crucial for the previous results, as the next proposition shows.

Let $V(c,g,n) = \frac{(c+\phi g)^{1-\rho}}{1-\rho} - n^{1+\chi}$, $\rho > 0$, $\rho \neq 1$, $\chi \geq 0, \phi \geq 0$, be an utility function similar to (1) except for the fact that it is separable in public consumption and leisure.

Proposition 6 *If preferences of the representative agent in the economy are represented by utility function $V(c,g,n)$, there is no point in the parameter space for which equilibrium is indeterminate.*

Proof. See Appendix. ■

4 Indeterminacy and the Labor Market

Most work discussing indeterminacy of equilibria in one sector models shows that it is associated with upward sloping demand curves, downward sloping supply curves, or both. The main result of Benett and Farmer (2000) is that the Benhabib-Farmer condition [Benhabib and Farmer (1994)], that labor demand and supply curves cross with the "wrong slopes" generalizes to the case when preferences are non-separables in private consumption and leisure. However, Benett-Farmer show that it is the Frisch labor supply curve (i.e., holding constant the marginal utility of consumption[3]) rather than the constant-consumption labor supply curve, which should be considered in this analysis. In this section, we relate our conditions for indeterminacy to the slopes of labor demand and both labor supply curves.

In our model, under the utility function $U(c,g,n)$, the Frisch labor supply curve and the constant-consumption labor supply curve differ. The Frisch labor supply curve (from (5)) is:

$$\omega = \frac{1+\chi}{(1-\tau)\lambda} g^{1-\rho} n^{\chi}, \tag{24}$$

and using again (15) to eliminate g from (24), we get the *reduced-form Frisch labor supply curve:*

[3] The use of this term follows Bennett and Farmer (2000) and Browning et al. (1985).

$$\omega = \frac{1+\chi}{(1-\tau)\lambda} \left(\tau A k^{\alpha}\right)^{1-\rho} n^{\chi+(1-\rho)(1-\alpha)} ,$$

having a slope, in the neighborhood of the steady state:

$$\left(\frac{d\omega}{dn}\right)_{F}^{s} = \frac{\omega_{ss}}{n_{ss}} \left[\chi + (1-\rho)(1-\alpha)\right] . \tag{25}$$

which will be negative when $\rho > \frac{1}{1-\alpha}$ (which implies $\rho > 1$) and $(\rho - 1)(1-\alpha) > \chi$, being positive otherwise.

From (12), the slope of the labor demand curve in the neighborhood of steady state is:

$$\left(\frac{d\omega}{dn}\right)^{d} = -\alpha \frac{\omega_{ss}}{n_{ss}},$$

which is always negative.

Proposition 7 *The competitive equilibrium is indeterminate in our economy if and only if the reduced-form Frisch labor supply curve is downward-sloping, and steeper than the labor demand curve.*

Proof. From Proposition 4 and its corollary, if equilibrium is indeterminate we have $\rho > \frac{1}{1-\alpha}$ and $\chi < -1 + \rho(1-\alpha)$. Since $(\rho - 1)(1-\alpha) > -1 + \rho(1-\alpha)$ for all α, ρ, the labor supply curve slopes downward. It will also be steeper than the labor demand curve, since the former condition also implies: $(\rho - 1)(1-\alpha) - \chi > \alpha$.

Conversely, if the latter condition holds, then $\chi < -1 + \rho(1-\alpha)$ and equilibrium is indeterminate, by Proposition 4. ■

Hence, the appropiate condition for indeterminacy of the competitive equilibrium to arise in the economy we consider is that the Frisch labor supply curve crosses the labor demand curve with the "wrong slope", the former being steeper (i.e., the Benhabib-Farmer condition holds).

The intuition behind the existence of stationary sunspot equilibria is as follows. If agents expect future public consumption to be above average, then future hours worked will increase for any given capital stock (from (24)), and the expected rate of return on capital will be higher, since the marginal product of capital is decreasing in the capital/labor ratio. The increase in the expected interest rate produces an increase in current labor supply and output. Since the income tax rate is fixed, current public consumption will also be higher. Thus, expectations of above-average public consumption next period lead to higher current public consumption.

Under the $V(c, g, n)$ utility function, it is easy to see that the Frisch labor supply curve does not depend on public consumption, and the previous argument breaks down. This is consistent with proposition 6, which shows that, under $V(.)$, our model does not display indeterminacy. Therefore, the non-separability of public consumption and leisure in the utility function plays a key role in generating expectations-driven fluctuations in this economy.

