OPTIMAL TIME-CONSISTENT FISCAL POLICY IN AN ENDOGENOUS GROWTH ECONOMY WITH PUBLIC CONSUMPTION AND CAPITAL

***Alfonso Novales*[1]**
***Rafaela Pérez*[2]**
***Jesus Ruiz*[3]**

This version: July 15, 2014

ABSTRACT

In an endogenous growth model where the fiscal authority cannot commit to policy decisions beyond the current period, we explore the time-consistent optimal choice for two policy instruments: the income tax rate and the split of government spending between utility bearing consumption and productive services to firms. We show that under the time-consistent Markov policy the economy lacks any transitional dynamics and there is local and global determinacy of equilibrium. For empirically plausible parameter values we find that the Markov-perfect policy implies a higher tax rate and a larger proportion of government spending allocated to consumption than those chosen under a commitment constraint. As a result, economic growth is slightly lower under the Markov-perfect policy than under the Ramsey policy, with growth under lump-sum taxes being highest.

The implication of our results is that if the private sector is aware of the government's inability to pledge future policy decisions, then the government should impose a slightly higher tax rate and devote a higher share of public resources to consumption, with a relatively low cost in terms of growth.

JEL classification: E61, E62, H21

Keywords: time-consistency, Markov-perfect optimal policy, Ramsey optimal policy, endogenous growth, income tax rate, government spending composition.

[1] Departamento de Economía Cuantitativa, Universidad Complutense de Madrid (Spain) and Instituto Complutense de Análisis Económico (ICAE), anovales@ccee.ucm.es

[2] Departamento de Análisis Económico, Universidad Complutense de Madrid (Spain) and Instituto Complutense de Análisis Económico (ICAE), rmperezs@ccee.ucm.es

[3] Departamento de Economía Cuantitativa, Universidad Complutense de Madrid (Spain) and Instituto Complutense de Análisis Económico (ICAE), jruizand@ccee.ucm.es

The authors would like to thank financial support from the Spanish Ministry of Science and Innovation through grant ECO2009-10398, the program of support to Research Groups from Universidad Complutense de Madrid and BSCH, the Xunta de Galicia through Grant 10PXIB300177PR and the Fundación Ramón Areces through its program of Research Grants in Economics.

1. Introduction

The relevance of time consistent policies stems from the fact that the government has no incentive to change its policy once private agents have made their decisions conditional on the policy announcement. Unfortunately, the difficulty in solving the time-consistent Markov policy optimization problem has generally led academic research into the characterization of the more limited Ramsey optimal policies. The latter assume commitment and are hence subject to potential deviations by the government from the previously announced policy rule. The same technical difficulty also explains that most research on time-consistent optimal policies has been done in exogenous growth environments.

As main examples, Ortigueira (2006) and Klein, Krusell and Rios-Rull (2008) consider an stylized exogenous growth model, with leisure and public consumption in the utility function, to characterize the optimal time-consistent tax policy under two different game designs. Klein, Krusell and Rios-Rull (2008) consider a game in which the government is a dominant player that takes the optimal reaction of private agents as given when deciding the optimal policy. Ortigueira (2006) compares the results obtained under the structure in Klein, Krusell and Rios-Rull with those from an alternative design of the game in which the government and private agents make their respective decisions simultaneously, characterizing the behavior of the economy along the transition to the optimal steady-state. These authors consider alternative fiscal structures, always with a single instrument: either a single tax levied on total income, a single tax on capital income or a single tax on labor income. Martin (2010) follows the same game structure as Klein, Krusell and Rios-Rull (2008), extending the analysis to the simultaneous consideration of different tax rates for capital and labor income and solving for the optimal time consistent choice for both fiscal instruments. A further exogenous growth analysis is done by Azzimonti et al. (2009), who characterize the Markovian tax rate raised on total income when used to finance public investment.

However, for the analysis of optimal taxation it is essential to overcome the two limitations mentioned above, by describing how to characterize the optimal time consistent fiscal policy under endogenous growth. Endogenous growth models not only allow for a more plausible representation of actual economies, but also for explicitly taking into account the effect of fiscal policy on the rate of growth. This is crucial when analyzing the growth effects of productive government spending, as in the seminal papers by Barro (1990),

Cazzavillan (1996) or Glomm and Ravikumar (1997). Recently, Jaimovich and Rebelo (2013) propose an endogenous growth model with heterogeneity in entrepreneurial ability in support of the empirical evidence on the highly non-linear effects of taxation on growth: when the tax rates are low or moderate, marginal increases in the tax rate have small growth impact whilst, when tax rates are high, further tax hikes have large, negative impact on growth, due to the disincentives to investment and innovation.

The two mentioned extensions, time-consistent policy in endogenous growth framework, have been considered by Malley et al. (2002), who characterize the Markov tax policy in an endogenous growth economy where the government raises tax revenues on total income, using the proceeds to finance public consumption and production services. However, their setup is still restrictive in two aspects: *i*) the split of government spending between consumption and production services is exogenously given, and *ii*) private agents are supposed to have a logarithmic utility function and physical capital is supposed to fully depreciate each period. Under these parametric restrictions, the Ramsey policy is not subject to a time consistency problem and it coincides with the Markov perfect solution, a result that we show later on.[4]

In our analysis we dispose of these two additional limitations: First, we consider an economic environment with a CRRA utility function defined on private and public consumption, with incomplete depreciation of capital. Second, we incorporate an endogenously time-varying split of government spending between public consumption and production services. We show that a time consistent optimal policy exists and it is described by the optimal choice of both, the income tax rate and the split of public spending between consumption and production activities. We prove that the dynamics of the model can be characterized by the ratio of productive services provided by the government over private capital. Under the Markov solution this ratio is always on the Balanced Growth Path. Additionally, we numerically show that there is not indeterminacy of equilibrium and hence, the Markov solution lacks any transitional dynamics.

Under this more general economic framework, when comparing the optimal Markov-perfect and Ramsey policies, we find that: *i*) the income tax rate is higher under the time consistent policy, since the Markov government cannot internalize the distortionary effects of the current tax on the level of investment undertaken in previous periods (as in Ortigueira (2006), in a neoclassical growth framework), *ii*) the proportion of public resources devoted

[4] Azzimonti et al. (2009) also show this result for an exogenous growth economy.

to consumption is higher under the Markov government than under the Ramsey government, since the former only commits to current policies, thereby giving priority to current consumption, with an immediate effect on utility, rather than to production activities, whose effects on welfare will mainly take place in future periods, and *iii*) as a result, economic growth is slightly lower under the Markov-perfect policy than under the Ramsey policy, with the growth rate under lump-sum taxes being the highest.

The implication is that a government that is aware that society knows its inability to pledge future policy decisions should impose a slightly higher tax rate and devote a higher share of public resources to consumption, with a relatively lower implied rate of growth.

2. The model economy

We assume in what follows that population does not grow, and also that the economy can be described by representative agents: firm, household and government. We further assume full employment.

The representative firm maximizes profits subject to a technology that produces the single consumption commodity. The stock of private capital, K_t, together with production services provided by the government, $i_{p,t}$, are used together with the labor force, L_t, as production inputs in a technology: $Y_t = BK_t^{\alpha}(L_t\, i_{p,t})^{1-\alpha}$. In line with Barro (1990) and Cazzavillan (1996) we assume here a flow of public services of rival nature, and hence it is the quantity of the public good assigned to each firm the relevant variable in the private production process. The representative firm pays rents $r_t K_t + w_t L_t$ to households for the use of private capital and labor, solving each period the static profit optimization problem:

$$\underset{\{K_t, L_t\}}{Max}\, \Pi_t = BK_t^{\alpha}(L_t\, i_{p,t})^{1-\alpha} - r_t K_t - w_t L_t .$$

We assume that labor is inelastically supplied by the household, and we normalize it to 1: $L_t = 1, \forall t$.

Markets for production inputs are competitive. At each point in time, optimality conditions imply that input prices are equal to their marginal product:

$$r_t = \alpha B\left(K_t / i_{p,t}\right)^{\alpha-1},$$
$$w_t = (1-\alpha)B\left(K_t / i_{p,t}\right)^{\alpha} i_{p,t} .$$

Apart from the production services to firms, $i_{p,t}$, the government also provides consumption services to households, g_t. We denote by η_t the proportion of the proceeds from income taxes that are used each period to finance public consumption services, g_t, the remaining tax revenues being used to pay for production services, $i_{p,t}$. The government budget constraint is $\tau_t\left(r_tK_t + w_t\right) = g_t + i_{p,t}$, where

$$g_t = \eta_t\tau_t\left(r_tK_t + w_t\right), \tag{1}$$

$$i_{p,t} = (1-\eta_t)\tau_t\left(r_tK_t + w_t\right). \tag{2}$$

From the government budget expenditure rules (1), (2), and the optimality conditions for the competitive firms we get,

$$g_t = \eta_t\tau_t BK_t^{\alpha} i_{p,t}^{1-\alpha},$$

$$i_{p,t} = (1-\eta_t)\tau_t BK_t^{\alpha} i_{p,t}^{1-\alpha},$$

so that production services are provided according to,

$$i_{p,t} = \left[(1-\eta_t)\tau_t B\right]^{1/\alpha} K_t \tag{3}$$

while consumption services are,

$$g_t \equiv \mathcal{G}(K_t;\tau_t,\eta_t) = \eta_t\tau_t^{1/\alpha}B^{1/\alpha}(1-\eta_t)^{(1-\alpha)/\alpha}K_t, \tag{4}$$

and equilibrium real interest rates and real wages become,

$$r_t \equiv r(K_t;\tau_t,\eta_t) = \alpha B^{1/\alpha}\left[(1-\eta_t)\tau_t\right]^{\frac{1-\alpha}{\alpha}}, \tag{5}$$

$$w_t \equiv w(K_t;\tau_t,\eta_t) = (1-\alpha)B^{1/\alpha}\left[(1-\eta_t)\tau_t\right]^{\frac{1-\alpha}{\alpha}}K_t. \tag{6}$$

The representative household maximizes his/her life-time discounted utility, $\sum_{t=0}^{\infty}\rho^t U(c_t,g_t)$, defined over private and public consumption, c_t, g_t, subject to a flat tax rate τ_t on total income.[5] They know the current values of τ_t and η_t, and expect future governments to follow policies $\tau_{t+1} = \mathcal{T}(K_{t+1})$ and $\eta_{t+1} = \mathcal{H}(K_{t+1})$. The typical household solves the problem:

$$\upsilon\left(k_t,K_t;\tau_t;\eta_t;\mathcal{T};\mathcal{H}\right) = \underset{\{c_t,k_{t+1}\}}{Max}\left[U\left(c_t,g_t\right) + \rho\tilde{\upsilon}(k_{t+1},K_{t+1};\mathcal{T};\mathcal{H})\right] \tag{7}$$

given k_0, and subject to the budget constraint,

[5] We also assume that consumption services provided by the government are of rival nature, so the argument in the utility function is the per-capita level of the public good.

$$c_t + k_{t+1} - (1-\delta)k_t = (1-\tau_t)\left[w_t(K_t;\tau_t,\eta_t) + r_t(K_t;\tau_t,\eta_t)k_t\right], \tag{8}$$

where k_t denotes the level of capital chosen by the household and K_t the economy-wide per capita stock of capital.

The solution leads to a consumption function $\mathcal{C}(K_t;\tau_t,\eta_t)$ satisfying the Euler equation,[6]

$$\begin{aligned} U_c\left(\mathcal{C}(K_t;\tau_t,\eta_t),\mathcal{G}(K_t;\tau_t,\eta_t)\right) = \rho U_c\left(\mathcal{C}(K_{t+1};\tau_{t+1},\eta_{t+1}),\mathcal{G}(K_{t+1};\tau_{t+1},\eta_{t+1})\right)\times \\ \left[1-\delta+\left(1-\tau_{t+1}\right)\alpha B^{1/\alpha}\left[(1-\eta_{t+1})\tau_{t+1}\right]^{(1-\alpha)/\alpha}\right]. \end{aligned} \tag{9}$$

With homogeneous households and firms, we have in equilibrium: $K_t = k_t$, and all the variables in the model can be regarded either as aggregate per capita or in individual terms.

Using (5), (6) and (8), the stock of capital k_t can be seen to evolve over time according to:

$$k_{t+1} = (1-\delta)k_t + (1-\tau_t)\left[(1-\eta_t)\tau_t\right]^{(1-\alpha)/\alpha} B^{1/\alpha}k_t - \mathcal{C}(k_t;\tau_t,\eta_t). \tag{10}$$

Substituting (3) in the production function, output is given by $y_t = B^{1/\alpha}\left[(1-\eta_t)\tau_t\right]^{\frac{1-\alpha}{\alpha}} k_t$. As a consequence, in the competitive equilibrium allocation, *i*) the ratio of production services to output, $i_{p,t}/y_t$ is equal to $(1-\eta_t)\tau_t$, an extension of the framework in Barro (1990), and *ii*) the ratio of private capital to output, k_t/y_t, is a function of $(1-\eta_t)\tau_t$ and structural parameters α and B. In line with Barro (1990), the constant returns to scale in private capital and public production services, jointly with the fact that the complementary public input expands in a parallel manner to the private capital, guarantees the existence of endogenous growth in our model economy.

