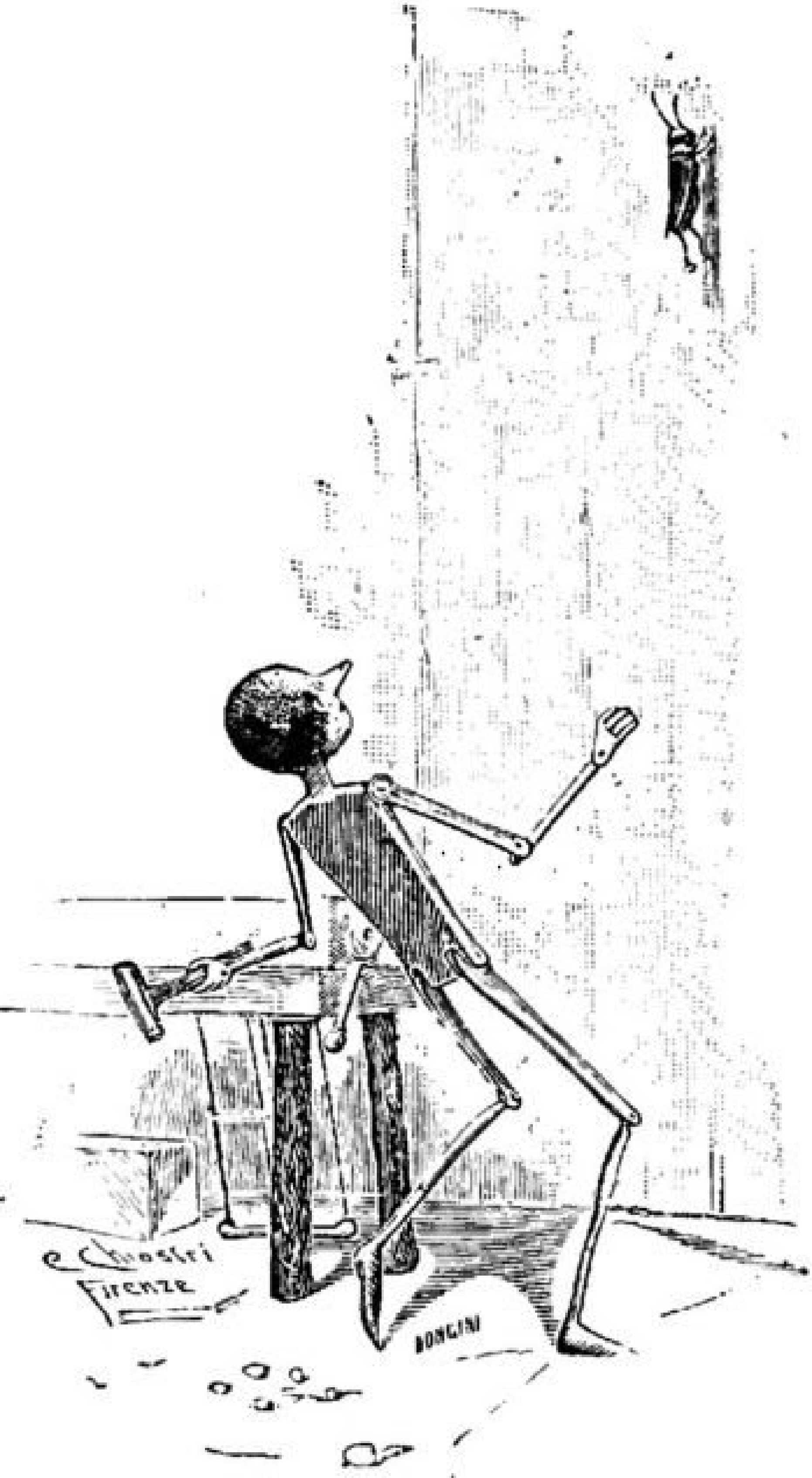
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STATIONARITY AND INVERTIBILITY OF A DYNAMIC CORRELATION MATRIX

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One of the most widely-used multivariate conditional volatility models is the dynamic conditional correlation (or DCC) specification. However, the underlying stochastic process to derive DCC has not yet been established, which has made problematic the derivation of asymptotic properties of the Quasi-Maximum Likelihood Estimators (QMLE). To date, the statistical properties of the QMLE of the DCC parameters have purportedly been derived under highly restrictive and unverifiable regularity conditions. The paper shows that the DCC model can be obtained from a vector random coefficient moving average process, and derives the stationarity and invertibility conditions of the DCC model. The derivation of DCC from a vector random coefficient moving average process raises three important issues, as follows: (i) demonstrates that DCC is, in fact, a dynamic conditional covariance model of the returns shocks rather than a dynamic conditional correlation model; (ii) provides the motivation, which is presently missing, for standardization of the conditional covariance model to obtain the conditional correlation model; and (iii) shows that the appropriate ARCH or GARCH model for DCC is based on the standardized shocks rather than the returns shocks. The derivation of the regularity conditions, especially stationarity and invertibility, may subsequently lead to a solid statistical foundation for the estimates of the DCC parameters. Several new results are also derived for univariate models, including a novel conditional volatility model expressed in terms of standardized shocks rather than returns shocks, as well as the associated stationarity and invertibility conditions.

Keywords: dynamic conditional correlation, dynamic conditional covariance, vector random coefficient moving average, stationarity, invertibility, asymptotic properties

Classification: C22, C52, C58, G32

1. INTRODUCTION

Among multivariate conditional volatility models, the dynamic conditional correlation (or DCC) specification of Engle [15] is one of the most widely used in practice. Both multivariate conditional correlations and the associated conditional covariance models are very useful for determining optimal hedging strategies, volatility spillovers and causality in volatility among financial commodities. Checking the underlying stochastic properties, where they might exist, is crucial in examining the internal consistency of

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the models, as well as in deriving asymptotic properties of the associated parameter estimates, for purposes of sensible empirical analysis.

These theoretical issues are especially important in empirical energy finance, where the relationships among the prices, returns and volatility of fossil fuels, such as oil, coal and gas, and the associated carbon emissions, are crucial for public and private policy making. In this context, Chang and McAleer [6], Chang, McAleer and Tansuchat [7, 8, 9, 10], Chang, McAleer and Zuo [12], and Chang, McAleer and Wang [11] have discussed important practical issues arising in empirical finance, especially as they relate to the pricing, returns and volatility of the primary sources of fossil fuel energy output, the resulting volatility in pricing carbon emissions, and in related stock prices.

In order to calculate optimal hedging strategies (or risk insurance) to mitigate financial risk, the two alternative models that have been used widely for estimating and forecasting multivariate conditional correlations and conditional covariances have been based on: (i) the diagonal and full BEKK models of Baba et al. [3] and Engle and Kroner [16], which have been derived from an m-dimensional vector random coefficient autoregressive process (see McAleer et al. [21] and Section 2 below); and (ii) the DCC model, which was presented without an underlying stochastic specification in Engle [15].

The basic DCC modelling approach has been as follows: (i) estimate the univariate conditional variances using the GARCH(1,1) model of Bollerslev [4], which are based on the returns shocks; and (ii) estimate what is purported to be the conditional correlation matrix of the standardized residuals.

The first step in the modelling approach is arbitrary as the conditional variances could just as easily be based on the standardized residuals themselves, as will be shown in Section 4 below. The second step is fatally flawed as the model can be derived from an appropriate underlying stochastic process as a conditional covariance model rather than as a conditional correlation model. However, as no regularity conditions were presented in the presentation of the DCC model in Engle [15], no statistical properties have yet been derived for the estimated parameters of the model.

A similar comment applies to the varying conditional correlation model of Tse and Tsui [23], where the first stage is based on a standard GARCH(1,1) model using returns shocks. The second stage is slightly different from the DCC formulation as the conditional correlations are defined appropriately. However, as no regularity conditions are presented, including invertibility, no statistical properties can be derived.