Under the $U(c,g,n)$ utility function, the *constant-consumption, reduced form labor supply curve* can be obtained using (15) to eliminate g from (8):

$$\omega = \frac{1+\chi}{1-\tau}\left(\phi + \frac{c}{\tau A k^{\alpha} n^{1-\alpha}}\right)^{\rho} \tau A k^{\alpha} n^{1-\alpha+\chi}, \tag{26}$$

having a slope, in the neighborhood of the steady state:

$$\left(\frac{d\omega}{dn}\right)^{s} = \frac{\omega_{ss}}{n_{ss}}\left[(1-\alpha+\chi) + \frac{\rho\,(\alpha-1)}{1+\phi\frac{\tau A (k_{ss}/n_{ss})^{(\alpha-1)}}{c_{ss}/k_{ss}}}\right]. \tag{27}$$

Proposition 8 *The constant-consumption, reduced form labor supply curve has a negative slope if and only if* $\chi < \varphi_3(\rho)$, *where* $\varphi_3(\rho) = (1-\alpha)\left[\frac{\rho}{1+\phi\frac{\tau}{1-\tau}\frac{\delta+\beta}{\delta(1-\alpha)+\beta}} - 1\right]$.

Proof. Easy to show from (16), (17) and (27). ■

Corollary 9 *A necessary condition for the constant-consumption, reduced form labor supply curve to have a negative slope is that* $\rho > \varphi_4 = 1 + \phi\frac{\tau}{1-\tau}\frac{\delta+\beta}{\delta(1-\alpha)+\beta} > 1$.

Proof. Only if this inequality holds we will have a non-trivial range of values for χ in Proposition 4. ■

The next two propositions show that, if public consumption enters together with private consumption as a composite good in the utility function then, contrary to what happens to the Frisch labor supply curve, the reduced-form labor supply curve obtained holding constant consumption can be upward-sloping when equilibrium is indeterminate. If public consumption enters the utility function just through the interaction with leisure, then the same necessary and sufficient conditions for indeterminacy apply to the labor supply curve obtained holding constant consumption than to the Frisch labor supply curve. This is because, as pointed out in Bennet and Farmer (2000), when preferences are separable in consumption and leisure, both supply curves coincide. This latter case is an illustration of the fact that non-separability between private consumption and leisure is not needed for indeterminacy to arise with a downward sloping labor supply curve.

Proposition 10 *If* $\phi > 0$, *there is a non-trivial set in the parameter space for which we have indeterminacy of equilibria with an upward sloping reduced form, constant-consumption, labor supply curve.*

Proof. The previously defined $\varphi_1(\rho)$ function satisfies: $\frac{\partial\varphi_1(\rho)}{\partial\rho} = 1-\alpha > \frac{1-\alpha}{1+\phi\frac{\tau}{1-\tau}\frac{\delta+\beta}{\delta(1-\alpha)+\beta}} = \frac{\partial\varphi_3(\rho)}{\partial\rho}$, for all $\phi, \tau, \delta, \beta, \alpha$ and $\varphi_1(0) < \varphi_3(0)$, since $\varphi_1(0) = -1$ and $\varphi_3(0) = -(1-\alpha)$. Being $\varphi_1(.)$ and $\varphi_3(.)$ linear functions of ρ, they have a single intersection at a given $\tilde{\rho} > 0$, with $\rho > \tilde{\rho} \Rightarrow \varphi_3(\rho) < \varphi_1(\rho)$. Hence, indeterminacy pairs (ρ, χ) with $\rho > \tilde{\rho}$ and $\varphi_3(\rho) < \chi < \varphi_1(\rho)$ produce an upward sloping reduced form, constant-consumption, labour supply curve, while for those with $\chi < \varphi_3(\rho)$ that labour supply curve is downward sloping. If there is any indeterminacy pair (ρ, χ) with $\rho < \tilde{\rho}$, the labor supply curve will be downward sloping. ■

Proposition 11 *If ϕ=0, there is no equilibrium indeterminacy with an upward sloping labor supply curve obtained holding constant consumption. The competitive equilibrium is then indeterminate if and only if the constant-consumption labor supply curve is downward sloping, but steeper than the labor demand curve.*

Proof. In this case, the Frisch labor supply curve and the constant-consumption labor supply curve are the same [see (25) and (27) under ϕ=0], and Proposition 7 shows the result. ■

Our final result shows that requiring non-separability between public and private consumption in the utility function ($\phi > 0$) to obtain indeterminacy of equilibria under a positively sloped constant-consumption labor supply curve, is not restrictive from the point of view of the sign of the elasticity of the labor supply function relative to public consumption. That is, in an indeterminate competitive equilibrium with upward sloped labor supply curve an increase in public consumption could shift the labor supply curve either to the left or the right.

