Lower Partial Moments under Gram Charlier Distribution: Performance Measures and Efficient Frontiers*

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Abstract

We derive closed-form expressions for the performance measures (PMs) based on the lower partial moments (LPMs), such as the Farinelli-Tibiletti and Kappa measures, with Gram-Charlier (GC) density for returns. It is verified that the LPMs can be obtained as a linear function on both higher moments, skewness and excess kurtosis. We also show that these PMs influence differently to the Sharpe ratio in ranking portfolios due to the effects of the higher moments. We also obtain the efficient frontiers (EFs) based on the mean-LPM framework. We find important differences between portfolios from different EFs regarding their stock compositions, portfolio skewness and excess kurtosis levels. Finally, we also obtain closed-form expressions for PMs under a more flexible density like the SNP density which nests the GC density.

JEL Classification: C10, C61, G11, G17.
Keywords: Downside risk, Performance measure, Rank correlation, Efficient frontier, Co-lower partial moment matrix, Copula, SNP distribution.

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1 Introduction

Whether the popular Sharpe ratio (SR), proposed by Sharpe (1966, 1994), is an adequate performance measure (reward-to-variability index) for ranking financial products still remains a controversial question among academics and practitioners. Although SR is fully compatible with normally distributed returns, and more general with elliptical distributions[1] of returns, it may lead to incorrect evaluations when stock returns exhibit asymmetry and / or heavy tails. Since the standard deviation as the two-sided variability measure in SR seems to be dubious as a risk measure, several one-sided type measures of risk have been proposed. Their corresponding performance measures (PMs) are known as one-sided PMs depending on a portfolio return threshold, τ. Some of these PMs are also characterized by one-sided type reward measures.

The one-sided PMs are mainly the following two groups: the Generalized Rachev ratio (GRR) based on the conditional VaR (CVaR), see Biglova et al. (2004), and the Farinelli-Tibiletti (FT) ratio based on the partial moments (upper and lower partial moments, respectively, for the reward and risk measures), see Farinelli and Tibiletti (2008). Both groups use one-sided measures to modeling the reward and risk measures. Finally, another performance measure known as the Sortino and Satchell (SS) ratio (2001), or Kappa ratio, verifies that only the risk measure is a one-sided measure (lower partial moment). Our work aims, among others, at the behaviour of those PMs based on partial moments under a static portfolio performance and assuming a certain portfolio return distribution so as to get closed-form expressions. More alternative reward-to-variability ratios are well documented in Caporin et al. (2013) and their references inside.

There is a stream of research concluding that the choice of the performance measure does not affect the ranking in portfolios. Eling and Schuhmacher (2007) and Eling (2008) find that rank correlations between SR and alternative PMs for hedge fund (HF) data are highly positively correlated. These empirical results are also supported by some theoretical studies such as Schuhmacher and Eling (2011, 2012). They show that under several conditions, the PM is a strictly increasing function in SR. Nevertheless, we show that rank correlations from FT measures and SR exhibit low rank correlations in most cases. Our results agree, among others, with the empirical evidence in Eling et al. (2011).

There are also some recent studies, based on a dynamic portfolio performance setting, concluding that the one-sided PMs outperform the benchmark SR. For instance, Biglova et al. (2004) and Farinelli et al. (2008, 2009) show that the cumulative final wealth is higher by implementing either GRR or FTR instead of SR. Another example is Caporin and Lisi (2011), who analyze the dynamic properties of rank correlations between PMs and SR based on a rolling window analysis. They find low rank correlations for some PMs from the FT group and so on. In short, this empirical evidence suggests that the selection of the performance measure matters.

We also study the rank correlation sensitivity when returns departure from the normality assumption (skewness, s, and / or excess kurtosis, ek, different from zero). We can find some PMs from the SS group (or Kappa measures), which are highly correlated

[1] See, for more datails, Owen and Rabinovitch (1983).

with SR when considering the whole sample. Nevertheless, if we divide the total sample of funds between a group with lower SR and another with higher SR values, it is verified that rank correlations tend to decrease for those funds from the higher SR group. These results can be explained through the levels of s and ek implied in the distributions of these portfolio returns. This evidence is also confirmed by Zakamouline (2011).

This paper has several goals. First, we try to understand the behaviour of PMs when the portfolio returns departure from the normal distribution and so, how important the selection of alternative PMs (against SR) can be in ranking portfolios. We obtain closed-form expressions of PMs based on the lower partial moments (LPMs), proposed by Bawa (1975) and Fishburn (1977),[2] as downside risk measures. We mainly concentrate on those one-sided PMs from the FT[3] and SS groups. We assume that the probability density function (pdf) for the portfolio returns is driven by the Gram-Charlier (GC) expansion. We also get the closed-form expression for the GC shortfall probability to obtain some PMs with downside risk measures based on the value at risk (VaR), such as both the Reward-to-VaR and the Reward-to-CVaR.

The GC distribution has been implemented, among others, by Corrado and Su (1996), Jondeau and Rockinger (2001) and Jurczenko and Maillet (2006). The advantage of the GC distribution over alternative distributions is that the higher moments s and ek directly appear as the pdf's parameters. Jondeau and Rockinger (2001) obtain the constraints that s and ek must hold in order to guarantee a well-defined pdf. We get easy closed-form expressions for the LPMs, which can be expressed as simple linear functions on both s and ek. Hence, we can easily understand the behaviour of these risk measures due to changes of s and ek. For instance, Passow (2005) obtains closed-form expressions, in particular, for the Sharpe-Omega ratio (or Kappa measure of order one) under the Johnson distribution (JD) family. Contrary to the GC pdf, the unbounded Johnson distribution from the JD family allows for higher levels for both s and ek. This result becomes an important advantage with respect to the GC distribution but it has the drawback that the Sharpe-Omega ratio under the JD family becomes more cumbersome and difficult to interpret for changes of s and ek than in our case.

The restriction of the GC pdf to capture some levels of s and ek of portfolio returns suggests other distributions trying to seize better these higher moments but getting more complex or hard tractable expressions for either the FT or SS measures under different distributions. Nevertheless, as a possible solution to the restricted higher moments under GC, we suggest and obtain the LPM closed-form expresions under the SNP density, introduced by Gallant and Nychka (1987), whose parametric properties have been studied by León et al. (2009).[4]

Second, we obtain the efficient frontiers (EFs) based on the LPMs. We compare the composition of the portfolios from the EFs under both the mean-LPM and the Markowitz

[2]Some other seminal references about LPMs are Bawa and Lindenberg (1977), Holthausen (1981), Harlow and Rao (1989) and Harlow (1991).

[3]Note that the reward measures from the FT ratios are just the upper partial moments, which can be expressed in terms of the LPMs.

[4]The SNP pdf allows a higher flexibility than GC in terms of s and ek. Its density is always positive and it nests the GC distribution.

(1959) mean-variance (MV) settings. For the minimization of the LPM of a portfolio return, we use the symmetric co-lower partial moment matrix from Nawrocki (1991). For the comparison between portfolio frontiers, we use the root mean squared dispersion index (RMSDI), see Grootveld et al. (1999). A sensitivity analysis is implemented by changing the return threshold, τ.

We also obtain the theoretical skewness and excess kurtosis levels implied in the portfolios from these EFs. Note that it is necessary to get previously both (theoretical) co-skewness and co-kurtosis matrices by assuming in this paper a Gaussian copula for the dependence among the different stock returns and the GC distribution for their marginal distributions. Our multivariate setting for stock returns agrees with the procedure to compute EFs under the mean-LPM (MLPM) approach by Nawrocki (1991). We find differences when obtaining both s and ek from a frontier portfolio under the MV approach against another frontier portfolio based on the LPM measure.

The rest of the paper is organized as follows. In Section 2 we present some portfolio performance measures based on the LPMs, that is, both FT and SS (or Kappa) measures. In Section 3 we show the GC pdf, implemented by JR (2001), for the behaviour of returns and some characteristics of this density. In Section 4 we obtain the closed-form expression for the LPM measure of order 0, which is also known as the shortfall probability. We also analyze here the behaviour of the shortfall probability under different values of τ. In Section 5 we can easily obtain the closed-form expressions for LPM measures under the GC distribution and hence, the corresponding FTR and Kappa measures. We show the behaviour of the Kappa ratios regarding the level of s and ek. We also get the iso-curves, mainly for the Kappa measures.

In Section 6 we conduct an intensive simulation study using the GC distribution for the performance evaluation. In Section 7 we get the EFs by using the LPMs under the GC density and compare with the traditional MV frontier. We also obtain both s and ek from the MLPM efficient frontiers in order to study the behaviour of them. Section 8 shows the SNP distribution and the closed-form expressions of LPMs. Finally, Section 9 summarizes and provides the main conclusions. The description of our data series for our simulation studies can be found in Appendix A. The proofs of propositions and corollaries are deferred to Appendix B.

2 Performance measures based on partial moments

The lower partial moments (LPMs) measure risk by negative deviations of the stock returns, r, in relation to a minimal acceptable return, or return threshold, τ. Fishburn (1977), among others, defines the LPM of order m for a stock return as

$$LPM_f(\tau, m) = E_f[(\tau - r)_+^m] = \int_{-\infty}^{\tau} (\tau - r)^m f(r)\, dr, \tag{1}$$

where $f(\cdot)$ denotes the probability density function (pdf) of r and $(y)_+ = \max(y, 0)$. As LPMs consider only negative deviations of returns from τ, they seem to be a more appropriate measure of risk than the standard deviation, which considers both negative and positive deviations from the expected return μ. Hence, returns below the threshold, τ, are seen by investors as loss. This (loss) threshold level, τ, might be the rate of inflation, the real interest rate, the return on a benchmark index, a risk-free rate, etc. About the parameter m, LPM with the order $0 < m < 1$ can express 'risk seeking', for $m = 1$ 'risk neutrality', and $m > 1$ 'risk aversion' behaviour for the investor. Thus, the higher m the more risk averse an investor. The LPMs of order 0 and 1 are, respectively, the *shortfall probability* and the *expected shortfall*. Finally, the upper partial moment (UPM) is defined as

$$UPM_f(\tau, q) = E_f[(r-\tau)_+^q] = \int_\tau^{-\infty} (r-\tau)^q f(r)\, dr. \tag{2}$$

Note that we can express $UPM_f(\tau, q)$ as a function of $LPM_f(\tau, q)$ and hence, we can speak about performance measures based on LPMs. Moreover, some performance measures based on the basis of the Value at risk (VaR) can also be expressed in terms of the LPMs, as shown at the end of this section.

Throughout this paper, we will obtain closed-form PM expressions by assuming the Gram-Charlier (GC) pdf for the standardised portfolio returns, z, and setting m and q from (1) and (2) to be integer numbers, i.e. $m, q \in \mathbb{N}_+ \equiv \{1, 2, 3, ...\}$.

2.1 The Kappa measures

The corresponding risk-adjusted return PMs based on LPMs are the *SS* or *Kappa* ratios, see Sortino and Satchell (2001), which are defined as

$$K_f(\tau, m) = \frac{\mu - \tau}{\sqrt[m]{LPM_f(\tau, m)}}, \tag{3}$$

where μ is the mean of the (portfolio) stock return r with $f(r)$ as pdf, i.e. $\mu = E_f[r]$. Note that $\mu - \tau$ is just an excess expected return.[5] Some popular measures which are nested in (3) are the Omega-Sharpe ratio (see Kaplan and Knowles, 2004) for $m = 1$, the *Sortino ratio* (see Sortino and van der Meer, 1991) for $m = 2$ and *Kappa 3* (see Kaplan and Knowles, 2004) for $m = 3$.

2.2 The Farinelli-Tibiletti ratio (FTR)

Next, we show an alternative performance measure (see Farinelli and Tibiletti, 2008) known as the Farinelli-Tibiletti ratio (FTR), which nests a family of PMs depending on both the LPMs in (1) and the UPMs in (2). The FTR is defined as

$$FTR_f(\tau, q, m) = \frac{\sqrt[q]{UPM_f(\tau, q)}}{\sqrt[m]{LPM_f(\tau, m)}}, \tag{4}$$

where q, $m > 0$. It is known that the higher q and m, the higher the agent's preference for (in the case of expected gains, parameter q) or dislike of (in the case of expected losses, parameter m) extreme events. Note that (4) nests two popular PMs: (i) the Omega ratio[6] (see Keating and Shadwick, 2002) when $q = m = 1$, (ii) the Upside Potential ratio (see

[5] This is a more general definition of excess expected return. If $\tau = r_f$, where r_f denotes the risk-free rate, we have the more usual expression of excess expected return.

[6] It is shown later that $K_f(\tau, 1) = FTR_f(\tau, 1, 1) - 1$. Finally, the Bernardo and Ledoit (2000) ratio is just the Omega ratio for $\tau = 0$ and hence, it represents the gain-to-loss ratio.

Sortino et al., 1999) when $q=1$ and $m=2$. Rewritting (2) as a function of (1), we have an alternative expression of FTR in the following corollary.

