

Probabilities of consistent election outcomes with majorities based on difference in support

Mostapha Diss ·
Patrizia Pérez-Asurmendi

Received: date / Accepted: date

Abstract Computer simulations are used to evaluate the likelihood of consistent outcomes under the class of majorities based on difference in support. These majorities require certain consensus in collective preferences to declare an alternative as the winner. More precisely, individuals show preference intensities in the unit interval among each pair of alternatives and it is required that the winner alternative obtains a difference in the sum of the intensities with respect to the loser alternative. This difference is a real number located between 0 and the total number of voters. We introduce the values of the required threshold for which majorities based on difference in support lead to transitive and triple-acyclic collective decisions with a probability of 1. Our results improve the previous theoretical ones since they require softer thresholds to reach consistent collective decisions.

Keywords: Computer simulations; Majorities based on difference in support; Probability; Transitivity; Triple-acyclicity.

1 Introduction

Many consistent conditions such as the transitivity are imposed as minimal requirement concerning any social choice rule used to aggregate individual preferences into a collective outcome. Transitivity represents the idea that, for an individual or collective preference, when alternative x_1 is preferred to x_2

Corresponding author: M. Diss
Université de Lyon, Lyon, F-69007, France; CNRS, GATE Lyon Saint-Etienne, Ecully, F-69130, France; Université Jean Monnet, Saint-Etienne, F-42000, France.
E-mail: mostapha.diss@univ-st-etienne.fr

P. Pérez-Asurmendi
PRESAD and SEED Research Groups, Department of Economic Analysis II (Quantitative Economics), Universidad Complutense de Madrid, Campus de Somosaguas, 28223, Pozuelo de Alarcón, Madrid, Spain.
E-mail: patrizip@ucm.es

and x_2 is preferred to x_3 then x_1 is preferred to x_3 . The literature related to this notion dates back 200 years to the great work of the Marquis de Condorcet (1785) which states that voters with individual transitive rankings can produce an election outcome which is not transitive if one chooses pairwise majority voting as aggregation. This result gave rise to numerous studies and promoted the probabilistic study of the occurrence of voting paradoxes and their consequences under different aggregation rules.

The probability of voting paradoxes has been the subject of a whole strand of literature. Under this approach, it is assumed an a priori probability model to describe the individual preferences, derived the conditions under which the paradox appears and reached probabilities through combinatoric calculus. In this context, stand out the studies about majority rule (Fishburn and Gehrlein 1980; Gehrlein 1983; Gehrlein and Fishburn 1976), supermajority rules (Balasko and Crès 1997; Tovey 1997) or scoring rules (Cervone et al 2005; Gehrlein and Fishburn 1980, 1981, 1983), among others. To circumvent the impossibility of having analytical results, other papers have undertaken a methodology using computer simulations of elections. Specifically, the study of the cyclical and intransitive collective decisions under the simple majority rule are carried out in Campbell and Tullock (1965); DeMeyer and Plott (1970); Jones et al (1995); Klahr (1966), among others.

Voting rules based on the majority principle are the most studied in the literature and that principle remains one of the most widely used voting rules in real life. The majority rule, and other classic voting systems, have the advantage of being simple to use and easy to understand by the voters but they have major drawbacks. One of them is based on the idea that the preferences expressed by the voters are assumed to be dichotomous (also called crisp or ordinal), i.e. individuals can only declare if an alternative is preferred to another, or if they are indifferent. As the Nobel laureate Amartya Sen (1970) has noted, “voters opinions can be misrepresented when the preferences are dichotomous in the sense that the collective decision does not take into consideration the intensities of each individual preference”. The importance of considering intensities in the individual preferences has also been noted in Cook and Kress (1985); Meek (1975); Nurmi (1981, 2008); Tanino (1984), and Morales (1997) (see translation in McLean and Urken (1995)), among others.

Reciprocal preferences have been introduced in the literature in order to deal with preference intensities. This framework allows individuals to show preference intensities among each pair of alternatives by a real number in the unit interval $[0, 1]$. It is clear that the information contained in reciprocal preferences is much richer than the case of dichotomous preferences. Indeed, for each voter, a reciprocal preference between two alternatives x_1 and x_2 expresses the degree by which the voter prefers x_1 to x_2 . More precisely, the closer the intensity is to 1, the more x_1 is preferred to x_2 , and the closer the intensity is to 0, the more x_2 is preferred to x_1 . In addition, an intensity of 0.5 stands for the exact indifference between the two alternatives x_1 and x_2 . Following this approach, when we extend the set of alternatives from two to three alternatives, we assume that each individual reciprocal preference fulfil

some properties of transitivity in order to guarantee individual rationality of each reciprocal preference.

Considering preference intensities, majorities based on difference in support \tilde{M}_k have been introduced in the literature as a possible aggregation system. These majorities require certain consensus in collective preferences to declare an alternative as the winner and depend on the idea of a threshold. More precisely, these aggregation rules suggest that alternative x_1 is preferred to alternative x_2 if and only if the sum of the intensities for x_1 exceeds the aggregated intensity of x_2 by a threshold k given by a real number located between 0 and the total number of voters. In other words, a higher value of k means that a stronger preference for x_1 over x_2 is needed in order to x_1 to be declared socially preferred to x_2 . Conversely, a low value of k means that a weak preference for x_1 over x_2 is needed to declare x_1 socially preferred to x_2 . The axiomatic characterization of majorities based on difference in support is introduced in García-Lapresta and Llamazares (2010). Notice that, for other kinds of operators which can be used for the aggregation of reciprocal preferences and their characterizations, the reader is referred to García-Lapresta and Llamazares (2000); Llamazares (2004, 2007); Llamazares and García-Lapresta (2003, 2008), among others.

This paper is devoted to analyze the probabilities of consistent collective decisions over the class of majorities based on difference in support. We specifically calculate the probabilities of transitive and triple-acyclic strict collective preferences and the corresponding ones of transitive weak collective preferences for these majorities. To calculate these probabilities, we apply the Monte-Carlo simulation methodology inspired by the studies in Campbell and Tullock (1965); DeMeyer and Plott (1970); Jones et al (1995); Klahr (1966), among others.

Specifically, we generate the individual reciprocal preference relations for the case of three alternatives. Each individual intensity of preference is understood as a continuous random variable in the unit interval consistently built with a specific transitivity condition over the individual's reciprocal preference relations. Then, we fix the required difference in support and aggregate these individual preferences with the corresponding majority based on difference in support. We derive the resultant collective ordering of alternatives and evaluate its consistency. Finally, we iterate that procedure to estimate desired probabilities as the number of consistent orderings over the total number of simulated collective orderings.

The objective of this paper is twofold. First, the methodology proposed here allows us to hypothesize about a relationship between the type of individual preference intensities and the likelihood of consistent collective decisions. Intuitively, we expected to find softer conditions to have consistent collective decisions with probability 1 as individuals are more rational, i.e. the strongest the assumed individual transitivity condition is. As we will see, our results refute that idea. Second, we compare our results on probabilities with the theoretical ones about the consistency of the collective preferences under majorities based on difference in support in Llamazares et al (2013)

and Llamazares and Pérez-Asurmendi (2015). In these articles, necessary and sufficient conditions for which these majorities provide transitive and triple-acyclic collective decisions have been introduced. The algebraic tools used in the proofs lead to several impossibility results. Specifically, for the case of the transitivity of the collective decisions they proved that it can be always found an example of intransitive collective preferences when the individual transitivity condition is weak. In the case of assuming stronger conditions to individual preferences, the collective preference is transitive only when individual preferences show a higher level of unanimity. Although they found less demanding conditions in their results about the triple-acyclicity of the collective decision, it is also required a high level of similarity in individual preferences in order to find consistent collective decisions. Under the probabilistic approach followed here, the required conditions for having consistent collective decisions are much softer than there promoting the use of these types of majorities whenever the consistency of the collective decision is a concern for the society or the policy maker.

The paper is organized as follows. Section 2 describes the theoretical framework followed in this paper and introduces majorities based on difference in support. Section 3 is devoted to the method of simulation. Sections 4 and 5 discuss the results and Section 6 concludes.