The DCC model has been analyzed critically in a number of papers as its underlying stochastic process has not yet been established, which has made problematic the derivation of the asymptotic properties of the Quasi-Maximum Likelihood Estimators (QMLE). To date, the statistical properties of the QMLE of the DCC parameters have been derived under highly restrictive and unverifiable regularity conditions, which in essence amounts to proof by assumption.

This paper shows that the DCC specification can be obtained from a vector random coefficient moving average process, and derives the sufficient conditions for stationarity and invertibility of the DCC model. The derivation of regularity conditions may subsequently lead to a solid statistical foundation for the estimates of the DCC parameters.

The derivation of DCC from a vector random coefficient moving average process raises three important issues: (i) demonstrates that DCC is, in fact, a dynamic conditional covariance model of the returns shocks rather than a dynamic conditional correlation model; (ii) provides the motivation, which is presently missing, for standardization of the conditional covariance model to obtain the conditional correlation model; and (iii) shows that the appropriate ARCH or GARCH model for DCC is based on the standardized shocks rather than the returns shocks.

The remainder of the paper is organized as follows. In Section 2, the standard ARCH model is derived from a random coefficient autoregressive process to provide a background for the remainder of the paper. The multivariate counterpart of ARCH is derived from a vector random coefficient autoregressive process, which will explain intuitively how the univariate results of Marek [20] on a random coefficient moving average process can be extended to an m-dimensional vector counterpart. In Section 3, the DCC model is presented and discussed. Section 4 presents and discusses a new vector random coefficient moving average process that will be used as an underlying stochastic process in order to derive DCC. Several new results are derived for the associated univariate models, including a novel conditional volatility model expressed in terms of standardized shocks rather than returns shocks, as well as the associated stationarity and invertibility conditions. In Section 5, DCC is demonstrated to be derived from the vector random coefficient moving average process. The conditions for stationarity and invertibility of DCC are derived in Section 6. Some concluding comments are given in Section 7.

2. RANDOM COEFFICIENT AUTOREGRESSIVE PROCESS

This section presents the underlying stochastic autoregressive processes for univariate and multivariate GARCH processes, as compared with the multivariate moving average process for the multivariate DCC process in the following section. Consider the following random coefficient autoregressive process of order one:

$$\varepsilon_t = \phi_t \varepsilon_{t-1} + \eta_t \tag{1}$$

where

$$\begin{aligned} &\phi_t \sim iid(0, \alpha), \\ &\eta_t \sim iid(0, \omega), \text{ independent of } \{\phi_t\}. \end{aligned}$$

The ARCH(1) model of Engle [14] can be derived as (see Tsay [22]):

$$h_t = E(\varepsilon_t^2 | I_{t-1}) = \omega + \alpha \varepsilon_{t-1}^2, \tag{2}$$

where h_t is conditional volatility, and I_{t-1} is the information set at time *t-1*. The use of an infinite lag length for the random coefficient autoregressive process in equation (1) leads to the Generalized ARCH (or GARCH) model of Bollerslev [4].

The diagonal version of the BEKK model of Baba et al. [3] and Engle and Kroner [16], though not the Hadamard BEKK and full BEKK models, can be derived from a vector random coefficient autoregressive process (see McAleer et al. [21]). As the statistical properties of vector random coefficient autoregressive processes are well known, the statistical properties of the parameter estimates of the ARCH, GARCH, and diagonal BEKK models are straightforward to establish.

3. DCC SPECIFICATION

This section presents the DCC model, as given in Engle [15], which does not have an underlying stochastic specification that leads to its derivation. Let the conditional mean of financial returns be given as:

$$y_t = E(y_t|I_{t-1}) + \varepsilon_t, \tag{3}$$

where $y_t = (y_{1t,\dots,}y_{mt})'$, $y_{it} = \Delta \log P_{it}$ represents the log-difference in stock prices (P_{it}), $i = 1,\dots,m$, I_{t-1} is the information set at time *t-1*, and ε_t is conditionally heteroskedastic. Without distinguishing between dynamic conditional covariances and dynamic conditional correlations, Engle [15] presented the DCC specification as:

$$Q_t = (1-\alpha-\beta)\overline{Q} + \alpha\eta_{t-1}\eta'_{t-1} + \beta Q_{t-1}, \tag{4}$$

where Q_t in (4) is purported to be a conditional correlation matrix, without proof, $\overline{Q}$ is assumed to be positive definite with unit elements along the main diagonal, the scalar parameters α and β are assumed to be non-negative and satisfy the stability condition, $\alpha+\beta < 1$, the standardized shocks, $\eta_t = (\eta_{1t,\dots,}\eta_{mt})'$, where $\eta_{it} = \varepsilon_{it}/\sqrt{h_{it}}$, are assumed to be *iid*, and D_t is a diagonal matrix with typical element $\sqrt{h_{it}}$, $i = 1,\dots,m$. If m is the number of financial assets, the multivariate definition of the relationship between ε_t and η_t is $\varepsilon_t = D_t\eta_t$.

Define the conditional covariance matrix of ε_t as Q_t. As the $m \times 1$ vector, η_t, is assumed to be *iid* for all m elements, the conditional correlation matrix of η_t is given by Γ_t. Therefore, the conditional expectation of the covariance matrix of ε_t is defined as:

$$Q_t = D_t\,\Gamma_t\,D_t. \tag{5}$$

Equivalently, the conditional correlation matrix, Γ_t, is defined as:

$$\Gamma_t = D_t^{-1}\,Q_t\,D_t^{-1}. \tag{6}$$

Equation (5) is useful if a model of Γ_t is available for purposes of estimating Q_t, whereas equation (6) is useful if a model of Q_t is available for purposes of estimating Γ_t. Ling and McAleer [19] and McAleer et al. [21] provide general proofs of the asymptotic properties of univariate and multivariate conditional volatility models based on satisfying the regularity conditions in Jeantheau [18] for consistency, and in Theorem 4.1.3 in Amemiya [2] for asymptotic normality.

In view of equations (5) and (6), as the matrix Q_t in equation (4) does not satisfy the definition of a correlation matrix, Engle [15] uses the following standardization for Q_t in equation (4):

$$R_t = (diag(Q_t))^{-1/2}Q_t(diag(Q_t))^{-1/2}. \tag{7}$$

There is no clear explanation given in Engle [15] for the standardization in equation (7) or, more recently, in Aielli [1], especially as equation (7) does not satisfy the definition of a correlation matrix, which is given in equation (6). The standardization in equation (7) might make sense if the matrix Q_t in equation (4) were the conditional covariance matrix of ε_t or η_t though this is not made clear. It is worth noting that the unconditional covariance matrix of ε_t is not analytically tractable.

Despite the title of the paper, Aielli [1] also does not provide any stationarity conditions for the DCC model, and does not mention invertibility. Indeed, in the literature on DCC, it is not clear whether equation (4) refers to a conditional covariance or a conditional correlation matrix, although the latter is presumed, without proof. Some caveats regarding DCC are given in Caporin and McAleer [5].

4. VECTOR RANDOM COEFFICIENT MOVING AVERAGE PROCESS

The random coefficient moving average process will be presented in its original univariate form in section 4.1, as in Marek [20], with an extension to its multivariate counterpart in section 4.2, in order to derive the corresponding univariate and multivariate conditional volatility models, respectively.

4.1. Univariate process

In an interesting and useful paper, Marek [20] proposed a linear moving average model with random coefficients (RCMA), and established the conditions for stationarity and invertibility. In this section, we extend the univariate results of Marek [20] using an m-dimensional vector random coefficient moving average process of order p, which is used as an underlying stochastic process to derive the DCC model, and prove the stationarity and invertibility conditions. Several new results are also derived for the associated univariate models, including a novel conditional volatility model expressed in terms of standardized shocks rather than returns shocks, as well as the associated stationarity and invertibility conditions.