Corollary 1 *Let* $\psi_f(\tau,q)=E_f[(r-\tau)^q]$ *and* $f(\cdot)$ *be the pdf of the portfolio stock return* r*, then* (4) *can be expressed as*

$$FTR_f(\tau,q,m)=\frac{\sqrt[q]{\psi_f(\tau,q)+(-1)^{q+1}LPM_f(\tau,q)}}{\sqrt[m]{LPM_f(\tau,m)}}. \tag{5}$$

Proof. It is straightforward. ■

Finally, we can also rewrite (5) in terms of the Kappa measures (3). Thus,

$$FTR_f(\tau,q,m)=\frac{\sqrt[q]{\psi_f(\tau,q)+(-1)^{q+1}(\mu-\tau)^q K_f(\tau,q)^{-q}}}{(\mu-\tau)K_f(\tau,m)^{-1}}. \tag{6}$$

Note that for $q=1$ in (6), we have $\psi_f(\tau,1)=\mu-\tau$ and then, $FTR_f(\tau,1,m)=K_f(\tau,m)\left[1+K_f(\tau,1)^{-1}\right]$.

2.3 Performance measures on the basis of VaR

The Value at risk is just the α-quantile of the distribution of the portfolio return r with $f(\cdot)$ as pdf and denoted as VaR_f^α. Since $LPM_f(VaR_f^\alpha,0)=\alpha$, we can get VaR_f^α by the inversion method. Connecting with this risk measure, we have as performance measure the *Reward-to-VaR* (see Dowd, 2000):

$$RVaR_f^\alpha(\tau)=\frac{\mu-\tau}{\left|VaR_f^\alpha\right|}, \tag{7}$$

where $|\cdot|$ denotes the absolute value function.

Another risk measure is the *Expected Shortfall* (ES), i.e. $ES_f^\alpha=-\frac{1}{\alpha}E_f\left[r\,\middle|\,r\le VaR_f^\alpha\right]$. It is also known as the conditional value at risk, $CVaR_f^\alpha$. We can relate $LPM_f(\tau,1)$ to ES_f^α with $\tau=VaR_f^\alpha$ and so, we obtain[7] $LPM_f(VaR_f^\alpha,1)=\alpha ES_f^\alpha+\alpha VaR_f^\alpha$. The connected performance measure with $CVaR_f^\alpha$ (or ES_f^α) is known as the *Reward-to-CVaR* (see Agarwal and Naik, 2004):

$$RCVaR_f^\alpha(\tau)=\frac{\mu-\tau}{CVaR_f^\alpha}=\frac{\mu-\tau}{(1/\alpha)LPM_f(VaR_f^\alpha,1)-VaR_f^\alpha}. \tag{8}$$

3 GC density and properties

The stock return r is a random variable defined as

$$r=\mu+\sigma z,\qquad z\sim GC(0,1,s,ek), \tag{9}$$

[7] See Jarrow and Zhao (2006).

such that the pdf of the random variable z is the Gram-Charlier expansion with zero mean, unit variance, s and ek as the levels, respectively, of skewness and excess kurtosis. Hence, the return in (9) is just an affine transformation of a random variable with GC expansion as pdf. The GC pdf denoted as $g(z)$, where z is a standardised random variable,[8] is defined as

$$g(z) = p(z)\,\phi(z)\,; \quad p(z) = 1 + \frac{s}{\sqrt{3!}} H_3(z) + \frac{ek}{\sqrt{4!}} H_4(z), \tag{10}$$

where $\phi(\cdot)$ denotes the pdf of the standard normal variable and $H_k(z)$ is the normalized Hermite polynomial of order k. These polynomials can be defined recursively for $k \geq 2$ as

$$H_k(z) = \frac{zH_{k-1}(z) - \sqrt{k-1}H_{k-2}(z)}{\sqrt{k}}, \tag{11}$$

with initial conditions $H_0(z) = 1$ and $H_1(z) = z$. It holds that $\{H_k(z)\}_{k\in\mathbb{N}}$ constitutes an orthonormal basis with respect to the weighting function $\phi(z)$, that is,

$$E_\phi[H_k(z)H_l(z)] = \mathbf{1}_{(k=l)}, \tag{12}$$

where $\mathbf{1}_{(\cdot)}$ is the usual indicator function and the operator $E_\phi[\cdot]$ takes the expectation of its argument regarding $\phi(\cdot)$. Note that $E_\phi[H_k(z)] = 0$ for $k \geq 1$ and then, (12) is just the covariance between $H_k(z)$ and $H_l(z)$.

3.1 Positivity of pdf

Note that $g(z)$ in (10) allows for additional flexibility over a normal density since it introduces both skewness and (excess) kurtosis of the distribution. It can lead to negative values for certain values of both centered moments. JR (2001) obtain numerically a restricted space Γ for possible values of (ek, s) to guarantee the positivity of $g(z)$. The constrained GC expansion restricted to Γ will be referred as the true GC density. Figure 1 exhibits Γ with a frontier (the envelope) delimiting the oval domain. Γ is a compact and convex set. Note that $ek \in [0, 4]$ while $|s| \leq 1.0493$ verifying that the range of s depends on the level of ek. For instance, if $|s| = 0.6$ then ek ranges from 0.6908 to 3.7590, while for $s = 0$, ek ranges from 0 to 4. The maximum size for skewness is reached for $ek = 2.4508$. Obviously, the case for the normal distribution corresponds to the origin.

[INSERT FIGURE 1 AROUND HERE]

Gallant and Tauchen (1989) suggest to square the polynomial $p(\cdot)$ in (10) to guarantee a true pdf under the CG framework. In this case, we lose the interpretation of some moments of the distribution. León et al. (2009) analyze the statistical properties of seminonparametric (SNP) distributions, which were introduced by Gallant and Nychka (1987). The SNP density (always positive by construction) can be expressed as a Gram-Charlier series of Type A, which is the product of a standard normal density times an infinite series of Hermite polynomials. So, the GC density in (10) is just a truncated GC expansion It is shown that Γ is contained into a higher space under SNP distributions.[9] Contrary to the GC density, the parameters implied in the SNP pdf do not correspond directly to the

[8] The pdf of r in (9) is obtained as $f(r) = g(z)/\sigma$.

[9] See the skewness-kurtosis frontiers under the SNP distribution in Figure 1 from León et al. (2009).

levels of skewness and kurtosis of the distribution. In Section 8 we get the closed-form expressions for the LPMs under the SNP distribution.

Finally, we select some data consisting in monthly return series for ten hedge fund (HF) index strategies. They are denoted as HF_j, $j = 1, \ldots, 10$. See Appendix A for more details about these data. Table 3 in Appendix A exhibits the constrained maximum likelihood (CML) estimation of the parameters implied in (9) and (10)for each monthly series. From now on, these estimations will be used in this paper to implement simulations.

3.2 Moments of r and z

To show that $z \sim GC(0, 1, s, ek)$ with pdf $g(\cdot)$ in (10), we need to obtain the first four (noncentral) moments of z. These moments can be obtained by using the relationship between the powers of z and the Hermite polynomials in (11) and the condition in (12):

$$\begin{aligned} E_g[z] &= E_g[H_1(z)] = 0, \\ E_g\left[z^2\right] &= \sqrt{2}E_g[H_2(z)] + 1 = 1, \\ E_g\left[z^3\right] &= \sqrt{3!}E_g[H_3(z)] + 3E_g[H_1(z)] = s, \\ E_g\left[z^4\right] &= \sqrt{4!}E_g[H_4(z)] + 6\sqrt{2}E_g[H_2(z)] + 3 = ek + 3. \end{aligned}$$

The following proposition aims to obtain a general expression for $E_g\left[z^k\right]$ where $k \in \mathbb{N}_+$.

Proposition 1 *The general expression for $E_g\left[z^k\right]$, where $k \geq 5$ and pdf $g(\cdot)$ in (10), is given as*

$$E_g\left[z^k\right] = \begin{cases} \lambda_{k,0} + \lambda_{k,1}\, ek, & k \text{ is even}, \\ \lambda_{k,2}\, s, & k \text{ is odd}, \end{cases} \tag{13}$$

where $\lambda_{k,l} \in \mathbb{R}$ can be seen in Appendix B. Since r in (9) is an affine transformation of z, the non-central moments of r are obtained as

$$E_f[r^k] = \sum_{n=0}^{k} \binom{k}{n} \mu^{k-n}\sigma^n E_g[z^n], \tag{14}$$

where $\binom{k}{n} = \frac{k!}{n!(k-n)!}$.

Proof. See Appendix B. ■

Note that, if k is even (odd) $E_g\left[z^k\right]$ depends only on the excess kurtosis (skewness).

4 Shortfall probability: LPM$(\tau, 0)$

The LPM equation in (1) for $m = 0$ is just the distribution function for the standardised return z in (9). The expression $LPM_f(\tau, 0)$ is also known as the "shortfall probability".

Proposition 2 *The shortfall probability is given by*

$$LPM_f(\tau, 0) = \Phi(\tau^*) - \frac{s}{3\sqrt{2!}}H_2(\tau^*)\,\phi(\tau^*) - \frac{ek}{4\sqrt{3!}}H_3(\tau^*)\,\phi(\tau^*), \tag{15}$$

where $\tau^* = (\tau - \mu)/\sigma$.

Proof. See Appendix B. ■

Remember that getting some performance measures based on VaR, e.g. (7) and (8), we need to compute VaR_f^α. As $LPM_f(VaR_f^\alpha, 0) = \alpha$, we get VaR_f^α by the inversion method. According to Proposition 2, $VaR_f^\alpha = \mu + \sigma F_{GC}^{-1}(\alpha; s, ek)$.[10] Note that, for a normal distribution, we get $VaR_f^\alpha = \mu + \sigma z_\alpha$, where z_α the α-quantile from a $N(0,1)$ distribution.

The following Corollary shows the behaviour of $LPM_f(\tau, 0)$ with respect to the parameters s, ek, τ, μ and σ.

Corollary 2 *Let $LPM_f(\tau, 0)$ in (15) and $\tau^* = (\tau - \mu)/\sigma$, it holds that*

i) $\partial LPM_f(\tau, 0)/\partial s > 0 \Leftrightarrow |\tau^*| < 1$.

ii) $\partial LPM_f(\tau, 0)/\partial ek > 0 \Leftrightarrow \tau^* \in \left(-\infty, -\sqrt{3}\right) \cup \left(0, \sqrt{3}\right)$.

iii) For $\tau^* \in \left(-\sqrt{3-\sqrt{6}}, 0\right) \cup \left(\sqrt{3+\sqrt{6}}, +\infty\right)$, *then* $\partial LPM_f(\tau, 0)/\partial \tau^* > 0$ *and so,*

iii.1) $\partial LPM_f(\tau, 0)/\partial \tau > 0$.

iii.2) $\partial LPM_f(\tau, 0)/\partial \mu < 0$.

iii.3) $\partial LPM_f(\tau, 0)/\partial \sigma > 0$ *iff* $\mu > \tau$.

Proof. See Appendix B. ■

As a result, for $\tau^* \in (0, 1)$, we have $\partial LPM_f(\tau, 0)/\partial s > 0$ and $\partial LPM_f(\tau, 0)/\partial ek > 0$. Next, we study the behaviour of the shortfall probability for different values of τ and $(ek, s) \in \Gamma$.

4.1 Value at Risk as threshold level

If $LPM_f(\tau, 0) = \alpha$, the threshold level τ is just the α-quantile of the distribution of the stock return. Thus, $\tau = VaR_f^\alpha$ for very small values of α. To start with, take as an example the HF_8 monthly returns whose parameters from Table 3 in Appendix A are: $\mu = 0.86\%$, $\sigma = 2.61\%$, $s = 0.4$, and $ek = 1.52$. For these returns, the threshold levels $\tau_{0.01} = -5.49\%$ and $\tau_{0.05} = -2.97\%$ correspond, respectively, to the 1% and 5% quantiles. Under the GC distribution, the same α-quantiles for the standardised stock return z are $\tau_{0.01}^* = -2.433$ and $\tau_{0.05}^* = -1.467$.

Figure 2 shows the shortfall probability in (15) for different values of $(ek, s) \in \Gamma$ subject to $\mu = 0.86\%$, $\sigma = 2.61\%$. The different lines in this Figure illustrate the possible combinations obtained from $\tau = \tau_{0.01}, \tau_{0.05}$ and $s = -0.7, 0, 0.4$. Fixing the skewness level, we can see that the lines for $\tau_{0.01}$ ($\tau_{0.05}$) exhibit a positive (negative) slope. This is due to the result *(ii)* in Corollary 2. Note also that the length of each line is different because the range of ek changes with the level of s for the GC pdf to be well defined.

[10] Let $z_{GC,\alpha} \equiv F_{GC}^{-1}(\alpha; s, ek)$ where $F_{GC}(\cdot)$ is another way to denote the LPM for $m = 0$ in (15). Then, $z_{GC,\alpha}$ is obtained through a numerical search based on the *bisection method* by using the Matlab function 'fzero'. For more details, see Brandimarte (2006).

We can see that, for any τ, the upper (lower) line corresponds to $s = -0.7$ ($s = 0.4$). This is a consequence of (i) from Corollary 2. That is, the shortfall probability increases (decreases) independently of ek since the skewness level decreases (increases) with respect to, for instance, the initial level of $s = 0$.[11] Two points for HF_8 can be seen in two different lines: $(1.52, 0.01)$ in line where $s = 0.4$, $\tau = -5.49\%$ and $(1.52, 0.05)$ in the line with $s = 0.4$, $\tau = -2.97\%$. Any other point can be considered as a hypothetical portfolio with the same μ and σ as HF_8.