2 Preliminaries

Consider a set of alternatives $X = \{x_1, x_2, x_3\}$ in an election with m individuals. Let S be a binary relation on X , i.e. a subset of the Cartesian product $X \times X$. In what follows, $x_i S x_j$ stands for $(x_i, x_j) \in S$, i.e. when x_i is in the relation S with x_j . S^{-1} is the inverse relation of S defined by $x_i S^{-1} x_j \Leftrightarrow x_j S x_i$ and S^c is the complement relation of S defined by $x_i S^c x_j \Leftrightarrow \neg(x_i S x_j)$. Given two binary relations S and T , the intersection of S and T is also a binary relation defined by $x_i(S \cap T)x_j \Leftrightarrow (x_i S x_j \wedge x_i T x_j)$. A binary relation S on X is

1. *reflexive* if $\forall x \in X, x S x$,
2. *symmetric* if $\forall x_i, x_j \in X, x_i S x_j \Rightarrow x_j S x_i$,
3. *asymmetric* if $\forall x_i, x_j \in X, x_i S x_j \Rightarrow \neg(x_j S x_i)$,
4. *antisymmetric* if $\forall x_i, x_j \in X, (x_i S x_j \wedge x_j S x_i) \Rightarrow x_i = x_j$,
5. *complete* if $\forall x_i, x_j \in X, x_i S x_j \vee x_j S x_i$,
6. *transitive* if $\forall x_i, x_j, x_l \in X, (x_i S x_j \wedge x_j S x_l) \Rightarrow x_i S x_l$,
7. *triple-acyclic* if $\forall x_i, x_j, x_l \in X, (x_i S x_j \wedge x_j S x_l) \Rightarrow \neg(x_l S x_i)$.

A *weak preference* R is a complete binary relation on the set of alternatives X . The *strict preference* P associated with R is the asymmetric binary relation on X defined by $P = (R^{-1})^c$ and the corresponding *indifference relation* I is the reflexive and symmetric binary relation on X defined by $I = R \cap R^{-1}$. $\mathcal{P}(X)$ is the set of strict preferences. A *weak ordering* is a transitive weak preference whereas a *linear ordering* is also antisymmetric.

From definitions above it is well known that any weak ordering implies a transitive strict preference relation and a transitive indifference relation. Moreover, any transitive strict preference is also a triple-acyclic preference relation. Notice that the converse is not true.

Given that the social decision between two alternatives is given by either a strict preference relation or an indifference relation, and that three alternatives are in contest, we consider the 27 cases in Table 1 as possible social outcomes.

Table 1 Possible social outcomes in a three-alternative election.

1. x_1Px_2 x_2Px_3 x_1Px_3	10. x_1Ix_2 x_2Px_3 x_1Px_3	19. x_1Ix_2 x_2Px_3 x_1Ix_3
2. x_1Px_3 x_3Px_2 x_1Px_2	11. x_2Ix_3 x_3Px_1 x_2Px_1	20. x_1Px_2 x_2Px_3 x_1Ix_3
3. x_2Px_1 x_1Px_3 x_2Px_3	12. x_1Ix_3 x_3Px_2 x_1Px_2	21. x_3Px_1 x_1Px_2 x_2Ix_3
4. x_2Px_3 x_3Px_1 x_2Px_1	13. x_1Ix_2 x_2Ix_3 x_1Ix_3	22. x_2Px_3 x_3Px_1 x_1Ix_2
5. x_3Px_1 x_1Px_2 x_3Px_2	14. x_1Px_2 x_2Ix_3 x_1Ix_3	23. x_3Px_2 x_1Px_3 x_1Ix_2
6. x_3Px_2 x_2Px_1 x_3Px_1	15. x_1Ix_2 x_2Ix_3 x_1Px_3	24. x_2Px_1 x_1Px_3 x_2Ix_3
7. x_1Px_2 x_2Ix_3 x_1Px_3	16. x_1Ix_2 x_2Ix_3 x_3Px_1	25. x_3Px_2 x_2Px_1 x_1Ix_3
8. x_2Px_1 x_1Ix_3 x_2Px_3	17. x_1Ix_3 x_3Px_2 x_1Ix_2	26. x_1Px_2 x_2Px_3 x_3Px_1
9. x_3Px_1 x_1Ix_2 x_3Px_2	18. x_2Px_1 x_1Ix_3 x_2Ix_3	27. x_2Px_1 x_1Px_3 x_3Px_2

Our interest focuses on the frequency of consistent social outcomes given the 27 possible outcomes above. We distinguish among three cases of consistent outcomes; the case of weak orderings corresponding to the first thirteen outcomes, the case of transitive strict preferences corresponding to the first nineteen and the case of triple-acyclic strict preferences corresponding to the first twenty-fifth outcomes.

2.1 Individual preferences

We consider that individuals compare the alternatives on X by pairs and declare their preferences by means of values r_{ij}^p , which belong to the unit interval $[0, 1]$ with the following interpretation: $r_{ij}^p > 0.5$ indicates that the individual p prefers the alternative x_i to the alternative x_j , the more the nearer is the value of r_{ij}^p to 1 that represents the maximum degree of preference for x_i over x_j ; conversely, $r_{ij}^p < 0.5$, means that individual p prefers alternative x_j to x_i , the more the nearer is the value of r_{ij}^p to 0 that represents the maximum degree of preference for x_j over x_i ; finally, $r_{ij}^p = 0.5$ stands for the indifference between x_i and x_j for individual p . The reciprocity of these preferences is described by the condition $r_{ij}^p + r_{ji}^p = 1$.

As noted in the Introduction, to avoid the possibility of having incoherent individual preferences, we need to assume some kind of rationality condition. But, in this framework, several concepts could be taken to ensure such rationality requirement (see, among others, Dasgupta and Deb 1996; Dubois and Prade 1980; García-Lapresta and Meneses 2005; Zadeh 1971). Here, we consider the following transitivity conditions for reciprocal preference relations.

Definition 1 We say that individual p is

1. *0.5-transitive* if $\forall i, j, l \in \{1, 2, 3\}$

$$(r_{ij}^p > 0.5 \wedge r_{jl}^p > 0.5) \Rightarrow r_{il}^p > 0.5,$$

2. *min-transitive* if $\forall i, j, l \in \{1, 2, 3\}$

$$(r_{ij}^p > 0.5 \wedge r_{jl}^p > 0.5) \Rightarrow r_{il}^p \geq \min\{r_{ij}^p, r_{jl}^p\},$$

3. *am-transitive* if $\forall i, j, l \in \{1, 2, 3\}$

$$(r_{ij}^p > 0.5 \wedge r_{jl}^p > 0.5) \Rightarrow r_{il}^p \geq (r_{ij}^p + r_{jl}^p)/2,$$

4. *max-transitive* if $\forall i, j, l \in \{1, 2, 3\}$

$$(r_{ij}^p > 0.5 \wedge r_{jl}^p > 0.5) \Rightarrow r_{il}^p \geq \max\{r_{ij}^p, r_{jl}^p\}.$$

The preferences of each individual over the alternatives in $X = \{x_1, x_2, x_3\}$ can be represented using a 3×3 matrix $R^p = (r_{ij}^p)$ as follows:

$$R^p = \begin{pmatrix} 0.5 & r_{12}^p & r_{13}^p \\ 1 - r_{12}^p & 0.5 & r_{23}^p \\ 1 - r_{13}^p & 1 - r_{23}^p & 0.5 \end{pmatrix}. \quad (1)$$

Individual preferences are collected in a vector where each vector-element represents the preferences of an individual. Assuming m individuals¹ a profile of reciprocal preferences is a vector $(R^1, \dots, R^m) \in \mathcal{R}(X)^m$, where $\mathcal{R}(X)$ the set of all reciprocal preference relations.

2.2 Majorities based on difference in support

In García-Lapresta and Llamazares (2010), majorities based on difference in support were introduced in the framework of reciprocal preferences allowing individuals to declare their degrees of preferences over pairs of alternatives. Majorities based on difference in support allow us to aggregate each profile of reciprocal preferences into a strict collective preference P_k over the set of alternatives. Under these rules, the winner alternative is required to reach a support that exceeds the support for the other alternative in a quantity, fixed before the voting process. Formal definition for these majorities is as follows.