Consider a univariate random coefficient moving average process given by:

$$\varepsilon_t = \theta_t \eta_{t-1} + \eta_t, \tag{8}$$

where $\eta_t \sim iid\ (0, \omega)$. The sequence $\{\theta_t\}$ is supposed to be independent of $\eta_{t-1}, \eta_t, \eta_{t+1}, \ldots$, which is called the Future Independence Condition, with mean zero and variance α. It is also assumed to be measurable with respect to I_t, where I_t is the information set generated by the random variable $\{\varepsilon_t, \varepsilon_{t-1,\ldots}\}$. Furthermore, it is assumed that the process $\{\varepsilon_t\}$ is stationary and invertible, such that $\eta_t \in I_t$. For further details, see Marek [20].

Without the measurability assumption on $\{\theta_t\}$ it would be difficult to obtain results on the invertibility of the model. However, an important special case of the model arises when $\{\theta_t\}$ is *iid*, that is, not measurable with respect to I_t, in which case the conditional and unconditional expectations of ε_t are zero, and the conditional variance of ε_t is given by:

$$h_t = E(\varepsilon_t^2 | I_{t-1}) = \omega + \alpha \eta_{t-1}^2 \tag{9}$$

which differs from the ARCH(1) model in equation (2) in that the returns shock is replaced by the standardized shock. This is a new result in the conditional volatility literature.

As $\eta_t \sim iid\ (0, \omega)$, the unconditional variance of ε_t is given as:

$$E(h_t) = (1 + \alpha)\, \omega.$$

The use of an infinite lag length for the random coefficient moving average process in equation (8), with appropriate restrictions on θ_t, would lead to a generalized ARCH model that differs from the GARCH model of Bollerslev [4] as it would replace the returns shock with a standardized shock.

The univariate ARCH(1) model in equation (9) is contained in the family of GARCH models proposed by Hentschel [17], and the augmented GARCH model class of Duan [13].

It can be shown from the results in Marek [20] that a sufficient condition for stationarity is that the vector sequence $\upsilon_t = (\eta_t, \theta_t \eta_{t-1})'$ is stationary. Moreover, by Lemma 2.1 of Marek [20], a new sufficient condition for invertibility is that:

$$E\left[\log|\theta_t|\right] < 0. \tag{10}$$

The stationarity of $\nu_t = (\eta_t, \theta_t, \eta_{t-1})$ and the invertibility condition in equation (10) are new results for the univariate ARCH(1) model given in equation (9), as well as its direct extension to GARCH models.

4.2. Multivariate process

Extending the analysis given above to the multivariate case and to a vector random coefficient moving average (RCMA) model of order p, we can derive a special case of DCC(p,q), namely DCC(p,0), as follows:

$$\varepsilon_t = \sum_{j=1}^{p} \theta_{jt} \eta_{t-j} + \eta_t, \tag{11}$$

where ε_i and η_i are both $m \times 1$ vectors and θ_{jt}, $j = 1,\ldots,p$ are random $m \times m$ matrices, independent of $\eta_{t-1}, \eta_t, \eta_{t+1}, \ldots$. Under Assumption 1, it is possible to derive the conditional covariance matrix of ε_i in equation (11):

Assumption 1.

1. $E(\eta_t|I_{t-1}) = 0$, $E(\eta_t \eta_t'|I_{t-1}) = \Omega$.
2. The random coefficient matrices θ_{jt} have the following properties: For all $j = 1,\ldots,p$, and $t = 1,\ldots,T$, it is assumed that $E(\theta_{jt}|I_{t-1})= 0$ and $E(\theta_{jt,kl}\theta'_{jt,mn}|I_{t-1}) = A_{j,kl}A'_{j,mn}$ for appropriate matrices $A_{j,kl}$ and $A_{j,mn}$ that form the conditional covariance matrix of θ_{jt}, and $E(\theta_{jt,kl}\theta'_{is,mn}|I_{t-1}) = 0$, $i \neq j$, and/or $s \neq t$.

This is similar to Proposition 1 of McAleer et al. [21] in that the conditional covariance matrix is given by:

$$H_t = E(\varepsilon_t \varepsilon_t'|I_{t-1}) = \Omega + \sum_{j=1}^{p} A_j \eta_{t-j} \eta_{t-j}' A_j'$$

such that:

$$E(vec(H_t)) = \left(I_m + \sum_{j=1}^{p} A_j \otimes A_j \right) vec(\Omega).$$

This approach can easily be extended to include autoregressive terms. For example, in a model analogous to GARCH(p,q), namely:

$$H_t = \Omega + \sum_{i=1}^{p} A_i \eta_{t-i} \eta_{t-i}^{'} A_i' + \sum_{j=1}^{q} B_j H_{t-j} B_j',$$

where the parameter matrices B_j are such that the maximum eigenvalue of $\sum_{j=1}^{q} B_j \otimes B_j$ is smaller than one in modulus, it follows that:

$$E(vec(H_t)) = \left(I_m - \sum_{j=1}^{q} B_j \otimes B_j \right)^{-1} \left(I_m + \sum_{j=1}^{p} A_j \otimes A_j \right) vec(\Omega).$$

The derivation given above shows that, as compared with the standard DCC formulation, which is not based on an underlying stochastic process that leads to its derivation, the formulation given above permits straightforward computation of the unconditional variances and covariances via the derived models in equations .

It should also be noted that in Aielli's [1] variation of the standard DCC model, it is possible to calculate the unconditional expectation of the Q_t matrix, as in equation (4), but this is not equal to the unconditional covariance matrix of ε_t , which is analytically intractable. This is an additional advantage of using the vector random coefficient moving average process given in the above equations, as will be shown explicitly in the following section

5. DERIVATION OF DCC

In this section, the DCC model will be derived from a vector random coefficient moving average process as the underlying stochastic process. If θ_{jt} in equation (11) is given as:

$$\theta_{jt} = \lambda_{jt} I_m, \text{ with } \lambda_{jt} \sim iid(0, \alpha_j),$$

$j = 1, \ldots, p$, where λ_{jt} is a scalar random variable, then the conditional covariance matrix can be shown to be:

$$H_t = E(\varepsilon_t \varepsilon_t^{'} | I_{t-1}) = \Omega + \sum_{j=1}^{p} \alpha_j \eta_{t-j} \eta_{t-j}^{'}. \tag{12}$$

The DCC model in equation (4) is obtained by letting $p \to \infty$ in equations (11) and (12), setting $\alpha_j = \alpha \, \beta^{j-1}$, and standardizing H_t in equation (12) to obtain a conditional correlation matrix. For the case $p = 1$ in equation (12), the appropriate univariate conditional volatility model is given in the new model in equation (9), which uses the standardized shocks, rather than standard ARCH in equation (2), which uses the returns shocks.

The derivation of DCC in equation (12) from a vector random coefficient moving average process is important as it: (i) demonstrates that DCC is, in fact, a dynamic conditional covariance model of the returns shocks rather than a dynamic conditional correlation model; (ii) provides the motivation, which is presently missing, for standardization of the conditional covariance model to obtain the conditional correlation model; and (iii) shows that the appropriate ARCH and GARCH models for DCC are based on the standardized shocks rather than the returns shocks. Point (iii) provides novel univariate ARCH and GARCH models.

It is worth noting that the derivation of the DCC model using the underlying vector random coefficient moving average process is not argued to be unique as the latter has not been shown to be a necessary condition. However, to date there has been no derivation of the DCC model from an underlying stochastic process that leads to its derivation.

6. DERIVATION OF STATIONARITY AND INVERTIBILITY OF DCC

The formulation of DCC given in the previous section is more natural than the standard treatment as it can be derived from an underlying stochastic process which leads to its derivation, and can be also analyzed in terms of mathematical and statistical properties, such as stationarity, invertibility, and existence of moments.