Finally, a point from a certain line, independently of s, shows that the higher its ek level the higher its value of $LPM_f(\tau_{0.01}, 0)$ while the opposite happens for $\tau_{0.05}$. More precisely, increasing ek and considering $s = -0.7$ (0.4), $LPM_f(\tau_{0.01}, 0)$ can increase from 2.49% (0.3%) to 4.17% (2.46%) while $LPM_f(\tau_{0.05}, 0)$ can decrease from 8.31% (5.81%) to 6.38% (3.32%) for $s = -0.7$ ($s = 0.4$). Now, the reason is (ii) from Corollary 2.

[INSERT FIGURE 2 AROUND HERE]

4.2 Non-negative threshold levels

Figure 3 shows the same analysis as Figure 2 for HF_8 but now we take *non-negative* thresholds such as $\tau = 0\%$ and $\tau = 0.39\%$. The last one is the risk-free monthly interest rate extracted from 10-year US Treasury bonds obtained as the sample mean from November 1998 to February 2008 (i.e., 4.72% per annum).[12] We highlight three features from this Figure. First, every line exhibits a negative slope meaning that the higher ek the lower the shortfall probability. Second, contrary to Figure 2, for any τ, the lines with positive skewness are above those for negative skewness.[13] Third, there are two points associated with HF_8 in two different lines: $(1.52, 0.37)$ in line with $s = 0.4$, $\tau = 0\%$ and $(1.52, 0.44)$ in the line with $s = 0.4$, $\tau = 0.39\%$. Again, any other point is just a different (hypothetical) portfolio with the same μ and σ as HF_8.

[INSERT FIGURE 3 AROUND HERE]

5 LPMs and related PMs with GC distribution

This section starts providing the general closed-form expressions of $LPM_f(\tau, m)$ and so the related expressions for $FTR_f(\tau, q, m)$ and $K_f(\tau, m)$ where the stock return is driven by (9). We show how these LPMs are linear functions of both s and ek, and the behaviour of the above PMs. We also obtain some Kappa (FT) iso-curves.

5.1 Closed-form expressions

Proposition 3 *Let z be the standardised return of r in* (9)*. The lower partial moment of order $m \in \mathbb{N}_+$ for the security return r can be expressed as*

$$LPM_f(\tau, m) = LPM_n(\tau, m) + \frac{s}{\sqrt{3!}}\theta_{2,m} + \frac{ek}{\sqrt{4!}}\theta_{3,m}, \tag{16}$$

[11] Note that (i) from Corollary 2 holds, in our case, that $\partial LPM_f(\tau, 0)/\partial s < 0$ since $|\tau_k^*| > 1$ for $k = 0.01, 0.05$.

[12] This period is the same as the HF database in Appendix A.

[13] Note that (i) from Corollary 2 holds now that $\partial LPM_f(\tau, 0)/\partial s > 0$ since the values of $\tau^* = (\tau - 0.86\%)/2.61\%$ for $\tau = 0\%, 0.39\%$ verify that $|\tau^*| < 1$.

where $LPM_n(\tau, m)$ is the LPM by assuming the normal distribution for r in (9), denoted as $n(\cdot)$, with

$$LPM_n(\tau, m) = \sum\nolimits_{k=0}^{m} (-1)^k \binom{m}{k} (\tau - \mu)^{m-k} \sigma^k B_k, \tag{17}$$

and

$$\theta_{j,m} = \sum\nolimits_{k=0}^{m} (-1)^k \binom{m}{k} (\tau - \mu)^{m-k} \sigma^k A_{kj}, \tag{18}$$

such that $B_k = B_k(\tau^*)$ and $A_{kj} = A_{kj}(\tau^*)$, with $\tau^* = (\tau - \mu)/\sigma$, can be seen in Appendix B.

Proof. See Appendix B. ■

As can be seen, the behaviour of $\theta_{j,m}$ depends on the expected return, the volatility and the return threshold, i.e. $\theta_{j,m} = \theta_{j,m}(\mu, \sigma, \tau)$. Note that $\theta_{2,m}$ and $\theta_{3,m}$ measure, respectively, the sensitivity of $LPM_f(\tau, m)$ to changes in s and ek. Note that the Kappa measures are easily obtained by using (3) and (16). The following Corollary shows the general expression for the FT measures in (5) under the GC density for the standardised return.

Corollary 3 *Let z be the standardised return of r in (9). The performance measure $FTR_f(\tau, q, m)$ in (5) for $q, m \in \mathbb{N}_+$ can be expressed as*

$$FTR_f(\tau, q, m) = \frac{\sqrt[q]{\psi_f(\tau, q) + (-1)^{q+1} LPM_f(\tau, q)}}{\sqrt[m]{LPM_f(\tau, m)}}, \tag{19}$$

where $LPM_f(\tau, \cdot)$ is given in (16) and $\psi_f(\tau, q) = E_f[(r - \tau)^q]$ is obtained as

$$\psi_f(\tau, q) = \sum\nolimits_{k=0}^{q} \binom{q}{k} (\mu - \tau)^{q-k} \sigma^k E_g\left[z^k\right], \tag{20}$$

where $E_g\left[z^k\right]$ is given in (13).

Proof. It is straightforward. ■

As many studies about performance evaluation focus on some popular Kappa measures, such as the Omega-Sharpe, Sortino and Kappa 3 ratios, and the Upside Potential ratio from the FT measures, we are very interested in the expression of $LPM_f(\tau, m)$ for $m = 1, 2, 3$ shown in the following Corollary.

Corollary 4 *The expressions of $\theta_{j,m}$ for $j = 2, 3$ and $LPM_n(\tau, m)$ for $m = 1, 2, 3$ in (16) are:*

$$\begin{aligned} \theta_{j,1} &= (\tau - \mu)A_{0j} - \sigma A_{1j}, \\ \theta_{j,2} &= (\tau - \mu)^2 A_{0j} - 2(\tau - \mu)\sigma A_{1j} + \sigma^2 A_{2j}, \\ \theta_{j,3} &= (\tau - \mu)^3 A_{0j} - 3(\tau - \mu)^2 \sigma A_{1j} + 3(\tau - \mu)\sigma^2 A_{2j} - \sigma^3 A_{3j}, \end{aligned} \tag{21}$$

and

$$\begin{aligned} LPM_n(\tau,1) &= (\tau-\mu)\,\Phi(\tau^*) + \sigma\phi(\tau^*)\,, \\ LPM_n(\tau,2) &= (\tau-\mu)^2\,\Phi(\tau^*) + (\tau-\mu)\,\sigma\phi(\tau^*) + \sigma^2\Phi(\tau^*)\,, \\ LPM_n(\tau,3) &= (\tau-\mu)^3\,\Phi(\tau^*) + (\tau-\mu)^2\,\sigma\phi(\tau^*) + 3\,(\tau-\mu)\,\sigma^2\Phi(\tau^*) + 2\sigma^3\phi(\tau^*)\,. \end{aligned} \tag{22}$$

where the values for $A_{kj} = A_{kj}(\tau^*)$ with $\tau^* = (\tau-\mu)/\sigma$, can be seen in the Appendix B.[14]

Proof. See Appendix B. ■

5.2 Behaviour of Kappa measures respecting s and ek

We analyze the effects of the higher moments on the performance ratios. We fix the parameter vector (μ,σ,τ) at (μ_0,σ_0,τ_0). Then, $LPM_f(\tau_0,m)$=LPM_m is a function, g_m, on both s and ek. Let ΔLPM_m and $dLPM_m$ denote, respectively, the increment and the total differential of LPM_m with respect to its arguments.[15] The next Corollary inmediately arises.

Corollary 5 *If we approximate ΔLPM_m by $dLPM_m$, we get*

$$\Delta LPM_m = \frac{\partial g_m}{\partial s}\Delta s + \frac{\partial g_m}{\partial ek}\Delta ek$$

holding that

$$\Delta LPM_m > 0 \Leftrightarrow \Delta s < \varphi_m \Delta ek, \quad \varphi_m = -\frac{\theta_{3,m}}{2\theta_{2,m}}. \tag{23}$$

Proof. It is straightforward. ■

Table 1 exhibits the behaviour of the popular Kappa measures by changing either s or ek, that is, we provide these measures for alternative portfolios with the same μ and σ but different values for s and ek such that $(ek,s) \in \Gamma$. We consider again the values related to HF_8 ($\mu = 0.86\%$, $\sigma = 2.61\%$), $\tau = r_f$ (i.e., 0.39%), and three possible values of skewness ($s = -0.7$, 0, 0.4). The Sharpe ratio, $SR = (\mu - r_f)/\sigma$, is equal to 0.1796 and it is constant across this Table. The SR is considered as the benchmark measure. Plugging these parameters into the expression for φ_m in (23), we get $\varphi_1 = -1.3471$, $\varphi_2 = 0.0499$ and $\varphi_3 = 0.2289$.

[INSERT TABLE 1 AROUND HERE]

The columns 2 to 10 show the three Kappa measures for the different point combinations (ek,s) according to different levels of skewness. The main results can be summarized as follows.

[14] Note that $LPM_n(\tau,m)$ in (22) could also be denoted, to shorten, as $\theta_{1,m}$ for the case of $j=1$ in (21) since both expressions coincide. Nevertheless, we have decided in this paper to use $LPM_n(\tau,m)$ instead of $\theta_{1,m}$. Thus, we denote the coefficients of s and ek in (16), respectively, as $\theta_{2,m}$ and $\theta_{3,m}$.

[15] That is, $\Delta LPM_m = g_m(s+\Delta s, ek+\Delta ek) - g_m(s,ek)$, where Δx represents a small increment in x.

1. Let be the portfolio π_1, with $(ek, s) = (0.8996, 0)$, and build new portfolios by only increasing ek. We can see that, if ek increases, $K_f(r_f, 1)$ increases but both $K_f(r_f, 2)$ and $K_f(r_f, 3)$ decrease.

2. Take a new portfolio π_2 with $(ek, s) = (0.8996, 0.4)$. It holds that $K_f(r_f, m)$ increases if we only increase s by changing π_1 for π_2. The same behaviour holds for alternative values of ek.

3. Suppose now that ek increases and s decreases. Consider either portfolio π_3, with $(ek, s) = (1.2048, -0.7)$, or π_4, with $(ek, s) = (2.1205, -0.7)$. It is verified that both $K_f(r_f, 2)$ and $K_f(r_f, 3)$ decrease when going from π_1 to either π_3 or π_4. Meanwhile, there are opposite effects about the behaviour of $LPM_f(r_f, 1)$. Note that $K_f(r_f, 1)$ decreases for π_3 while it increases for π_4.

4. Finally, we can see that (23) holds under these examples and so, the behaviour of their related Kappa measures is verified. Thus, changing π_1 for π_2 leads to $\Delta s = 0.4$, $\Delta ek = 0$. The case of changing π_1 for π_3 leads to $\Delta s = -0.7$, $\Delta ek = 0.3052$. Finally, changing π_1 for π_4 leads $\Delta s = -0.7$, $\Delta ek = 1.2209$.

In short, we can suggest from the above results that $\partial LPM_f(r_f, m)/\partial s < 0$ for $m = 1, 2, 3$, $\partial LPM_f(r_f, 1)/\partial ek < 0$ and $\partial LPM_f(r_f, m)/\partial ek > 0$ for $m = 2, 3$. These results are also supported by studying the behaviour of $\theta_{2,m}$ and $\theta_{3,m}$ in (21) from many simulated parameters of μ and σ. The simulation results confirm the previous conclusions.[16] Thus, it is held that $\theta_{2,m} < 0$, $\theta_{3,1} < 0$, $\theta_{3,2} > 0$ and $\theta_{3,3} > 0$.

5.3 Iso-curves for performance measures

We obtain the points (ek, s) that provide the same value for the selected Kappa measure given fixed levels of τ, μ and σ. To shorten, let Ψ denote the vector (μ, σ, τ) and let Ψ_0 be a fixed value for Ψ. Thus, the iso-curve associated for any Kappa measure, or iso-Kappa, corresponds to the following set of points Π defined as

$$\Pi(m, \Psi_0) = \left\{ (ek, s) \in \Gamma : \overline{K}_f(\tau_0, m) = \frac{\mu_0 - \tau_0}{\sqrt[m]{LPM_n(\tau_0, m) + \frac{s}{\sqrt{3!}}\theta_{2,m} + \frac{ek}{\sqrt{4!}}\theta_{3,m}}} \right\}, \quad (24)$$

where $\overline{K}_f(\tau_0, m)$ denotes a fixed value for the Kappa ratio given by equations (3) and (16). These spaces are easily obtained according to the following Corollary.

Corollary 6 *The iso-Kappa* (24) *implies a linear relation between* s *and* ek. *Thus,* $s = a_m + \varphi_m ek$ *such that the slope* φ_m *is defined in* (23) *and*

$$a_m = \frac{\sqrt{6}}{\theta_{2,m}} \left[\xi_{0,m} - LPM_n(\tau_0, m)\right], \quad \xi_{0,m} = \left[\frac{\mu_0 - \tau_0}{\overline{K}_f(\tau_0, m)}\right]^m, \quad (25)$$

with $LPM_n(\tau_0, m)$ *given in* (17).

[16] We obtain pairs (μ_i, σ_i), $i \leq 10{,}000$ where each parameter is obtained randomly (uniform distribution) such that $\mu_i \in [0.49\%, 0.96\%]$ and $\sigma_i \in [0.96\%, 2.61\%]$. The minimum and maximum values for μ and σ are selected from Table 3 in Appendix A. We also choose different values for τ. More details are available upon request. Finally, see footnote 18.