A potential problem with optimal policy design under endogenous growth is the possibility of having indeterminacy of equilibria. In that situation, equilibrium trajectories are undetermined, being dependent on the initial value of some control variables. Policy recommendations emerging from economic models may then lack any real meaning.

Since this model shares some characteristics of the AK-family of models, it may also lack any transitional dynamics, which would be very relevant for the characterization of optimal policy as well as for welfare evaluation.

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[6] Along the paper we denote partial derivatives by $F_v \equiv \frac{\partial F}{\partial v}$.

Along the paper we argue[7] that in the Markov and Ramsey solutions η_t and τ_t, as well as the ratios of private and public consumption to private capital, $c_t / k_t, g_t / k_t$, are constant from the initial period. This has two implications: the model lacks any transitional dynamics and there is no indeterminacy of equilibria, with the economy being on the single balanced growth path from the initial period onwards.

3. Optimal policy

3.1 The time-consistent optimal policy

We use the same equilibrium concept as Klein, Krusell and Ríos-Rull (2008) and the "government-moves-first" case in Ortigueira (2006).[8] We consider a dynamic game played by a sequence of governments, each one of them choosing current period policies on the basis of the state of the economy, defined by the stock of private capital. Hence, each government chooses the current tax rate τ_t and the proportion of revenues used to purchase public consumption, η_t, before the household decides on consumption and savings. When making optimal policy choices, the government knows the household decision rule $\mathcal{C}(k_t;\tau_t,\eta_t)$ that describes the reaction of the household to policy decisions.

Acting as a leader, the government chooses the current tax rate and the split of public resources taking as given the policies followed by future governments and taking into account that reaction of the household to the policy choices, as follows:

$$V\left(k_t\right) = \underset{\{\tau_t,\eta_t\}}{Max}\left[U\left(\mathcal{C}\left(k_t;\tau_t,\eta_t\right),\mathcal{G}\left(k_t;\tau_t,\eta_t\right)\right)+\rho\tilde{V}(k_{t+1})\right] \qquad \text{[P1]}$$

where $\mathcal{G}\left(k_t;\tau_t,\eta_t\right)$ and k_{t+1} are given by (4) and (10). $V\left(k_t\right)$ and $\tilde{V}\left(k_{t+1}\right)$ denote the value function for the current government and the continuation value function, respectively. Although they will be treated as different functions when characterizing optimality conditions in what follows, in equilibrium the two functions will be the same.

Proposition 1. *The time consistent policy corresponding to the Markov equilibrium is the solution to the set of Generalized Euler Equations (GEE):*

[7] We have analytical proofs of these results for the case of logarithmic utility and full depreciation of capital, and numerical arguments for the general case.

[8] Which is also used in Krusell and Rios-Rull (1999), and Krusell, Quadrini and Rios-Rull (1996).

$$\frac{U_{c_t}\mathcal{C}_{\tau_t}+U_{g_t}\mathcal{G}_{\tau_t}}{\mathcal{C}_{\tau_t}+\Lambda(\tau_t)\,\Omega(\tau_t,\eta_t)\,k_t}=\frac{U_{c_t}\mathcal{C}_{\eta_t}+U_{g_t}\mathcal{G}_{\eta_t}}{\mathcal{C}_{\eta_t}+\dfrac{1-\alpha}{\alpha}\dfrac{1}{(1-\eta_t)}\Omega(\tau_t,\eta_t)\,k_t}, \tag{11}$$

and

$$\frac{U_{c_t}\mathcal{C}_{\tau_t}+U_{g_t}\mathcal{G}_{\tau_t}}{\Omega(\tau_t,\eta_t)\Lambda(\tau_t)k_t+\mathcal{C}_{\tau_t}}=\rho\left[\begin{array}{c}U_{c_{t+1}}\mathcal{C}_{k_{t+1}}+U_{g_{t+1}}\mathcal{G}_{k_{t+1}}+\dfrac{U_{c_{t+1}}\mathcal{C}_{\tau_{t+1}}+U_{g_{t+1}}\mathcal{G}_{\tau_{t+1}}}{\Omega(\tau_{t+1},\eta_{t+1})\Lambda(\tau_{t+1})k_{t+1}+\mathcal{C}_{\tau_{t+1}}}\times\\ \left[1-\delta+\Omega(\tau_{t+1},\eta_{t+1})-\mathcal{C}_{k_{t+1}}\right]\end{array}\right], \tag{12}$$

where $\Lambda(\tau)\equiv\dfrac{\tau-(1-\alpha)}{\alpha\tau(1-\tau)}$, $\Omega(\tau,\eta)\equiv(1-\tau)\left[(1-\eta)\tau\right]^{(1-\alpha)/\alpha}B^{1/\alpha}$.

Proof.- *See Appendix* 1.□

Equation (11) is a condition relating the optimal choice of the two policy instruments at a given point in time, while equation (12) characterizes the optimal intertemporal choice of income tax rates.

From (10) we get the size of the reduction in time *t* investment from an increase in taxes is: $\partial(k_{t+1}-k_t)/\partial\tau_t=-\left[\mathcal{C}_{\tau_t}+\Lambda(\tau_t)\,\Omega(\tau_t,\eta_t)\,k_t\right]$. Hence, the left hand side at (11) gives the change in utility produced by a tax increase, per unit of crowded-out investment. This is what Ortigueira (2006) calls today's marginal value of taxation. By a similar argument, the right hand side at (11) is the change in utility from an increase in the share of resources devoted to public consumption, per unit of crowded-out investment. The optimal choices of the two policy instruments must satisfy the equality between these two marginal effects on utility.

The left hand side at (12) is again the marginal change in utility per unit of crowded out investment implied by a decrease in the tax rate. Lower taxes at *t*+1 stimulate investment, and an additional unit of capital at *t*+1 has a direct effect on utility of $U_{c_{t+1}}\mathcal{C}_{k_{t+1}}+U_{g_{t+1}}\mathcal{G}_{k_{t+1}}$ through its effect on private and public consumption and an indirect effect through its impact on time *t*+2 capital stock, $\dfrac{\partial k_{t+2}}{\partial k_{t+1}}=1-\delta+\Omega(\tau_{t+1},\eta_{t+1})-\mathcal{C}_{k_{t+1}}$, which needs to be appropriately discounted. The total effect is given by the square bracket at the right hand side of (12). It shows that the change in utility per unit of crowded-out investment at time *t* implied by a marginal change in the optimal tax rate must be equal to the discounted change in utility resulting at time *t*+1.

Definition.- A Markov-Perfect equilibrium is a set of functions $\mathcal{C}(k_t;\tau_t,\eta_t)$, $\mathcal{T}(k_t)$, $\mathcal{H}(k_t)$ and $V(k_t)$ such that:

i) Given government rules (3) and (4), $\mathcal{C}(k_t;\tau_t,\eta_t)$ solves the Euler equation (9) subject to the law of motion of the stock of capital (10),

ii) $\mathcal{T}(k_t)$, $\mathcal{H}(k_t)$ satisfy conditions (3), (4), the law of motion of the stock of capital (10), as well as the Generalized Euler Equations (11) and (12); and

iii) $V(k_t)$ is the value function of government obtained as a solution to [P1]:

$$V(k_t)=U\left(\mathcal{C}\left(k_t;\mathcal{T}(k_t),\ \mathcal{H}(k_t)\right),\mathcal{G}\left(k_t;\mathcal{T}(k_t),\ \mathcal{H}(k_t)\right)\right)+\rho V(k_{t+1}).$$

3.2 The Ramsey policy

As usual, we define the benchmark "Ramsey equilibrium" as the solution to an optimal-policy problem where the government can commit to future policies. The Ramsey optimal policy is then the solution to the problem of maximizing the time aggregate, discounted utility of the household, subject to the equilibrium conditions (4), (9) and (10) as constraints:

$$\begin{aligned}
&\underset{\{c_t,g_t,\tau_t,\eta_t,k_{t+1}\}}{Max}\sum_{t=0}^{\infty}\rho^t U\left(c_t,g_t\right)\\
&\text{subject to:}\\
&g_t=\eta_t\tau_t\left[(1-\eta_t)\tau_t\right]^{\frac{1-\alpha}{\alpha}}B^{\frac{1}{\alpha}}k_t \qquad\qquad [P2]\\
&U_{c_t}=\rho U_{c_{t+1}}\left[1-\delta+\alpha\,\Omega\left(\tau_{t+1},\eta_{t+1}\right)\right]\\
&k_{t+1}=(1-\delta)k_t+\Omega(\tau_t,\eta_t)k_t-c_t\ .
\end{aligned}$$

where $\Omega\left(\tau_t,\eta_t\right)\equiv(1-\tau_t)\left[(1-\eta_t)\tau_t\right]^{(1-\alpha)/\alpha}B^{1/\alpha}$. In Appendix 2 we characterize the first order conditions and the balanced growth path for the Ramsey problem.

The Ramsey policy takes into account the optimal reactions of private agents. However, it is time inconsistent, since once private agents adjust their decisions to the announced economic policy it will be optimal for the government to change policy.

Given the complexity involved in characterizing optimal policy under lack of commitment, attention has often been restricted to Ramsey policies, in spite of their well-known limitation of assuming commitment on the part of the current government on future periods. It is therefore important to evaluate to what extent the Markov-perfect fiscal policy differs from the Ramsey policy in our setup. We will perform such analysis in Section 6.

4. An analytical solution: logarithmic utility and full depreciation of private capital

We consider in this section the special case of logarithmic preferences that are separable in private and public consumption, $U\left(c_t, g_t\right) = \ln c_t + \theta \ln g_t$, together with full depreciation of private capital. The two assumptions together allow us to obtain an analytical characterization of the time consistent optimal fiscal policy that we can compare with the Ramsey solution as well as with the allocation that would be obtained under lump-sum taxes.

Under this utility function, the competitive equilibrium allocation is characterized by the system:

$$\left.\begin{aligned} k_{t+1} &= \Omega(\tau_t, \eta_t) k_t - c_t, \\ \frac{c_{t+1}}{c_t} &= \rho\left[\alpha\, \Omega(\tau_{t+1}, \eta_{t+1})\right] \end{aligned}\right\} \tag{13}$$

Proposition 2. *Under full depreciation of private capital and a logarithmic utility function, the competitive equilibrium allocations are given by:*

$$k_{t+1} = \rho\alpha\, \Omega(\tau_t, \eta_t)\, k_t, \tag{14}$$

$$c_t = \left(1 - \rho\alpha\right) \Omega(\tau_t, \eta_t)\, k_t. \tag{15}$$

Proof. *Plugging in the previous system* (13) *a guess for the functional form for the competitive equilibrium allocation as:* $k_{t+1} = A\Omega_t k_t$, *it is easy to show that* $A = \rho\alpha$. □

Expressions (4) and (15) for g_t, c_t allow us to compute the partial derivatives that enter into the Generalized Euler equations (11)-(12), to find an analytical solution to the time consistent optimal policy problem.

The next set of results shows that under the Markov and Ramsey solutions the model lacks any transitional dynamics and there is no indeterminacy of equilibria, with the economy being on the single balanced growth path from the initial period onwards.

Proposition 3. *Under full depreciation of private capital and a logarithmic utility function, separable in private and public consumption, the optimal time-consistent fiscal policy satisfies:*

$$\tau_t^M = \frac{1-\alpha}{1-\eta_t^M} \forall t \tag{16}$$

Proof.- *See Appendix* 3.

Corollary 1.- *Under the optimal Markov policy the ratio of productive public services to private capital is constant for all t.*

Proof.- *Using (16) in (9), we get:* $\frac{i_{p,t}}{k_t} = (1-\alpha)^{1/\alpha} B^{1/\alpha}, \forall t$.

Proposition 4.- *Under full depreciation of private capital and a logarithmic utility function, separable in private and public consumption,*

i) There is no local indeterminacy of equilibria,

ii) The economy lacks transitional dynamics,

iii) The optimal Markov policy is:

$$\tau_t^M = \tau^M = 1 - \frac{\alpha(1+\rho\theta)}{1+\theta}, \quad \forall t, \tag{17}$$

$$\eta_t^M = \eta^M = \frac{\alpha\theta(1-\rho)}{1-\alpha+\theta(1-\alpha\rho)}, \quad \forall t. \tag{18}$$

Proof-. *See Appendix 3.* □

Notice that the optimal split of resources between public consumption and productive services is well defined, taking values between 0 and α, while the optimal income tax rate is always between 1-α and one.

We now characterize the optimal allocation of resources in terms of the ratios of private and public consumption to the stock of private capital: $\chi_t \equiv \frac{c_t}{k_t}, \phi_t \equiv \frac{g_t}{k_t}$. These ratios must remain constant along the balanced growth path.