Definition 2 (Majorities based on difference in support or \widetilde{M}_k majorities García-Lapresta and Llamazares (2010)) Given $k \in [0, m)$,

¹ To calculate the probabilities presented here, m takes the following values: 3, 4, 5, 10, 100, 1,000 and 100,000.

the majority based on difference in support or \widetilde{M}_k majority is the mapping $\widetilde{M}_k : \mathcal{R}(X)^m \rightarrow \mathcal{P}(X)$ defined by $\widetilde{M}_k(R^1, \dots, R^m) = P_k$, where

$$x_i P_k x_j \Leftrightarrow \sum_{p=1}^m r_{ij}^p > \sum_{p=1}^m r_{ji}^p + k. \quad (2)$$

Using the fact that $r_{ij}^p = 1 - r_{ji}^p$ for each voter p , (2) is equivalent to:

$$x_i P_k x_j \Leftrightarrow \frac{\sum_{p=1}^m r_{ij}^p}{m} > 0.5 + \frac{k}{2m}. \quad (3)$$

The indifference relation associated with P_k is defined by:

$$x_i I_k x_j \Leftrightarrow \left| \sum_{p=1}^m r_{ij}^p - \sum_{p=1}^m r_{ji}^p \right| \leq k. \quad (4)$$

Putting the relation $r_{ij}^p = 1 - r_{ji}^p$ into (4), one obtains:

$$x_i I_k x_j \Leftrightarrow \left| \frac{\sum_{p=1}^m r_{ij}^p}{m} - 0.5 \right| \leq \frac{k}{2m}. \quad (5)$$

Example 1 Let R^I and R^{II} be the following reciprocal preference relations over the alternatives on $X = \{x_1, x_2, x_3\}$.

$$R^I = \begin{pmatrix} 0.5 & 1 & 0.9 \\ 0 & 0.5 & 0.6 \\ 0.1 & 0.4 & 0.5 \end{pmatrix}, \quad R^{II} = \begin{pmatrix} 0.5 & 0.8 & 1 \\ 0.2 & 0.5 & 0.7 \\ 0 & 0.3 & 0.5 \end{pmatrix}.$$

Consider the profile $(R^1, R^2, R^3, R^4, R^5)$ where

$$R^p = \begin{cases} R^I & \text{if } p = 1, 2, 3, \\ R^{II} & \text{if } p = 4, 5. \end{cases}$$

Assuming a required difference in support k equal to 1.75 and applying the corresponding $\widetilde{M}_{1.75}$ majority we have

$$\begin{aligned} \sum_{p=1}^5 r_{12}^p &= 4.6 > \sum_{p=1}^5 r_{21}^p + 1.75 = 0.4 + 1.75 \Rightarrow x_1 P_{1.75} x_2, \\ \left| \sum_{p=1}^5 r_{23}^p - \sum_{p=1}^5 r_{32}^p \right| &= |3.2 - 1.8| \leq 1.75 \Rightarrow x_2 I_{1.75} x_3, \\ \sum_{p=1}^5 r_{13}^p &= 4.7 > \sum_{p=1}^5 r_{31}^p + 1.75 = 0.3 + 1.75 \Rightarrow x_1 P_{1.75} x_3. \end{aligned}$$

As noticed in the Introduction, in the case of crisp preferences, given a pair of alternatives, individuals declare if they prefer an alternative to another one or if they are indifferent between them. In other words, the values of r_{ij}^p are restricted to the set of discrete values $\{0, 0.5, 1\}$. If $r_{ij}^p = 1$, individual p prefers alternative x_i to alternative x_j , whereas if $r_{ij}^p = 0$, individual p prefers x_j to x_i . If $r_{ij}^p = 0.5$, individual p is indifferent between both alternatives. In this framework, the concept of majorities based on difference of votes was introduced in García-Lapresta and Llamazares (2001) and was later axiomatically characterized in Llamazares (2006), and subsequently in Houy (2007). Under these majorities, an alternative, say x_i , is declared the winner if the number of individuals who prefer that alternative, to the other one, say x_j , exceeds the number of individuals who prefer x_j to x_i in a difference of votes, fixed before the election process. With m individuals, that difference could take any integer value in $\{0, \dots, m - 1\}$. It is clear that these majorities are located between simple majority rule when the difference of votes is zero and unanimity when the difference of votes is the total number of individuals m minus one.

Assuming weak or linear individual orderings and using the well-known Impartial Anonymous Culture (IAC) condition (Gehrlein and Fishburn 1976), calculations of the probabilities of consistent outcomes (transitivity and triple-acyclicity) under majorities based on difference in votes have been conducted in Diss and Pérez-Asurmendi (2015). The IAC condition assumes that all voting situations are drawn independently and uniformly given that each voting situation indicates a specific combination of the number of voters associated with each order of the alternatives. The objective of Diss and Pérez-Asurmendi (2015) was twofold. First, the needed thresholds which guarantee that the probability of consistent outcomes is close to 1 have been found. Second, the authors have set forth the impact of weak orderings and linear orderings for the individual preferences on the probability of consistent outcomes. It is shown for instance that in the cases of transitive and triple-acyclic strict preferences, the probabilities are higher considering weak than linear orderings.

In the present paper, we extend the study in Diss and Pérez-Asurmendi (2015) to the majorities based on difference in support \widetilde{M}_k . However, as long as the intensities of preference between each pair of alternatives can take any value in the continuous interval $[0, 1]$, the IAC model can not be applied. Our difficulty is that there is not any equivalent model to IAC in the framework of reciprocal preferences since the set of voting situations is not numerable. In other words, the probabilistic analysis carried out in Diss and Pérez-Asurmendi (2015) turns impossible to study in the case of majorities based on difference in support \widetilde{M}_k . Consequently, we perform a computer simulation to estimate these probabilities. Our simulation method will follow the same spirit as the Impartial Culture (IC) condition (Guilbaud 1952), a well-known model which considers the set of all preference profiles as a sample space and where a voter preference profile identifies the specific preference ranking that each voter has for the candidates. Notice that individual voter's preferences are not anonymous under IC condition while they are under IAC assumption.

3 Simulation method

In this section, we detail the simulation method used in this paper in order to provide the probabilities of reaching consistent collective decisions under \widetilde{M}_k majorities. Our probabilities are estimated as the proportion of the number of consistent outcomes in the simulation over the total number of simulated outcomes. We generate 100,000 outcomes to guarantee our results with a confidence level of 99% and a sampling error of less than a 0.0041%².

In the following, we describe the methodology applied in the simulations to estimate the probability for the considered three types of consistent collective decisions under \widetilde{M}_k majorities, i.e. transitive weak preferences, transitive and triple-acyclic strict preferences. We follow that scheme taking into account each type of individual transitive reciprocal relations, i.e. 0.5-transitive, min-transitive, am-transitive and max-transitive reciprocal preference relations. Notice that the matrix in (1) representing a reciprocal preference relation is determined by the vector composed of the intensities r_{12}, r_{23} and r_{13} .

1. At the beginning of the evaluation, we randomly generate m vectors representing the reciprocal preferences of the m individuals. For each individual, each component of the triplet (r_{12}, r_{23}, r_{13}) is drawn from an uniform distribution on $[0; 1]$. Such vectors are built bearing in mind one of the considered transitivity conditions for reciprocal preference relations. In other words, if the resulting individual reciprocal preference (r_{12}, r_{23}, r_{13}) is not transitive for the function under consideration (i.e. 0.5-transitivity, min-transitivity, am-transitivity and max-transitivity), it is rejected. This first step ends when exactly m individual transitive reciprocal preferences are generated.
2. We compute the sum of the individuals' intensities of preference over each pair of alternatives relative to the number of voters m through a vector $\mathbf{S} = (\frac{S_{12}}{m}, \frac{S_{23}}{m}, \frac{S_{13}}{m})$ where $S_{ij} = \sum_{p=1}^m r_{ij}^p$.
3. Having in mind the conditions in equations (3) and (5) and the value of the threshold $a = \frac{k}{2m}$, the collective decision is evaluated over each pair of alternatives in the vector \mathbf{S} .
4. The collective decision in \mathbf{S} is classified following the cases of possible collective outcomes displayed in Table 1. For instance, the case 27 corresponds

² Assuming a proportion of consistent outcomes P on the population of a 50%, the proportion p in a random sample of size $n \geq 30$ for a confidence level of 99%, diverges from the one of the population in an error of less than ϵ :

$$Prob(|P - p| \leq \epsilon) \geq 0.99.$$

Taken into account that the sample proportion p is distributed as $N(P, \sqrt{P(1-P)/n})$, the sampling error ϵ is as follows:

$$\epsilon = z_{\alpha/2} \sqrt{P(1-P)/n}.$$

In our case, $n = 100,000$ and the corresponding percentile of the normal distribution for a confidence level of 99% is $z_{\alpha/2} = 2.57$. Thus, $\epsilon \leq 0.00407$.

to $\frac{S_{12}}{m} < 0.5 - a$, $\frac{S_{23}}{m} < 0.5 - a$, and $\frac{S_{13}}{m} > 0.5 + a$. Indeed, if the collective decision is one of the cases 26 or 27, the strict preference P_k is not triple-acyclic. If it is one of the cases from 19 to 27, the strict preference P_k is not transitive. Finally, if it is one of the cases from 14 to 27, the weak preference R_k is not transitive.