Proof. It is straightforward. ■

The iso-Kappa in (25) will be labeled as 'iso-Omega-Sharpe', 'iso-Sortino' and 'iso-Kappa 3' respectively for $m = 1, 2, 3$. Since $\partial\xi_{0,m}/\partial\overline{K}_f(\tau_0, m) < 0$ for $\mu_0 > \tau_0$, then $\partial a_m/\partial\overline{K}_f(\tau_0, m) > 0$ iff $\theta_{2,m} < 0$. Let $\Psi_0 = (0.86\%, 2.61\%, 0.39\%)$ be the parameter set used to obtain Table 1, then the slopes φ_m for the different iso-Kappas (see the values of φ_m in Subsection 5.2) verify that $\varphi_1 < 0$, $\varphi_2 > 0$, $\varphi_3 > 0$. So, an increase in ek leads to a decrease (increase) in s when moving along the iso-Omega-Sharpe (iso-Sortino or iso-Kappa 3) curve.

By setting $s = -0.7$ and taking higher levels of ek, we can see in Table 1 that $K_f\,(0.39\%, 1)$ increases ($\xi_{0,1}$ decreases) but $K_f\,(0.39\%, m)$ decreases ($\xi_{0,m}$ increases) for $m = 2, 3$. This means that the iso-Omega-Sharpe curves with negative slopes move in paralell to the right with higher levels of $\overline{K}_f(0.39\%, 1)$ since a_1 in (25) increases because $\theta_{2,1} < 0$. Nevertheless, both the iso-Sortino and iso-Kappa3 curves with positive slopes move in paralell to the right with lower levels of $\overline{K}_f(0.39\%, m)$ since a_1 decreases because $\theta_{2,m} < 0$.[17]

Note that, on the one hand the iso-Kappas from Corollary 6 can be very restrictive since we are fixing both the mean and volatility parameters for the portfolio returns, but on the other hand we get linear equations which can contribute to a better understanding the behaviour of the iso-Kappas.[18]

Finally, outside the Kappa measures, if we get the iso-curves under the FT measures (except for $q = m$ in (19) such as, for instance, the Omega measure for $q = m = 1$), they do not hold a linear relation between s and ek. Nevertheless, we could obtain a linear approximation. For instance, consider the 'iso-Upside potential ratio' corresponding to the following set of points Υ defined as:

$$\Upsilon(\Psi_0) = \left\{ (ek, s) \in \Gamma : \overline{FTR}_f\,(\tau_0, 1, 2) = \frac{\mu_0 - \tau_0 + LPM_n\,(\tau_0, 1) + \frac{s}{\sqrt{3!}}\theta_{2,1} + \frac{ek}{\sqrt{4!}}\theta_{3,1}}{\sqrt{LPM_n\,(\tau_0, 2) + \frac{s}{\sqrt{3!}}\theta_{2,2} + \frac{ek}{\sqrt{4!}}\theta_{3,2}}} \right\}, \tag{26}$$

where Ψ_0 is defined previously and $\overline{FTR}_f\,(\tau_0, 1, 2)$ denotes a fixed value of the performance measure in (19) from Corollary 3. Thus, using a multivariate Taylor expansion of first order around the point $(s, ek) = (0, 0)$ in the denominator of $FTR_f\,(\tau_0, 1, 2)$, we can get a linear relation between s and ek.

6 Simulation analysis

We implement a simulation analysis based on the closed-form expressions for the performance measures by assuming the GC distribution for the standardised stock returns in (9).

[17] The values for $\theta_{2,m}$ are, respectively, $\theta_{2,1} = -7.53 \times 10^{-4}$, $\theta_{2,2} = -2.18 \times 10^{-4}$ and $\theta_{2,3} = -1.87 \times 10^{-5}$. The curves for the iso-Kappas are not exhibited here for the sake of brevity.

[18] Suppose that we do not fix μ in Corollary 6. Then, the iso-Kappas depend on μ, s and ek. By using the implicit function Theorem, we can obtain the corresponding partial derivatives (evaluated at a certain point) to analyze the behavior of μ with respect to s and ek. This extension is not shown here but it is available upon request.

The advantage of using the GC distribution is that the simulated value of any performance measure exhibitd in Section 2 is obtained immediately. That is, once we obtain randomly a parameter vector $(\mu_i, \sigma_i, s_i, ek_i)$ representing the return distribution in (9) of a hypothetical portfolio i, we do not need to simulate paths of a certain length for monthly returns to get the alternative performance measures since we have their closed-form expressions in Section 5. We start our analysis by obtaining the Spearman's rank correlation between the Sharpe ratio and any alternative performance measure for different portfolios. The higher this rank correlation, the lower difference in ranking between the Sharpe ratio and the selected performance measure. We also study how skewness and excess kurtosis can affect the evaluation of portfolios.

6.1 Simulation of parameters and performance ratios

We generate randomly the parameter vector $(\mu_i, \sigma_i, s_i, ek_i)$, $i = 1, ..., N_T$.[19] The range of these parameters is obtained from their CML estimations of these parameters for the hedge funds,[20] see Table 3 in Appendix A. Let $x_{\min}$ ($x_{\max}$) denote the minimum (maximum) value for the CML estimation of the parameter $x = \sigma$, s, SR. Then, we have $\sigma_{\min} = 0.963\%$, $\sigma_{\max} = 2.163\%$, $s_{\min} = -0.798$, $s_{\max} = 0.987$, $SR_{\min} = 1\%$ and $SR_{\max} = 22.3\%$.

We simulate four independent uniform random variables U_j, $j = 1, ..., 4$ on the interval $(0, 1)$, each with sample size, N, of $10,000$. The realizations of these variables will be denoted as $u_{ji}, i = 1, ..., N$. We implement the following steps for i :

1. We compute $\sigma_i = \sigma_{\min} + (\sigma_{\max} - \sigma_{\min})\, u_{1i}$ and $SR_i = SR_{\min} + (SR_{\max} - SR_{\min})\, u_{2i}$. Then, the mean is obtained as $\mu_i = r_f + \sigma_i SR_i$.

2. The skewness is obtained as $s_i = s_{\min} + (s_{\max} - s_{\min})\, u_{3i}$. The corresponding excess kurtosis is $ek_i = ek_{i,\min} + (ek_{i,\max} - ek_{i,\min})\, u_{4i}$ such that both $(ek_{i,\min}, s_i)$ and $(ek_{i,\max}, s_i)$ belong to the restricted space Γ, that is, $(ek_i, s_i) \in \Gamma$.[21]

3. After simulating $\vartheta_i = (\mu_i, \sigma_i, s_i, ek_i)$, we compute all the PMs described previously by plugging ϑ_i into the corresponding formulas. To set a fair comparison among these measures, we select $\tau = r_f$. Thus, given the portfolio i defined by ϑ_i, we get SR_i and any other performance measure, denoted as π_i, that can be $K_i(r_f, m)$, $FTR_i\,(r_f, q, m)$, $RVaR_i^{\alpha}\,(r_f)$ or $RCVaR_i^{\alpha}\,(r_f)$.

6.2 Rank correlations

We obtain the average of the Spearman's correlation over a sample size of one-hundred rank correlations between π_i and SR_i, such that each correlation is obtained through N vectors (π_i, SR_i) computed for portfolios characterised by the vector ϑ_i according to Subsection 6.1. For instance, we obtain the correlation between $FTR\,(r_f, q, m)$ in (19), for (positive) integer values of $q, m \leq 6$, and SR. It is shown (not reported here but available upon request) that the larger q and m, the lower the rank correlation. If $q \geq 2, m \geq 3$,

[19] $N_T = N \times T$ where as we will see later, N denotes the sample size per regression and T is the number of regressions. We will set $N = 10,000$ and $T = 100$.

[20] We rule out HF_6 since its Sharpe ratio is negative, see Subsubsection 6.3.1.

[21] We use the notation $ek_{i,\min}$ and $ek_{i,\max}$ to emphasize that the range for the possible values of ek depends on s_i.

the correlation never exceeds 25%. Then, these PMs lead to different portfolio rankings respecting the Sharpe ratio. Nevertheless, the Omega or Kappa of order one exhibits a very high correlation (97.70%).[22] This evidence suggests that there is no ranking difference with regard to the Sharpe ratio. All the above results are also supported by Eling et al. (2011) who analyze, among others, the FTRs for different values of q and m. The following Subsection is about a more robust analysis by splitting the total sample $N = 10,000$ in two subsamples depending on the size of SR. It will be shown, for instance, that there can be even more ranking difference between $K(r_f, 1)$ and SR the higher SR.

6.3 Effects of skewness and kurtosis on portfolio evaluation

Here, we will mainly concentrate on studying those PMs with higher rank correlations according to our previous results in Subsection 6.2. The next analysis is rather similar to the one by Zakamouline (2011).

6.3.1 The models

We consider the following two models:

1. The first model is defined as

$$\pi_i = \alpha_\pi SR_i^{\beta_\pi}, \quad \alpha_\pi, \beta_\pi > 0, \quad i = 1, \cdots, N, \tag{27}$$

   where π is a specific performance measure (and the same PM for all portfolios i in the above equation) such that $\pi_i > 0$.[23] The portfolio i is defined regarding the parameter vector ϑ_i. As α_π and β_π are positive, π_i is equivalent to SR_i in the sense that both produce the same ranking. If testing with data the evidence of a high equivalence between π_i and SR_i, in that case there would be a high goodness of fit through the (adjusted) R^2 statistic, denoted by $R^2_{\pi,0}$, from estimating by ordinary least squares (OLS) the equation (27) in logarithmics.

2. The second model is given as

$$\pi_i = \alpha_\pi SR_i^{\beta_\pi} \exp\left(\beta_\pi^s s_i + \beta_\pi^{ek} ek_i + \varepsilon_{\pi,i}\right), \quad i = 1, ..., N, \tag{28}$$

   where $\varepsilon_{\pi,i}$ is the error term according to π and ϑ_i. Note that β_π^s and β_π^{ek} are, respectively, the (relative) sensitivity of π to the skewness and excess kurtosis from the portfolio return distribution. That is, $\beta_\pi^s = (\partial\pi/\partial s)/\pi$ and $\beta_\pi^{ek} = (\partial\pi/\partial ek)/\pi$. We estimate by OLS the following expression, obtained after taking logarithms in (28), given by

$$\log(\pi_i) = \log(\alpha_\pi) + \beta_\pi \log(SR_i) + \beta_\pi^s s_i + \beta_\pi^{ek} ek_i + \varepsilon_{\pi,i}. \tag{29}$$

   Let $R^2_{\pi,1}$ denote the (adjusted) R^2 statistics of (29). If the estimates of β_π^s and β_π^{ek} were statistically significant, then both s and ek can affect π and hence, π could

[22] The same happens to $K(r_f, 2)$ and $K(r_f, 3)$ with correlations of, respectively, 94.29% and 91.18% but lower than that of $K(r_f, 1)$. For the upside potential ratio, $FTR(r_f, 1, 2)$, the rank correlation is 63.23%. Finally, for $\alpha = 1\%, 5\%$, the rank correlations for $RVaR$ ($RCVaR$) are, respectively, 79.11% and 74.35% (87.88% and 79.82%)

[23] We just concentrate on those portfolios having positive PMs as the relevant ones in our study.

produce a different ranking among different portfolios than SR. It also means that $R^2_{\pi,1} - R^2_{\pi,0}$ would become large and the (Spearman's) rank correlation coefficient between π and SR, denoted as $R_S(\pi, SR)$, would be small. Otherwise, if the estimates of both parameters were not significant, then $R^2_{\pi,0}$ and $R^2_{\pi,1}$ would be rather the same value and the above rank correlation would be larger.

6.3.2 Estimation results

Table 2 provides the OLS estimates of β_π, β^s_π and β^{ek}_π, the rank correlation $R_S(\pi, SR)$ and the statistics $R^2_{\pi,0}$ and $R^2_{\pi,1}$ obtained from a total of nine regressions from (29). Zakamouline (2011) shows[24] that a larger SR implies a lower $R_S(\pi, SR)$. A possible reason might be the larger the Sharpe ratio the larger the adjustment for non-normality of the portfolio return distribution by the selected PM. So, our simulation analysis aims to test this behaviour by using two non-overlapping ranges for SR for each PM. Specifically, we take $SR_{\min}$ and $SR_{\max}$ from Subsection 6.1 and then, consider the intervals $J_1 \equiv [1\%, 16.14\%]$ and $J_2 \equiv (16.14\%, 22.3\%]$.[25] We split each sample size N in two parts, N_1 and N_2, and run two regressions with π_i as dependent variable. In the first regression, the independent variables are the vectors $(1, SR_i, s_i, ek_i)$ such that both s_i and ek_i come from ϑ_i and $SR_i = (\mu_i - r_f)/\sigma_i \in J_1$ where μ_i and σ_i also belong to ϑ_i. The second regression is based on the remaining N_2 points such that each $SR_i \in J_2$. The last column of this Table displays the Chow test (and its p-value).That is, testing the null hypothesis of no structural break (one regression) against the alternative one of structural break (two regressions). In short, this experiment is repeated 100 times and Table 2 exhibits the *mean* values of the above parameter estimates, rank correlations, etc. The main results from this Table are as follows:

- **R^2 statistics, rank correlation and Chow test**

In most cases, it is better running two regressions than one regression since the p-values for the Chow test are zero for all PMs except $RVaR$ and $RCVaR$ with $\alpha = 1\%$. In other words, the larger SR the larger the sensitivity of any π to the higher moments. For any π, the rank correlation $R_S(\pi, SR)$ becomes lower under J_2. $R^2_{\pi,1} - R^2_{\pi,0}$ becomes pretty higher under J_2. In short, these results and (most of) the next ones agree with those of Zakamouline (2011).