This section derives the stationarity and invertibility conditions for the DCC model in Theorem 1, based on Assumption 2:

Assumption 2.

$$E\ \left[\log\ \|\Theta_{t-k}\|\right] < \ \log\sqrt{pm} \tag{13}$$

where $\|\Theta_t\|$ is the Frobenius norm, and Θ_t is given by:

$$\Theta_t = \begin{pmatrix} -\theta_{1t} & -\theta_{2t} & \dots & -\theta_{pt} \\ 1 & 0 & \dots & 0 \\ . & . & . & . \\ 0 & \dots & 1 & 0 \end{pmatrix}.$$

Theorem 1. A sufficient condition for stationarity is that the vector sequence:

$$\upsilon_t = \left(\eta_t, \theta_{1t}\eta_{t-1}, \dots, \theta_{pt}\eta_{t-p}\right)'$$

is stationary. Furthermore, under Assumption 2, the vector random coefficient moving average process, ε_t, is invertible.

P r o o f. The proof of stationarity is similar to the sufficient condition for stationarity of the univariate random coefficient moving average process, namely that the vector sequence $\upsilon_t = (\eta_t, \theta_t\eta_{t-1})'$ is stationary. For invertibility, note that:

$$\eta_t = \varepsilon_t - \sum_{j=1}^{p} \theta_{jt}\eta_{t-j}$$

which can be written as:

$$\tilde{\eta}_t = \Theta_t\tilde{\eta}_{t-1} + \tilde{\varepsilon}_t$$

where $\tilde{\eta}_t = (\eta_t, \eta_{t-1}, \ldots, \eta_{t-p+1})'$ and $\tilde{\varepsilon}_t = (\varepsilon_t, \varepsilon_{t-1}, \ldots, \varepsilon_{t-p+1})'$.
Hence,

$$\tilde{\eta}_t = \sum_{j=0}^{n-1} \left(\prod_{k=1}^{j} \Theta_{t-k+1} \right) \tilde{\varepsilon}_{t-j} + \left(\prod_{k=0}^{n-1} \Theta_{t-k} \right) \tilde{\eta}_{t-n}.$$

Now let:

$$\tilde{\eta}_t^{(n)} = \sum_{j=0}^{n} \left(\prod_{k=1}^{j} \Theta_{t-k+1} \right) \tilde{\varepsilon}_{t-j}.$$

Consider

$$\frac{1}{n} \log \frac{1}{\sqrt{pm}} \| \tilde{\eta}_t - \tilde{\eta}_t^n \| = \frac{1}{n} \log \frac{1}{\sqrt{pm}} \left\| \left(\prod_{k=1}^{n-1} \Theta_{t-k} \right) \tilde{\eta}_{t-n} \right\|$$

$$\leq \frac{1}{n} \log \frac{1}{\sqrt{pm}} \left\| \prod_{k=1}^{n-1} \Theta_{t-k} \right\| + \frac{1}{n} \log \frac{1}{\sqrt{pm}} \| \tilde{\eta}_{t-n} \|$$

$$\leq \frac{1}{n} \sum_{k=1}^{n} \log \frac{1}{\sqrt{pm}} \| \Theta_{t-k} \| + \frac{1}{n} \log \frac{1}{\sqrt{pm}} \| \tilde{\eta}_{t-n} \|$$

$$\underset{a.s.}{\longrightarrow} E \ \log \frac{1}{\sqrt{pm}} \| \Theta_{t-k} \| < 0$$

as $E \ \log \| \Theta_{t-k} \| < \sqrt{pm}$, by assumption. This implies that $\eta_t - \eta_t^n \underset{a.s.}{\longrightarrow} 0$ and, hence, η_t is asymptotically measurable with respect to $\{\varepsilon_{t-1}, \varepsilon_{t-2}, \ldots\}$, and ε_t is invertible.

$\square$

The derivation of the sufficient conditions for stationarity and invertibility of the DCC model in Theorem 1 makes it more viable and understandable in practice, and contributes toward a statistical analysis of the model for practical purposes, as discussed in Section 1.

Note that a sufficient condition for equation (13) is that:

$$\sum_{j=1}^{p} E \ \|\theta_{jt}\|^2 < m \tag{14}$$

as

$$\begin{aligned}
& E\log\frac{1}{\sqrt{pm}}\parallel\Theta_{t-k}\| \le \log E\frac{1}{\sqrt{pm}}\parallel\Theta_{t-k}\| \\
= \quad & \log E\frac{1}{\sqrt{pm}}\sqrt{\sum_{j=1}^{p}\|\theta_{jt}\|^2+(p-1)m} \\
= \quad & \log E\sqrt{\frac{1}{\sqrt{pm}}\sum_{j=1}^{p}\|\theta_{jt}\|^2+(p-1)/p} \\
\le \quad & \log\sqrt{\frac{1}{\sqrt{pm}}\sum_{j=1}^{p}E\,\|\theta_{jt}\|^2+(p-1)/p} \\
< \quad & 0.
\end{aligned}$$

The condition given in equation (14) may be easier to check in practice than the condition given in equation (13). The simplicity and convenience of equation (13) may be important for the practical implementation of the DCC model.

For the special case $\theta_{jt} = \lambda_{jt} I_m$, with $\lambda_{jt} \sim iid(0, \alpha_j)$, $j = 1, \ldots, p$, discussed in Section 5 above, the condition in equation (14) simplifies to the well-known condition on the long-run persistence to returns shocks, namely:

$$\sum_{j=1}^{p} E\lambda_{jt}^2 = \sum_{j=1}^{p}\alpha_j \; < \; 1.$$

7. CONCLUSION

The paper was concerned with one of the most widely-used multivariate conditional volatility models, namely the dynamic conditional correlation (or DCC) specification. As the underlying stochastic process to derive the DCC model has not yet been established, this has made problematic the derivation of the asymptotic properties of the Quasi-Maximum Likelihood Estimators (QMLE). To date, the statistical properties of the QMLE of the DCC parameters have been derived under highly restrictive and unverifiable regularity conditions.

The paper showed that the DCC specification could be obtained from a vector random coefficient moving average process, and derived the sufficient stationarity and invertibility conditions of the DCC model. The derivation of the regularity conditions may eventually lead to a solid foundation for the statistical analysis of the QMLE estimates of the DCC parameters.

The derivation of DCC demonstrated that DCC is, in fact, a dynamic conditional covariance model of the standardized shocks rather than a dynamic conditional correlation model based on returns shocks, as presumed in Engle [15]. Moreover, the derivation of the DCC model provided the motivation, which is presently missing, for standardizing the conditional covariance model to obtain the conditional correlation model. The standardization of the estimated DCC models in practice does not satisfy the definition of a correlation matrix, which has always been problematic in interpreting the DCC model (see, for example, Caporin and McAleer [5]).

The derivation of the DCC model also showed that the appropriate ARCH and GARCH models underlying the DCC model are based on the standardized shocks rather than the returns shocks. Several new results were also derived for univariate models, including a novel conditional volatility model that was derived from an underlying univariate random coefficient moving average process.

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The Fiction of Full BEKK:

Pricing Fossil Fuels and Carbon Emissions*

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Abstract

The purpose of the paper is to (i) show that univariate GARCH is not a special case of multivariate GARCH, specifically the Full BEKK model, except under parametric restrictions on the off-diagonal elements of the random coefficient autoregressive coefficient matrix, that are not consistent with Full BEKK, and (ii) provide the regularity conditions that arise from the underlying random coefficient autoregressive process, for which the (quasi-) maximum likelihood estimates (QMLE) have valid asymptotic properties under the appropriate parametric restrictions. The paper provides a discussion of the stochastic processes that lead to the alternative specifications, regularity conditions, and asymptotic properties of the univariate and multivariate GARCH models. It is shown that the Full BEKK model, which in empirical practice is estimated almost exclusively compared with Diagonal BEKK (DBEKK), has no underlying stochastic process that leads to its specification, regularity conditions, or asymptotic properties, as compared with DBEKK. An empirical illustration shows the differences in the QMLE of the parameters of the conditional means and conditional variances for the univariate, DBEKK and Full BEKK specifications.