5. This four steps are iterated 100,000 times to obtain the number of inconsistent collective decisions. Specifically, the number of simulated outcomes in which the weak preference R_k is transitive, in which the strict preference P_k is transitive and in which P_k is triple-acyclic, respectively.
6. Finally, each of the desired probabilities, i.e. the probability of transitive R_k and the probability of transitive and triple-acyclic P_k , is calculated as the number of consistent outcomes over the total number of simulated outcomes.

Below, the code used for the evaluation of our probabilities is presented for the case of am-transitive individual reciprocal preferences, a number of voters $m = 3$ and a required threshold $k = 0.1$.

Simulation code: the case of am-transitive individual reciprocal preferences, $m = 3$, and $k = 0.1$.

```

disp('Start');
m=3;
s=100000;
k=0.1;
a=(k/(2*m));
A=zeros (m,3);
success=0;
for i=1:s
    while (success < m)
        R=rand (1,3);
        if (( (R(1))>=0.5)&&(R(2))>=0.5)
            if ( (R(3))>=((R(1)+R(2))/2))
                success=success+1;
                A(success,1)=R(1);
                A(success,2)=R(2);
                A(success,3)=R(3);
            end;
        elseif (( (R(3))>=0.5)&&(1-R(2))>=0.5)
            if ( (R(1))>=((R(3)+1-R(2))/2))
                success=success+1;
                A(success,1)=R(1);
                A(success,2)=R(2);
                A(success,3)=R(3);
            end;
        elseif (( (1-R(1))>=0.5)&&(R(3))>=0.5)
            if ( (R(2))>=((1-R(1)+R(3))/2))
                success=success+1;
                A(success,1)=R(1);
                A(success,2)=R(2);
                A(success,3)=R(3);
            end;
        elseif (( (R(2))>=0.5)&&(1-R(3))>=0.5)
            if ( (1-R(1))>=((R(2)+1-R(3))/2))
                success=success+1;
                A(success,1)=R(1);
                A(success,2)=R(2);
                A(success,3)=R(3);
            end;
        end;
    end;
end;

```

```

    elseif (( (R(1))≥0.5)&&(1-R(3))≥0.5)
        if ( (1-R(2))≥((R(1)+1-R(3))/2))
            success=success+1;
            A(success,1)=R(1);
            A(success,2)=R(2);
            A(success,3)=R(3);
        end;
    elseif (( (1-R(2))≥0.5)&&(1-R(1))≥0.5)
        if ( (1-R(3))≥((1-R(2)+1-R(1))/2))
            success=success+1;
            A(success,1)=R(1);
            A(success,2)=R(2);
            A(success,3)=R(3);
        end;
    else
        disp ('Non matchee');
    end;
end;
S=(sum(A,1)/m);
cond27=(S(1)<(0.5-a)&&S(2)<(0.5-a)&&S(3)>(0.5+a));
cond26=(S(1)>(0.5+a)&&S(2)>(0.5+a)&&S(3)<(0.5-a));
cond25=(S(1)<(0.5-a)&&S(2)<(0.5-a)&&abs(S(3)-0.5)≤a);
cond24=(S(1)<(0.5-a)&&abs(S(2)-0.5)≤a&&S(3)>(0.5+a));
cond23=(abs(S(1)-0.5)≤a&&S(2)<(0.5-a)&&S(3)>(0.5+a));
cond22=(abs(S(1)-0.5)≤a&&S(2)>(0.5+a)&&S(3)<(0.5-a));
cond21=(S(1)>(0.5+a)&&abs(S(2)-0.5)≤a&&S(3)<(0.5-a));
cond20=(S(1)>(0.5+a)&&S(2)>(0.5+a)&&abs(S(3)-0.5)≤a);
cond19=(abs(S(1)-0.5)≤a&&S(2)>(0.5+a)&&abs(S(3)-0.5)≤a);
cond18=(S(1)<(0.5-a)&&abs(S(2)-0.5)≤a&&abs(S(3)-0.5)≤a);
cond17=(abs(S(1)-0.5)≤a&&S(2)<(0.5-a)&&abs(S(3)-0.5)≤a);
cond16=(abs(S(1)-0.5)≤a&&abs(S(2)-0.5)≤a &&S(3)<(0.5-a));
cond15=(abs(S(1)-0.5)≤a&&abs(S(2)-0.5)≤a&&S(3)>(0.5+a));
cond14=(S(1)>(0.5+a)&&abs(S(2)-0.5)≤a&&abs(S(3)-0.5)≤a);
    if (cond26||cond27)
        outtc=1;
    else
        outtc=0;
    end;
M(1,i)=outtc;
success=0;
S=[];
if (outtc==1||cond20||cond21||cond22||cond23||cond24||cond25)
    outnt=1;
else
    outnt=0;
end;
M(2,i)=outnt;
success=0;
S=[];
if (outnt==1||cond14||cond15||cond16||cond17||cond18||cond19)
    outcw=1;
else
    outcw=0;
end;
M(3,i)=outcw;
success=0;
S=[];
end;
D=sum(M,2);
E=s-D;
Probta=E(1)/s
Probtr=E(2)/s
Probcw=E(3)/s
disp('Finish');

```

In the appendix, the simulated probabilities of consistent collective decisions under M_k majorities are listed in tables. Main results are introduced in Section 4.

4 Results: Probabilities of consistent collective decisions under majorities based on difference in support

From our results, it is obvious to show that the probability of having transitive weak preferences R_k , the probability of having transitive strict preferences P_k , and the probability of having triple-acyclic strict preferences P_k are the same when $k = 0$ regarding each type of g -transitivity. For instance, for $k = 0$ and 0.5-transitivity, Table A.1, Table A.5, and Table A.9 show that this probability is equal to 0.8835 for $m = 3$. Indeed, when $k = 0$, the indifference in each collective preference is not allowed and the possible social outcomes in Table 1 are reduced to cases from 1 to 6 and the two cycles 26 and 27. As a consequence, the three probabilities are the same. However, when the threshold k increases, Table 2 illustrates that the lower the consistency condition over the collective decision is, the higher the simulated probability of consistency is, regarding each type of g -transitivity. To illustrate this result, we first take into account the threshold $k = 0.1$ for $m = 1,000$. For 0.5-transitive individual preferences, we deduce from Table 2 that the probability of having transitive weak preferences R_k (0.8679) is smaller than the one of having transitive strict preferences P_k (0.8680), which is in turn smaller than the probability of having triple-acyclic strict preferences P_k (0.8733). This is true for the other types of g -transitivity (i.e. min-transitivity, am-transitivity, and max-transitivity). We also take into account the threshold $k = 3$ in order to illustrate that this variation holds for other values of k . In addition, Table 2 illustrates that as well as the threshold k increases the difference between the three probabilities increases. For instance, for $k = 0.1$ and 0.5-transitivity, the difference between the probability of having transitive weak preferences R_k and the probability of having triple-acyclic strict preferences P_k is equal to 0.0054 whereas this difference is given by 0.1512 for $k = 3$.

Table 2 Simulated probabilities of consistent decisions with $m = 1,000$.

		0.5-trans.	min-trans.	am-trans.	max-trans.
$k = 0.1$	R_k transitive	0.8679	0.9459	0.9657	0.9713
	P_k transitive	0.8680	0.9460	0.9658	0.9714
	P_k triple-acyclic	0.8733	0.9491	0.9685	0.9739
$k = 3$	R_k transitive	0.7816	0.8802	0.9113	0.9184
	P_k transitive	0.8267	0.9245	0.9536	0.9609
	P_k triple-acyclic	0.9328	0.9827	0.9923	0.9945

One of the main facts we can observe from the simulations is that, for a given number of voters m , when the threshold k increases, we find that:

- The probability of having transitive weak preferences R_k first decreases and then increases.
- The probability of having transitive strict preferences P_k first decreases and then increases.
- The probability of having triple-acyclic strict preferences P_k increases. This is true for each value of the threshold k , even for small values. Indeed, when $k = 0$ the probability of triple-acyclic P_k is equal to 0.8835, 0.9563, 0.9751, and 0.9777 for 0.5-transitivity, min-transitivity, am-transitivity, and max-transitivity, respectively. However, even for a small value $k = 0.001$, the probability to observe a triple-acyclic P_k increases to 0.8838, 0.9566, 0.9753, and 0.9778.
- These three facts hold with independence of the assumed individual transitivity condition.