- **Behaviour of OLS beta estimates**

  - For all performance measures, the OLS estimates for the three betas, $\widehat{\beta}$, are statistically significant at the 1% level and so, both skewness and excess kurtosis play significant roles in these measures.
  - For any π, $\widehat{\beta}_\pi$ is always positive and becomes larger for the regression under J_2, although there is no much difference between both regressions. So, we can conclude that a higher SR leads to a higher π.

[24] She implements a simulation analysis by assuming the Normal-Inverse-Gaussian for the return distribution.

[25] The number 16.14% is the mean of the Sharpe ratios from nine hedge funds (except HF_6).

- $\widehat{\beta}_\pi^s$ is larger in the regression under J_2 and always positive suggesting that all measures appreciate positive skewness.
- $\widehat{\beta}_\pi^{ek}$ is negative in most cases except the Omega-Sharpe ratio and $RVaR$ with $\alpha = 5\%$. It suggests that most PMs penalize positive excess kurtosis.
- $\left|\widehat{\beta}_\pi^s\right|$ is much greater than $\left|\widehat{\beta}_\pi^{ek}\right|$ in most PMs except the Omega-Sharpe ratio.

[INSERT TABLE 2 AROUND HERE]

7 Efficient frontiers under LPMs

Each portfolio from the efficient frontier (EF) shows either the highest level of expected return at a given level of risk, measured through the LPM or variance, or the lowest risk for a given expected return.[26] The portfolios include n securities where the marginal distribution for the standardised stock returns in (9) follow the GC distribution in (10). This Section is structured as follows. First, we present the optimization problem under this new framework. For simplicity, we assume a multivariate Gaussian copula to capture the dependence among the stock returns. Second, we compare the portfolios in the EF under the LPMs with those from the traditional mean-variance (MV) approach. This comparison is interesting since it allows to understand which sort of portfolios, in terms of skewness and excess kurtosis, are selected in the EF frontier according to the LPM risk measure in (16).

7.1 Optimization method

The optimization program, OP_m, is as follows:

$$\min_{w} LPM_p(\tau, m) = w' \cdot CLPM \cdot w, \tag{30}$$

subject to

$$\begin{aligned} w' \cdot \mu &= \mu_p, & (31)\\ w' \cdot l &= 1, & (32)\\ w &\geq \mathbf{0}, & (33) \end{aligned}$$

where $LPM_p(\tau, m)$ is the LPM measure of a return portfolio p, $w = (w_1, \ldots, w_n)'$ and $\mu = (\mu_1, \cdots, \mu_n)'$ denote, respectively, the weight and expected stock return vectors, and l is the $n \times 1$ vector of ones. Note that we only allow for long positions in the stocks. Finally, $CLPM \equiv CLPM(\tau, m)$ in (30) is the co-lower partial moment matrix of order $n \times n$. To simplify the computations, in a similar way to Nawrocki (1991) and Huang et al. (2001), we assume symmetry in CLPM.[27] The elements of the CLPM are defined as

$$\begin{aligned} CLPM_{jj} &= LPM_j(\tau, m)^{2/m}, \quad j = 1, \cdots, n & (34)\\ CLPM_{jk} &= \rho_{jk} LPM_j(\tau, m)^{1/m} LPM_k(\tau, m)^{1/m}, \quad j \neq k & (35) \end{aligned}$$

[26] We do not obtain EFs under the UPM-LPM framework. This issue is left for future research. See Cumova and Nawrocki (2014) for more details.

[27] The asymmetric CLPM matrix is implemented, among others, in Cumova and Nawrocki (2011).

where $LPM_j(\tau, m)$ denotes the LPM in (16) for the stock return j and ρ_{jk} represents the correlation coefficient between the stock returns j and k. Since $CLPM(\tau, m)$ is symmetric, OP_m turns to be a quadratic (convex) optimization problem such that each local minimum is a global one. Finally, the traditional variance-covariance Markowitz algorithm, OP_0, is the same as the above OP_m but now the objective function (30) is the variance of the portfolio stock return, σ_p^2. That is, $\sigma_p^2 = w' M_2 w$ where M_2 is the covariance matrix.

7.2 MV versus MLPM comparison

Consider again as true parameters the CML estimations of (μ, σ, s, ek) for each hedge fund (see Table 3 in Appendix A). We will implement both OP_m and OP_0 algorithms to compare the mean-LPM (MLPM) and the traditional Markowitz efficient frontiers. Let w^{MV} and w^{MLPM} denote the weigth vector of the portfolio included, respectively, in a mean-variance (MV) and a MLPM efficient portfolio. As Grootveld and Hallerbach (1999), we measure the difference in the portfolio compositions in both EFs by using the root mean squared dispersion index:

$$RMSDI(\tau, m) = \sqrt{\frac{1}{N_p \times n} \sum_{i=1}^{N_p} \sum_{j=1}^{n} \left(w_{ij}^{MLPM} - w_{ij}^{MV} \right)^2}, \tag{36}$$

where N_p is the total number of selected efficient portfolios, n is the number of assets included in a portfolio, and w_{ij}^{MV} and w_{ij}^{MLPM} denote the investment fractions of, respectively, the MV and MLPM efficient portfolio i for the security j. We select the portfolio i such that the expected portfolio returns under the two different risk measures coincide, that is, $\sum_{j=1}^{n} w_{ij}^{MV} \mu_j = \sum_{j=1}^{n} w_{ij}^{MLPM} \mu_j$.

[INSERT FIGURE 4 AROUND HERE]

Figure 4 includes the dispersion index in (36) for the EF portfolios by solving the OP_m algorithm under different values of m and τ, and comparing each with the EF portfolios provided by the OP_0 algorithm.[28] The right-hand graph exhibits, for one hundred EF portfolios ($N_p = 100$) obtained by using the OP_m algorithm,[29] the values of $RMSDI(\tau, m)$ as a function of $\tau \in [0\%, 0.39\%]$. We see large differences between the portfolio composition under the selected $LPM(\tau, m)$ and MV across the different values of τ. Note that the higher τ, the lower $RMSDI(\tau, m)$ verifying that MLPM portfolios under $m = 3$ tend to approach more to the EF portfolio composition under the MV framework. For example, for $\tau = 0.39\%$, we have $RMSDI(0.39\%, 1) = 3.66\%$, $RMSDI(0.39\%, 2) = 3.18\%$, and $RMSDI(0.39\%, 3) = 2.65\%$. Our results agree, for example, with those from Marmer and Ng (1993) and Jarrow and Ng (2006) who showed that, for non-Gaussian return distributions, the mean-semivariance optimal portfolios are significantly different from mean-variance optimal portfolios.

The left-hand graph shows for each m the values of $RMSDI(\tau, m)$ as a function of the mean of the monthly portfolio return such that the selected return threshold is the

[28] The correlation matrix is the sample one obtained from the ten HF monthly return series ($n = 10$) introduced in Appendix A.

[29] The expected portfolio returns range from $\mu_1 = 0.5182\%$ to $\mu_{100} = 0.9553\%$ since $N_p = 100$. This interval is obtained from the intersection of the different ranges for the mean of the portfolio returns from the four EFs ($m = 1, 2, 3$ and MV).

monthly risk-free rate, $\tau = 0.39\%$. For instance, each point in the Omega-Sharpe line ($m = 1$) is the root of the mean of the squared errors $(w_{ij}^{MLPM} - w_{ij}^{MV})^2$, $j \leq 10$ of the portfolio i (where $i \leq 100$) in the EF with mean μ_i (see footnote 7.2). In short, we get the $RMSDI\,(0.39\%, 1)$ for each portfolio i with mean μ_i by using (36) with $N_p = 1$. For any m, we can see significant differences between the portfolio composition under both $LPM\,(0.39\%, m)$ and MV for different values of μ_i.

7.3 Skewness and kurtosis of portfolio stock returns

According to the previous assumptions, we will compute the theoretical skewness and excess kurtosis, denoted as s_p and ek_p, for the portfolio returns in the different MLPM efficient frontiers. These portfolios include n stocks with GC as the marginal distribution for each standardised stock return and a multivariate Gaussian copula for the dependence among the stock returns.[30] Note that we first have to obtain the co-skewness and co-kurtosis matrices, denoted as M_3 and M_4 respectively.

7.3.1 Co-skewness and co-kurtosis matrices

The co-skewness and co-kurtosis among asset returns are defined[31] as

$$s_{ijk} = E\left[(r_i - \mu_i)(r_j - \mu_j)(r_k - \mu_k)\right], \quad i, j, k = 1, \ldots, n \tag{37}$$

and

$$k_{ijkl} = E\left[(r_i - \mu_i)(r_j - \mu_j)(r_k - \mu_k)(r_l - \mu_l)\right], \quad i, j, k, l = 1, \ldots, n. \tag{38}$$

Hence, the co-skewness matrix of order $\left(n, n^2\right)$ is

$$M_3 = \left[S_{1jk} \cdots S_{njk}\right],$$

where $S_{ijk},\ i = 1, \cdots, n$ denotes the short notation for the (n, n) submatrix $\left(s_{ijk}\right)_{j,k=1,\ldots,n}$ with elements in (37).

The co-kurtosis matrix of order $\left(n, n^3\right)$ is

$$M_4 = \left[K_{11kl}\, K_{12kl} \ldots K_{1nkl} \cdots\ K_{n1kl}\, K_{n2kl} \ldots K_{nnkl}\right],$$

where $K_{ijkl},\ i, j = 1, \cdots, n$ is the short notation for the (n, n) submatrix $\left(k_{ijkl}\right)_{k,l=1,\cdots,n}$ with elements in (38). Because of certain symmetries, not all the elements of these matrices need to be computed. Thus, M_3 and M_4 involve only $\binom{n+2}{3}$ and $\binom{n+3}{4}$ elements, respectively. Since we assume a Gaussian copula, these matrices are easily computed as shown in the next Lemma.

Lemma 1 *Let $(r_1, \ldots, r_n)$ denote the random vector of n stock returns such that $r_i = \mu_i + \sigma_i z_i$ where $z_i \sim GC\,(0, 1, s_i, ek_i)$ with pdf in (10). The dependence among these returns is captured by a Gaussian copula with elements ρ_{ij}. Then, the elements of M_3 in (37) are given by*

$$s_{ijk} = \begin{cases} \sigma_i^3 s_i, & i = j = k, \\ 0, & \textit{otherwise}, \end{cases} \tag{39}$$

[30] See Cherubini et al. (2004) for more details.

[31] We adopt the notation by Jondeau and Rockinger (2006).

where σ_i and s_i are, respectively, the standard deviation and skewness of the return r_i. The elements of M_4 in (38) *are given as*

$$k_{ijkl} = \begin{cases} \sigma_i^4 \left(3 + ek_i\right), & i = j = k = l, \\ \sigma_{ij}\sigma_{kl} + \sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk}, & \text{otherwise,} \end{cases} \tag{40}$$

where ek_i is the excess kurtosis of the return r_i and $\sigma_{ij} = \sigma_i\sigma_j\rho_{ij}$ denotes the covariance between r_i and r_j.

Proof. See, among others, Isserlis (1918) or Holmsquit (1988) for the higher-order moments of the multivariate normal distribution. ■

Note that the GC marginal distribution is easily imposed in the above equations (39) and (40) verifying that $s_{iii} = \sigma_i^3 s_i^3$ and $k_{iiii} = \sigma_i^4 \left(3 + ek_i\right)$. Finally, the skewness and kurtosis of the portfolio stock return are obtained as

$$s_p = \frac{w' M_3 \left(w \otimes w\right)}{\sigma_p^3}, \tag{41}$$

and

$$k_p = \frac{w' M_4 \left(w \otimes w \otimes w\right)}{\sigma_p^4}, \tag{42}$$

where $\otimes$ denotes the "Kronecker product". Finally, $ek_p = k_p - 3$ is the portfolio excess kurtosis.

7.3.2 Getting s_p and ek_p from MLPM efficient frontiers

Figure 5 exhibits s_p and ek_p for the one hundred EF portfolios and return thresholds $\tau = 0\%, 0.39\%$ as a function of the monthly expected return $\mu_j \in [0.5182\%, 0.9553\%]$.[32] The left-hand graphics include the values of s_p under $\tau = 0\%$ (upper graph) and $\tau = 0.39\%$ (lower graph) while the values of ek_p are shown on the right side.[33] Several results arise. First, for any τ and m, there are differences beween MV and MLPM. These differences become more important for $m = 1$. Second, there are no significant differences between $LPM\left(\tau, 2\right)$ and $LPM\left(\tau, 3\right)$. Third, there are significant differences between $LPM\left(0\%, 1\right)$ and $LPM\left(0\%, m\right)$ where $m = 2, 3$ for lower expected portfolio returns.

[INSERT FIGURE 5 AROUND HERE]

8 The SNP density and LPMs

We show here the SNP density already introduced in Subsection 3.1 to model stock returns. The main advantages of this density are the following. First, it nests the GC distribution in (10), and so, it allows a higher flexibility in terms of skewness and excess kurtosis. Second, it is always positive and no restrictions on its parameters are needed to guarantee the positivity of the pdf. Third, it provides easily closed-form expressions for the LPMs

[32] This interval is the same as that used to obtain the dispersion index in Figure 4.