Proposition 5. *The optimal allocation of resources under the Markov-perfect optimal policy is given by*:

$$\gamma_t^M \equiv \left(\frac{k_{t+1}}{k_t}\right)^M = \gamma^M = \rho\alpha\, \Omega(\tau^M, \eta^M) = \frac{\rho\alpha^2(1+\rho\theta)}{1+\theta}(1-\alpha)^{\frac{1-\alpha}{\alpha}} B^{1/\alpha} \quad \forall t,$$

$$\chi_t^M \equiv \left(\frac{c_t}{k_t}\right)^M = \chi^M = (1-\rho\alpha)\Omega(\tau^M, \eta^M) = \frac{(1-\rho\alpha)\alpha(1+\rho\theta)}{1+\theta}(1-\alpha)^{\frac{1-\alpha}{\alpha}} B^{1/\alpha} \quad \forall t,$$

$$\phi_t^M \equiv \left(\frac{g_t}{k_t}\right)^M = \phi^M = \frac{\eta^M \tau^M}{1-\tau^M}\Omega(\tau^M, \eta^M) = (1-\alpha)^{\frac{1-\alpha}{\alpha}} B^{1/\alpha} \quad \forall t .$$

Proof. *Their expressions can readily be obtained from* (4), (14) *and* (15). □

The three following corollaries can be readily shown from (17) and (18):

***Corollary* 2.** *When public consumption does not enter as an argument into the utility function (θ=0), the Markov-perfect optimal tax rate coincides with that in Barro* (1990)*:* $\tau = 1-\alpha$ *. In that situation, public resources are fully devoted to production.*

Corollary 3. *The Markov-perfect optimal tax rate converges to the Barro tax as the discount rate approaches 1, with public resources again being fully devoted to production.*

Corollary 4. *i*) *The proportion of public resources devoted to public consumption increases with θ and α, and it decreases with ρ; ii) the optimal time consistent income tax increases with θ, and it decreases with α and ρ.*

As expected, the proportion of public resources devoted to consumption increases with the relative importance of public consumption in the utility function. It also increases with the output elasticity of private capital. A more productive private capital, relative to public production services, allows for a higher share of public resources being devoted to consumption. Turning the argument around, the more productive is the public input relative to private capital, the more interesting it is to allocate public resources to productive activities rather than to consumption. The share of public resources dedicated to consumption decreases for a larger ρ. We then tend to value future consumption almost as much as current consumption and it becomes more interesting to shift resources to the future by increasing productive services.

As public consumption is more appreciated by consumers for higher values of θ and lower values of ρ, it is appropriate to raise higher tax revenues to finance that component of public spending. On the contrary, a high elasticity of private capital, α, leads the private sector to allocate more resources to investment, and taxes can be lower.

4.1 Comparing Ramsey and Markov policies under logarithmic utility and full depreciation of private capital

The following proposition shows that, for this special case, the Ramsey and Markov policies coincide.

Proposition 6. *Under a logarithmic utility function and full depreciation, the optimal Ramsey policy, becomes:*

$$\tau^R = 1 - \frac{\alpha(1+\rho\theta)}{1+\theta},$$
$$\eta^R = \frac{\alpha\theta(1-\rho)}{1-\alpha+\theta(1-\alpha\rho)}.$$

Proof: *See Appendix 3.* □

The income tax and the proportion of public resources devoted to public consumption under the Ramsey policy coincide with the values obtained under the time-consistent policy, so the properties analyzed in Proposition 4 and Corollaries 2 to 4 for the Markov-perfect optimal policy apply to the Ramsey policy as well. The equality of solutions arises because under a logarithmic utility and complete depreciation of physical capital the Ramsey policy is time consistent, a result shown by Azzimonti et al. (2009) in a neoclassical growth model.

5. Optimal time-consistent fiscal policy under CRRA preferences and incomplete depreciation of private capital

The Generalized Euler conditions (11) and (12) should incorporate the consumption decision rule of private agents, which is characterized as the solution to the Euler equation (9) of the competitive equilibrium. Unfortunately, it is not possible to find the analytical solution to (9) in general, and that precludes us from obtaining an analytical characterization of the transition towards the balanced growth path.

Assuming a CRRA utility: $U(c_t, g_t) = \frac{c_t^{1-\sigma} g_t^{\theta(1-\sigma)} - 1}{1-\sigma}, \sigma > 0$ the Euler condition of the competitive equilibrium becomes,

$$c_t^{-\sigma} g_t^{\theta(1-\sigma)} = \rho c_{t+1}^{-\sigma} g_{t+1}^{\theta(1-\sigma)} \left[1 - \delta + \alpha\left(1-\tau_{t+1}\right)\left[\left(1-\eta_{t+1}\right)\tau_{t+1}\right]^{\frac{1-\alpha}{\alpha}} B^{1/\alpha} \right],$$

So that the relevant system of equations for $\phi_t, \chi_t, \gamma_t, \tau_t, \eta_t$, written in ratios, becomes:

- Euler condition of the competitive equilibrium:

$$\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\gamma_t^{\sigma-\theta(1-\sigma)} = \rho\chi_{t+1}^{-\sigma}\phi_{t+1}^{\theta(1-\sigma)}\left[1-\delta+\alpha\left(1-\tau_{t+1}\right)\left[\left(1-\eta_{t+1}\right)\tau_{t+1}\right]^{\frac{1-\alpha}{\alpha}}B^{1/\alpha}\right], \tag{19}$$

- law of motion of physical capital:

$$\gamma_t = \Omega(\tau_t,\eta_t)+1-\delta-\chi_t \tag{20}$$

- the ratio of public consumption to capital, from (4):

$$\phi_t = \eta_t\tau_t^{1/\alpha}(1-\eta_t)^{(1-\alpha)/\alpha}B^{1/\alpha}, \tag{21}$$

- and the two Generalized Euler equations:

$$\frac{\chi_{\tau_t}+\theta\chi_t\dfrac{1}{\alpha\tau_t}}{\chi_{\tau_t}+\Lambda\left(\tau_t\right)\Omega\left(\eta_t,\tau_t\right)} = \frac{\chi_{\eta_t}+\theta\chi_t\dfrac{\alpha-\eta_t}{\alpha\eta_t(1-\eta_t)}}{\chi_{\eta_t}+\dfrac{1-\alpha}{\alpha}\dfrac{\Omega\left(\eta_t,\tau_t\right)}{1-\eta_t}} \tag{22}$$

$$\frac{\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\gamma_t^{-\sigma+\theta(1-\sigma)}\left(\chi_{\tau_t}+\chi_t\dfrac{\theta}{\alpha}\dfrac{1}{\tau_t}\right)}{\chi_{\tau_t}+\Lambda\left(\tau_t\right)\Omega(\tau_t,\eta_t)} = \rho\chi_{t+1}^{-\sigma}\phi_{t+1}^{\theta(1-\sigma)}\times$$
$$\left[(1+\theta)\chi_{t+1}+\left(\chi_{k_{t+1}}+\frac{\chi_{t+1}}{\phi_{t+1}}\phi_{k_{t+1}}\right)+\frac{\chi_{\tau_{t+1}}+\chi_{t+1}\dfrac{\theta}{\alpha}\dfrac{1}{\tau_{t+1}}}{\chi_{\tau_{t+1}}+\Lambda\left(\tau_{t+1}\right)\Omega(\tau_{t+1},\eta_{t+1})}\left(\gamma_{t+1}-\chi_{k_{t+1}}\right)\right]. \tag{23}$$

Note that to obtain (22) and (23) we have used $C_{\tau_t} = \chi_{\tau_t}k_t; C_{\eta_t} = \chi_{\eta_t}k_t; G_{\tau_t} = \phi_{\tau_t}k_t; G_{\eta_t} = \phi_{\eta_t}k_t$, and the partial derivatives:

$$\phi_{\eta_t} = \phi_t\frac{\alpha-\eta_t}{\alpha\eta_t(1-\eta_t)}; \qquad \phi_{\tau_t} = \frac{1}{\alpha\tau_t}\phi_t;$$
$$\gamma_{\tau_t} = \frac{1-\alpha-\tau_t}{\alpha\tau_t(1-\tau_t)}\Omega\left(\eta_t,\tau_t\right); \quad \gamma_{\eta_t} = -\frac{1-\alpha}{\alpha}\frac{\Omega\left(\eta_t,\tau_t\right)}{1-\eta_t}; \quad \gamma_{\chi_t} = -1,$$

that emerge from (20) and (21).

The right-hand side at (19) involves values at time t+1 of policy variables, η_{t+1}, τ_{t+1}, and ratios of decision variables to the stock of private capital, χ_{t+1}, ϕ_{t+1}. Each one of these

must be a function of the single state variable in the economy,[9] k_t, so that we can think of the right-hand side at (19) as a function $F(k_t)$.

$$\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\gamma_t^{\sigma-\theta(1-\sigma)} = F(k_t). \tag{24}$$

We can now characterize the optimal Markov policy in the more general set up considered in this Section. We start by showing that the relationship between the two policy instruments is the same we found under logarithmic preferences and full depreciation of private capital. As shown in Appendix 4, the function $F(k_t)$ cancels out in the Generalized Euler equation (11) that relates both policy choices and, as a consequence, it does not play any role in the characterization of the relationship between τ_t and η_t.

Proposition 7.- *The time-consistent optimal choice of the two policy instruments satisfies:*

$$\tau_t = \frac{1-\alpha}{1-\eta_t}, \forall t. \tag{25}$$

Proof.- *See Appendix 4.*

Again, optimal income tax rates will be above $1-\alpha$, whereas the optimal proportion of public resources devoted to consumption will always be below α.

We now argue that this economy lacks any transitional dynamics. Taking the result from Proposition 7 to (3) implies that the ratio i_{pt}/k_t is constant, the output to capital ratio $(y_t/k_t) = B^{1/\alpha}(1-\alpha)^{(1-\alpha)/\alpha}$ is also constant, and so is the real rate of interest. Under the Markov policy, this economy shares the same characteristics of standard AK-models. These results also imply that the ratios $\{\chi_t, \phi_t\}$ are not functions of k_t, so that $\partial\chi_t/\partial k_t = 0$, $\partial\phi_t/\partial k_t = 0$.

Hence, equations (20), (21) and (25) allow us to write equations (19) and (23) as a nonlinear dynamic system in $\{\chi_t, \eta_t\}$:

$$\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\gamma_t^{\sigma-\theta(1-\sigma)} = \rho\chi_{t+1}^{-\sigma}\phi_{t+1}^{\theta(1-\sigma)}\left(1-\delta+\alpha\Omega_{t+1}\right), \tag{26}$$

[9] They will be functions of the state k_t if there is transitional dynamics. Later on, we will show that this economy lacks any transitional dynamics so that η_{t+1}, τ_{t+1} and χ_{t+1}, ϕ_{t+1} end up being just functions of structural parameters.

$$\frac{\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\gamma_t^{-\sigma+\theta(1-\sigma)}\left(\chi_{\tau_t}+\chi_t\frac{\theta(1-\eta_t)}{\alpha(1-\alpha)}\right)}{\chi_{\tau_t}+\Lambda_t\,\Omega_t}=\rho\chi_{t+1}^{-\sigma}\phi_{t+1}^{\theta(1-\sigma)}\times$$
$$\left[(1+\theta)\chi_{t+1}+\frac{\chi_{\tau_{t+1}}+\chi_{t+1}\frac{\theta(1-\eta_{t+1})}{\alpha(1-\alpha)}}{\chi_{\tau_{t+1}}+\Lambda_{t+1}\,\Omega_{t+1}}\gamma_{t+1}\right], \tag{27}$$

where

$$\phi_t=\frac{\eta_t}{1-\eta_t}\left[(1-\alpha)B\right]^{1/\alpha},\ \gamma_t=\Omega_t+1-\delta-\chi_t,$$
$$\Omega_t=\frac{\alpha-\eta_t}{1-\eta_t}(1-\alpha)^{\frac{1-\alpha}{\alpha}}B^{1/\alpha},\ \Lambda_t=\frac{(1-\eta_t)\eta_t}{\alpha(\alpha-\eta_t)}.$$

The nonlinearity of this system forces us to analyzing the potential equilibrium local indeterminacy in this model numerically. The dynamic properties of this system can be analyzed through the two eigenvalues of its linearized version. An eigenvalue below 1 would suggest an indetermination of equilibrium since in that case, we would need an initial condition for either η or χ in order to compute the time series for $\{\chi_t,\eta_t\}$. Any arbitrary choice would yield a valid Markov equilibrium then producing a situation of indeterminacy of equilibria. That would in turn generate transitional dynamics, as the trajectory followed by the economy would depend on the initial choice for either η or χ. On the other hand, having both eigenvalues greater than one would imply that the only stable solution is obtained with η_t and χ_t being constant over time: $\eta_t=\eta,\chi_t=\chi,\ \forall t$, without indeterminacy of equilibria.

We have numerically computed such eigenvalues for wildly different parameterizations, obtaining always both eigenvalues above one, even for empirically implausible parameter values. Lacking an analytical proof, our numerical analysis suggests that there is not indeterminacy of equilibria, with $\eta_t=\eta,\chi_t=\chi,\ \forall t$, and hence, $\tau_t=\tau,\gamma_t=\gamma,\phi_t=\phi\ \forall t$, the economy being at each point in time on its balanced growth path.