Proposition 12 *If the competitive equilibrium is indeterminate and $\frac{1}{1-\alpha}$>φ_4 then the elasticity of the constant consumption labor supply curve relative to public consumption is always positive. When the competitive equilibrium is indeterminate and $\frac{1}{1-\alpha}$<φ_4 , the elasticity of the constant consumption labor supply curve relative to public consumption is positive if ρ>φ_4, being negative if $\frac{1}{1-\alpha} < \rho$<$\varphi_4$.*

Proof. Diferentiating in (8) with $d\omega = dc = 0$, we get, in the neighborhood of the steady state:

$$\begin{aligned}\varepsilon_{n,g} &= \frac{dn}{dg}\frac{g_{ss}}{n_{ss}} = -\frac{\rho\phi}{\chi}\frac{1}{c_{ss}/g_{ss}+\phi} + \frac{\rho-1}{\chi} \\ &= -\frac{\rho\phi}{\chi}\frac{1}{\frac{1-\tau}{\tau}\frac{\delta(1-\alpha)+\beta}{\delta+\beta}+\phi} + \frac{\rho-1}{\chi},\end{aligned}$$

where we have used (18) to reach the last equality. Hence, $\varepsilon_{n,g} \gtreqless 0$ if and only if $\rho \gtreqless \varphi_4$. If $\frac{1}{1-\alpha} > \varphi_4$, under indeterminacy of equilibria we will have, from corollary 5: $\rho > \frac{1}{1-\alpha} > \varphi_4$, and, consequently, $\varepsilon_{n,g} > 0$. If $\frac{1}{1-\alpha} < \varphi_4 < \rho$, we will again have $\varepsilon_{n,g} > 0$, while if $\frac{1}{1-\alpha} < \rho < \varphi_4$, we will have $\varepsilon_{n,g} < 0$. ■

5 A Numerical Example

In this section, we provide a numerical illustration of our analytical results identifying, for a given parameterization, the (χ, ρ)-regions where the competitive equilibrium is either determinate or indeterminate. Parameter values are: $\beta = 0.01$, $\alpha = 0.35$, $\phi = 0.9$ (the latter in the range estimated by Ni(1995)), $\tau = 0.2$, $A = 1$, $\delta = 0.0252$. Assuming quarterly data, these values are compatible with annual depreciation of 10% and a steady-state real interest rate of 4%. In Figure 1.A we also identify (χ, ρ)-pairs for which the constant-consumption labor supply curve is positively sloped. Figures 1.B and 1.C reproduce Figure 1.A for alternative parameter values. In Figure 1.B, the only difference is $\phi = 0.6$. In Figure 1.C we use the same values as in Figure 1.A except for the tax rate, which is taken to be $\tau = 0.4$. Note that the indeterminacy condition is the same in the three graphs in Figure 1 since it is independent of ϕ and τ. Comparing the three figures we see that the bigger the substitution between private and public consumption (ϕ) and the bigger the tax rate, the larger the set of (ρ, χ)-combinations compatible with indeterminacy of equilibria and an upward sloping labor supply curve.

Finally, Figure 2 shows, among the (ρ, χ)-combinations for which there is indeterminacy of equilibria, those for which the two eigenvalues are complex, implying an endogenous mechanism for the propagation of business cycles. Large values of either χ or ρ ($\chi > 0.31$, $\rho > 3.92$) preclude this possibility. Parameter values are the same as in Figure 1.A.

6 Conclusions

For a class of utility functions characterized by non-separability between public consumption and leisure, we have shown that the competitive equilibrium can be indeterminate for plausible values of the intertemporal elasticity of substitution, under endogenous government expenditures (financed through a fixed income tax rate) and constant returns to scale in production. The region in the parameter space leading to indeterminacy is characterized by:

i) it is compatible with the constant-consumption labor supply curve and the labor demand curve having the standard slope signs, provided that private and public consumption are substitutes. However, the Frisch labor supply curve is downward sloping whenever there is indeterminacy,

ii) the intertemporal elasticity of substitution must be below the labor income share on output, which is usually calibrated in the literature to be around 0.35. Equivalently, the relative risk aversion parameter must be above the inverse of that share. This condition only excludes values for the risk aversion parameter below 1.54, indeterminacy being a possibility for values above that threshold,

iii) the tax rate and, hence, the size of the public sector, measured by the participation of public consumption on output, does not enter the condition characterizing indeterminacy. This means that the government cannot do anything to stabilize this economy, contrary to the result reached by Schmitt-Grohé and Martin Uribe in a different environment under constant government expenditures. In our case, a bigger public sector only increases the likelihood that indeterminacy will arise with the standard slopes for the constant-consumption labor supply and demand curves.