Keywords: Random coefficient stochastic process, Off-diagonal parametric restrictions, Diagonal BEKK, Full BEKK, Regularity conditions, Asymptotic properties, Conditional volatility, Univariate and multivariate models, Fossil fuels and carbon emissions.

JEL: C22, C32, C52, C58.

1. Introduction

The most widely estimated univariate and multivariate models of time-varying volatility for financial data, as well as any high frequency data that are measured in days, hours and minutes, is the conditional volatility model. The underlying stochastic processes that lead to the specifications, regularity conditions and asymptotic properties of the most popular univariate conditional volatility models, such as GARCH (see Engle (1982) and Bollerslev (1986)) and GJR (see Glosten et al. (1993)) are well established in the literature, though McAleer and Hafner (2014) have raised caveats regarding the existence of the stochastic process underlying exponential GARCH (EGARCH) (see Nelson (1990, 1991)).

However, the same cannot be said about multivariate conditional volatility models, specifically Full BEKK (see Baba et al. (1985) and Engle and Kroner (1995)), for which the underlying stochastic process that leads to the specification, regularity conditions and asymptotic properties have either not been established, or are simply assumed rather than derived. These conditions are essential for forecasting and valid statistical analysis of the empirical estimates, which are the primary purposes of the models.

The purpose of the paper is to show that the stochastic process underlying univariate GARCH is not a special case of that underlying multivariate GARCH, except under parametric restrictions on the off-diagonal elements of the random coefficient autoregressive coefficient matrix that are not consistent with Full BEKK. The paper provides the regularity conditions that arise from the underlying random coefficient autoregressive process, and for which the (quasi-) maximum likelihood estimates (QMLE) have valid asymptotic properties under the appropriate parametric restrictions.

The Full BEKK model is estimated almost exclusively in empirical practice, to the exclusion of Diagonal BEKK (DBEKK), despite the fact that Full BEKK has no underlying stochastic process that leads to its specification, regularity conditions, or asymptotic properties, as shown in the proposition and four corollaries, as compared with DBEKK.

The plan of the paper is as follows. Section 2 provides a discussion of the stochastic processes, regularity conditions, and asymptotic properties of univariate and multivariate GARCH models. Section 3 shows that the Full BEKK model has no underlying stochastic process that leads to its specification, regularity conditions, or asymptotic properties, as compared with DBEKK. In Section 4, an empirical illustration for the financial returns on spot and futures prices of fossil fuels and carbon emissions for the European Union and USA shows the differences that can arise in the QMLE of the parameters of the conditional means and conditional variances for the univariate, DBEKK and Full BEKK specifications. Section 5 gives some concluding comments.

2. Univariate and Multivariate GARCH Models

2.1 Univariate Conditional Volatility Models

Consider the conditional mean of financial returns for commodity i, in a financial portfolio of m assets, as follows:

$$y_{it} = E(y_{it}|I_{t-1}) + \varepsilon_{it} , \qquad i = 1, 2, \dots, m, \tag{1}$$

where the returns, $y_{it} = \Delta log P_{it}$, represent the log-difference in financial commodity prices, P_t, I_{t-1} is the information set for all financial assets at time *t-1*, $E(y_{it}|I_{t-1})$ is the conditional expectation of returns, and ε_{it} is a conditionally heteroskedastic error term.

In order to derive conditional volatility specifications, it is necessary to specify the stochastic processes underlying the returns shocks, ε_{it}. The most popular univariate conditional volatility model, GARCH model, is discussed below.

Consider the random coefficient autoregressive process underlying the returns shocks, ε_{it}, as follows:

$$\varepsilon_{it} = \phi_{it}\varepsilon_{it-1} + \eta_{it} , \qquad i = 1, 2, \dots, m, \tag{2}$$

where

$\phi_{it} \sim iid(0, \alpha_i)$, $\alpha_i \geq 0$,

$\eta_{it} \sim iid(0, \omega_i)$, $\omega_i \geq 0$,

$\eta_{it} = \varepsilon_{it}/\sqrt{h_{it}}$ is the standardized residual,

h_{it} is the conditional volatility of financial asset *i*.

Tsay (1987) derived the following conditional volatility of financial asset *i* as an ARCH process (see Engle, 1982):

$$E(\varepsilon_{it}^2 | I_{t-1}) \equiv h_{it} = \omega_i + \alpha_i \varepsilon_{it-1}^2 , \tag{3}$$

where h_t represents conditional volatility, and I_{t-1} is the information set available at time *t-1*. A lagged dependent variable, h_{t-1}, is typically added to equation (3) to improve the sample fit:

$$h_{it} \equiv E(\varepsilon_{it}^2 | I_{t-1}) = \omega_i + \alpha_i \varepsilon_{it-1}^2 + \beta_i h_{t-1}, \beta_i \in (-1, 1). \tag{4}$$

From the specification of equation (2), it is clear that both ω_i and α_i should be positive as they are the unconditional variances of two different stochastic processes. In equation (4), which is a GARCH(1,1) model for commodity *i* (see Bollerslev, 1986), the stability condition requires that $\beta_i \in (-1, 1)$.

The stochastic process can be extended to asymmetric conditional volatility models (see, for example, McAleer (2014)), and to give higher-order lags and a larger number of alternative commodities, namely up to *m*-1. However, the symmetric process considered here is sufficient to focus the key ideas associated with the purpose of the paper.

As the stochastic process in equation (2) follows a random coefficient autoregressive process, under normality (non-normality) of the random errors, the maximum likelihood estimators (quasi-maximum likelihood estimators, QMLE) of the parameters will be consistent and asymptotically normal. It is worth emphasizing that the regularity conditions include invertibility, which is obvious from equation (2), as:

$$\varepsilon_{it} - \phi_{it}\varepsilon_{it-1} = \eta_{it}.$$

The standardized residuals, η_{it} , can be expressed in terms of the empirical data through equations (1) and (2), as ε_{it} can be estimated using equation (1), ε_{it-1} is the lagged value, which has already been estimated, and the random coefficient can be generated under appropriate explicit assumptions regarding its underlying stochastic process. In short, η_{it} can be related directly to the data, y_{it} , using equations (1) and (2).

Ling and McAleer (2003) and McAleer et al. (2008) provide general proofs of the asymptotic properties of univariate and multivariate conditional volatility models based on satisfying the regularity conditions in Jeantheau (1998) for consistency, and in Theorem 4.1.3 in Amemiya (1985) for asymptotic normality.

2.2 Multivariate Conditional Volatility Models

The multivariate extension of the univariate ARCH and GARCH models is given in Baba et al. (1985) and Engle and Kroner (1995). It is useful to define the multivariate extension of the relationship between the returns shocks and the standardized residuals, that is, $\eta_{it} = \varepsilon_{it}/\sqrt{h_{it}}$. The multivariate extension of equation (1), namely:

$$y_t = E(y_t|I_{t-1}) + \varepsilon_t \,, \tag{5}$$

can remain unchanged by assuming that each of the three components in equation (5) is an $m \times 1$ vector, where m is the number of financial assets.

The following two definitions are intended to elaborate on the discussion below:

Definition 1: Each marginal of ε_{it} should be a univariate counterpart of the multivariate returns vector, ε_t.

Definition 2: An underlying stochastic process of a univariate returns shock, or multivariate returns shocks, is one that leads to the regularity conditions, likelihood function, and asymptotic properties of the resulting quasi- maximum likelihood estimators.