5.1 Solving for the Markov equilibrium

Since the optimal policy instruments $\{\tau_t, \eta_t\}$ and the consumption to capital ratio $\frac{C(k_t;\tau_t,\eta_t)}{k_t} = \chi(\tau,\eta)$ are constant over time, we can compute the Markov equilibrium along the balanced growth path.

Evaluating the Euler equation (19) along the balanced growth path and using (25), we get:

$$\gamma = \left[\rho\left(1-\delta+\alpha\ \Omega(\tau,\eta)\right)\right]^{1/(\sigma-\theta(1-\sigma))}. \tag{28}$$

Taking this expression to (20), we get:

$$\chi(\tau,\eta) = 1-\delta+\Omega(\tau,\eta)-\left[\rho\left(1-\delta+\alpha\ \Omega(\tau,\eta)\right)\right]^{1/(\sigma-\theta(1-\sigma))}. \tag{29}$$

with partial derivatives χ_τ and χ_η:

$$\chi_\tau \equiv \frac{\partial\chi}{\partial\tau} = \Lambda(\tau)\Omega(\tau,\eta)\left(\frac{\rho\alpha\gamma^{1-\sigma+\theta(1-\sigma)}}{\sigma-\theta(1-\sigma)}-1\right), \tag{30}$$

$$\chi_\eta \equiv \frac{\partial\chi}{\partial\eta} = \frac{1-\alpha}{\alpha(1-\eta)}\Omega(\tau,\eta)\left(\frac{\rho\alpha\gamma^{1-\sigma+\theta(1-\sigma)}}{\sigma-\theta(1-\sigma)}-1\right). \tag{31}$$

Hence, equation (27) becomes

$$\frac{\gamma^{-\sigma+\theta(1-\sigma)}\left(\chi_\tau+\chi\frac{\theta(1-\eta)}{\alpha(1-\alpha)}\right)}{\chi_\tau+\Lambda\,\Omega} = \rho\left[(1+\theta)\chi+\frac{\chi_\tau+\chi\frac{\theta(1-\eta)}{\alpha(1-\alpha)}}{\chi_\tau+\Lambda\,\Omega}\gamma\right]. \tag{32}$$

Finally, the Markov equilibrium $\{\gamma^M, \chi^M, \tau^M, \eta^M\}$ is obtained as the solution to the system (25), (28), (29), and (32). The system can only be solved numerically and the following section is devoted to analyze its properties. Under all parameterizations considered, the system has been shown to have a single solution,[10] suggesting that the equilibrium is globally determined.

[10] When solving the nonlinear system of equations, we have tried very different sets of initial conditions, always reaching the same solution shown in the Tables.

6. Comparing the Ramsey and Markov solutions in the general case

Let us now compare the Markov and Ramsey solutions between themselves, as well as with the allocation of resources that would be achieved by a benevolent planner using lump-sum taxes, which is characterized in Appendix 5. We will use $\tau_t^P = \dfrac{g_t + i_{p,t}}{y_t}$ as a measure of the size of the public sector in the planner solution and we will use $\eta_t^P = \dfrac{g_t}{g_t + i_{p,t}}$ for the composition of public expenditures. Both of them will be used in the graphs and tables we present below.

Let us now examine the values taken by the main variables in the economy along the balanced growth path under the three alternative fiscal policies: *i*) the planner's policy under lump-sum taxes, *ii*) the Ramsey policy and *iii*) the time-consistent policy, all of them under the more general setup, with a CRRA utility function and incomplete depreciation of private capital. Unfortunately, our results are not readily comparable with those in the literature because numerical results are usually derived using a logarithmic, separable utility function, whereas our results correspond to general CRRA utility functions, and also because of our consideration of endogenous growth.

The Markov equilibrium is obtained as explained in section 5.1. As shown in Appendix 2, the solution to the Ramsey problem [P2] is characterized by a system of 8 dynamic equations in $\{\gamma, \chi, \phi, \eta, \tau, \tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3\}$ that allows us to compute the balanced growth path for the Ramsey policy (τ^R, η^R) as well as the implied allocation of resources, characterized by $(\gamma^R, \chi^R, \phi^R)$ and three multipliers, $\{\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3\}$. That system is made up only by control variables, with no participation of any state variable. Hence, in the absence of local indeterminacy of equilibrium, the only possible solution is that control variables stay on the balanced growth path (BGP) from the initial period, with no transition.[11]

Under incomplete depreciation of private capital, the choice of parameter values: $\theta = 0.4$, $1-\alpha = 0.20$, $\rho = 0.99$, $\delta = 0.10$, $B = 0.4555$, when generating annual data lead to sensible properties of the Markov solution. Parameter values are standard in the literature for annual data except for θ, which is chosen so that the ratio of public consumption to private

[11] For the parameterizations used in the paper we have numerically checked the absence of local indeterminacy of equilibria.

consumption for the Markov solution is in line with data for the postwar US economy (g/c=0.25). For instance, for $\sigma = 2$, we get a ratio of public to private consumption around 0.25, an annual growth rate γ=1.5%, and a gross real interest rate: $1/\left(\rho\gamma^{(1-\sigma)(1+\theta)}\right) \approx 1.03$. The value chosen for α is consistent with a broad concept of capital that includes both physical and human components, as it is commonly established in endogenous growth models with public production services and private capital (see Cazzavillian, 1996). As to the elasticity of output with respect to public capital, 1-α, we set a value which is in line with previous literature: Azzimonti et al. (2009) takes a benchmark elasticity of 0.25, but the range of values varies significantly across authors between the 0.03 estimated by Eberts (1986), and the 0.39 estimated by Aschauer (1989).

Figure 1 shows values for the main variables in the economy under the three equilibrium concepts as a function of the risk aversion parameter, σ. Over the whole range of values considered, the optimal income tax increases with risk aversion. It always falls between 20% and 30%, being higher under the Markov-perfect policy than under the time-inconsistent Ramsey policy. The proportion of public resources devoted to consumption, relative to production, is also increasing in σ, staying between 6% and 32%. It is also higher under the Markov-perfect solution than under the Ramsey policy.

Steady state growth is slightly higher under the Ramsey policy. Growth rates are large for low values of the risk aversion parameter, but they become quite realistic for values of σ above 1.5. As a proportion of output, private consumption is higher under the Ramsey policy, while public consumption is higher under the Markov policy. In terms of specific values, private consumption never exceeds 35% of output under either policy, while public consumption remains below 10% of output, both observations below the levels observed in actual economies. However, the ratio of public to private consumption is around 25%, as in observed data. For the Markov and Ramsey solutions we could obtain ratios of public and private consumption to output similar to those in actual data, at the expense of getting income tax rates implausibly high.

A planner with access to lump-sum taxes under commitment would devote to consumption an even higher proportion of public resources than the Markov and Ramsey solutions, and the growth rate would be considerably higher than under the alternative solutions.

That the income tax is higher under the Markov-perfect policy than under the Ramsey solution is consistent with the result obtained by Ortigueira (2006) in an exogenous

growth economy under inelastic labor supply.[12] This result arises because the Markovian government cannot internalize the distortionary effects of current taxation on past investment, while in the Ramsey solution, the government takes fully into account the negative effect of the income tax on future investment. A similar argument explains that the Markov government devotes a higher proportion of public resources to consumption, which has a direct impact on current utility, to the expense of productive services, which would have a positive effect mainly on future utility. Together with a higher share of resources devoted to production, a lower income tax rate leads to a higher growth rate under the Ramsey than under the Markov solution.

Figure 2 presents results for σ = 2, and values of the relative weight of public consumption in the utility function, θ, between 0.2 and 1.5, the remaining parameters being as in Figure 1. As expected, public consumption as a share of total public spending increases with θ. Qualitative results stay the same, with the Markov-perfect policy imposing a higher income tax than the Ramsey policy and devoting a higher proportion of public resources to consumption. The growth rate is again higher under the Ramsey than under the Markov policy.

Table 1 summarizes the results by displaying a single point from Figure 1 and Figure 2. Table 2 analyzes the effects of a change in α. The value of B has been chosen to guarantee positive growth rates under the Markov and Ramsey solutions.

Since the ratio of productive services to output is the same for the three solutions implies that the product $(1-\eta)\tau$ and hence, the ratio of private capital to output, are also the same for the three solutions under any parameterization. The common value of $(1-\eta)\tau$ turns out to be equal to the elasticity of output with respect to the public input, again an extension of the result obtained by Barro (1990).

The solution under lump-sum taxes leads to the largest public sector and devotes a lowest share of public resources to production. Since taxes are not distortionary under the planner's solution, a larger proportion of resources extracted by the public sector can be made compatible with a higher rate of growth.

The comparison between the two panels in Table 1 shows what happens as public consumption becomes more important in the utility function: while the ratios of both types of capital to output remain unchanged, the optimal tax rate increases, as it does the

[12] Even though the two results are not strictly comparable, since one of them refers to an exogenous growth economy and the other to an endogenous growth economy.

proportion of public resources devoted to consumption. These two changes lead to a lower rate of growth.[13]

Table 1. Values for the main variables under the three solution concepts. Effects of a change in θ

	$B = 0.4555$, $\sigma = 2.00$, $\theta = 0.40$, $\alpha = 0.80$, $\delta = 0.10$, $\rho = 0.99$			$B = 0.4555$, $\sigma = 2.00$, $\theta = 1.00$, $\alpha = 0.80$, $\delta = 0.10$, $\rho = 0.99$		
	Planner	*Markov*	*Ramsey*	*Planner*	*Markov*	*Ramsey*
η (%)	26.7	24.9	20.4	41.6	38.7	30.9
τ (%)	27.3	26.6	25.1	34.2	32.6	28.9
γ (%)	3.6	1.5	1.6	2.9	0.8	1.1
c/y(%)	18.3	27.4	28.4	14.3	24.2	26.9
g/y(%)	7.3	6.6	5.1	14.3	12.6	8.9
i_p/y(%)	20.0	20.0	20.0	20.0	20.0	20.0
k/y	4.0	4.0	4.0	4.0	4.0	4.0

Table 2. Values for the main variables under the three solution concepts. Effects of a change in α

	$B = 0.658$, $\sigma = 2.00$, $\theta = 0.40$, $\alpha = 0.80$, $\delta = 0.10$, $\rho = 0.99$			$B = 0.658$, $\sigma = 2.00$, $\theta = 0.40$, $\alpha = 0.70$, $\delta = 0.10$, $\rho = 0.99$		
	Planner	*Markov*	*Ramsey*	*Planner*	*Markov*	*Ramsey*
η (%)	32.9	32.1	28.9	19.2	16.9	13.1
τ (%)	29.8	29.4	28.1	37.1	36.1	34.5
γ (%)	8.1	4.5	4.7	4.8	1.5	1.6
c/y(%)	24.6	33.9	34.8	17.9	28.9	30.0
g/y(%)	9.8	9.5	8.1	7.1	6.1	4.5
i_p/y(%)	20.0	20.0	20.0	30.0	30.0	30.0
k/y	2.5	2.5	2.5	3.0	3.0	3.0

Note to the tables: for the planner solution $\tau_t^P = \dfrac{g_t + i_{p,t}}{y_t}$ and $\eta_t^P = \dfrac{g_t}{g_t + i_{p,t}}$.

Table 2 shows that an increase in the productivity of the public input (lower α) leads to higher tax rates. The government then detracts more aggregate resources from the economy and devotes a larger proportion of them to production. Because of the increase in the tax rate generated by a lower α parameter, the productivity of private capital and hence, the rate of growth, both decrease.

Rates of growth under the Ramsey and Markov policies in Tables 1 and 2 are very similar. However, for many alternative parameterizations, they may easily differ in close to one percent point.[14]

[13] Rates of growth under the Ramsey and Markov policies in Tables 1 and 2 are very similar. However, for many alternative parameterizations, they may easily differ in close to one percent point.

7. Welfare

In this section we compute the level of welfare that would arise along the balanced growth path under the time consistent Markov policy and compare it with the level of welfare that would be obtained under lump-sum taxes.[15] As in Lucas (1987), what we compute is the consumption compensation (as a percentage of output) that would be needed under the Markov rule to achieve the same level of welfare than under the resource allocation of the planner with non-distortionary taxation.

Under a CRRA utility, welfare can be written,

$$W_i = \sum_{t=0}^{\infty} \rho^t \frac{c_{t,i}^{1-\sigma} g_{t,i}^{\theta(1-\sigma)} - 1}{1-\sigma} = \frac{1}{1-\sigma}\left[\frac{\chi_i^{1-\sigma}\phi_i^{1-\sigma}}{1-\rho\,\gamma_i^{(1-\sigma)(1+\theta)}} - \frac{1}{1-\rho}\right], \quad i = P,\ M\ .$$

Let $\{c_{t,i},\ g_{t,i}\}$, with i=P, M, be the optimal path for private and public consumption for the planner's solution (P) and the Markov solution (M), respectively, that is:

$$c_{t,i} = \chi_i k_{t,i} = \chi_i k_0 \gamma_i^t \underbrace{=}_{k_0=1} \chi_i \gamma_i^t,$$

$$g_{t,i} = \phi_i k_{t,i} = \phi_i k_0 \gamma_i^t \underbrace{=}_{k_0=1} \phi_i \gamma_i^t, \qquad i = P,\ M,\ R$$

where we have indicated the normalization k_0=1.