These results are shown by the probabilities of each table in the Appendix when $m \geq 100$. However, for small values of m and the considered thresholds, these simulations only show the increase in the probability of transitive R_k and transitive P_k . That seems to contradict above remarks about the behavior of the probabilities. In Table 3 we set $m = 3$ and consider several values of k to clarify that.

Table 3 The impact of the threshold k on the probability of consistent outcomes for a given number of voters. The example of $m = 3$.

		$k = 0$	$k = 0.1$	$k = 0.5$	$k = 1$	$k = 2$
0.5-trans.	R_k transitive	0.8835	0.8339	0.6110	0.5438	0.8883
	P_k transitive	0.8835	0.8497	0.8626	0.9587	0.9994
	P_k triple-acyclic	0.8835	0.9216	0.9890	0.9997	1
min-trans.	R_k transitive	0.9563	0.9222	0.7444	0.6551	0.8978
	P_k transitive	0.9563	0.9383	0.9642	0.9950	1
	P_k triple-acyclic	0.9563	0.9779	0.9997	1	1
am-trans.	R_k transitive	0.9751	0.9476	0.7987	0.7036	0.8913
	P_k transitive	0.9751	0.9629	0.9844	0.9988	1
	P_k triple-acyclic	0.9751	0.9899	1.0000	1	1
max-trans.	R_k transitive	0.9777	0.9518	0.8029	0.7102	0.9015
	P_k transitive	0.9777	0.9679	0.9896	0.9993	1
	P_k triple-acyclic	0.9777	0.9915	1.0000	1	1

Going deeper in the probability of the transitive collective weak preference, we can not establish a clear relationship between the strength of the considered g -transitivity condition and that probability as we can see in Table 4. For example, for a number of voters equal to 100, the required threshold for a simulated probability equal to 1 is 26.40 in the case of 0.5-transitive reciprocal

preference relations, 27.23 for min-transitive reciprocal preference relations, 27.90 for am-transitive ones and 25.47 in the case of max-transitive reciprocal preference relations.

Table 4 Thresholds k for simulated probabilities of R_k being transitive equal to 1.

	$m = 3$	$m = 10$	$m = 100$	$m = 1,000$
0.5-trans.	2.97	7.95	26.40	84.83
min-trans.	2.96	8.32	27.23	95.32
am-trans.	2.97	7.48	27.90	83.01
max-trans.	2.98	7.56	25.47	80.00

However, looking at having the probability of the strict preference being transitive equal to 1 (see Table 5), we can see that the higher the g -transitivity condition is, the lower the required thresholds k are. We find an exception to that in the case of $m = 100$, given that the threshold when regarding am-transitive reciprocal preference relations is 9.17 whereas the one considering max-transitive reciprocal preference relations equals 9.72 and therefore exceeds the previous one.

Table 5 Thresholds k for simulated probabilities of P_k being transitive equal to 1.

	$m = 3$	$m = 10$	$m = 100$	$m = 1,000$
0.5-trans.	2.58	5.17	14.51	47.79
min-trans.	1.96	3.47	12.51	35.24
am-trans.	1.56	3.10	9.17	33.10
max-trans.	1.51	2.78	9.72	26.10

Considering the strict preference relation, we find that the higher the g -transitive condition is, the lower the thresholds k that provide a simulated probability of having triple-acyclic strict preferences equal to 1 are, with the exception of the case of max-transitive reciprocal preference relations (see Table 6).

Table 6 Thresholds k for simulated probabilities of P_k being triple-acyclic equal to 1.

	$m = 3$	$m = 4$	$m = 5$	$m = 10$	$m = 100$	$m = 1,000$
0.5-trans.	1.32	1.50	1.91	2.65	8.29	26.60
min-trans.	0.67	0.88	0.90	1.40	4.56	14.99
am-trans.	0.56	0.67	0.83	1.11	3.34	11.68
max-trans.	0.67	0.72	0.75	0.99	3.58	9.95

Furthermore, as we can see in Table 7, regarding the value of the threshold k that provides a simulated probability of the strict preference relation P_k being transitive equal to 1, the percentage k/m decreases when the value of m increases. For example, look at the case of max-transitive reciprocal preference relations. The percentage that the threshold represents over the number of voters is a 50,3 percent in the case of $m = 3$, a 27.8 percent for $m = 10$, a 9.72 percent for $m = 100$ and a 2.61 percent in the instance of $m = 1,000$.

Table 7 k/m (%) and P_k transitive.

m	0.5-trans.		min-trans.		am-trans.		max-trans.	
	k	k/m (%)	k	k/m (%)	k	k/m (%)	k	k/m (%)
3	2.58	86	1.96	65.3	1.56	52	1.51	50.3
10	5.17	51.7	3.47	34.7	3.10	31	2.78	27.8
100	14.51	14.51	12.51	12.51	9.17	9.17	9.72	9.72
1,000	47.79	4.78	35.24	3.52	33.10	3.31	26.10	2.61

Finally, considering the value of k that provides a simulated probability of P_k being triple-acyclic equal to 1, the percentage k/m decreases when the value of m increases, considering min-transitive and am-transitive reciprocal preference relations. In Table 8, we have those percentages. As we can see, this is not the case for 0.5-transitive reciprocal preference relations when regarding $m = 4$ and $m = 5$ and max-transitive reciprocal preference relations when considering $m = 5$, $m = 100$ and $m = 1,000$.

Table 8 k/m (%) and P_k triple-acyclic.

m	0.5-trans.		min-trans.		am-trans.		max-trans.	
	k	k/m (%)	k	k/m (%)	k	k/m (%)	k	k/m (%)
3	1.32	44	0.67	22.3	0.56	18.6	0.67	22.3
4	1.50	37.5	0.88	22	0.67	16.75	0.72	18
5	1.91	38.2	0.90	18	0.83	16.6	0.75	0.15
100	8.29	26.5	4.56	4.56	3.34	3.34	3.58	3.58
1,000	26.60	2.66	14.99	1.499	11.68	1.168	9.95	0.995

5 Discussion

Recently some analytical studies about the consistency of majorities based on difference in support have been developed. These theoretical results rely on the needed threshold to ensure transitive and triple-acyclic strict preferences for different types of transitive reciprocal preference relations.

On the one hand, the case of transitive strict preferences is studied in Llamazares et al (2013). The results can be summarized as follows:

1. The transitivity of the strict preference can not be ensured for any threshold of support k less than $m - 1$.
2. The transitivity of the strict preference can not be ensured for any threshold of support k less than m if the reciprocal preference relations are less demanding than am-transitive ones.
3. The strict preference is transitive for any threshold of support such that $k \in [m - 1, m)$ if the reciprocal preference relations are at least am-transitive ones.

On the other hand, the case of triple-acyclic strict preferences is analysed in Llamazares and Pérez-Asurmendi (2015) with the following results:

1. The triple-acyclicity of the strict preference, in the case of 0.5-transitive reciprocal preference relations, can be guaranteed if the threshold of support k is located in $[\lfloor 2m/3 \rfloor, m)$ where $\lfloor a \rfloor$ stands for the integer part of a .
2. The triple-acyclicity of the strict preference, in the case of min-transitive and max-transitive reciprocal preference relations, can be guaranteed if the threshold of support k belongs to $[m/3, m)$.
3. In the case of max-transitive reciprocal preference relations, it conjectures that strict preference relations are triple-acyclic if the threshold k belongs to $[\lfloor 2m/3 \rfloor / 2, m)$.

The probabilistic results setting here complement the above theoretical ones by the following reasons. First, thresholds with associated probabilities of consistent strict preferences equal to 1 are found for all the considered types of transitive reciprocal preference relations. Second, reasonable thresholds are required to certify the consistency of the strict preference with a probability value of 1 in those cases where theoretical results asked a very high threshold to guarantee such consistency. Third, the conjecture about the needed thresholds in the case of max-transitive reciprocal preference relations seems to be true.