Consider the vector random coefficient autoregressive process of order one, which is the multivariate extension of the univariate process given in equation (2):

$$\varepsilon_t = \Phi_t \varepsilon_{t-1} + \eta_t, \tag{6}$$

where
ε_t and η_t are $m \times 1$ vectors,
Φ_t is an $m \times m$ matrix of random coefficients,
$\Phi_t \sim iid(0, A)$, A is positive definite,
$\eta_t \sim iid(0, C)$, C is an $m \times m$ matrix.

Vectorization of a full matrix A to *vec A* can have dimension as high as $m^2 \times m^2$, whereas vectorization of a symmetric matrix A to *vech A* can have a smaller dimension of $\mathrm{m(m+1)/2 \times m(m+1)/2}$.

In the case where A is a diagonal matrix, with $a_{ii} > 0$ for all $i = 1,\ldots,m$ and $|b_{jj}| < 1$ for all $j = 1,\ldots,m$, so that A has dimension $m \times m$, McAleer et al. (2008) showed that the multivariate extension of GARCH(1,1) from equation (6) is given as the Diagonal BEKK (DBEKK) model, namely:

$$Q_t = CC' + A\varepsilon_{t-1}\varepsilon_{t-1}'A' + BQ_{t-1}B', \tag{7}$$

where A and B are both diagonal matrices. The diagonality of the positive definite matrix A is essential for matrix multiplication as $\varepsilon_{t-1}\varepsilon_{t-1}'$ is an $m \times m$ matrix; otherwise equation (7) could not be derived from the vector random coefficient autoregressive process in equation (6).

McAleer et al. (2008) showed that the QMLE of the parameters of the DBEKK model were consistent and asymptotically normal, so that standard statistical inference on testing hypotheses is valid (or further details, see Chang et al., 2018). It should be emphasized that the QMLE of the parameters in the conditional means, namely equations (1) and (5), and the conditional variances, namely equations (4) and (7), will differ as the multivariate models, (5) and (7), respectively, are estimated jointly, whereas the univariate models, (1) and (4), respectively, are estimated individually.

3. Full BEKK

Consider element *i* of equation (6), that is:

$$\varepsilon_{it} = \sum_{j=1}^{m} \phi_{ijt}\varepsilon_{ijt-1} + \eta_{it} , \quad i = 1, 2, \dots , m, \tag{8}$$

which is not equivalent to equation (2) unless $\phi_{ijt} = 0 \;\; \forall \; i \neq j$. Such parametric restrictions are not consistent with the Full BEKK specification, which assumes $\phi_{ijt} \neq 0$ for at least one $i \neq j$, $i, j = 1, 2, \dots , m$.

The stochastic process given in equation (8) is not a random coefficient autoregressive process because of the presence of an additional m-1 random coefficients, ϕ_{ijt}, $i \neq j$. Importantly, equation (8) is **not** invertible as the standardized residual, η_{it}, cannot be connected to the data, y_{it}, as m equations are required, as in equation (6). Consequently, the stochastic process underlying univariate ARCH is not a special case of the stochastic process underlying multivariate ARCH unless $\phi_{ijt} = 0 \;\; \forall \; i \neq j$.

The same condition holds $\forall$ *i*, *j* = 1,…,*m*, which leads to the following proposition:

<u>Proposition</u>: The stochastic process underlying univariate ARCH in equation (2) is a special case of the stochastic process underlying multivariate ARCH in equation (8) if and only if:

$$\phi_{ijt} = 0 \;\; \forall \, i \neq j, \, i, j = 1, 2, \ldots, m.$$

Proof: If $\phi_{ijt} = 0 \;\; \forall \, i \neq j$, equation (8) collapses to equation (2), with $\phi_{iit} = \phi_{it}$. If $\phi_{ijt} \neq 0$ for at least one $i \neq j$, equation (2) is not a special case of equation (8).

A similar condition holds for univariate GARCH and multivariate GARCH.

The Proposition leads to the following corollaries:

Corollary 1: The $m \times m$ matrix of random coefficients, Φ_t, is a diagonal matrix.

Corollary 2: From Corollary 1, it follows that the $m \times m$ weight matrix of (co-)variances, A, is a diagonal matrix, which is not consistent with Full BEKK.

Corollary 3: Corollaries 1 and 2 show that a Full BEKK model, namely where there are no restrictions on the off-diagonal elements in Φ_t, and hence no restrictions in the off-diagonal elements in A, is not possible if univariate ARCH is to be a special case of its multivariate counterpart, Full BEKK.

Corollary 4: As there are no underlying regularity conditions for Full BEKK, including invertibility, the model cannot be estimated using an appropriate likelihood function. Therefore, it is not possible to derive the asymptotic properties of the QMLE of the unknown parameters in the Full BEKK soecification.

Corollary 4 is consistent with the proof in McAleer et al. (2008) that the QMLE of Full BEKK has no asymptotic properties, whereas the QMLE of Diagonal BEKK can be shown to be consistent and asymptotically normal.

For all intents and purposes, the statistical properties of Full BEKK cannot be derived from an underlying stochastic process, except by assumption.

It should be emphasized that the QMLE of the parameters in the conditional means and the conditional variances for univariate GARCH, DBEKK and Full BEKK will differ as the multivariate models are estimated jointly, whereas the univariate models are estimated individually. The QMLE of the parameters of the conditional means and the conditional variances of DBEKK and Full BEKK will differ as DBEKK imposes parametric restrictions on the off-diagonal terms of the conditional covariance matrix of Full BEKK.

4. An Empirical Illustration for Fossil Fuels and Carbon Emissions

The data for the empirical analysis are given in Chang et al. (2017), who evaluated the financial returns on spot and futures prices for fossil fuels and carbon emissions for the European Union and USA using the DBEKK and Full BEKK models. The authors did not provide the estimates for the univariate GARCH models, or compare the differences in the conditional means and conditional variances of the univariate, DBEKK and Full BEKK specifications. The purpose of the empirical illustration in this section is to show the differences that can arise in the QMLE of the parameters of the conditional means and conditional variances of the univariate, DBEKK and Full BEKK specifications.

The carbon emission trading market of the European Union (EU) has daily data only on futures prices, whereas only daily spot prices are available for carbon emissions for the USA. Daily data for EU carbon emission, crude oil, and coal futures are available from 2 April 2008 to 19 May 2017, while daily data for US carbon, coal, and oil spot prices are available from 6 January 2016 to 19 May 2017. The data sources and definitions are given in Table 1, where "fr" denotes futures returns, "sr" denotes spot returns, and daily returns are calculated as obtained as the first difference in the natural logarithm of the relevant daily price data.

The descriptive statistics for the returns of the six variables are given in Table 2 (for a detailed discussion of the data, see Chang et al., 2017). Table 3 presents the ADF test of Dickey and Fuller (1979, 1982) and Said and Dickey (1984), the DF-GLS test of Elliott et al. (1996), and the KPSS test of Kwiatkowski et al. (1992) to test for unit roots in the individual returns series (see Chang et al., 2017).

The univariate GARCH estimates for EU carbon, coal and oil futures returns are given in Table 4. The QMLE of the parameters of the conditional means are standard in that there is not a lot of explanatory power. However, the QMLE of the parameters of the conditional variances are highly significant, with the short run responses to shocks being around 0.1 or less, and the long run responses to shocks lying between 0.996 and 0.997.

The univariate GARCH estimates for US carbon, coal and oil spot returns are given in Table 5. The QMLE of the parameters of the conditional means are similar to those in Table 4 in that there is not a lot of explanatory power. However, the QMLE of the parameters of the conditional variances are highly significant. The short run responses to shocks are surprisingly large for carbon at 0.462, while those for coal and oil are more standard at 0.073 and 0.130, respectively. Give these estimates, the long run responses to shocks are 0.936, 0.982 and 0.954 for carbon, coal and oil, respectively, all of which are considerably lower than their counterparts for EU futures returns.