[33] Note that, in the MV setup, both s_p and ek_p in the upper graphs are the same as in the lower ones since OP_0 does not depend on τ.

since we can apply some properties already obtained under the GC density. A possible drawback respecting the GC density is that both skewness and excess kurtosis are not directly the SNP parameters but a function of them. In short, the SNP density of a random variable x is defined as

$$h(x) = \frac{\phi(x)}{v'v}\left(\sum_{i=0}^{p} v_i H_i(x)\right)^2, \tag{43}$$

where $v = (v_0, v_1, ..., v_p)' \in \mathbb{R}^{p+1}$, $\phi(\cdot)$ denotes the pdf of a standard normal variable and $H_i(x)$ are the normalized Hermite polynomials in (11). Since $h(\cdot)$ is homogeneous of degree zero in v, we can either impose $v'v = 1$ or $v_0 = 1$ to solve the scale indeterminacy. By expanding the squared expression in (43), we arrive at Proposition 1 in León et al. (2009). For $p = 2$, we get

$$h(x) = \phi(x)\sum_{k=0}^{4}\gamma_k(v) H_k(x), \tag{44}$$

where

$$\begin{array}{lll} \gamma_0(v) = 1, & \gamma_1(v) = \frac{2v_1\left(v_0+\sqrt{2}v_2\right)}{v'v}, & \gamma_2(v) = \frac{\sqrt{2}\left(v_1^2+2v_2^2+\sqrt{2}v_0v_2\right)}{v'v}, \\ \gamma_3(v) = \frac{2\sqrt{3}v_1v_2}{v'v}, & \gamma_4(v) = \frac{\sqrt{6}v_2^2}{v'v}. & \end{array} \tag{45}$$

We are interested in an affine transformation $z^* = a(v) + b(v)x$ with $g(\cdot)$ as the density of z^* verifying that $E_g[z^*] = 0$, $E_g\left[z^{*2}\right] = 1$. Hence, the location parameter $a(v)$ and the scale one $b(v)$ are obtained as

$$a(v) = -\frac{E_h[x]}{\sqrt{V_h[x]}}, \qquad b(v) = \frac{1}{\sqrt{V_h[x]}}, \tag{46}$$

where $E_h[x]$ and $V_h[x]$ denote, respectively, the mean and variance of x with $h(\cdot)$ in (44) as pdf. Finally, we can express the stock return r as

$$r = \mu + \sigma z^* = \mu + a\sigma + b\sigma x, \tag{47}$$

such that $f(r) = h(x)/(b\sigma)$ is the pdf of r. The mean and variance of r are, respectively, $E_f[r] = \mu$ and $V_f[r] = \sigma^2$. The next Proposition shows the general LPM expression of r with the SNP density.

Proposition 4 *Let r be the stock return in (47) with pdf defined as $f(r) = h(x)/(b\sigma)$ such that $h(x)$ is the SNP density in (44). The lower partial moment $LPM_f(\tau, m)$ of r can be expressed as*

$$LPM_f(\tau, m) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}\kappa_0^{m-j}\kappa_1^j C_j, \tag{48}$$

where $\kappa_0 = \tau - \mu - a\sigma$, $\kappa_1 = b\sigma$, the parameters a and b are defined in (46) and

$$C_j = \sum_{i=0}^{4}\xi_i(v) B_{j+i}, \tag{49}$$

such that $B_l = \int_{-\infty}^{\tau_+} x^l \phi(x) dx$, *with general solution in* (64) *from Appendix B, where* $\tau_+ = \kappa_0/\kappa_1$ *and* $\xi_i(v)$ *is given by the following expressions:*

$$\begin{aligned}
\xi_0(v) &= 1 - \gamma_2(v)/\sqrt{2} + 3\gamma_4(v)/\sqrt{4!}, \\
\xi_1(v) &= \gamma_1(v) - 3\gamma_3(v)/\sqrt{3!}, \\
\xi_2(v) &= \gamma_2(v)/\sqrt{2} - 6\gamma_4(v)/\sqrt{4!}, \\
\xi_3(v) &= \gamma_3(v)/\sqrt{3!}, \\
\xi_4(v) &= \gamma_4(v)/\sqrt{4!},
\end{aligned} \tag{50}$$

where the coefficients $\gamma_k(v)$ *are defined in* (45).

Proof. See Appendix B. ■

Finally, we could also obtain the efficient frontiers by assuming now a different SNP pdf as marginal distribution for each stock return. Thus, a different parameter vector v in (44) for each stock return.

9 Conclusions

We have obtained the closed-form formulas for the partial moments, that is, both upper (UPMs) and lower partial moments (LPMs) when the implied distribution for the stock returns is driven by the Gram-Charlier (GC) density with restrictions on the skewness, s, and excess of kurtosis, ek, parameters to guarantee the probability density function (pdf) is well-defined, see Jondeau and Rockinger (2001). We can express the UPMs as functions of the LPMs. It is verified that the LPMs under GC become linear functions on both s and ek. Because of this, we can easily understand the behaviour of this kind of downside-risk measures with respect to changes in s and ek and hence, the behaviour of the related performance measures (PMs). These PMs are mainly the Farinelli-Tibiletti ratios (FTRs) and the Kappa measures. We also study the PMs based on the Value at Risk and Expected Shortfall (or CVaR) under the GC density.

Our simulation study concludes that the choice of the performance measure (PM) can affect the evaluation of portfolios differently to the Sharpe ratio (SR) because of the sensitivity of the selected PM to the levels of s and ek implied in the portfolio returns. Our results agree with those from Zakamouline (2011). Thus, the selection of PMs becomes relevant in ranking portfolios.

We have also obtained the efficient frontiers (EFs) based on the LPMs as an alternative risk measure taking the standard deviation as benchmark. We have compared the portfolio composition from the EFs under both the mean-LPM and mean-variance settings. We have shown large differences in both compositions through a sensitivity analysis by changing the return threshold. As a result, we can conclude that the choice of the risk measure affects the portfolio composition from the EF. Finally, we have also found some differences among the skewness and excess kurtosis levels implied in the portfolios from these EFs.

Several issues are left for further research. First, we can study the portfolio optimization but using LPMs under alternative dependence structure between financial returns using alternative copula functions with marginal stock return distributions capturing the autocorrelation and GARCH structure. Boubaker and Sghaier (2013) show this kind of analysis but restricted to the mean-variance EFs setup under the presence of long-memory for daily hedge fund returns. Harris and Mazibas (2013) consider alternative optimization frameworks such as the Mean-CVaR, Omega optimization model under both AR(1)-GARCH family and copula modeling for monthly hedge fund returns. Finally, it would be interesting to study the optimal combination of performance measures, see Billio et al. (2012).

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Appendix A. Data and ML estimation

The ten hedge fund (HF) indices used here are available from *www.hedgefundresearch.com*. The same dataset has been used in Mausser et al. (2006) to optimise a portfolio's omega ratio based on a linear programming method. We collect monthly index HF returns for the period from November 1999 to February 2008 with a total of 100 observations. Table 3 reports the constrained maximum likelihood (CML) parameter estimates and the ML standard errors for these monthly return series by assuming that the data come from (9). Thus, $r_{j,t} = \mu_j + \sigma_j z_t$ where $z_t \sim iid\ GC(0, 1, s_j, ek_j)$, $j = 1, \ldots, 10$ represents the hedge fund HF_j and $t = 1, ..., 100$. The names implied by the index symbols (starting with "HFRX") in the first row of the Table are the following: HFRXGL (Global Hedge Fund), HFRXEW (Equal Weighted Strategies), HFRXCA (Convertible Arbitrage), HFRXDS (Distressed Securities), HFRXEH (Equity Hedge), HFRXEMN (Equity Market Neutral), HFRXED (Event-Driven), HFRXM (Macro Index), HFRXMA (Merger Arbitrage) and HFRXRVA (Relative Value Arbitrage). The sample correlation matrix for the ten monthly return series of HFs for the above period is not exhibitd for the sake of brevity. It is shown that all sample correlations are positive. The maximum correlation is between HF_1 and HF_5 (0.887) and the minimum one is between HF_6 and HF_8 (0.019).

[INSERT TABLE 3 AROUND HERE]

Appendix B. Proofs

Proof of Proposition 1

Kendall and Stuart (1977) shows that

$$H_k(z) = \sum_{n=0}^{[k/2]} a_{k,n}\, z^{k-2n}, \tag{51}$$

where $[\cdot]$ rounds its argument to the nearest (smaller) integer and

$$a_{k,n} = \left(-\frac{1}{2}\right)^n \frac{\sqrt{k!}}{(k-2n)!n!}.$$

Taking expectations in (51) with pdf $g(\cdot)$, see equation (10), we obtain

$$E_g[H_k(z)] = \sum_{n=0}^{[k/2]} a_{k,n}\, E_g\left[z^{k-2n}\right]. \tag{52}$$

Given (10) and (12), we get $E_g[H_k(z)] = 0$ for $k \geq 5$ and so, we obtain recursively the expression for $E_g\left[z^k\right]$ in (13). For instance, setting $k = 5$ and plugging $E_g[z] = 0$ and $E_g\left[z^3\right] = s$ into (52), we get $E_g\left[z^5\right] = 10s$. For $k = 6$, we have $E_g\left[z^6\right] = 15 + 15ek$, etc. Finally, regarding the moments of r in (9), with pdf $f(r) = g(z)/\sigma$, we obtain (14) by using the binomial expansion.

Proof of Proposition 2

Let z be the standardised return of r in (9) with pdf $g(z)$ in (10). Its distribution function is

$$\begin{aligned} F_{GC}(z; s, ek) &= \int_{-\infty}^{z} g(x)\,dx = \int_{-\infty}^{z} p(x)\phi(x)\,dx \\ &= \Phi(z) + \frac{s}{\sqrt{3!}}\int_{-\infty}^{z} H_3(x)\,\phi(x)\,dx + \frac{ek}{\sqrt{4!}}\int_{-\infty}^{z} H_4(x)\,\phi(x)\,dx \\ &= \Phi(\tau^*) - \frac{s}{3\sqrt{2!}}H_2(\tau^*)\,\phi(\tau^*) - \frac{ek}{4\sqrt{3!}}H_3(\tau^*)\,\phi(\tau^*). \end{aligned} \tag{53}$$

where the last equality arises from the following relationship (see León et al. (2009) for details):

$$\int_{-\infty}^{a} H_k(x)\phi(x)dx = -\frac{1}{\sqrt{k}}H_{k-1}(a)\phi(a), \quad k \geq 1. \tag{54}$$

Let $f(r) = g(z)/\sigma$ be the pdf of r. Noting that $LPM_f(\tau, 0) = F_{GC}(\tau^*; s, ek)$ where $\tau^* = (\tau - \mu)/\sigma$ completes the proof.

Proof of Corollary 2

We can rewrite (15) as

$$LPM_f(\tau, 0) = B_0 + \frac{s}{\sqrt{3!}}A_{02} + \frac{ek}{\sqrt{4!}}A_{03} \tag{55}$$

where

$$B_0 = \Phi(\tau^*), \quad A_{02} = -\tfrac{1}{\sqrt{3}} H_2(\tau^*)\phi(\tau^*), \quad A_{03} = -\tfrac{1}{\sqrt{4}} H_3(\tau^*)\phi(\tau^*), \tag{56}$$

It holds that $B_0 > 0 \quad \forall \tau^*$, $A_{02} > 0$ iff $|\tau^*| < 1$ and $A_{03} > 0$ iff $\tau^* \in \left(-\infty, -\sqrt{3}\right) \cup \left(0, \sqrt{3}\right)$. Moreover, using the relations (a) $d\phi\,(x)\,/dx = -x\phi\,(x)$ and (b) $dH_k\,(x)\,/dx = \sqrt{k} H_{k-1}\,(x)$, we get

$$\begin{aligned}
\frac{\partial A_{02}}{\partial \tau^*} &> 0 \Leftrightarrow \tau^* \in \left(-\sqrt{3}, 0\right) \cup \left(\sqrt{3}, +\infty\right), \\
\frac{\partial A_{03}}{\partial \tau^*} &> 0 \Leftrightarrow \tau^* \in \left(-\infty, -\tau_1^*\right) \cup \left(-\tau_2^*, \tau_2^*\right) \cup \left(\tau_1^*, +\infty\right),
\end{aligned}$$

where $\tau_1^* = \sqrt{3 + \sqrt{6}}$ and $\tau_2^* = \sqrt{3 - \sqrt{6}}$. Hence,

$$\begin{aligned}
\frac{\partial LPM_f(\tau, 0)}{\partial s} &> 0 \Leftrightarrow |\tau^*| < 1, \\
\frac{\partial LPM_f(\tau, 0)}{\partial ek} &> 0 \Leftrightarrow \tau^* \in \left(-\infty, -\sqrt{3}\right) \cup \left(0, \sqrt{3}\right),
\end{aligned}$$

and $\partial LPM_f(\tau, 0)/\partial \tau^* > 0$ if $\tau^* \in (-\tau_2^*, 0) \cup (\tau_1^*, +\infty)$. Finally, the signs of the partial derivatives of $LPM_f(\tau, 0)$ with respect to τ, μ and σ are obtained by applying the chain rule and using that $\partial \tau^*/\partial \tau > 0$, $\partial \tau^*/\partial \mu < 0$ and $\partial \tau^*/\partial \sigma > 0$ iff $\mu > \tau$.