The consumption compensation λ needed for the Markov and Ramsey solutions to achieve the same level of welfare as under the planner's allocation can be obtained by solving the following equation:

$$W_P = \sum_{t=0}^{\infty} \rho^t \frac{(1+\lambda)^{1-\sigma} c_{t,j}^{1-\sigma} g_{t,j}^{\theta(1-\sigma)} - 1}{1-\sigma}, j = M, R$$

that is,

$$\frac{1}{1-\sigma}\left[\frac{\chi_P^{1-\sigma}\phi_P^{1-\sigma}}{1-\rho\,\gamma_P^{(1-\sigma)(1+\theta)}} - \frac{1}{1-\rho}\right] = \frac{1}{1-\sigma}\left[\frac{(1+\lambda)^{1-\sigma}\chi_j^{1-\sigma}\phi_j^{1-\sigma}}{1-\rho\,\gamma_j^{(1-\sigma)(1+\theta)}} - \frac{1}{1-\rho}\right], j = M, R$$

and finally,

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[14] For instance, for $B =$ 0.90, $\sigma =$ 2.00, $\theta =$ 2.00, $\alpha =$ 0.85, $\delta =$ 0.10, $\rho =$ 0.96, growth rates under Markov and Ramsey policies become 2.76% and 3.43%.

[15] We do not consider the level of welfare under the Ramsey solution because of its time-inconsistent nature.

$$1+\lambda=\left[\frac{1-\rho\,\gamma_j^{(1-\sigma)(1+\theta)}}{1-\rho\,\gamma_P^{(1-\sigma)(1+\theta)}}\right]^{\frac{1}{1-\sigma}}\frac{\chi_P}{\chi_j}\left(\frac{\phi_P}{\phi_j}\right)^{\theta},\ j=M,R. \tag{33}$$

To translate this compensation into output units, we have to compute $100\lambda\frac{c_{t,j}}{y_{t,j}}, j=M,R$, which is the compensation shown in Figure 3.

As the risk aversion parameter changes between 1 and 5, the Markov consumption compensation falls from 45% to 3% of output. In particular, for σ=2, the compensation that would be necessary to achieve the planner's welfare is around 8% of output. By and large, the decrease in consumption compensation is due to the decline in the value of the first factor in (33).[16]

The consumption compensation increases with θ. For σ=2, the Markov consumption compensation increases from 6% to 23% of output. Again, this increase in the consumption compensation is mainly due to the first factor in (33).[17] So, the difference in growth rates is the main determinant of the welfare loss under the Markov solution relative to the planner's solution, over and above the effects of differences in the ratios of private or public consumption to output.

In both cases, the Ramsey policy, if it could be maintained over time, it would lead to a slightly bigger loss of welfare relative to the planner policy. Both policies are not strictly comparable, and the welfare comparisons could go either way. In Klein et al. (2008), the Ramsey policy leads to a bigger welfare loss when the economy is subject to a total income tax. In Ortigueira (2006) the same result arises when the only source of revenue is a tax on labor income.[18]

[16] The first factor, which depends on growth rates, falls from 17.13 for σ=1.1, to 1.23 for σ=5. The second factor increases from 0.29 to 0.86, while the third factor initially increases from its starting value of 1.018 to 1.054, and it decreases after that to essentially its same initial level.

[17] The first factor increases from 1.72 to 3.02 as θ changes from 0.2 to 1.5. The second factor gradually decreases from 0.70 to 0.54, and the third factor shows a moderate increase, from 1.13 to 1.23.

[18] These authors do not report welfare levels. We have used the steady-state values they provide to compute steady-state welfare levels. Our statements above are valid under the assumption that the policy rules in both papers would position the economies on their balanced growth paths from the starting period.

8. Conclusions

We have characterized the optimal Markov-perfect fiscal policy in an endogenous growth economy where the fiscal authority cannot commit to policy choices beyond the current period. Tax revenues are used to finance public consumption and public production services, and we have considered two policy variables: a single tax on total income and the split of public resources between consumption and productive services.

Under logarithmic preferences and full depreciation of private capital, we can analytically characterize the optimal values of the two policy variables. With that particular specification, we show that the Markov-perfect policy coincides with the optimal Ramsey policy that would arise by imposing commitment.

For the more general case of a CRRA utility function and less than perfect depreciation of capital, we show the economy to be on its balanced growth path from the initial period onwards. In this case there is no closed form solution, but we compute numerical values for the Markov-perfect and the Ramsey optimal policies under parameter values calibrated to the US economy. We also explore the sensitivity of the numerical solutions to the values of three parameters: the intertemporal elasticity of substitution of consumption, the relative weight of public consumption in agents' utility function and the elasticity of output with respect to private capital. For empirically plausible parameter values, the income tax is higher under the Markov policy than under the Ramsey solution, and a higher proportion of public resources are devoted to consumption. Consequently, the growth rate is lower under the Markov policy than under the Ramsey solution.

The welfare loss of the Markov solution relative to the planner's allocation is mainly determined by the differences in growth rates, more than by differences in the ratios of private or public consumption to output.

The implication of our results is that if the private sector is aware of the government's inability to pledge future policy decisions, then the government should impose a slightly higher tax rate and devote a higher share of public resources to consumption, with a relatively low cost in terms of growth.

Considering a more complex tax structure, as well as non-trivial transitional dynamics in an endogenous growth model with public debt, are left as future extensions of this work.

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Appendix 1: *Proof of Proposition* 1

First order optimality conditions for the government's problem are:

- with respect to τ:

$$U_{c_t}\mathcal{C}_{\tau_t} + U_{g_t}\mathcal{G}_{\tau_t} + \rho\,\tilde{V}_{k_{t+1}}\left(\frac{\partial\Omega(\tau_t,\eta_t)}{\partial\tau_t}k_t - \mathcal{C}_{\tau_t}\right) = 0\,,$$

where:

$$\frac{\partial\Omega(\tau_t,\eta_t)}{\partial\tau_t} = (1-\tau_t)\left[(1-\eta_t)\tau_t\right]^{\frac{1-\alpha}{\alpha}} B^{1/\alpha}\,\frac{\tau_t-(1-\alpha)}{\alpha\tau_t(1-\tau_t)} = -\Omega(\tau_t,\eta_t)\Lambda(\tau_t)\,,$$

so that:

$$U_{c_t}\mathcal{C}_{\tau_t} + U_{g_t}\mathcal{G}_{\tau_t} = \rho\,\tilde{V}_{k_{t+1}}\left(\Omega(\tau_t,\eta_t)\,\Lambda(\tau_t)k_t + \mathcal{C}_{\tau_t}\right)$$

- with respect to η:

$$U_{c_t}\mathcal{C}_{\eta_t} + U_{g_t}\mathcal{G}_{\eta_t} + \rho\,\tilde{V}_{k_{t+1}}\left(\frac{\partial\Omega(\tau_t,\eta_t)}{\partial\eta_t}k_t - \mathcal{C}_{\eta_t}\right) = 0\,,$$

where:

$$\frac{\partial\Omega(\tau_t,\eta_t)}{\partial\eta_t} = -(1-\tau_t)\left[(1-\eta_t)\tau_t\right]^{\frac{1-\alpha}{\alpha}}\frac{1-\alpha}{\alpha}\frac{1}{1-\eta_t}B^{1/\alpha} = -\Omega(\tau_t,\eta_t)\frac{1-\alpha}{\alpha}\frac{1}{1-\eta_t}\,,$$

so that:

$$U_{c_t}\mathcal{C}_{\eta_t} + U_{g_t}\mathcal{G}_{\eta_t} = \rho\,\tilde{V}_{k_{t+1}}\left(\Omega(\tau_t,\eta_t)\frac{1-\alpha}{\alpha}\frac{1}{1-\eta_t}k_t + \mathcal{C}_{\eta_t}\right).$$

The envelope condition is:

$$V_{k_t} = U_{c_t}\mathcal{C}_{k_t} + U_{c_t}\left(\mathcal{C}_{\tau_t}\frac{\partial\tau_t}{\partial k_t} + \mathcal{C}_{\eta_t}\frac{\partial\eta_t}{\partial k_t}\right) + U_{g_t}\mathcal{G}_{k_t} + U_{g_t}\left(\mathcal{G}_{\tau_t}\frac{\partial\tau_t}{\partial k_t} + \mathcal{G}_{\eta_t}\frac{\partial\eta_t}{\partial k_t}\right) +$$
$$\rho\,\tilde{V}_{k_{t+1}}\left[1-\delta+\Omega(\tau_t,\eta_t)-\mathcal{C}_{k_t} + \left(\frac{\partial\Omega(\tau_t,\eta_t)}{\partial\tau_t}k_t - \mathcal{C}_{\tau_t}\right)\frac{\partial\tau_t}{\partial k_t} + \left(\frac{\partial\Omega(\tau_t,\eta_t)}{\partial\eta_t}k_t - \mathcal{C}_{\eta_t}\right)\frac{\partial\eta_t}{\partial k_t}\right],$$

which, after using the first order conditions derived above, it can be written as

$$V_{k_t} = U_{c_t}\mathcal{C}_{k_t} + U_{g_t}\mathcal{G}_{k_t} + \rho\,\tilde{V}_{k_{t+1}}\left[1-\delta+\Omega(\tau_t,\eta_t)-\mathcal{C}_{k_t}\right].$$

From the optimality conditions above we get,

$$\rho \tilde{V}_{k_{t+1}} = \frac{U_{c_t}\mathcal{C}_{\tau_t} + U_{g_t}\mathcal{G}_{\tau_t}}{\Omega(\tau_t,\eta_t)\Lambda(\tau_t)k_t + \mathcal{C}_{\tau_t}},$$

$$\rho \tilde{V}_{k_{t+1}} = \frac{U_{c_t}\mathcal{C}_{\eta_t} + U_{g_t}\mathcal{G}_{\eta_t}}{\Omega(\tau_t,\eta_t)\frac{1-\alpha}{\alpha}\frac{1}{1-\eta_t}k_t + \mathcal{C}_{\eta_t}},$$

which leads to condition (11).

Plugging the first equation into the envelope condition we get,

$$V_{k_t} = U_{c_t}\mathcal{C}_{k_t} + U_{g_t}\mathcal{G}_{k_t} + \frac{U_{c_t}\mathcal{C}_{\tau_t} + U_{g_t}\mathcal{G}_{\tau_t}}{\Omega(\tau_t,\eta_t)\Lambda(\tau_t)k_t + \mathcal{C}_{\tau_t}}\left[1-\delta+\Omega(\tau_t,\eta_t)-\mathcal{C}_{k_t}\right],$$

and, finally, we get equation (12):

$$\frac{U_{c_t}\mathcal{C}_{\tau_t} + U_{g_t}\mathcal{G}_{\tau_t}}{\Omega(\tau_t,\eta_t)\Lambda(\tau_t)k_t + \mathcal{C}_{\tau_t}} = \rho\left[U_{c_{t+1}}\mathcal{C}_{k_{t+1}} + U_{g_{t+1}}\mathcal{G}_{k_{t+1}} + \frac{U_{c_{t+1}}\mathcal{C}_{\tau_{t+1}} + U_{g_{t+1}}\mathcal{G}_{\tau_{t+1}}}{\Omega(\tau_{t+1},\eta_{t+1})\Lambda(\tau_{t+1})k_{t+1} + \mathcal{C}_{\tau_{t+1}}}\times \left[1-\delta+\Omega(\tau_{t+1},\eta_{t+1})-\mathcal{C}_{k_{t+1}}\right]\right].$$

Appendix 2: *Optimal Ramsey policy under a CRRA utility function and incomplete depreciation of private capital*

The Ramsey optimal policy is the solution to the utility maximization problem, subject to the equilibrium conditions as constraints. Under the CRRA utility function, the Lagrangian for the Ramsey problem becomes:

$$L = \sum_{t=0}^{\infty}\rho^t \frac{c_t^{1-\sigma}g_t^{\theta(1-\sigma)}-1}{1-\sigma} + \rho^t\mu_{1t}\left[\left(1-\delta+\Omega(\tau_t,\eta_t)\right)k_t - c_t - k_{t+1}\right] +$$
$$\rho^t\mu_{2t}\left[\eta_t\left(1-\eta_t\right)^{\frac{1-\alpha}{\alpha}}\tau_t^{1/\alpha}B^{1/\alpha}k_t - g_t\right] +$$
$$\rho^t\mu_{3t}\left[\rho c_{t+1}^{-\sigma}g_{t+1}^{\theta(1-\sigma)}\left(1-\delta+\alpha\,\Omega(\tau_{t+1},\eta_{t+1})\right) - c_t^{-\sigma}g_t^{\theta(1-\sigma)}\right].$$