Appendix

Part1: Deriving the elements of Γ :

From (14) and (15) we get:

$$\dot{k} = (1-\tau)\,Ak^{\alpha}n(k,\lambda)^{1-\alpha} - \delta k - c(k,\lambda). \tag{28}$$

where $c\,(k,\lambda)$ is given by (22) and $n\,(k,\lambda)$ is given by (21), while from (6), (10) and (11) we get:

$$\dot{\lambda} = -\lambda\left[(1-\tau)\,\alpha Ak^{\alpha-1}n\,(k,\lambda)^{1-\alpha} - (\delta+\beta)\right]. \tag{29}$$

The dynamics of system (28)-(29) in (k,λ) around steady state can be characterized through the linear approximation:

$$\begin{pmatrix} \dot{k} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix}\begin{pmatrix} k-k_{ss} \\ \lambda-\lambda_{ss} \end{pmatrix}, \tag{30}$$

with:

$$\Gamma_{11} = (1-\tau)\,A\alpha\left(\frac{k_{ss}}{n_{ss}}\right)^{\alpha-1} - \delta + (1-\tau)(1-\alpha)A\left(\frac{k_{ss}}{n_{ss}}\right)^{\alpha}\frac{\partial n}{\partial k}|_{ss} - \frac{\partial c}{\partial k}|_{ss} \tag{31}$$

$$\Gamma_{12} = (1-\tau)(1-\alpha)A\left(\frac{k_{ss}}{n_{ss}}\right)^{\alpha}\frac{\partial n}{\partial\lambda}|_{ss} - \frac{\partial c}{\partial\lambda}|_{ss}\quad, \tag{32}$$

$$\Gamma_{21} = -\lambda_{ss}(1-\tau)\alpha A\left[(\alpha-1)\,k_{ss}^{\alpha-2}n_{ss}^{1-\alpha} + (1-\alpha)\,k_{ss}^{\alpha-1}n_{ss}^{-\alpha}\frac{\partial n}{\partial k}|_{ss}\right]\quad, \tag{33}$$

$$\Gamma_{22} = -\lambda_{ss}(1-\tau)\alpha A\,(1-\alpha)\,k_{ss}^{\alpha-1}n_{ss}^{-\alpha}\frac{\partial n}{\partial\lambda}|_{ss}\quad, \tag{34}$$

with $\frac{\partial x}{\partial z}|_{ss}$, $x = n, c$ and $z = k, \lambda$, denoting partial derivatives, evaluated at steady state.

From (21) and (22), and using (15), (16) and (20), the partial derivatives in (31)-(34) are:

$$
\begin{aligned}
\frac{\partial n}{\partial \lambda}|_{ss} &= \frac{1}{D}\frac{n_{ss}}{\lambda_{ss}} \ , \\
\frac{\partial c}{\partial \lambda}|_{ss} &= \frac{1}{\lambda_{ss} D}\left[c_{ss}\left(1-\alpha-\frac{1+\chi}{\rho}\right)-\frac{1+\chi}{\rho}\phi g_{ss}\right] \ , \\
\frac{\partial n}{\partial k}|_{ss} &= \frac{\alpha\rho}{D}\frac{n_{ss}}{k_{ss}} \ , \\
\frac{\partial c}{\partial k}|_{ss} &= -\phi\frac{\tau}{1-\tau}\frac{(\delta+\beta)(1+\chi)}{D} \ ,
\end{aligned}
$$

where: $D = 1+\chi-\rho(1-\alpha) = \chi - \varphi_1(\rho)$.