Specifically, in the case of transitive strict preferences with 0.5-transitive and min-transitive reciprocal preference relations, the probabilities achieve the value of 1 for the considered values of m (see Table 9) whereas as it is said before, the theoretical result asserts that no threshold guarantees the transitivity of the strict preference for such types of reciprocal preference relations.

Table 9 Thresholds k such that the simulated probability of P_k being transitive is equal to 1 with 0.5- and min-transitive reciprocal preference relations.

	$m = 3$	$m = 4$	$m = 5$	$m = 10$	$m = 100$	$m = 1,000$
k (0.5-trans.)	2.58	2.97	3.24	5.17	14.51	47.79
k (min-trans.)	1.96	2.49	2.61	3.47	12.51	35.24

In the cases of am-transitive and max-transitive reciprocal preference relations, the thresholds that provide a probability value of transitive strict

preference relations equal to 1 are lower than the ones that guarantee the transitivity of the strict preference in the theoretical framework. Table 10 shows the theoretical minimum threshold required, i. e. $m - 1$, and the thresholds that provide a simulated probability value equal to 1. To illustrate, look at the case of $m = 1,000$. The theoretical result asserts that the threshold k has to belong to $[999, 1,000)$. By contrast, a probability value of 1 is achieved with a threshold of 33.10 in the case of am-transitive reciprocal preference relations and of 26.10 in the case of max-transitive reciprocal preference relations.

Table 10 Theoretical threshold k vs. thresholds k such that the simulated probability of P_k being transitive is equal to 1 with am- and max-transitive reciprocal preference relations.

	$m = 3$	$m = 4$	$m = 5$	$m = 10$	$m = 100$	$m = 1,000$
$m - 1$	2	3	4	9	99	999
k (am-trans.)	1.56	1.85	2.07	3.10	9.17	33.10
k (max-trans.)	1.51	1.66	1.93	2.78	9.72	26.10

In the case of triple-acyclic strict preferences, the thresholds to reach a probability value of 1 again are much lower than the ones required in the theoretical setting. Tables 11, 12 and 13 illustrate that fact in the cases of 0.5- and min-transitive reciprocal preference relations. In these cases we set forth the minimum theoretical threshold required and the thresholds that provide a simulated probability value equal to 1.

Table 11 Theoretical threshold k vs. thresholds k such that the simulated probability of P_k being triple-acyclic is equal to 1 with 0.5-transitive reciprocal preference relations.

	$m = 3$	$m = 4$	$m = 5$	$m = 10$	$m = 100$	$m = 1,000$
$\lfloor 2m/3 \rfloor$	2	2	3	3	66	666
k	1.32	1.50	1.91	2.65	8.29	26.60

Table 12 Theoretical threshold k vs. thresholds k such that the simulated probability of P_k being triple-acyclic is equal to 1 with min-transitive reciprocal preference relations.

	$m = 3$	$m = 4$	$m = 5$	$m = 10$	$m = 100$	$m = 1,000$
$m/3$	1	$\widehat{1.3}$	$\widehat{1.6}$	$\widehat{3.3}$	$\widehat{33.3}$	$\widehat{333.3}$
k	0.67	0.88	0.90	1.40	4.56	14.99

The case of max-transitive reciprocal preference relations deserves a special attention. In Llamazares and Pérez-Asurmendi (2015), authors conjecture

that the needed threshold to guarantee the triple-acyclicity of the collective strict preference when individuals are endowed with max-transitive reciprocal relations is lower or equal³ than in the case of individual min-transitive reciprocal relations. Unfortunately, they could not provide a formal proof for that assertion. Comparing the thresholds k such that the simulated probability of P_k is triple-acyclic with a probability of 1 from Tables 12 and 13, we can corroborate that conjecture. The thresholds in the case of max-transitive reciprocal preference relations are smaller or equal than the ones in min-transitive case.

Table 13 Theoretical threshold k vs. thresholds k such that the simulated probability of P_k being triple-acyclic is equal to 1 with max-transitive reciprocal preference relations.

	$m = 3$	$m = 4$	$m = 5$	$m = 10$	$m = 100$	$m = 1,000$
$\lfloor [2m/3]/2, m \rfloor$	1	1	1.5	3	33	333
k	0.67	0.72	0.75	0.99	3.58	9.95

A quite natural question is to wonder if the consideration of reciprocal preference relations has an impact on the probability of consistent decisions when compared with the case of crisp preferences, i.e. when voters only declare if they prefer an alternative to another one or if they are indifferent between both alternatives.

To highlight this idea, we consider two cases: First, we focus on voters who show linear preference orderings on the candidates. In other words, $r_{ij}^p \in \{0, 1\}$ for each pair of alternatives x_i and x_j and each voter p . Second, we also consider the case where individuals are endowed with weak preference orderings and declare their preferences by means of values $r_{ij}^p \in \{0, 0.5, 1\}$. If $r_{ij}^p = 0.5$, individual p is indifferent between both alternatives. Notice that in the first case, these majorities are equivalent to supermajorities (see for a formal proof of that Diss and Pérez-Asurmendi (2015)); in the second case, these voting rules are majorities based on difference of votes.

For that aim, we find the thresholds which guarantee a probability equal to 1 of P_k being triple-acyclic (Table 14), P_k being transitive (Table 15), and R_k being transitive (Table 16) taking into account individual linear and weak orderings and compare them with the corresponding ones when considering individual reciprocal preference relations.

³ Notice that $m/3 = \lfloor (2 \cdot 100)/3 \rfloor / 2$ when m is multiple of three.

Table 14 Crisp preferences *versus* reciprocal preferences: Thresholds k such that the simulated probability of P_k being triple-acyclic is equal to 1.

$m \downarrow$	Crisp preferences		Reciprocal preferences			
	$r_{ij}^p \in \{0, 1\}$	$r_{ij}^p \in \{0, 0.5, 1\}$	0.5-trans.	min-trans.	am-trans.	max-trans.
3	1	1	1.32	0.67	0.56	0.67
4	0	1	1.50	0.88	0.67	0.72
5	1	1	1.91	0.90	0.83	0.75
10	2	2	2.65	1.40	1.11	0.99
100	10	8	8.29	4.56	3.34	3.58
1000	36	30	26.60	14.99	11.68	9.95

Table 15 Crisp preferences *versus* reciprocal preferences: Thresholds k such that the simulated probability of P_k being transitive is equal to 1.

$m \downarrow$	Crisp preferences		Reciprocal preferences			
	$r_{ij}^p \in \{0, 1\}$	$r_{ij}^p \in \{0, 0.5, 1\}$	0.5-trans.	min-trans.	am-trans.	max-trans.
3	1	1	2.58	1.96	1.56	1.51
4	2	2	2.97	2.49	1.85	1.66
5	3	3	3.24	2.61	2.07	1.93
10	6	7	5.17	3.47	3.10	2.78
100	26	21	14.51	12.51	9.17	9.72
1000	78	89	47.79	35.24	33.10	26.10

Table 16 Crisp preferences *versus* reciprocal preferences: Thresholds k such that the simulated probability of R_k being transitive is equal to 1.

$m \downarrow$	Crisp preferences		Reciprocal preferences			
	$r_{ij}^p \in \{0, 1\}$	$r_{ij}^p \in \{0, 0.5, 1\}$	0.5-trans.	min-trans.	am-trans.	max-trans.
3	-	-	2.97	2.96	2.97	2.98
4	-	-	3.81	3.84	3.91	3.84
5	-	-	4.70	4.80	4.91	4.72
10	-	-	7.95	8.32	7.48	7.56
100	48	43	26.40	27.23	27.90	25.47
1000	147	172	84.83	95.32	83.01	80.00

We can deduce the following facts:

- For big electorates, required thresholds are lower when considering reciprocal preference relations than in the case of crisp preferences with independence of the analysed collective consistent decision (see the case of $m = 1,000$ in Tables 14, 15, and 16).