Taking the derivatives with respect to $c_t, g_t, k_{t+1}, \tau_t, \eta_t$ to be equal to zero, we obtain the optimality conditions for the Ramsey problem:

$$c_t^{-\sigma}g_t^{\theta(1-\sigma)} = \mu_{1t} - \mu_{3t}\sigma c_t^{-\sigma-1}g_t^{\theta(1-\sigma)} + \sigma\mu_{3,t-1}c_t^{-\sigma-1}g_t^{\theta(1-\sigma)}\left(1-\delta+\alpha\,\Omega(\tau_t,\eta_t)\right),$$

$$\theta c_t^{-\sigma}g_t^{\theta(1-\sigma)} = \mu_{2t} - (1-\sigma)\theta c_t^{-\sigma-1}g_t^{\theta(1-\sigma)}\left[\mu_{3t} - \mu_{3,t-1}\left(1-\delta+\alpha\,\Omega(\tau_t,\eta_t)\right)\right],$$

$$\mu_{1t} = \rho\left[\mu_{1,t+1}\left(1-\delta+\Omega(\tau_{t+1},\eta_{t+1})\right) + \mu_{2,t+1}B^{1/\alpha}\tau_{t+1}^{1/\alpha}\eta_{t+1}(1-\eta_{t+1})^{\frac{1-\alpha}{\alpha}}\right],$$

$$\mu_{1t}k_t\left(\frac{1-\alpha}{\alpha}\frac{1-\tau_t}{\tau_t}-1\right) + \mu_{2t}\eta_t k_t\frac{1}{\alpha} + \mu_{3t-1}c_t^{-\sigma}g_t^{\theta(1-\sigma)}\alpha\left(\frac{1-\alpha}{\alpha}\frac{1-\tau_t}{\tau_t}-1\right) = 0,$$

$$-\mu_{1t}\frac{1-\alpha}{\alpha}k_t+\mu_{2t}\frac{\tau_t}{1-\tau_t}k_t\left(1-\frac{\eta_t}{\alpha}\right)-\mu_{3t-1}c_t^{-\sigma}g_t^{\theta(1-\sigma)}\left(1-\alpha\right)=0\,.$$

Transforming the multipliers by: $\tilde{\mu}_{1t}\equiv\mu_{1t}k^{\sigma-\theta(1-\sigma)}$, $\tilde{\mu}_{2t}\equiv\mu_{2t}k^{\sigma-\theta(1-\sigma)}$, $\tilde{\mu}_{3t}\equiv\frac{\mu_{3t}}{k_t}$, and defining the rate of growth $\gamma_{t+1}=\frac{k_{t+1}}{k_t}$, and the ratios of private and public consumption to capital $\chi_t=\frac{c_t}{k_t}$, $\phi_t=\frac{g_t}{k_t}$, we can get a system of equations in stationary ratios. First, from the global constraint of resources, we get an expression for the growth rate:

$$\gamma_{t+1}=1-\delta+\Omega(\tau_t,\eta_t)-\chi_t\,.$$

Whereas from the government budget constraint, we can write the ratio of public to private capital:

$$\phi_t=B^{1/\alpha}\tau_t^{1/\alpha}\eta_t(1-\eta_t)^{\frac{1-\alpha}{\alpha}}\,.$$

From the Euler equation for the competitive equilibrium:

$$x_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\gamma_{t+1}^{\sigma-\theta(1-\sigma)}=\rho x_{t+1}^{-\sigma}\phi_{t+1}^{\theta(1-\sigma)}\left[\left(1-\delta+\alpha\,\Omega(\tau_{t+1},\eta_{t+1})\right)\right],$$

and from the set of optimality conditions above, we finally get the system of equations characterizing the optimal Ramsey policy represented in stationary ratios:

$$x_t^{-\sigma}\phi_t^{\theta(1-\sigma)}=\tilde{\mu}_{1t}-\sigma x_t^{-\sigma-1}\phi_t^{\theta(1-\sigma)}\left\{\tilde{\mu}_{3t}-\tilde{\mu}_{3t-1}\frac{1}{\gamma_t}\left[\left(1-\delta+\alpha\,\Omega(\tau_t,\eta_t)\right)\right]\right\},$$

$$\theta x_t^{1-\sigma}\phi_t^{\theta(1-\sigma)-1}=\tilde{\mu}_{2t}-(1-\sigma)\theta x_t^{-\sigma}\phi_t^{\theta(1-\sigma)-1}\left\{\tilde{\mu}_{3t}-\tilde{\mu}_{3t-1}\frac{1}{\gamma_t}\left[\left(1-\delta+\alpha\,\Omega(\tau_t,\eta_t)\right)\right]\right\},$$

$$\tilde{\mu}_{1t}=\rho\gamma_{t+1}^{\theta(1-\sigma)-\sigma}\left[\tilde{\mu}_{1t+1}\left(1-\delta+\Omega(\tau_{t+1},\eta_{t+1})\right)+\tilde{\mu}_{2t+1}B^{1/\alpha}\tau_{t+1}^{1/\alpha}(1-\eta_{t+1})^{\frac{1-\alpha}{\alpha}}\right],$$

$$\tilde{\mu}_{1t}\left(\frac{1-\alpha}{\alpha}\frac{1-\tau_t}{\tau_t}-1\right)+\tilde{\mu}_{2t}\eta_t\frac{1}{\alpha}+\tilde{\mu}_{3t-1}\frac{1}{\gamma_t}x_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\alpha\left(\frac{1-\alpha}{\alpha}\frac{1-\tau_t}{\tau_t}-1\right)=0,$$

$$-\tilde{\mu}_{1t}\frac{1-\alpha}{\alpha}+\tilde{\mu}_{2t}\frac{\tau_t}{1-\tau_t}\left(1-\frac{\eta_t}{\alpha}\right)-\tilde{\mu}_{3t-1}\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\left(1-\alpha\right)\frac{1}{\gamma_t}=0.$$

Along the balanced growth path, the system of equations for the Ramsey equilibrium becomes:

$$\gamma^{\sigma-\theta(1-\sigma)}=\rho\left(1-\delta+\alpha\,\Omega(\tau,\eta)\right),$$

$$\chi=1-\delta+\Omega(\tau,\eta)-\gamma,$$

$$\phi = B^{1/\alpha}\tau^{1/\alpha}\eta(1-\eta)^{\frac{1-\alpha}{\alpha}},$$

$$1 = \tilde{\mu}_1 x^{\sigma}\phi^{-\theta(1-\sigma)} + \tilde{\mu}_3\sigma\frac{1}{\chi}\left[\frac{1}{\gamma}\left(1-\delta+\alpha\,\Omega(\tau,\eta)\right)-1\right],$$

$$1 = \tilde{\mu}_2\frac{1}{\theta}x_t^{\sigma-1}\phi^{1-\theta(1-\sigma)} - \tilde{\mu}_3\frac{1-\sigma}{\chi}\left[\frac{1}{\gamma}\left(1-\delta+\alpha\,\Omega(\tau,\eta)\right)-1\right],$$

$$\tilde{\mu}_1\left[1-\rho\gamma^{\theta(1-\sigma)-\sigma}\left(1-\delta+\Omega(\tau,\eta)\right)\right] = \rho\gamma^{\theta(1-\sigma)-\sigma}\tilde{\mu}_2 B^{1/\alpha}\tau^{1/\alpha}\eta(1-\eta)^{\frac{1-\alpha}{\alpha}},$$

$$\left(\frac{1-\alpha}{\alpha}\frac{1-\tau}{\tau}-1\right) + \tilde{\mu}_2\eta\frac{1}{\alpha} + \tilde{\mu}_3\frac{1}{\gamma}x^{-\sigma}\phi^{\theta(1-\sigma)}\alpha\left(\frac{1-\alpha}{\alpha}\frac{1-\tau}{\tau}-1\right) = 0,$$

$$-\tilde{\mu}_1\frac{1-\alpha}{\alpha} + \tilde{\mu}_2\frac{\tau}{1-\tau}\left(1-\frac{\eta}{\alpha}\right) - \tilde{\mu}_3\chi^{-\sigma}\phi^{\theta(1-\sigma)}\left(1-\alpha\right)\frac{1}{\gamma} = 0\,.$$

Denoting by:

$\Psi = \frac{1}{\gamma}\left[1-\delta+\alpha\,\Omega(\tau,\eta)\right]-1$, $F = 1-\delta+\Omega(\tau,\eta)$, and $\Gamma = \frac{1-\rho\gamma^{\theta(1-\sigma)-\sigma}F}{\rho\gamma^{\theta(1-\sigma)-\sigma}\phi}$,

we characterize the balanced growth path of the Ramsey equilibrium by particularizing the system of equations above to:

$$\gamma = \left\{\rho\left[\left(1-\delta+\alpha\,\Omega(\tau,\eta)\right)\right]\right\}^{\frac{1}{\sigma-\theta(1-\sigma)}},$$

$$\chi = 1-\delta+\Omega(\tau,\eta)-\gamma,$$

$$\phi = B^{1/\alpha}\tau^{1/\alpha}\eta(1-\eta)^{\frac{1-\alpha}{\alpha}},$$

$$\tilde{\mu}_1 = \frac{1/\sigma}{\chi^{\sigma}\phi^{-\theta(1-\sigma)}\left[\frac{1}{\theta}\frac{\phi\Gamma}{\chi}+\frac{1-\sigma}{\sigma}\right]},$$

$$\tilde{\mu}_2 = \Gamma\tilde{\mu}_1,$$

$$\tilde{\mu}_3 = \frac{1-\chi^{\sigma}\phi^{-\theta(1-\sigma)}\tilde{\mu}_1}{\Psi\sigma/\chi},$$

$$\left[\tilde{\mu}_1 + \tilde{\mu}_3\frac{1}{\gamma}\chi^{-\sigma}\phi^{\theta(1-\sigma)}\alpha\right]\left(\frac{1-\alpha}{\alpha}\frac{1-\tau}{\tau}-1\right) + \tilde{\mu}_2\eta\frac{1}{\alpha} = 0,$$

$$-\tilde{\mu}_1\frac{1-\alpha}{\alpha} + \tilde{\mu}_2\frac{\tau}{1-\tau}\left(1-\frac{\eta}{\alpha}\right) - \tilde{\mu}_3\chi^{-\sigma}\phi^{\theta(1-\sigma)}(1-\alpha)\frac{1}{\gamma} = 0,$$

a system of 8 equations in $\{\gamma,\chi,\phi,\eta,\tau,\tilde{\mu}_1,\tilde{\mu}_2,\tilde{\mu}_3\}$ that allows us to compute the balanced growth path for the Ramsey policy (τ^R,η^R) as well as the implied allocation of resources, characterized by (γ^R,χ^R,ϕ^R).

Appendix 3.- *Proofs of propositions 3, 4 and 6*

Proof of Proposition 3:

The problem solved by the government is:

$$V(k_t) = \underset{\{\tau_t,\eta_t\}}{\text{Max}}\left[\ln \mathcal{C}(k_t;\tau_t,\eta_t) + \theta \ln \mathcal{G}(k_t;\tau_t,\eta_t) + \rho\,\tilde{V}(k_{t+1})\right]$$

where $k_{t+1} = \Omega(\tau_t,\eta_t)k_t - \mathcal{C}(k_t;\tau_t,\eta_t)$,

$$\mathcal{C}(k_t;\tau_t,\eta_t) = (1-\rho\alpha)\,\Omega(\tau_t,\eta_t)k_t,$$

$$\mathcal{G}(k_t;\tau_t,\eta_t) = \frac{\eta_t\tau_t}{1-\tau_t}\Omega(\tau_t,\eta_t)\,k_t.$$

The first order conditions for this problem are:

$$\tau_t:\ -(1+\theta)\,\Lambda(\tau_t) + \frac{\theta}{\tau_t(1-\tau_t)} - \rho\,\tilde{V}_{k_{t+1}}\left[\rho\alpha\,\Omega(\tau_t,\eta_t)\,\Lambda(\tau_t)k_t\right] = 0, \qquad \text{(A3.1)}$$

$$\eta_t:\ -(1+\theta)\frac{1-\alpha}{\alpha}\frac{1}{1-\eta_t} + \frac{\theta}{\eta_t} - \rho\,\tilde{V}_{k_{t+1}}\left[\rho\alpha\,\Omega(\tau_t,\eta_t)\frac{1-\alpha}{\alpha}\frac{1}{1-\eta_t}k_t\right] = 0, \qquad \text{(A3.2)}$$