Using (16) to eliminate $\left(\frac{k_{ss}}{n_{ss}}\right)^{\alpha-1} = \frac{\delta+\beta}{(1-\tau)A\alpha}$, together with these expressions, yields:

$$
\begin{aligned}
\Gamma_{11} &= (1-\tau)A\alpha\left(\frac{k_{ss}}{n_{ss}}\right)^{\alpha-1}\left[1+(1-\alpha)\frac{\rho}{D}\right]-\delta+\phi\frac{\tau}{1-\tau}\frac{(\delta+\beta)(1+\chi)}{D} = \\
&= \frac{(\delta+\beta)(1+\chi)}{D}-\delta+\phi\frac{\tau}{1-\tau}\frac{(\delta+\beta)(1+\chi)}{D} = \frac{(\delta+\beta)(1+\chi)}{D}\left(1+\phi\frac{\tau}{1-\tau}\right)-\delta
\end{aligned}
$$

and using (17)-(18) to eliminate $\frac{c_{ss}}{k_{ss}}, \frac{g_{ss}}{k_{ss}}$, we have:

$$
\Gamma_{12} = (1-\tau)(1-\alpha)A\left(\frac{k_{ss}}{n_{ss}}\right)^{\alpha}\frac{1}{D}\frac{n_{ss}}{\lambda_{ss}} - \frac{1}{D\lambda_{ss}}\left[c_{ss}\left(1-\alpha-\frac{1+\chi}{\rho}\right)-\phi\frac{1+\chi}{\rho}g_{ss}\right] =
$$

$$
= \frac{k_{ss}}{\lambda_{ss}}\frac{1}{D}\left[\frac{1-\alpha}{\alpha}(\delta+\beta)+\frac{1}{\rho}D\frac{c_{ss}}{k_{ss}}+\frac{1+\chi}{\rho}\phi\frac{g_{ss}}{k_{ss}}\right] =
$$

$$
= \frac{k_{ss}}{\lambda_{ss}}\frac{1}{D}\left[\frac{1-\alpha}{\alpha}(\delta+\beta)+\frac{1}{\rho}D\frac{\delta+\beta-\alpha\delta}{\alpha}+\frac{1+\chi}{\rho}\phi\frac{\delta+\beta}{\alpha}\frac{\tau}{1-\tau}\right] \ ,
$$

and, combining terms in $\delta+\beta$:

$$
\Gamma_{12} = \frac{k_{ss}}{\lambda_{ss}}\frac{1}{D}\frac{(\delta+\beta)(1+\chi)\left(1+\frac{\phi\tau}{1-\tau}\right)-\alpha\delta D}{\alpha\rho} \ ,
$$

$$
\Gamma_{21} = \frac{\lambda_{ss}}{k_{ss}}(1-\tau)\alpha A(1-\alpha)(k_{ss}/n_{ss})^{\alpha-1}\frac{1+\chi-\rho}{D} = \frac{\lambda_{ss}}{k_{ss}}\frac{(\delta+\beta)(1-\alpha)}{D}(1+\chi-\rho) \ ,
$$

$$
\Gamma_{22} = -\frac{(\delta+\beta)(1-\alpha)}{D}. \tag{35}
$$

Part 2: Proof of Proposition 1:
Denote $F = (\delta+\beta)\,\phi\frac{\tau}{1-\tau} > 0$ and $N_1 = (\delta+\beta)(1-\alpha) > 0$,

$$\begin{aligned}
Det(\Gamma) &= \Gamma_{11}\Gamma_{22} - \Gamma_{12}\Gamma_{21} \\
&= -\left(\frac{(\delta+\beta+F)(1+\chi)}{D} - \delta\right)\frac{N_1}{D} - \left(\frac{(\delta+\beta+F)(1+\chi)}{\alpha\rho D} - \frac{\delta}{\rho}\right)\frac{N_1}{D}(1+\chi-\rho) \\
&= -\frac{N_1}{D}\left(\frac{(\delta+\beta+F)(1+\chi)}{\alpha\rho} - \delta\frac{1+\chi}{\rho}\right) = -\frac{N_1(1+\chi)}{\alpha\rho D}\left[\delta(1-\alpha)+\beta+F\right],
\end{aligned}$$

so that $Det(\Gamma) > 0$ if and only if $D < 0$, i.e., if and only if $\chi < \varphi_1(\rho)$.

Part 3: Proof of Proposition 2:

First, notice that:

$$\chi - \varphi_2(\rho) = \chi - \frac{1}{F+\beta}\left[\delta - (\delta+\beta)\alpha - F - \delta(1-\alpha)\rho\right] .$$

Then,

$$\begin{aligned}
Tr(\Gamma) &= \Gamma_{11} + \Gamma_{22} = \frac{1}{D}\left[(\delta+\beta+F)(1+\chi) - \delta(1+\chi-\rho(1-\alpha)) - (\delta+\beta)(1-\alpha)\right] = \\
&= \frac{1}{D}\left[F + (\beta+F)\chi - \delta + \delta\rho(1-\alpha) + (\delta+\beta)\alpha\right] = \frac{1}{D}(\chi-\varphi_2(\rho))(F+\beta) = \\
&= (F+\beta)\frac{\chi-\varphi_2(\rho)}{\chi-\varphi_1(\rho)} ,
\end{aligned}$$

so that $Tr(\Gamma) < 0$ if and only if $\frac{\chi-\varphi_2(\rho)}{\chi-\varphi_1(\rho)} < 0$.