- In the case of the probability of R_k being transitive equal to 1 (Table 16), we find thresholds to reach that probability when considering reciprocal preference relations for all the considered number of individuals whilst in the case of crisp preferences we cannot provide some of them. Notice that, even for a threshold equal to $k = m - 1$, the probability of transitive R_k is non null when considering crisp preferences. For instance, for $m = 10$ and a threshold $k = 9$, the probability of transitive R_k is equal to 0.9944 and 0.9993 for $r_{ij}^p \in \{0, 1\}$ and $r_{ij}^p \in \{0, 0.5, 1\}$, respectively.
- In the case of the probability of P_k being triple-acyclic equal to 1 (Table 14), we find lower thresholds for min-transitive, am-transitive and max-transitive reciprocal preference relations than for crisp preferences.
- The case of the probability of P_k being transitive equal to 1 (Table 15) is more complex to analyse. On the one hand, the thresholds are lower when considering crisp preferences and $m = 3$. On the other hand, reciprocal preferences provide lower thresholds for bigger electorates than that. Specifically, greater than $m = 3$ when considering am-transitive and max-transitive reciprocal preferences relations, greater than $m = 4$ in the case of min-transitive reciprocal preference relations and greater than $m = 10$ when regarding 0.5-transitive reciprocal preference relations.

Therefore, we can ensure that considering reciprocal preference relations has an unequivocal effect on the needed thresholds to reach a probability of collective consistent decisions equal to 1 when considering big electorates.

Finally, we set forth the probabilities of consistent collective decisions with crisp preferences and the ones with reciprocal preferences for a given number of voters m and a given threshold k . Now our goal is to know if the likelihood of consistent collective decisions is higher or smaller when the voters show their preferences reciprocally than in the case where they are endowed with linear or weak orderings. In Table 17, we set the probabilities when $m = 3$ and $m = 1,000$ with the aim of distinguishing between the cases of small and big electorates.

Table 17 The impact of the individual type of preferences in the probability. The examples of $m = 3$ and $m = 1,000$.

		$m = 3$			$m = 1,000$		
		$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 10$
$r_{ij}^p \in \{0, 1\}$	R_k transitive	0.9511	0.6772	0.6908	0.8739	0.9004	0.8047
	P_k transitive	0.9511	1	1	0.8907	0.9160	0.9137
	P_k triple-acyclic	0.9511	1	1	0.9492	0.9643	0.9897
$r_{ij}^p \in \{0, 0.5, 1\}$	R_k transitive	0.8736	0.6943	0.7827	0.9399	0.9339	0.8893
	P_k transitive	0.9599	1	1	0.9442	0.9411	0.9272
	P_k triple-acyclic	0.9944	1	1	0.9727	0.9761	0.9910
0.5-trans.	R_k transitive	0.8835	0.5438	0.8883	0.7816	0.7479	0.5794
	P_k transitive	0.8835	0.9587	0.9994	0.8267	0.8241	0.8767
	P_k triple-acyclic	0.8835	0.9997	1	0.9328	0.9477	0.9905
min-trans.	R_k transitive	0.9563	0.6551	0.8978	0.8802	0.8534	0.7092
	P_k transitive	0.9563	0.9950	1	0.9245	0.9262	0.9659
	P_k triple-acyclic	0.9563	1	1	0.9827	0.9886	0.9996
am-trans.	R_k transitive	0.9751	0.7036	0.8913	0.9113	0.8883	0.7640
	P_k transitive	0.9751	0.9988	1	0.9536	0.9572	0.9857
	P_k triple-acyclic	0.9751	1	1	0.9923	0.9959	1.0000
max-trans.	R_k transitive	0.9777	0.7102	0.9015	0.9184	0.8971	0.7726
	P_k transitive	0.9777	0.9993	1	0.9609	0.9657	0.9904
	P_k triple-acyclic	0.9777	1	1	0.9945	0.9970	1

For a small electorate ($m = 3$) we have found the following:

- For $k = 0$, the probabilities of transitive R_k are greater when considering min-transitive, am-transitive and max-transitive reciprocal preference relations than when considering crisp and 0.5-transitive reciprocal preferences. The probabilities of transitive P_k are also greater bearing in mind am-transitive and max-transitive reciprocal preference relations than any other considering preferences. Nevertheless, the probability of triple-acyclic P_k is greater in the case of weak orderings than in the other possible cases.
- For $k = 1$, the probabilities of transitive R_k are greater when regarding am-transitive and max-transitive reciprocal preference relations than when considering crisp preferences and these last ones are also greater than when taking into account min-transitive and 0.5-transitive reciprocal preference relations. The probabilities of transitive P_k are greater when regarding crisp preferences than when considering reciprocal preference relations. In the case of the probabilities of triple-acyclic P_k , crisp preferences and reciprocal preference relations provide same probabilities with the exception of the case of 0.5-reciprocal preference relations attached with a lower probability.
- For $k = 2$, the probabilities of consistent outcomes are greater or equal when considering reciprocal preference relations than when regarding crisp preferences with the exception of the probability of transitive P_k when having 0.5 reciprocal preference relations.

For a big electorate ($m = 1,000$) we have the following:

- For $k = 3$ the lowest probabilities of consistent outcomes are reached when considering 0.5-transitive reciprocal preference relations. The probabilities of transitive R_k are greater when considering weak orderings than max-transitive, am-transitive, min-transitive reciprocal preference relations and those ones are also greater than when regarding linear orderings. The probabilities of transitive P_k are greater when considering am-transitive and max-transitive reciprocal relations than when regarding weak orderings. These last ones are greater than when considering min-transitive reciprocal relations which in turn are also greater than when regarding linear orderings. In the case of the probabilities of triple-acyclic P_k , the greatest ones corresponds to max-transitive, am-transitive and min-transitive reciprocal preference relations followed by the ones attached to crisp preferences.
- For $k = 4$ the probabilities of transitive R_k are greater when considering crisp than reciprocal preference relations. The probabilities of transitive and triple-acyclic P_k behave in the same way than in the case of $k = 3$ analysed above.
- For $k = 10$ the probabilities of transitive R_k are greater when considering crisp than reciprocal preference relations. The probabilities of transitive P_k are greater when considering max-transitive, am-transitive and min-transitive reciprocal preference relations than when considering crisp preferences. The lowest probabilities are reached when regarding 0.5-transitive reciprocal preference relations. The probabilities of triple-acyclic P_k are greater when considering reciprocal preference relations than when regarding crisp preferences.

Therefore, we can not draw a clear conclusion about the relationship between the probabilities and the considered type of individual preferences. For instance, 0.5-transitive reciprocal preference relations give rise to the lowest probabilities of consistent outcomes except in the cases of triple-acyclic P_k with $k = 2$ for small electorates and with $k = 10$ for big electorates, respectively.

For big electorates weak orderings provide greater probabilities of transitive R_k than the remaining considered individual preferences. Instead, in the case of small electorates, am-transitive and max-transitive reciprocal preference relations promote the greatest probabilities.

Analysing the case of the probabilities of transitive P_k considering big electorates, the greatest ones are reached when regarding max-transitive and am-transitive reciprocal preference relations. That is also the case for $m = 3$ when considering $k = 0$ but not for the other considered values of k (notice that in the case of $k = 1$ crisp preferences give rise to the highest probabilities and in the case of $k = 2$ the probabilities are the same for all the considered individual preferences with the exception of 0.5-transitive reciprocal preference relations).

Finally, when regarding the probabilities of triple-acyclic P_k we found that the greatest probabilities are attached to weak orderings when $m = 3$ whereas

to max-transitive, am-transitive and min-transitive reciprocal preference relations when $m = 1,000$.

6 Conclusion

In this paper, we estimate through simulations the probabilities of consistent preferences under majorities based on difference in support: the probability of having transitive weak preferences R_k , the probability of having transitive strict preferences P_k , and the probability of having triple-acyclic strict preferences P_k . We start our simulations by assuming four types of transitivity conditions for individual reciprocal preferences: 0.5-transitivity, min-transitivity, am-transitivity, and max-transitivity.

This paper contains new contributions to the literature of reciprocal preferences. First, we have found the thresholds which guarantee that the probability of consistent outcomes is equal to 1. These thresholds are much lower than the ones required in the theoretical setting. In fact, our results show a dramatic difference between these two thresholds: For instance, in some cases, the simulated threshold represents less than a 3% of the total number of voters whereas the theoretical one reaches the 67%. Given that the needed thresholds to reach consistent decisions with probability equal to 1 are not too demanding, the implementation of majorities based in difference of support have sense whenever the flexibility attached to individual preferences by the consideration of reciprocal preference relations and the consistency of the collective decision is a concern for the policy maker or for the society as a whole. Second, we have set forth the impact of the threshold k on the probability of consistent outcomes for a given number of voters. The probability of having transitive weak preferences R_k as well as the probability of having transitive strict preferences P_k exhibit a similar behaviour: for the lowest values of the threshold k , it decreases as the value of the threshold k increases whereas for higher values of k , it increases with the value of the threshold k . However, the probability of having triple-acyclic strict preferences P_k increases when the value of the threshold k does, for any number of voters. Third, we have compared the needed thresholds to reach a probability of collective consistent decisions equal to 1 regarding reciprocal preference relations with the ones considering crisp preferences and concluded that the thresholds are significantly lower in the case of reciprocal preference relations than in the case of crisp preferences. Finally, we have studied if the consideration of reciprocal preference relations instead of crisp preferences has an impact in the probability of collective consistent decisions. We have found that the probability depends on the concrete considered individual preference relation and on the size of the electorate.