Proof of Proposition 3

Let f and g denote, respectively, the pdfs for r and z in (9). So, it is verified that $f\,(r) = g\,(z)\,/\sigma$ where $g\,(\cdot)$ denotes the GC pdf and $z = (r - \mu)\,/\sigma$. Then, we can rewrite (1) as

$$LPM_f(\tau, m) = \int_{-\infty}^{\tau} (\tau - r)^m f(r)\, dr = \int_{-\infty}^{\tau^*} (\tau - \mu - \sigma z)^m g(z)\, dz \tag{57}$$

where $\tau^* = (\tau - \mu)\,/\sigma$. If we apply the binomial expansion on $(\tau - \mu - \sigma z)^m$ in (57), then

$$LPM_f(\tau, m) = \sum\nolimits_{k=0}^{m} (-1)^k \binom{m}{k} (\tau - \mu)^{m-k} \sigma^k \boldsymbol{I}_k, \tag{58}$$

where $\boldsymbol{I}_k = E_g\left[z^k \,|z < \tau^*\right]$ denotes the conditional expected value. Thus, for $k \geq 1$ we obtain

$$\boldsymbol{I}_k = \int_{-\infty}^{\tau^*} z^k g(z)\, dz = B_k + \frac{s}{\sqrt{3!}} A_{k2} + \frac{ek}{\sqrt{4!}} A_{k3}, \tag{59}$$

where

$$B_k = \int_{-\infty}^{\tau^*} z^k \phi(z) dz, \quad A_{k2} = \int_{-\infty}^{\tau^*} z^k H_3(z)\phi(z) dz, \quad A_{k3} = \int_{-\infty}^{\tau^*} z^k H_4(z)\phi(z) dz. \tag{60}$$

Both A_{k2} and A_{k3} can be expressed, by using (54), as

$$A_{k2} = \tfrac{1}{\sqrt{3!}}(B_{k+3} - 3B_{k+1}), \quad A_{k3} = \tfrac{1}{\sqrt{4!}}(B_{k+4} - 6B_{k+2} + 3B_k). \tag{61}$$

Note that $\boldsymbol{I}_0$ is just the equation of $LPM_f(\tau, 0)$ in (55) with B_0, A_{02}, and A_{03} given in (56). By plugging (59) into (58), we get

$$LPM_f(\tau, m) = LPM_n(\tau, m) + \frac{s}{\sqrt{3!}}\theta_{2,m} + \frac{ek}{\sqrt{4!}}\theta_{3,m},$$

where

$$\theta_{j,m} = \sum\nolimits_{k=0}^{m} (-1)^k \binom{m}{k} (\tau - \mu)^{m-k} \sigma^k A_{kj},$$

where $LPM_n(\tau, m) = E_n[(\tau - r)_+^m]$ such that $n(\cdot)$ is the pdf of the normal distribution with μ and σ, respectively, as the mean and standard deviation. Thus,

$$LPM_n(\tau, m) = \int_{-\infty}^{\tau^*} (\tau - \mu - \sigma z)^m \phi(z)\, dz = \sum\nolimits_{k=0}^{m} (-1)^k \binom{m}{k} (\tau - \mu)^{m-k} \sigma^k B_k. \quad (62)$$

Finally, to get the expression for B_k in (60), we need previously the following result:

$$z^k = \sum_{n=0}^{[k/2]} c_{k,n} H_{k-2n}(z), \quad (63)$$

where $c_{k,n} \in \mathbb{R}$ and $H_i(\cdot)$ is a Hermite polynomial. Note that (63), that is available upon request, is just the inversion formula of (51). If we take (54) and (63), then

$$\begin{aligned} B_k &= \int_{-\infty}^{\tau^*} z^k \phi(z) dz = \sum_{n=0}^{[k/2]} c_{k,n} \int_{-\infty}^{\tau^*} H_{k-2n}(z)\, \phi(z) dz \\ &= \sum_{n=0}^{[k/2]} c_{k,n} \left\{ \Phi(\tau^*) \mathbf{1}_{(k-2n=0)} + \frac{1}{\sqrt{k-2n}} H_{k-2n-1}(\tau^*)\, \phi(\tau^*)\, \mathbf{1}_{(k-2n \geq 1)} \right\}, \end{aligned} \quad (64)$$

where $\mathbf{1}_{(\cdot)}$ is the usual indicator function.

Proof of Corollary 4

The expressions of $\theta_{j,m}$ in (21) and $LPM_n(\tau, m)$ in (22) are easily obtained for $m \leq 3$ by using, respectively, the equations (18) and (17) from Proposition 3. Since we need to obtain A_{k2} and A_{k3} in (61) for $k = 0, 1, 2, 3, 4$, then we must previously get B_j in (60) for $j = 1, ..., 7$. By applying the condition (54),

$$\begin{aligned} B_1 &= -\phi(\tau^*), \\ B_2 &= -H_1(\tau^*)\, \phi(\tau^*) + \Phi(\tau^*), \\ B_3 &= -\sqrt{2} H_2(\tau^*)\, \phi(\tau^*) - 3\phi(\tau^*), \\ B_4 &= -\sqrt{3!} H_3(\tau^*)\, \phi(\tau^*) - 6 H_1(\tau^*)\, \phi(\tau^*) + 3\Phi(\tau^*), \\ B_5 &= -\sqrt{4!} H_4(\tau^*)\, \phi(\tau^*) - 10\sqrt{2} H_2(\tau^*)\, \phi(\tau^*) - 15\phi(\tau^*), \\ B_6 &= -\sqrt{5!} H_5(\tau^*)\, \phi(\tau^*) - 15\sqrt{3!} H_3(\tau^*)\, \phi(\tau^*) - 45 H_1(\tau^*)\, \phi(\tau^*) + 15\Phi(\tau^*), \\ B_7 &= -\sqrt{6!} H_6(\tau^*)\, \phi(\tau^*) - 21\sqrt{4!} H_4(\tau^*)\, \phi(\tau^*) - 105\sqrt{2} H_2(\tau^*)\, \phi(\tau^*) - 105\phi(\tau^*). \end{aligned}$$

Note that the above expressions agree with the general formula for B_j in (64).

Proof of Proposition 4

Let h and f denote, respectively, the pdfs for x and r in (44) and (47). So, it is verified that $f(r) = h(x)/b\sigma$. Then, we can rewrite (1) as

$$LPM_f(\tau,m) = \int_{-\infty}^{\tau} (\tau - r)^m f(r)\,dr = \int_{-\infty}^{\tau_+} (\kappa_0 - \kappa_1 x)^m h(x)\,dx, \tag{65}$$

where $\kappa_0 = \tau - \mu - a\sigma, \kappa_1 = b\sigma > 0$ and $\tau_+ = \kappa_0/\kappa_1$. The expressions of a and b in (46) can be obtained from the first two unconditional moments of x, denoted as $\mu'_x(1)$ and $\mu'_x(2)$, from Lemma 1 in León et al. (2009). Thus, $E_h[x] = \mu'_x(1)$ is just $\gamma_1(v)$ in equation (45) and

$$\mu'_x(2) = \frac{2\left(v_1^2 + 2v_2^2 + \sqrt{2}v_2 v_0\right)}{v'v} + 1,$$

then $V_h[x] = \mu'_x(2) - \mu'_x(1)^2$. If we apply the binomial expansion on $(\kappa_0 - \kappa_1 x)^m$ in (65) and consider (44), we have

$$\begin{aligned} LPM_f(\tau,m) &= \int_{-\infty}^{\tau_+} \left[\sum\nolimits_{j=0}^{m} (-1)^j \binom{m}{j} \kappa_0^{m-j}\kappa_1^j x^j\right] h(x)\,dx \\ &= \sum\nolimits_{j=0}^{m} (-1)^j \binom{m}{j} \kappa_0^{m-j}\kappa_1^j \left[B_j + \sum\nolimits_{k=1}^{4} \gamma_k(v)\, G_{jk}\right] \end{aligned} \tag{66}$$

where $B_j = \int_{-\infty}^{\tau_+} x^j \phi(x)dx$, with general solution in (64), and $G_{jk} = \int_{-\infty}^{\tau_+} x^j H_k(x)\phi(x)\,dx$. If we consider (51), then the expressions of G_{jk}, for $0 \le j \le m$ and $1 \le k \le 4$, are:

$$\begin{aligned} &G_{j1} = B_{j+1}, && G_{j2} = \tfrac{1}{\sqrt{2}}\left(B_{j+2} - B_j\right), \\ &G_{j3} = \tfrac{1}{\sqrt{3!}}(B_{j+3} - 3B_{j+1}), && G_{j4} = \tfrac{1}{\sqrt{4!}}(B_{j+4} - 6B_{j+2} + 3B_j). \end{aligned} \tag{67}$$

Note that B_l in the above expressions are evaluated at point τ_+. By plugging (67) into (66), we finally arrive at the expressions (48) – (50) in Proposition 4.

Appendix of Tables

Table 1: Sensitivity analysis for the Kappa measures under GC distribution. The effects of skewness (s) and excess kurtosis (ek)

	Panel A: Omega			**Panel B: Sortino**			**Panel C: Kappa 3**		
ek	$s=-0.7$	$s=0$	$s=0.4$	$s=-0.7$	$s=0$	$s=0.4$	$s=-0.7$	$s=0$	$s=0.4$
0.8996	0.5801	0.5959	0.6054	0.2622	0.2923	0.3150	0.1825	0.2070	0.2292
1.2048	0.5893	0.6057	0.6154	0.2617	0.2916	0.3141	0.1806	0.2040	0.2246
1.5100	0.5988	0.6157	0.6258	0.2612	0.2909	0.3133	0.1788	0.2011	0.2204
1.8153	0.6086	0.6261	0.6365	0.2607	0.2903	0.3124	0.1771	0.1983	0.2165
2.1205	0.6187	0.6368	0.6476	0.2602	0.2896	0.3116	0.1754	0.1957	0.2128
2.4257	0.6292	0.6479	0.6591	0.2598	0.2889	0.3107	0.1738	0.1932	0.2094
2.7309	0.6400	0.6594	0.6710	0.2593	0.2882	0.3099	0.1723	0.1909	0.2062
3.0361	0.6513	0.6713	0.6833	0.2588	0.2876	0.3091	0.1708	0.1887	0.2031
3.3414	0.6629	0.6836	0.6961	0.2583	0.2869	0.3083	0.1693	0.1865	0.2003
3.6466	0.6749	0.6964	0.7094	0.2578	0.2863	0.3074	0.1680	0.1845	0.1976

This table exhibits the values of the closed-form formulas for the Kappa measures (Omega-Sharpe, Sortino and Kappa 3) by using the LPM expressions from Corollary 4 for monthly returns. All portfolios in the table have $\mu = 0.86\%$ and $\sigma = 2.61\%$ but different values for s and ek such that $(ek, s) \in \Gamma$. The return threshold is $r_f = 0.39\%$. The Sharpe ratio is constant over this table, SR=0.1796.

Table 2: Some results from regression analyses by using simulated monthly returns from GC distribution

log (Perf. Measure)	Sharpe ratio interval	beta log(Sharpe)	beta Sk	beta Ek	Rank corre	R2	R20	Chow Test (p-value)
Omega-Sharpe	$[1\%, 16.14\%]$	1.1888	0.0286	0.0536	0.9509	0.9996	0.9000	10484.4060
	$(16.14\%, 22.3\%]$	1.2855	0.0458	0.0570	0.8965	0.9993	0.7941	(0.0000)
Sortino	$[1\%, 16.14\%]$	1.0993	0.1672	-0.0046	0.8728	0.9984	0.7541	1012.2615
	$(16.14\%, 22.3\%]$	1.1480	0.1820	-0.0077	0.7489	0.9966	0.5626	(0.0000)
Kappa 3	$[1\%, 16.14\%]$	1.0685	0.1974	-0.0417	0.8059	0.9916	0.6439	95.2884
	$(16.14\%, 22.3\%]$	1.1017	0.2060	-0.0457	0.6583	0.9853	0.4430	(0.0000)
Upside Potential	$[1\%, 16.14\%]$	0.2665	0.1473	-0.0421	0.3738	0.9938	0.1559	2166.0951
	$(16.14\%, 22.3\%]$	0.3895	0.1552	-0.0413	0.3588	0.9925	0.1443	(0.0000)
RVaR (1%)	$[1\%, 16.14\%]$	1.0561	0.2766	-0.1025	0.6272	0.8639	0.3630	4.0309
	$(16.14\%, 22.3\%]$	1.0844	0.2849	-0.1058	0.5013	0.8291	0.2058	(0.0752)
RVaR (5%)	$[1\%, 16, 14\%]$	1.1098	0.3826	0.0382	0.5453	0.8648	0.3142	11.1504
	$(16.14\%, 22.3\%]$	1.1736	0.4028	0.0412	0.4018	0.8373	0.1751	(0.0000)
RCVaR (1%)	$[1\%, 16, 14\%]$	1.0458	0.1643	-0.0883	0.7544	0.8883	0.5311	3.5877
	$(16.14\%, 22.3\%]$	1.0692	0.1684	-0.0906	0.6305	0.8418	0.3396	(0.1050)
RCVaR (5%)	$[1\%, 16, 14\%]$	1.0693	0.3272	-0.0645	0.6125	0.9599	0.3902	16.3899
	$(16.14\%, 22.3\%]$	1.1062	0.3387	-0.0668	0.4590	0.9488	0.2269	(0.0000)

This table exhibits the results of running by OLS the model given by (29). All regressions contain the same explanatory variables: the constant, the skewness (s), the excess of kurtosis (ek) and the log of the Sharpe ratio (log(SR)). The beta estimates corresponding to s, ek and $log(SR)$ are shown, respectively, in columns 3, 4, and 5. Consider a certain dependent variable (performance measure) in column 1, then the regression is run twice. The total sample size, N=10,000, is divided in two parts having one regression per subsample. The criterion followed to split N depends on the size of the Sharpe ratio (SR) of portfolio $i \in \{1, \cdots, N\}$, denoted as SR_i. The column 2 is labeled as the SR interval, with elements J_1 and J_2. If $SR_i \in J_1 \equiv [1\%, 16.14\%]$, then portfolio i belongs to subsample 1, otherwise it belongs to subsample 2 such that $SR_i \in J_2 \equiv (16.14\%, 22.3\%]$. This table shows a total of 20 different regressions. The column 6 represents the Spearman's rank correlation between any performance measure from column 1 and the SR for each subsample. Columns 7 and 8, labeled as R2 and R20, correspond, respectively, to the R^2 statistic of (29) and the log of (27). The last column represents both the value and the p-value (in parenthesis) for the Chow test with null hypothesis of no structural break (one regression) against the alternative one of structural break (two regressions). All portfolios, each defined by the vector $(\mu_i, \sigma_i, s_i, ek_i)$ where $i = 1, \cdots, N$, are simulated according to the procedure in Subsection 6.1. Note that we repeat the above process a total of 100 times, so any value or estimation of the table (from columns 3 to 9) is really the *mean over 100 estimates*. The beta estimates (columns 3 to 5) are all statistically significant at the 1% level for all the alternative performance measures.