Part 4. Proof of Proposition 6:

Proof: For utility function $V(c, g, n)$, the control variables (c, n), as functions of (k, λ) are given by:

$$n = \left[\frac{(1-\alpha)(1-\tau)}{1+\chi}A\lambda k^{\alpha}\right]^{\frac{1}{\chi+\alpha}} ,$$

$$c = \lambda^{-1/\rho} - \phi\tau\left[\frac{(1-\alpha)(1-\tau)}{1+\chi}A^{\frac{1+\chi}{1-\alpha}}k^{\alpha\frac{1+\chi}{1-\alpha}}\lambda\right]^{\frac{1-\alpha}{\chi+\alpha}} ,$$

which are the analogue of (21), (22), with partial derivatives:

$$
\begin{aligned}
\frac{\partial n}{\partial k}|_{ss} &= \frac{\alpha}{\chi+\alpha}\frac{n_{ss}}{k_{ss}} \ , \\
\frac{\partial c}{\partial k}|_{ss} &= -(1+\chi)\frac{\alpha}{\chi+\alpha}\frac{\lambda_{ss}^{-1/\rho}-c_{ss}}{k_{ss}} \ , \\
\frac{\partial n}{\partial \lambda}|_{ss} &= \frac{1}{\chi+\alpha}\frac{n_{ss}}{\lambda_{ss}} \ ,
\end{aligned}
$$

and, again, $\left(\frac{k_{ss}}{n_{ss}}\right)^{\alpha-1} = \frac{\delta+\beta}{(1-\tau)A\alpha}$, so that (31) and (34) become:

$$
\begin{aligned}
\Gamma_{11} &= \beta + (1-\alpha)\frac{1}{\chi+\alpha}(\delta+\beta) + (1+\chi)\frac{\alpha}{\chi+\alpha}\frac{\lambda_{ss}^{-1/\rho}-c_{ss}}{k_{ss}} \ , \\
\Gamma_{22} &= -(1-\alpha)\frac{1}{\chi+\alpha}(\delta+\beta) \ ,
\end{aligned}
$$

and: $Tr(\Gamma) = \Gamma_{11} + \Gamma_{22} = \beta + (1+\chi)\frac{\alpha}{\chi+\alpha}\left(\lambda_{ss}^{-1/\rho} - c_{ss}\right)$. But, from (20), optimality implies: $\phi g_{ss} = \lambda_{ss}^{-1/\rho} - c_{ss} > 0$. Hence, $Tr(\Gamma) > 0$ for any feasible set of parameter values, without the possibility of having indeterminacy.

References

1. Aiyagari, S.R., The econometrics of indeterminacy: an applied study, a comment, *Carnegie-Rochester Conf. Ser. Public Policy* 43 (1995), 273-282.

2. Barinci, J.P. and A. Chéron, Sunspots and the business cycle in a finance constrained economy, *Journal of Economic Theory* **97** (2001), 30-49.

3. Benhabib, J. and R.E.A. Farmer, Indeterminacy and increasing returns, *Journal of Economic Theory* **63** (1994), 19-41.

4. Benhabib, J. and R.E.A. Farmer, Indeterminacy and sector specific externalities, *Journal of Monetary Economics* **37** (1996), 421-443.

5. Benhabib, J. and K. Nishimura, Indeterminacy and sunspots with constant returns, *Journal of Economic Theory* **81** (1998), 58-96.

6. Benhabib, J. and R. Perli, Uniqueness and indeterminacy: transitional dynamics with multiple equilibria", *Journal of Economic Theory* **63** (1994), 113-142.

7. Benett, R.L. and R.E.A. Farmer, Indeterminacy with non-separable utility, *Journal of Economic Theory* **93** (2000), 118-143.

8. Browning, M.J., A. Deaton, and M. Irish, A profitable approach to labor supply and commodity demands over the life-cycle, *Econometrica* **53** (1985), 503-543.