From (A3.1) and (A3.2) we obtain a relationship between the optimal values of the tax rate and the government spending split in the Markov-perfect equilibrium:

$$\tau_t^M = \frac{1-\alpha}{1-\eta_t^M}\ \forall t\,. \qquad \text{(A3.3)}$$

Proof of Proposition 4:

i) To examine the dynamic properties of the Markov solution, we consider the envelope condition :

$$V_{k_t} = (1+\theta)\frac{1}{k_t} + (1+\theta)\frac{1}{\Omega(\tau_t,\eta_t)}\frac{\partial\Omega(\tau_t,\eta_t)}{\partial\tau_t}\left(\frac{\partial\tau_t}{\partial k_t} + \frac{\partial\eta_t}{\partial k_t}\right) +$$
$$\theta\left(\frac{1}{\tau_t(1-\tau_t)}\frac{\partial\tau_t}{\partial k_t} + \frac{1}{\eta_t}\frac{\partial\eta_t}{\partial k_t}\right) + \rho\,\tilde{V}_{k_{t+1}}\left[\rho\alpha\left(\Omega(\tau_t,\eta_t) + \frac{\partial\Omega(\tau_t,\eta_t)}{\partial\tau_t}\left(\frac{\partial\tau_t}{\partial k_t} + \frac{\partial\eta_t}{\partial k_t}\right)\right)\right],$$

which, using conditions (A3.1) and (A3.2), it can be written as,

$$V_{k_t} = (1+\theta)\frac{1}{k_t} + \rho\,\tilde{V}_{k_{t+1}}\left[\rho\alpha\,\Omega(\tau_t,\eta_t)\right]. \qquad \text{(A3.4)}$$

Using (A3.2) and (A3.3) in (A3.4), we obtain the dynamic equation:

$$\tilde{\eta}_t - \frac{1}{\rho}\tilde{\eta}_{t-1} + \frac{1+\theta}{\rho} = 0, \qquad \text{(A3.5)}$$

where $\tilde{\eta}_t \equiv \frac{\theta\alpha}{(1-\alpha)}\frac{1-\eta_t}{\eta_t}$. The solution to the difference equation (A3.5) is unstable, since $1/\rho > 1$. Hence the only stable solution is that $\tilde{\eta}_t$ stays constant over time, and the same applies to η_t, that is, $\eta_t = \eta, \forall t$.

ii) Since η_t is constant, condition (A3.3) implies that τ_t is also constant. From the two conditions:

$$\mathcal{C}(k_t;\tau_t,\eta_t) = (1-\rho\alpha)\,\Omega(\tau_t,\eta_t)k_t,$$

$$\mathcal{G}(k_t;\tau_t,\eta_t) = \frac{\eta_t\tau_t}{1-\tau_t}\Omega(\tau_t,\eta_t)\,k_t,$$

implying that the ratios χ_t and ϕ_t will also be constant. Together with the absence of indeterminacy, this result implies that the economy lacks transitional dynamics, being on the balanced growth path from the initial period on.

iii) From (A3.5) we obtain the value of η:

$$\eta^M = \frac{\alpha\theta(1-\rho)}{1-\alpha+\theta(1-\alpha\rho)},$$

and using (A3.3), we obtain the Markov perfect optimal tax rate:[19]

$$\tau^M = 1 - \frac{\alpha(1+\rho\theta)}{1+\theta}. \square$$

Proof of proposition 6:

Particularizing the system of equations for the balanced growth path under the optimal Ramsey policy obtained in Appendix 2 to the case of a logarithmic utility function (σ=1) and full depreciation (δ=1), we obtain:

[19] Malley et al. (2002) obtain a similar expression for the Markov perfect tax rate.

$$\gamma = \rho B^{1/\alpha}(1-\tau)\alpha\left[\tau(1-\eta)\right]^{\frac{1-\alpha}{\alpha}} \Rightarrow \gamma = \rho\alpha F,$$

$$\chi = B^{1/\alpha}(1-\tau)\left[\tau(1-\eta)\right]^{\frac{1-\alpha}{\alpha}} - \gamma,$$

$$\phi = B^{1/\alpha}\tau^{1/\alpha}\eta(1-\eta)^{\frac{1-\alpha}{\alpha}},$$

$$\Omega = \frac{1}{\gamma}\left(B^{1/\alpha}(1-\tau)\alpha\left[\tau(1-\eta)\right]^{\frac{1-\alpha}{\alpha}}\right) - 1 = \frac{1}{\rho} - 1,$$

$$F = B^{1/\alpha}(1-\tau)\left[\tau(1-\eta)\right]^{\frac{1-\alpha}{\alpha}},$$

$$\Gamma = \frac{1-\rho\gamma^{-1}F}{\rho\gamma^{-1}\phi} = \frac{1-1/\alpha}{1-\rho\gamma^{-1}F},$$

$$\tilde{\mu}_1 = \frac{1}{\chi\frac{1}{\theta}\frac{\phi\Gamma}{\chi}} = \frac{\theta\rho\gamma^{-1}}{1-\rho\gamma^{-1}F} = -\frac{\theta}{B^{1/\alpha}(1-\tau)(1-\alpha)\left[\tau(1-\eta)\right]^{\frac{1-\alpha}{\alpha}}},$$

$$\tilde{\mu}_2 = \Gamma\tilde{\mu}_1 = \frac{\theta}{\phi} = -\frac{\theta}{B^{1/\alpha}\tau^{1/\alpha}\eta(1-\eta)^{\frac{1-\alpha}{\alpha}}},$$

$$\tilde{\mu}_3 = \frac{1-\chi\tilde{\mu}_1}{\left(\frac{1}{\rho}-1\right)\frac{1}{\chi}},$$

together with:

$$\left(\tilde{\mu}_1 + \tilde{\mu}_3\frac{1}{\gamma\chi}\alpha\right)\frac{1-\alpha-\tau}{\alpha\tau} + \tilde{\mu}_2\eta\frac{1}{\alpha} = 0,$$

$$-\tilde{\mu}_1\frac{1-\alpha}{\alpha} + \tilde{\mu}_2\frac{\tau}{1-\tau}\left(1-\frac{\eta}{\alpha}\right) - \tilde{\mu}_3\chi(1-\alpha)\frac{1}{\gamma} = 0.$$

Substituting the expressions for the Lagrange multipliers into the last two equations gives us:

$$\left(-\frac{\theta}{B^{1/\alpha}(1-\tau)(1-\alpha)\left[\tau(1-\eta)\right]^{\frac{1-\alpha}{\alpha}}} + \frac{1-\chi\tilde{\mu}_1}{\frac{1}{\rho}-1}\frac{1}{\gamma}\alpha\right)\frac{1-\alpha-\tau}{\alpha\tau} + \frac{\theta}{B^{1/\alpha}\tau^{1/\alpha}\eta(1-\eta)^{\frac{1-\alpha}{\alpha}}}\eta\frac{1}{\alpha} = 0,$$

$$\frac{\theta}{B^{1/\alpha}(1-\tau)(1-\alpha)\left[\tau(1-\eta)\right]^{\frac{1-\alpha}{\alpha}}}\left(\frac{1-\alpha}{\alpha} - \frac{1-\frac{1}{\alpha}}{\rho\chi^{-1}\phi}\frac{\tau}{1-\tau}\left(1-\frac{\eta}{\alpha}\right)\right) - \frac{1-\chi\tilde{\mu}_1}{\left(\frac{1}{\rho}-1\right)\gamma}(1-\alpha) = 0.$$

Finally leading to the system:

$$-\frac{\theta}{1-\alpha}+\frac{1+\frac{1-\rho\alpha}{1-\alpha}\theta}{1-\rho}(1-\alpha-\tau^R)+\theta(1-\tau^R)=0,$$

$$\eta^R=\frac{\theta\alpha(1-\tau^R)}{\frac{(1+\rho\theta)(1-\alpha)}{1-\rho}+(1-\tau^R)(1-\alpha)}.$$

The first equation yields the Ramsey-optimal tax rate as a function of the structural parameters *α, θ, ρ,* while the second equation gives us the associated optimal split of public resources. It is easy to see that the solution to this system is given by,

$$\tau^R=1-\frac{\alpha(1+\rho\theta)}{1+\theta},$$

$$\eta^R=\frac{\alpha\theta(1-\rho)}{1-\alpha+\theta(1-\alpha\rho)}.$$

Appendix 4.- *Optimal Markov policy*

Proof of Proposition 7:

Taking into account the generalized Euler equation (22):

$$\frac{\chi_{\tau_t}+\theta\chi_t\frac{1}{\alpha\tau_t}}{\chi_{\tau_t}+\Lambda(\tau_t)\Omega(\eta_t,\tau_t)}=\frac{\chi_{\eta_t}+\theta\chi_t\frac{\alpha-\eta_t}{\alpha\eta_t(1-\eta_t)}}{\chi_{\eta_t}+\frac{1-\alpha}{\alpha}\frac{\Omega(\eta_t,\tau_t)}{1-\eta_t}}, \tag{A4.1}$$

we need to compute the partial derivatives of χ_t with respect to the two policy variables τ_t,η_t. To that end, we differentiate in (24) to obtain:

$$\left[-\sigma\chi_t^{-\sigma-1}\phi_t^{\theta(1-\sigma)}\gamma_t^{\sigma-\theta(1-\sigma)}-\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\left(\sigma-\theta(1-\sigma)\right)\gamma_t^{\sigma-\theta(1-\sigma)-1}\right]d\chi_t+$$
$$\left[\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)-1}\gamma_t^{\sigma-\theta(1-\sigma)}\theta(1-\sigma)\frac{1}{\alpha}\frac{\phi_t}{\tau_t}+\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\left(\sigma-\theta(1-\sigma)\right)\gamma_t^{\sigma-\theta(1-\sigma)-1}\frac{1-\alpha-\tau_t}{\alpha\tau_t(1-\tau_t)}\Omega\left(\tau_t,\eta_t\right)\right]d\tau_t=0$$

so that,

$$\chi_{\tau_t}=\frac{d\chi_t}{d\tau_t}=\frac{\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\gamma_t^{\sigma-\theta(1-\sigma)}\left[\theta(1-\sigma)\frac{1}{\alpha\tau_t}+\frac{\sigma-\theta(1-\sigma)}{\gamma_t}\frac{1-\alpha-\tau_t}{\alpha\tau_t(1-\tau_t)}\Omega\left(\tau_t,\eta_t\right)\right]}{\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\gamma_t^{\sigma-\theta(1-\sigma)}\left[\sigma\frac{1}{\chi_t}+\left(\sigma-\theta(1-\sigma)\right)\frac{1}{\gamma_t}\right]}, \tag{A4.2}$$

and, similarly, we would obtain:

$$\chi_{\eta_t} = \frac{d\chi_t}{d\eta_t} = \frac{\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\gamma_t^{\sigma-\theta(1-\sigma)}\left[\theta(1-\sigma)\frac{\alpha-\eta_t}{\alpha\eta_t(1-\eta_t)} - \frac{\sigma-\theta(1-\sigma)}{\gamma_t}\frac{1-\alpha}{\alpha}\frac{\Omega(\tau_t,\eta_t)}{1-\eta_t}\right]}{\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\gamma_t^{\sigma-\theta(1-\sigma)}\left[\sigma\frac{1}{\chi_t} + (\sigma-\theta(1-\sigma))\frac{1}{\gamma_t}\right]}. \quad \text{(A4.3)}$$

As we can see, the product $\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\gamma_t^{\sigma-\theta(1-\sigma)}$ cancels out again in both partial derivatives. As a consequence, the characterization of that product that we made at (24) as a function $F(k_t)$ of the state of the economy does not play any role in the first Generalized Euler equation that relates the optimal choice of the two policy variables τ_t, η_t .