Table A.7 Probabilities of transitive P_k for am-transitive reciprocal preference relations.

$m \rightarrow$ $k \downarrow$	3	4	5	10	100	1,000	100,000
0	0.9751	0.9710	0.9712	0.9694	0.9676	0.9670	0.9692
1.56	1	0.9999	0.9997	0.9973	0.9605	0.9551	0.9672
1.85		1	1.0000	0.9990	0.9655	0.9542	0.9668
2.07			1	0.9995	0.9697	0.9538	0.9665
3.10				1	0.9845	0.9538	0.9653
9.17					1	0.9828	0.9600
33.10						1	0.9570
47.79							0.9629

Table A.8 Probabilities of transitive P_k for max-transitive reciprocal preference relations.

$m \rightarrow$ $k \downarrow$	3	4	5	10	100	1,000	100,000
0	0.9777	0.9766	0.9751	0.9741	0.9727	0.9726	0.9744
1.51	1	1.0000	0.9999	0.9981	0.9689	0.9619	0.9727
1.66		1	0.9999	0.9989	0.9712	0.9614	0.9725
1.93			1	0.9996	0.9758	0.9608	0.9723
2.78				1	0.9870	0.9607	0.9715
9.72					1	0.9897	0.9662
26.10						1	0.9626
47.79							0.9690

Table A.9 Probabilities of triple-acyclic P_k for 0.5-transitive reciprocal preference relations.

$m \rightarrow$ $k \downarrow$	3	4	5	10	100	1,000	100,000
0	0.8835	0.8781	0.8763	0.8751	0.8706	0.8705	0.8728
1.32	1	0.9999	0.9998	0.9967	0.9479	0.9013	0.8763
1.50		1	1.0000	0.9983	0.9546	0.9052	0.8768
1.91			1	0.9998	0.9681	0.9135	0.8778
2.65				1	0.9837	0.9270	0.8794
8.29					1	0.9834	0.8926
26.60						1	0.9291

References

- Balasko Y, Crès H (1997) The probability of Condorcet cycles and super majority rules. *J Econ Theory* 75:237–270
- Campbell CD, Tullock G (1965) A measure of the importance of cyclical majorities. *Econ J* 75(300):853–857
- Cervone DP, Gehrlein WV, Zwicker WS (2005) Which scoring rule maximizes Condorcet efficiency under IAC? *Theor Decis* 58(2):145–185
- Condorcet Md (1785) *Essai sur l'Application de l'Analyse à la Probabilité des Décisions Rendues à la Pluralité des Voix*. Imprimerie Royale, Paris
- Cook WD and Kress M (1985) Ordinal Ranking with Intensity of Preference. *Manage Sci* 31:26–32
- Dasgupta M, Deb R (1996) Transitivity and fuzzy preferences. *Soc Choice Welfare* 13(3):305–318
- DeMeyer F, Plott C (1970) The probability of a cyclical majority. *Econometrica* 38(2):345–354
- Diss M, Pérez-Asurmendi P (2015) Consistent collective decisions under Majorities based on difference of votes, *Theory Dec*, Forthcoming
- Dubois D, Prade H (1980) *Fuzzy Sets and Systems: Theory and Applications*. Mathematics in Science and Engineering, Academic Press, New York
- Fishburn PC, Gehrlein WV (1980) The Paradox of Voting: Effects of individual indifference and intransitivity. *J Public Econ* 14(1):83–94
- García-Lapresta JL, Llamazares B (2000) Aggregation of fuzzy preferences: Some rules of the mean. *Soc Choice Welfare* 17(4):673–690
- García-Lapresta JL, Llamazares B (2001) Majority decisions based on difference of votes. *J Math Econ* 35(3):463–481
- García-Lapresta JL, Llamazares B (2010) Preference intensities and majority decisions based on difference of support between alternatives. *Group Decis Negot* 19(6):527–542
- García-Lapresta JL, Meneses LC (2005) Individual-valued preferences and their aggregation: Consistency analysis in a real case. *Fuzzy Sets Syst* 151(2):269–284
- Gehrlein WV (1983) Condorcet's Paradox. *Theor Decis* 15(2):161–197
- Gehrlein WV (1997) Condorcet Paradox and the Condorcet efficiency of voting rules. *Math Jpn* 40:173–199
- Gehrlein WV, Fishburn PC (1976) Condorcet's Paradox and anonymous preference profiles. *Public Choice* 26:1–18
- Gehrlein WV, Fishburn PC (1980) Robustness of positional scoring over subsets of alternatives. *Appl Math Optim* 6:241–255
- Gehrlein WV, Fishburn PC (1981) Constant scoring rules for choosing one among many alternatives. *Qual Quant* 15:203–210
- Gehrlein WV, Fishburn PC (1983) Scoring rule sensitivity to weight selection. *Public Choice* 40:249–261
- Guilbaud G -TH. (1952) Les théories de l'intérêt général et le problème logique de l'agrégation. *Economie Appliquée* 5:501–584
- Houy N (2007) Some further characterizations for the forgotten voting rules. *Math Soc Sci* 53(1):111–121
- Jones B, Radcliff B, Taber C, Timpono R (1995) Condorcet winners and the Paradox of Voting: Probability calculations for weak preference orders. *Am Polit Sci Rev* 89(1):133–144
- Klahr D (1966) A computer simulation of the Paradox of Voting. *Am Polit Sci Rev* 60(2):384–390
- Llamazares B (2004) Simple and absolute special majorities generated by OWA operators. *Eur J Oper Res* 158:707–720
- Llamazares B (2006) The forgotten decision rules: Majority rules based on difference of votes. *Math Soc Sci* 51(3):311–326
- Llamazares B (2007) Choosing OWA operator weights in the field of Social Choice. *Inf Sci* 177:4745–4756

- Llamazares B, García-Lapresta JL (2003) Voting systems generated by quasiarithmetic means and OWA operators, in: J. Fodor, B. De Baets (Eds.), Principles of Fuzzy Preference Modelling and Decision Making, Academia, Press, Ghent, pp. 195–213
- Llamazares B, García-Lapresta JL (2008) Extension of some voting systems to the field of gradual preferences, in: H. Bustince, F. Herrera, J. Montero (Eds.), Fuzzy Sets and Their Extensions: Representation, Aggregation and Models, Springer-Verlag, Berlin, pp. 297–315
- Llamazares B, Pérez-Asurmendi P (2015) Triple-acyclicity in majorities based on difference in support. *Inf Sci* 299:209–220
- Llamazares B, Pérez-Asurmendi P, García-Lapresta JL (2013) Collective transitivity in majorities based on difference in support. *Fuzzy Sets Syst* 216:3–15
- McLean I and Urken AB (1995) *Classics of Social Choice*. Ann Arbor, The University of Michigan Press
- Meek BL (1975) A transferable voting system including intensity of preference. *Mathématique et Sciences Humaines* 50:23–29
- Morales JI (1797) *Memoria Matemática sobre el Cálculo de la Opinion en las Elecciones*. Imprenta Real, Madrid
- Nurmi H (1981) Approaches to collective decision making with fuzzy preference relations. *Fuzzy Sets Syst* 6:249–259
- Nurmi H (2008) Fuzzy social choice: a selective retrospect. *Soft Computing* 12:281–288
- Sen AK (1970) *Collective Choice and Social Welfare*, Holden-Day, San Francisco
- Tanino T (1984) Fuzzy preference orderings in group decision making. *Fuzzy Sets Syst* 12:117–131
- Tovey CA (1997) Probabilities of preferences and cycles with super majority rules. *J Econ Theory* 75:271–279
- Zadeh LA (1971) Similarity relations and fuzzy orderings. *Inform Sci* 3(2):177–200