9. Cazzavillan, G., Public spending, endogenous growth, and endogenous fluctuations, *Journal of Economic Theory* **71** (1996)**,** 394-415

10. Farmer, R.E.A., Money in a real business cycle model, *Journal of Money Credit and Banking* **29** (1997), 568-611.

11. Farmer, R.E.A. and J.T. Guo, Real business cycles and the animal spirits hypothesis, *Journal of EconomicTheory* **63** (1994), 42-73.

12. Guo, J. and K.J. Lansing, Indeterminacy and stabilization policy, *Journal of Economic Theory* **82** (1998), 481-490.

13. Ni, S., An empirical analysis on the substitutability between private consumption and government purchases, *Journal of Monetary Economics* **36** (1995), 593-605.

14. Perli, R., Indeterminacy, home production and the business cycle, *Journal of Monetary Economics* **41** (1998), 105-125.

15. Schmitt-Grohé, S. And M. Uribe, Balanced-budget rules, distortionary taxes, and aggregate inestability, *Journal of Political Economy* **105** (1997), 976-1000.

16. Weder, M., Fickle consumers, durable goods and business cycles, *Journal of Economic Theory* **81** (1998), 37-57.

17. Wen, Y., Capacity utilization under increasing returns to scale, *Journal of Economic Theory* **81** (1998), 7-36.

18. Xie, D., Divergence in economic performance: transitional dynamics with multiple equilibria, *Journal of Economic Theory* **63** (1994), 97-112.