The corresponding estimates for the DBEKK and Full BEKK models for EU carbon, coal and oil futures returns are given in Tables 6 and 7, respectively. The QMLE of the conditional means for DBEKK and Full BEKK are different from each other, and are also different from their univariate counterparts in Table 4. The QMLE of the elements of the weighting matrix A and stability matrix B, namely a11, a22, a33, b11, b22 and b33, respectively, are substantially different between both DBEKK (especially a22 and b33) and Full BEKK (especially a22, a33 and b33), and even more so in comparison with their univariate counterparts in Table 4. These results provide strong support for the theoretical analysis in Sections 2 and 3.

The corresponding estimates for the DBEKK and Full BEKK models for US carbon, coal and oil spot returns are given in Tables 8 and 9, respectively. The QMLE of the conditional means for DBEKK and Full BEKK are different from each other, and are also different from their univariate counterparts in Table 5. The QMLE of the elements of the weighting matrix A and stability matrix B, namely a11, a22, a33, b11, b22 and b33, respectively, are substantially different between both DBEKK (especially a22, a33 and b33) and Full BEKK (especially a22, a33 and b33), which reflect the findings in Tables 6 and 7, and even more so in comparison with their univariate counterparts in Table 4. These results also strongly support the theoretical analysis in Sections 2 and 3.

5. Conclusion

The Full BEKK model in Baba et al. (1985) and Engle and Kroner (1995), who do not derive the model from an underlying stochastic process, was presented as equation (6), with A and B given as full matrices, with no restrictions on the off-diagonal elements. The Full BEKK model is estimated almost exclusively in empirical practice, to the exclusion of Diagonal BEKK, despite the fact that Full BEKK has no underlying stochastic process that leads to its specification, regularity conditions, or asymptotic properties, as shown in the proposition and four corollaries.

The full BEKK model can be replaced by the triangular or Hadamard (element-by-element multiplication) BEKK models, with similar problems of identification and (lack of) existence. The full, triangular and Hadamard BEKK models cannot be derived from any known underlying stochastic processes that lead to their respective specifications, which means there are no regularity conditions (except by assumption) for checking the internal consistency of the alternative models, and consequently no valid asymptotic properties of the QMLE of the associated parameters (except by assumption).

Moreover, as the number of parameters in a full BEKK model can be as much as $3m(m+1)/2$, the "curse of dimensionality" will be likely to arise, which means that convergence of the estimation algorithm can become problematic and less reliable when there is a large number of parameters to be estimated. As a matter of fact, estimation of the full BEKK can be problematic even when m is as low as 5 financial assets. Such computational difficulties do not arise for the diagonal BEKK model. Convergence of the estimation algorithm is more likely when the number of commodities is less than 4, though this is nevertheless problematic in terms of interpretation.

The purpose of the paper was to show that univariate GARCH is not a special case of multivariate GARCH, specifically the Full BEKK model, except under parametric restrictions on a random coefficient autoregressive coefficient matrix that are not consistent with Full BEKK. The paper provided the regularity conditions that arise from the underlying random coefficient autoregressive process, and for which the (quasi-) maximum likelihood estimates have valid asymptotic properties under the appropriate parametric restrictions, for the univariate and multivariate GARCH models.

It was shown that the Full BEKK model has no underlying stochastic process that leads to its specification, regularity conditions, or asymptotic properties, as compared with the Diagonal BEKK (DBEKK) specification. It would seem that the purported statistical properties of Full BEKK exist by assumption.

An empirical illustration for the financial returns on spot and futures prices of fossil fuels and carbon emissions for the European Union and USA showed the significant differences that can arise in the QMLE of the parameters of the conditional means and conditional variances for the univariate, DBEKK and Full BEKK specifications, which gave strong support for the theoretical analysis demonstrated in the paper.

Table 1

Data Sources and Definitions

Variable name	Definitions	Transaction market	Description
$EUcarbon_{fr}$	EU carbon futures return	ICE-ICE Futures Europe Commodities	ICE EUA Futures Contract EUR/MT
$EUcoal_{fr}$	EU coal futures return	ICE-ICE Futures Europe Commodities	ICE Rotterdam Monthly Coal Futures Contract USD/MT
$EUoil_{fr}$	EU oil futures return	ICE-ICE Futures Europe Commodities	Current pipeline export quality Brent blend as supplied at Sullom Voe USD/bbl
$UScarbon_{sr}$	US carbon spot return	over the counter	United States Carbon Dioxide RGGI Allowance USD/Allowance
$UScoal_{sr}$	US coal spot return	over the counter	Dow Jones US Total Market Coal Index USD
$USoil_{sr}$	US oil spot return	over the counter	West Texas Intermediate Cushing Crude Oil USD/bbl

Notes: ICE is the Intercontinental Exchange; EUA is the EU allowance; MT is metric ton; RGGI (Regional Greenhouse Gas Initiative) is a CO2 cap-and-trade emissions trading program comprised of ten New England and Mid-Atlantic States that will commence in 2009 and aims to reduce emissions from the power sector. RGGI will be the first government mandated CO2 emissions trading program in USA.

Table 2

Descriptive Statistics

2 April 2008 – 19 May 2017 for EU

6 January 2016 – 19 May 2017 for USA

Variable	Mean	Median	Max	Min	SD	Skewness	Kurtosis	Jarque-Bera
$\mathbf{EUcarbon_{f}}$	-0.078	-0.038	24.561	-42.457	3.349	-0.708	17.624	21434.2
$\mathbf{EUcoal_{fr}}$	-0.022	0	17.419	-22.859	1.599	-1.268	44.924	175155.8
$\mathbf{EUoil_{fr}}$	-0.026	-0.015	12.707	-10.946	2.246	0.054	6.522	1232.8
$\mathbf{UScarbon_{s}}$	-0.248	0	13.937	-36.446	2.986	-5.236	66.269	61346.8
$\mathbf{UScoal_{sr}}$	0.177	0.104	17.458	-14.183	4.041	0.047	5.343	81.99
$\mathbf{USoil_{sr}}$	0.094	0.037	11.621	-8.763	2.712	0.431	4.690	53.69

Table 3

Unit Root Tests

2 April 2008 – 19 May 2017 for EU

6 January 2016 – 19 May 2017 for USA

Variables	ADF	DF-GLS	KPSS
$\mathbf{EUcarbon_{fr}}$	-37.79*	-3.09*	0.05*
$\mathbf{EUcoal_{fr}}$	-35.48*	-10.34*	0.12*
$\mathbf{EUoil_{fr}}$	-51.97*	-1.53	0.10*
$\mathbf{UScarbon_{sr}}$	-10.64*	-1.46	0.06*
$\mathbf{UScoal_{sr}}$	-19.30*	-0.43	0.18*
$\mathbf{USoil_{sr}}$	-20.96*	-0.78	0.07*

Notes: * denotes the null hypothesis of a unit root is rejected at 1%.

Table 4

Univariate GARCH for EU $\mathrm{CARBON_{fr}}$, $\mathrm{COAL_{fr}}$, $\mathrm{OIL_{fr}}$

2 April 2008 – 19 May 2017

Explained variables	$\mathrm{CARBON_{fr}}$ (1)	$\mathrm{COAL_{fr}}$ (2)	$\mathrm{OIL_{fr}}$ (3)
Constant	0.032	-0.040*	0.003
	(0.050)	(0.024)	(0.033)
θ_1	0.017	0.097***	-0.039*
	(0.024)	(0.023)	(0.021)
θ_2	-0.090**	0.003	0.008
	(0.040)	(0.007)	(0.008)
θ_3	-0.055**	0.010	-0.008
	(0.023)	(0.013)	(0.028)
ω	-0.116***	0.009***	0.020***
	(0.037)	(0.002)	(0.007)
GARCH α	0.101***	0.016***	0.060***
	(0.015)	(0.002)	(0.010)
GARCH β	0.895***	0.980***	0.937***
	(0.016)	(0.002)	(0.010)
Log Likelihood	-5874.33	-4030.45	-4872.13

Notes: (1) : $\mathrm{CARBON_{fr}} = (\theta_1\,\mathrm{CARBON_{fr}}(-1),\, \theta_2\,\mathrm{COAL_{fr}}(-1),\, \theta_3\,\mathrm{OIL_{fr}}(-1))$

(2): $\mathrm{COAL_{fr}} = (\ \theta_1\,\mathrm{COAL_{fr}}(-1),\, \theta_2\,\mathrm{CARBON_{fr}}(-1),\, \theta_3\,\mathrm{OIL_{fr}}(-1))$

(3): $\mathrm{OIL_{fr}} = (\ \theta_1\,\mathrm{OIL_{fr}}(-1),\, \theta_2\,\mathrm{CARBON_{fr}}(-1),\, \theta_3\,\mathrm{COAL_{fr}}(-1))$

Standard errors are in parentheses, *** denotes significant at 1%, ** denotes significant at 5%, * denotes significant at 10%.