Using now the partial derivatives $\chi_{\tau_t}, \chi_{\eta_t}$ in the first Generalized Euler equation (A4.1), we finally get:

$$\frac{1+\frac{\sigma-\theta(1-\sigma)}{\theta\gamma_t}\left[\frac{1-\alpha-\tau_t}{1-\tau_t}\Omega(\eta_t,\tau_t)+\chi_t\theta\right]}{1-\sigma+\frac{\sigma}{\theta\chi_t}\frac{\tau_t-(1-\alpha)}{1-\tau_t}\Omega(\eta_t,\tau_t)} = \frac{1+\frac{\sigma-\theta(1-\sigma)}{\theta\gamma_t}\left[\chi_t\theta+\frac{(1-\alpha)\eta_t}{\alpha-\eta_t}\Omega(\eta_t,\tau_t)\right]}{1-\sigma+\frac{\sigma}{\theta\chi_t}\frac{(1-\alpha)\eta_t}{\alpha-\eta_t}\Omega(\eta_t,\tau_t)}$$

that can only hold if:

$$\tau_t = \frac{1-\alpha}{1-\eta_t}, \forall t. \square$$

Appendix 5. *The planner's problem under lump-sum taxes*

A planner with access to lump-sum taxes would allocate resources so as to maximize time aggregate utility with the global constraint of resources as its sole restriction, thereby solving the problem,

$$\underset{\{c_t,k_{t+1},k_{pt},g_t\}}{Max}\sum_{t=0}^{\infty}\rho^t\frac{c_t^{1-\sigma}g_t^{\theta(1-\sigma)}-1}{1-\sigma}$$

subject to:

$$k_{t+1}-(1-\delta)k_t+c_t+g_t+k_{p,t} = Bk_t^{\alpha}k_{p,t}^{1-\alpha},$$

leading to optimality conditions:

$$\frac{c_{t+1}}{c_t} = \left\{\rho\left[\alpha(1-\alpha)^{\frac{1-\alpha}{\alpha}}B^{1/\alpha}+(1-\delta)\right]\right\}^{\frac{1}{\sigma-\theta(1-\sigma)}},$$

that defines the rate of growth γ_P, and

$$\frac{k_{t+1}}{k_t} - (1-\delta) + \chi_t + \theta\chi_t + \left[(1-\alpha)B\right]^{1/\alpha} = B^{1/\alpha}(1-\alpha)^{\frac{1-\alpha}{\alpha}}.$$

These relationships lead to expressions for the ratios of private and public consumption to capital:

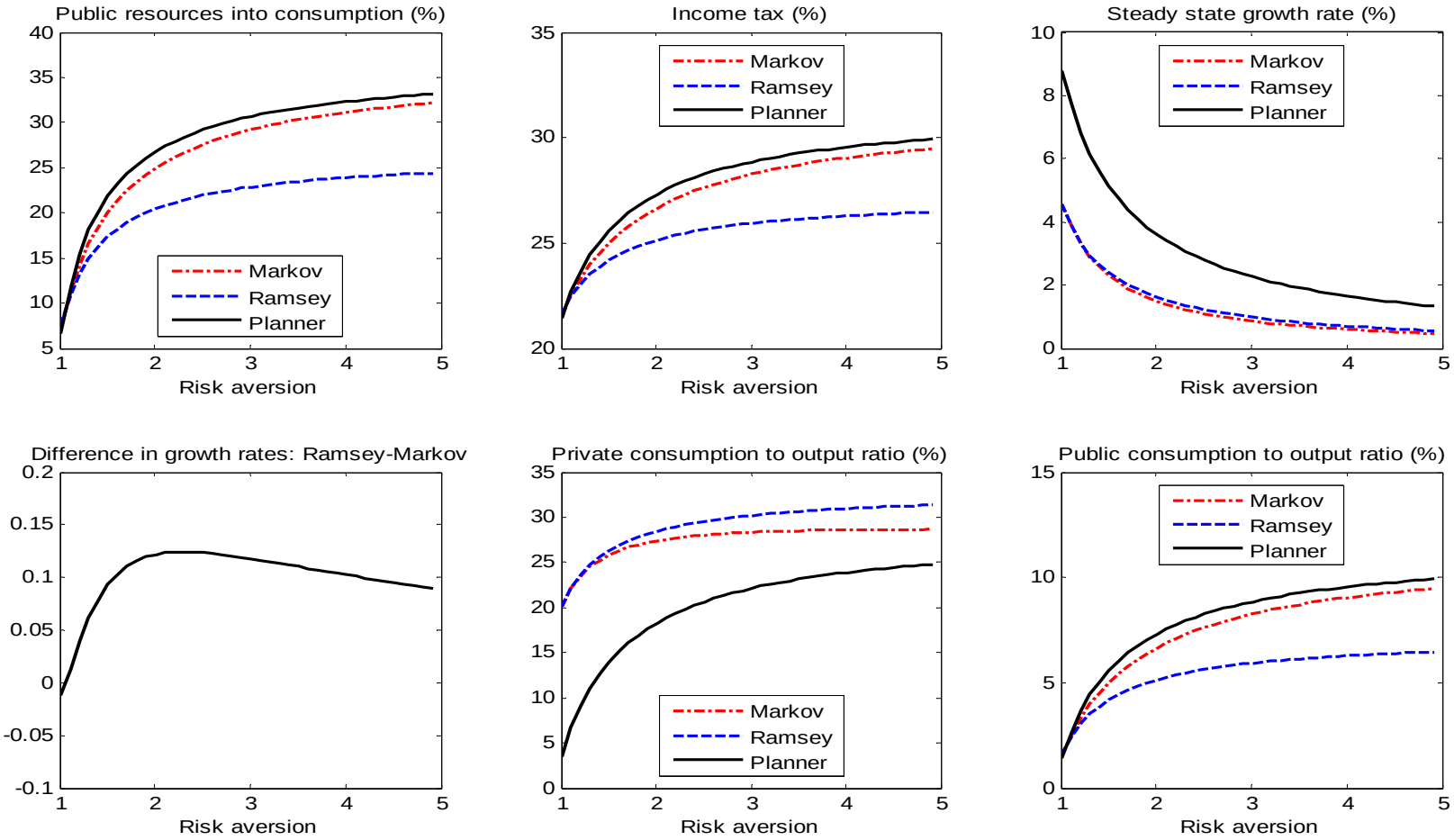
$$\chi_P = \frac{1}{1+\theta}\left[\alpha B^{1/\alpha}(1-\alpha)^{\frac{1-\alpha}{\alpha}} + (1-\delta) - \gamma_P\right],$$

$$\phi_P = \frac{g_t}{k_t} = \theta\chi_P.$$

For the purpose of comparison with the Markov and Ramsey equilibria, we can introduce a measure of the size of the public sector, as $\tau_t^P = \frac{g_t + i_{p,t}}{y_t}$ and the composition of public expenditures, $\eta_t^P = \frac{g_t}{g_t + i_{p,t}}$.

Figure 1

Values for the main variables in the economy under the three equilibrium concepts, for different values of the risk aversion parameter

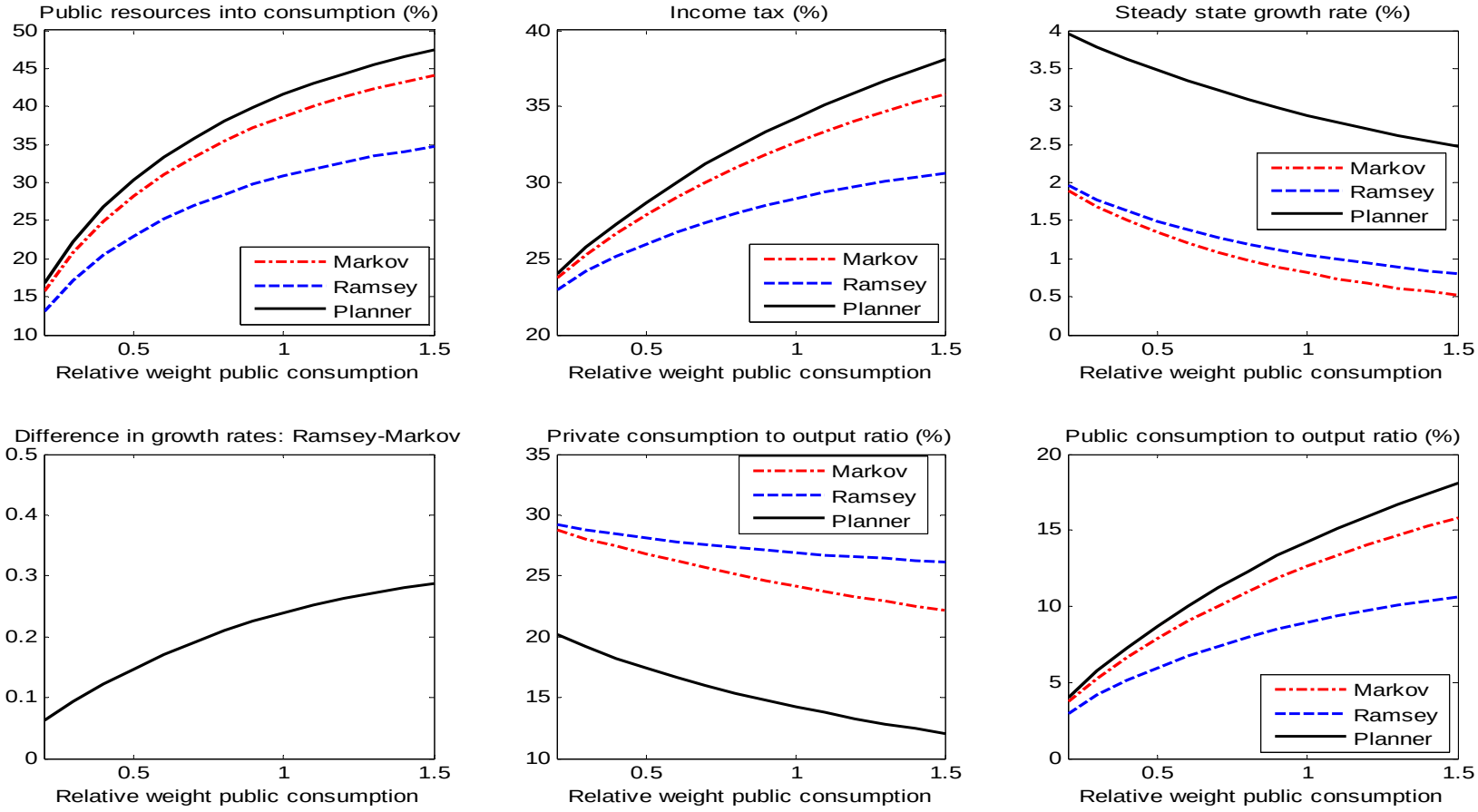


From left to right and from above to below, the graphs display: the share of public resources devoted to public consumption, the optimal income tax rate, the growth rate along the balanced path, the difference between the growth rates under the Ramsey and the Markov policies, and the ratios of private and public consumption to output.

$\theta = 0.40$	Relative weight of public consumption in utility function
$\alpha = 0.80$	Elasticity of private capital in production function
$\rho = 0.99$	Discount rate
$\delta = 0.10$	Depreciation rate
$B = 0.4555$	Productivity level

Figure 2

Values for the main variables in the economy under the three equilibrium concepts, for different values of the relative weight of public consumption in the utility function

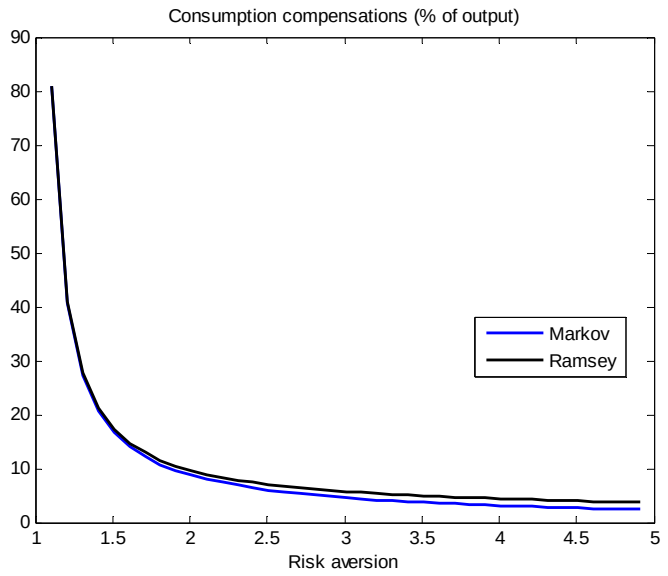


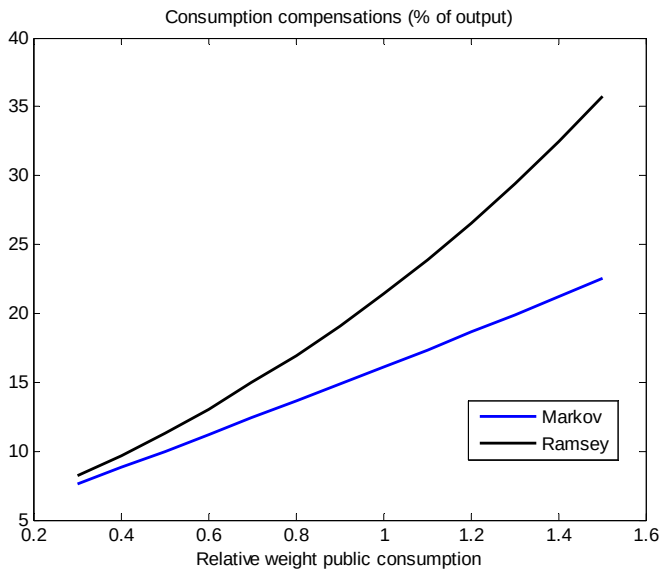
From left to right and from above to below, the graphs display: the share of public resources devoted to public consumption, the optimal income tax rate, the growth rate along the balanced path, the difference between the growth rates under the Ramsey and the Markov policies, and the ratios of private and public consumption to output.

$\sigma = 2.0$	Relative risk aversion
$\alpha = 0.80$	Elasticity of private capital in production function
$\rho = 0.99$	Discount rate
$\delta = 0.10$	Depreciation rate
$B = 0.4555$	Productivity level

Figure 3

Consumption compensation needed for the Markov policy to achieve the same level of welfare as the planner's allocation of resources





$\sigma = 2.00$	Relative risk aversion (in the second graph)
$\theta = 0.40$	Relative weight of public consumption in preferences (in the first graph)
$\alpha = 0.80$	Elasticity of private capital in production function
$\rho = 0.99$	Discount rate
$\delta = 0.10$	Depreciation rate
$B = 0.4555$	Productivity level