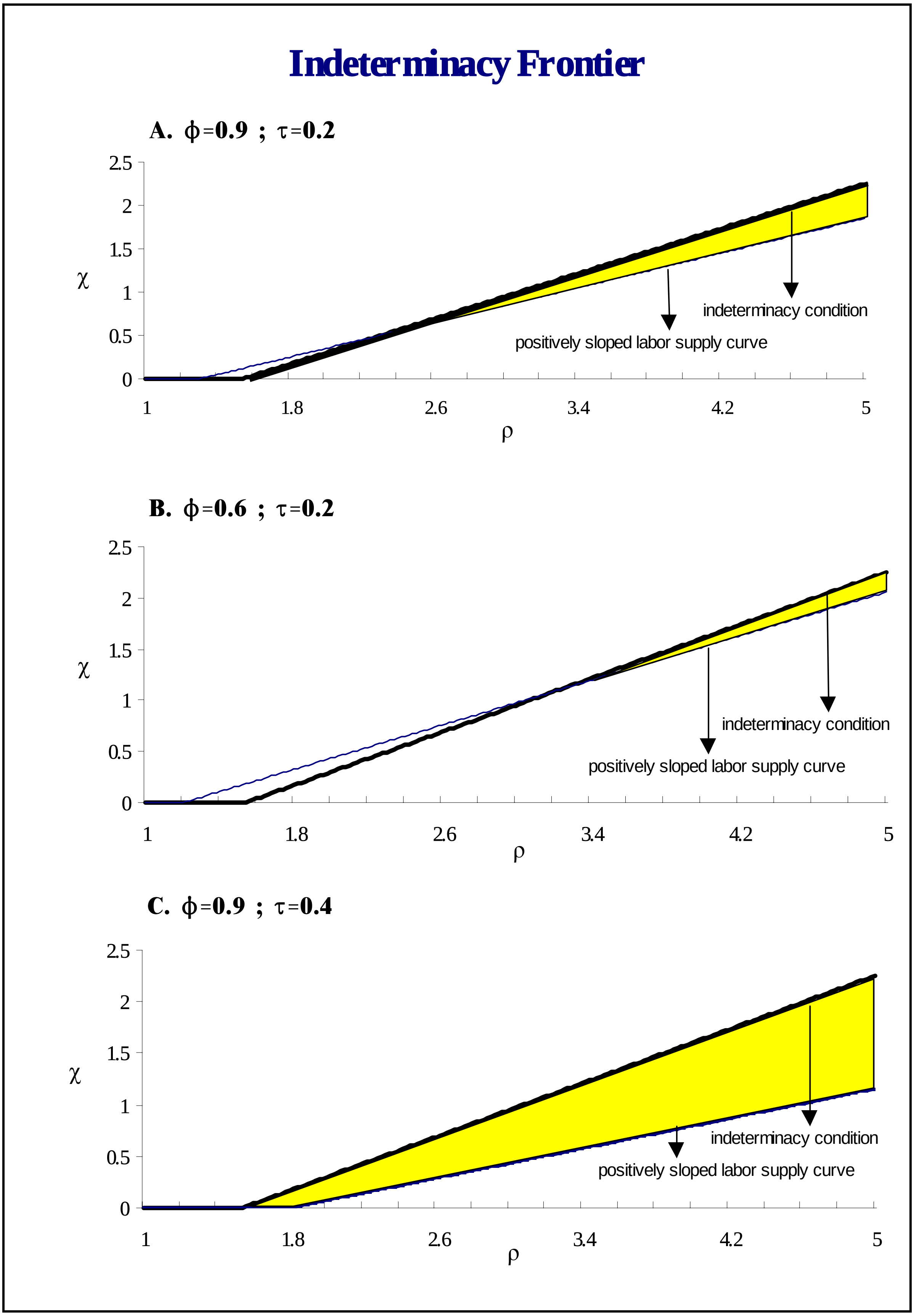

Fig. 1.Parameter regions for indeterminacy of equilibria and for a positively sloped labor supply curve. Points below each line satisfy the associated condition. The shaded region presents the (ρ , χ)-pairs for which equilibrium is indeterminate and the reduced-form labor supply curve obtained holding constant consumption has the standard slope.

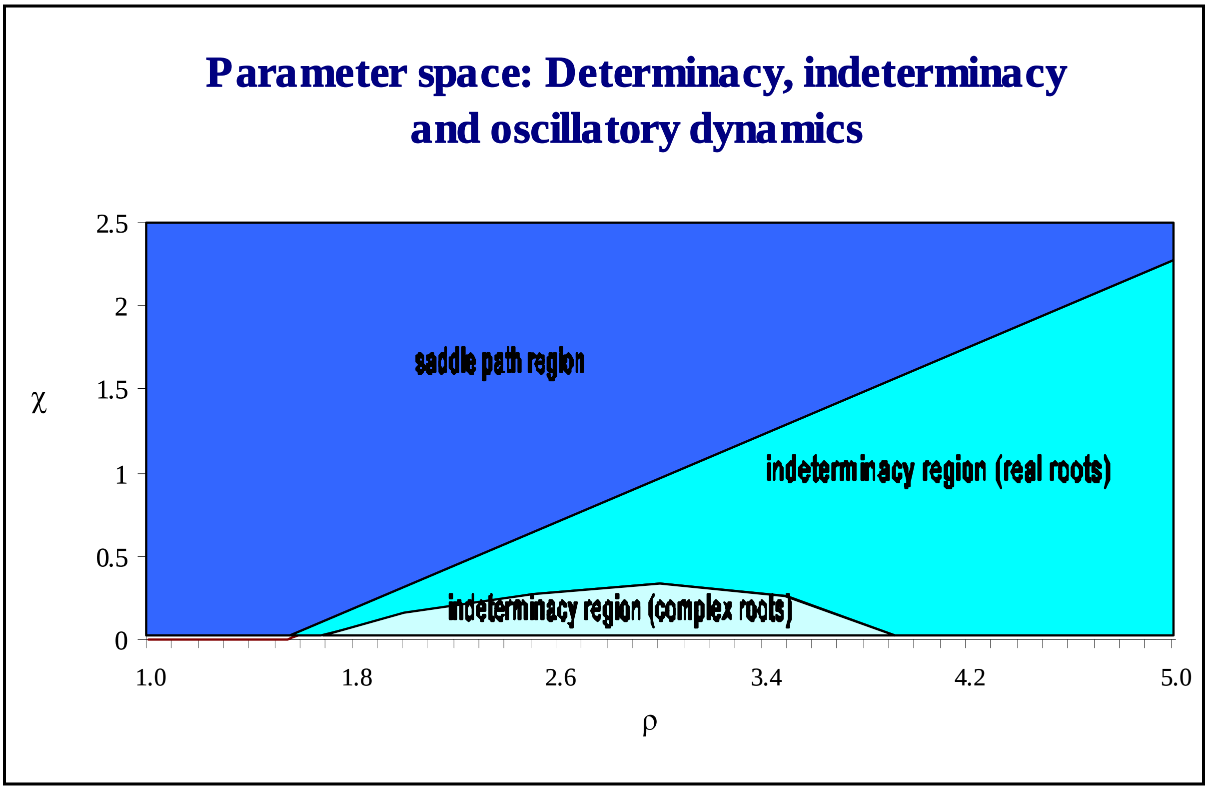
Parameter space: Determinacy, indeterminacy
and oscillatory dynamics
2.5
2
1.5
1
0.5
0
χ
saddle path region
indeterminacy region (real roots)
indeterminacy region (complex roots)
1.0
1.8
2.6
3.4
4.2
5.0
ρ

Fig. 2.