Table 5

Univariate GARCH for US $CARBON_{sr}$, $COAL_{sr}$, OIL_{sr}

6 January 2016 – 19 May 2017

Explained variables	$CARBON_{sr}$ (4)	$COAL_{sr}$ (5)	OIL_{sr} (6)
Constant	0.049	0.029	0.074
	(0.096)	(0.174)	(0.116)
θ_1	0.100	0.020	-0.082
	(0.100)	(0.058)	(0.060)
θ_2	0.012	0.038	-0.097*
	(0.025)	(0.078)	(0.056)
θ_3	-0.081**	-0.238***	0.038
	(0.038)	(0.080)	(0.038)
ω	0.729***	0.211	0.274*
	(0.170)	(0.147)	(0.147)
GARCH α	0.462***	0.073**	0.130**
	(0.091)	(0.030)	(0.044)
GARCH β	0.574***	0.909***	0.824***
	(0.052)	(0.034)	(0.055)
Log Likelihood	-759.38	-952.67	-816.74

Notes: (4) : $CARBON_{sr} = (\theta_1 CARBON_{sr}(-1), \theta_2 COAL_{sr}(-1), \theta_3 OIL_{sr}(-1))$

(5): $COAL_{sr} = (\theta_1 COAL_{sr}(-1), \theta_2 CARBON_{sr}(-1), \theta_3 OIL_{sr}(-1))$

(6): $OIL_{sr} = (\theta_1 OIL_{sr}(-1), \theta_2 CARBON_{sr}(-1), \theta_3 COAL_{sr}(-1))$

Standard errors are in parentheses, *** denotes significant at 1%, ** denotes significant at 5%, * denotes significant at 10%.

Table 6

DBEKK for EU Carbon, Coal, and Oil Futures

2 April 2008 – 19 May 2017

Mean equation	$CARBON_{fr}$	$COAL_{fr}$	OIL_{fr}
$CARBON_{fr}$	0.010	0.005	0.009
	(0.023)	(0.008)	(0.009)
$COAL_{fr}$	-0.078**	0.096***	0.073
	(0.038)	(0.023)	(0.023)
OIL_{fr}	-0.057**	0.009	0.002
	(0.024)	(0.014)	(0.027)
C	0.021	-0.034	-0.045*
	(0.053)	(0.024)	(0.022)

DBEKK	C			A			B		
$CARBON_{fr}$	0.379***	0.024**	0.128***	0.311***			0.947***		
	(0.055)	(0.010)	(0.024)	(0.025)			(0.009)		
$COAL_{fr}$		0.088***	0.022		0.118***			0.991***	
		(0.010)	(0.075)		(0.007)			(0.001)	
OIL_{fr}			0.000			-0.205***			-0.977***
			(0.077)			(0.013)			(0.003)

Notes: 1. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$, $C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$

2. Standard errors are in parentheses, *** denotes significant at 1%, ** denotes significant at 5%, * denotes significant at 10%.

Table 7

Full BEKK for EU Carbon, Coal, and Oil Futures

2 April 2008 – 19 May 2017

Mean equation	$CARBON_{fr}$	$COAL_{fr}$	OIL_{fr}
$CARBON_{fr}$	0.023	-0.003	0.013
	(0.02)	(0.007)	(0.011)
$COAL_{fr}$	-0.082**	0.086***	0.005
	(0.039)	(0.023)	(0.031)
OIL_{fr}	-0.045*	0.016	-0.018
	(0.023)	(0.015)	(0.023)
C	0.031	-0.016	-0.010
	(0.053)	(0.023)	(0.037)

Full BEKK	C			A			B		
$CARBON_{fr}$	0.435*** (0.055)	-0.067* (0.038)	0.077 (0.072)	0.331*** (0.023)	-0.014*** (0.004)	0.007 (0.006)	0.936*** (0.009)	0.009 (0.007)	-0.005 (0.010)
$COAL_{fr}$		0.000 (0.068)	0.000 (0.103)	0.037 (0.029)	-0.086*** (0.011)	0.120*** (0.017)	0.274*** (0.036))	0.737*** (0.015)	1.110*** (0.023)
OIL_{fr}			-0.000 (0.101)	-0.104*** (0.026)	-0.032** (0.013)	-0.168*** (0.010)	-0189*** (0.024)	-0.052*** (0.011)	0.054*** (0.015)

Notes : As in Table 4.

Table 8

DBEKK for US Carbon, Coal, and Oil Spot

6 January 2016 – 19 May 2017

Mean equation	$CARBON_{sr}$	$COAL_{sr}$	OIL_{sr}
$CARBON_{sr}$	0.122	-0.010	-0.070
	(0.106)	(0.078)	(0.053)
$COAL_{sr}$	0.034	0.037	0.050
	(0.024)	(0.057)	(0.041)
OIL_{sr}	-0.097***	-0.235***	-0.103*
	(0.036)	(0.083)	(0.060)
C	0.085	0.048	0.010
	(0.090)	(0.170)	(0.122)

DBEKK	C			A			B		
$CARBON_{sr}$	0.854***	-0.276	0.129	0.707***			0.757***		
	(0.105)	(0.294)	(0.332)	(0.073)			(0.038)		
$COAL_{sr}$		0.256	0.299*		-0.199***			0.972***	
		(0.314)	(0.154)		(0.034)			(0.008)	
OIL_{sr}			0.000			-0.222***			-0.964***
			(1.029)			(0.0035)			(0.010)

Note: As in Table 4.

Table 9

Full BEKK for US Carbon, Coal, and Oil Spot

6 January 2016 – 19 May 2017

Mean equation	$CARBON_{sr}$	$COAL_{sr}$	OIL_{sr}
$CARBON_{sr}$	0.079	-0.027	-0.105**
	(0.089)	(0.074)	(0.049)
$COAL_{sr}$	-0.006	-0.012	0.022
	(0.028)	(0.060)	(0.039)
OIL_{sr}	-0.048	-0.231***	-0.049
	(0.038)	(0.087)	(0.062)
C	0.043	0.139	0.010
	(0.089)	(0.166)	(0.118)

Full BEKK	C			A			B		
$CARBON_{sr}$	0.772***	0.119	0.685***	0.632***	-0.023	-0.077	0.791***	0.004	-0.034
	(0.092)	(0.606)	(0.178)	(0.054)	(0.089)	(0.064)	(0.025)	(0.112)	(0.063)
$COAL_{sr}$		0.000	0.000	0.002	-0.320***	0.036	-0.042	0.900***	0.578***
		(0.528)	(0.715)	(0.033)	(0.058)	(0.041)	(0.046)	(0.056)	(0.044)
OIL_{sr}			0.000	-0.028	-0.072	-0.252***	0.010	-1.267***	0.140*
			(0.721)	(0.049)	(0.092)	(0.060)	(0.080)	(0.074)	(0.082)

Note: As in Table 4.

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