Table 3: CML estimation of monthly return series from HFRX Tradeable Indices

	HFRXGL	HFRXEW	HFRXCA	HFRXDS	HFRXEH	HFRXEMN	HFRXED	HFRXM	HFRXMA	HFRXRVA
Mean (%)	0.702	0.605	0.533	0.671	0.955	0.229	0.699	0.859	0.491	0.554
SE	0.162	0.095	0.138	0.149	0.236	0.098	0.145	0.266	0.100	0.098
Std. Dev. (%)	1.656	0.963	1.199	1.445	2.575	0.972	1.518	2.613	1.011	0.998
SE	0.154	0.080	0.123	0.126	0.391	0.082	0.121	0.227	0.085	0.092
Skewness	0.360	-0.275	-0.798	0.408	0.987	-0.280	-0.571	0.401	-0.594	-0.233
SE	0.462	0.278	0.533	0.272	0.416	0.274	0.249	0.316	0.261	0.351
Exc. Kurt.	2.526	0.954	1.121	1.510	3.038	1.219	1.254	1.519	1.508	1.700
SE	0.662	0.689	1.096	0.689	1.320	0.660	0.628	0.599	0.646	0.708

This table shows the constrained maximum likelihood (CML) estimates for the different monthly hedge fund (HF) return series according to (9), that is, the standardised stock returns are driven by the Gram-Charlier (GC) density. The period goes from November 1999 to February 2008, i.e. a total sample size of 100 observations per each data series. The CML estimates for the mean and standard deviation parameters are given in percentages. The standard errors (SE) are calculated by using the Quasi-Maximum likelihood (QML) method.

Appendix of Figures

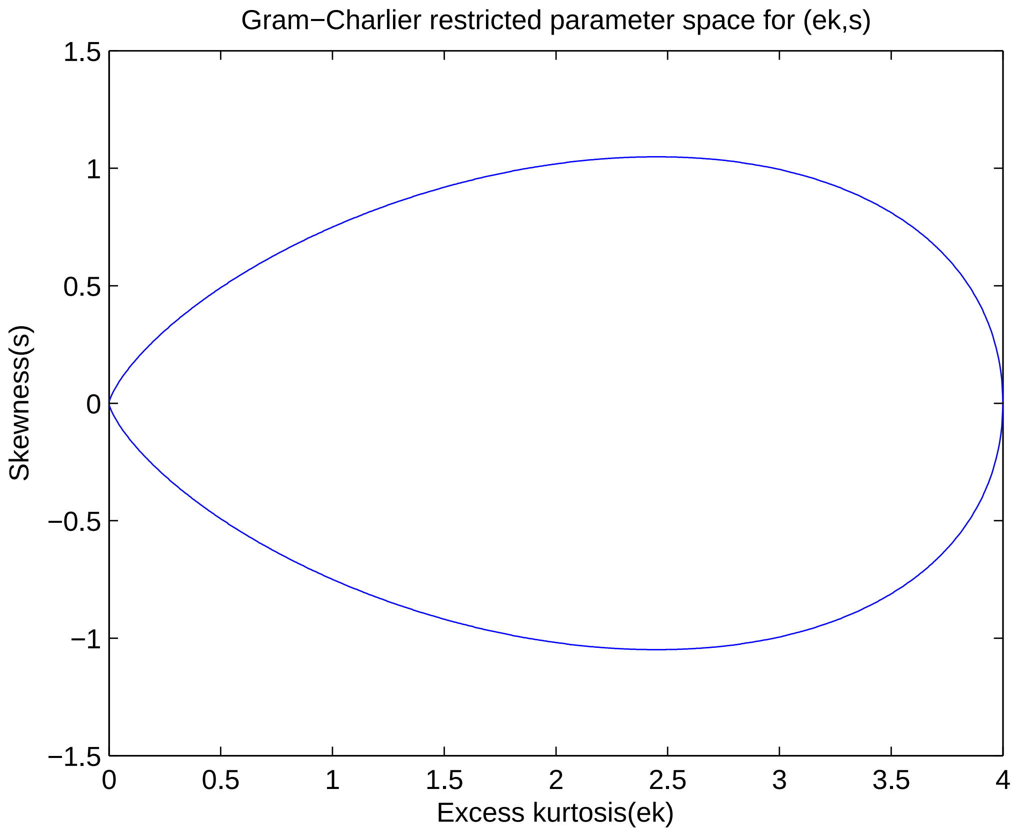

Figure 1: Space containing for stock returns the points (ek, s) with the "excess kurtosis" level, ek, in the x-axis and the "skewness" level, s, in the y-axis. This space is limited by a frontier (envelope) verifying that the Gram-Charlier (GC) density is well defined for the points on and inside the envelope. Thus, the GC density will be restricted to this space for (ek, s). Note that $ek \in [0, 4]$ while $s \in [-1.0493, 1.0493]$. The range of s depends on the level of ek. See Jondeau and Rockinger (2001) for more details about how to obtain this frontier.

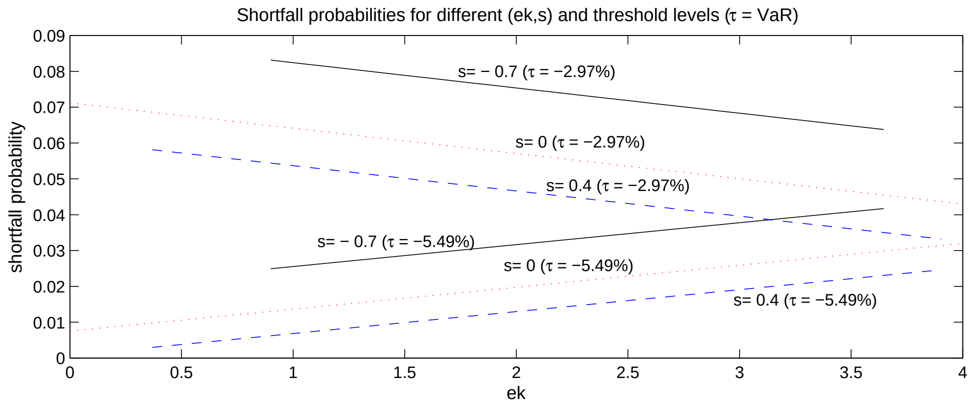

Figure 2: Each line contains the points $(ek, LPM(\tau, 0))$ where ek is the "excess kurtosis" level of the monthly stock return r and $LPM(\tau, 0)$ is the "shortfall probability" in (15). Thus, r is given in (9) with the standardised stock return, z, distributed under the GC density in (10). Note that across each line the skewness level, s, is fixed and also the return threshold, τ. Specifically, τ is just the 5% (1%)-quantile under the GC density. Thus, τ is the VaR and the skewness levels are $s = -0.7, 0, 0.4$. Note that the length of each line is different since each point must belong to the set exhibited in Figure 1. The mean and standard deviation for any monthly stock return are $\mu = 0.86\%$ and $\sigma = 2.61\%$. For the behaviour of these lines, see Corollary 2.

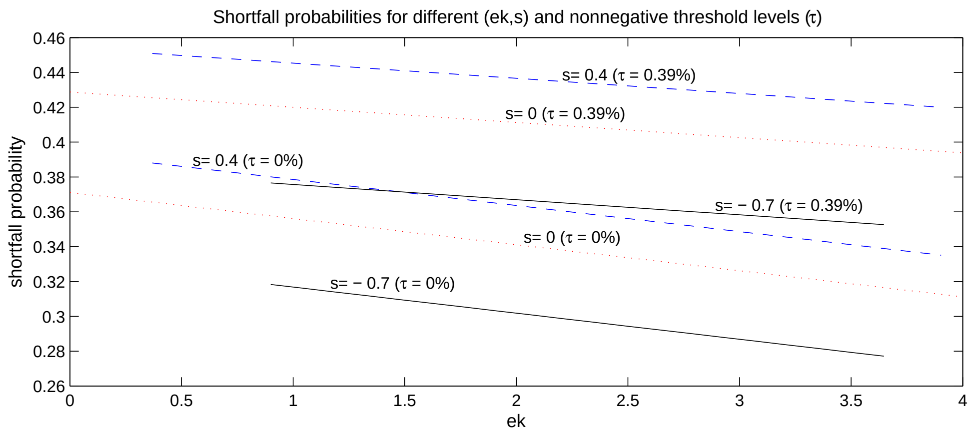

Figure 3: This figure is the same as Figure 2 but here the return thresholds are non-negative. These values are $\tau = 0\%$ and $\tau = 0.39\%$, where the last one is just the monthly risk-free interest rate on 10-year US Treasury bonds (i.e., the sample mean from November 1998 to December 2008). Each line contains the points $(ek, LPM(\tau, 0))$ where ek is the "excess kurtosis" level of the monthly stock return r, and $LPM(\tau, 0)$ is the "shortfall probability" in (15). Thus, r is given in (9) with the standardised stock return, z, distributed under the GC density in (10). By setting a level for τ, we show three lines corresponding to the skewness levels are $s = -0.7, 0, 0.4$. Note again that the length of each line is different since each point (ek, s) must belong to the set exhibited in Figure 1. The mean and standard deviation for any monthly stock return are $\mu = 0.86\%$ and $\sigma = 2.61\%$. For the behaviour of these lines, see Corollary 2.

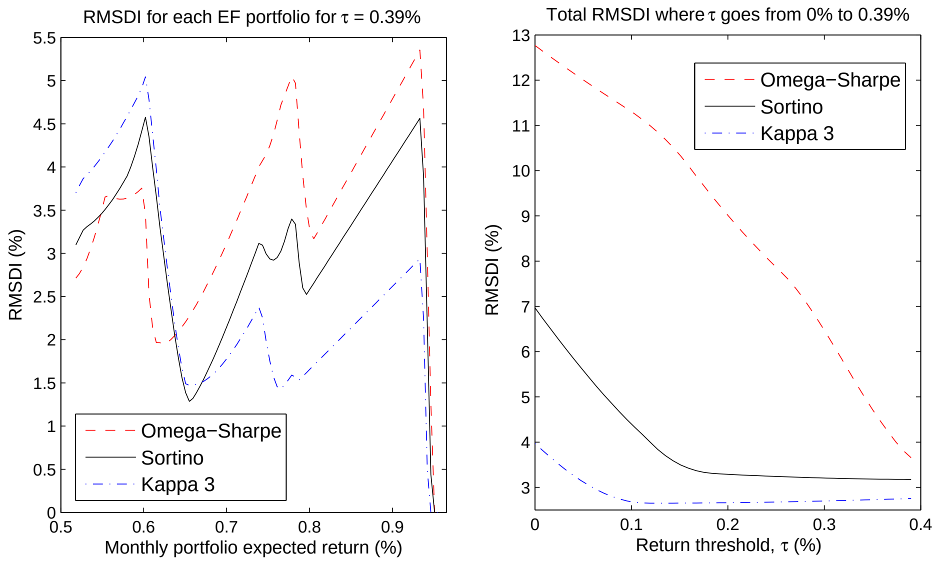

Figure 4: This figure contains two graphs. The *left-hand graph* shows one hundred efficient frontier (EF) portfolios by solving the optimization program driven by (30) to (33) for each $m = 1, 2, 3$ and also, the case under the MV approach with sample covariance matrix obtained from the hedge fund (HF) data in Appendix A. Other parameters such as $(\mu_j, \sigma_j, s_j, ek_j)$ for each HF return series $(j = 1, \cdots, 10)$ are exhibited in Table 3 in the same Appendix. The return threshold is $\tau = 0.39\%$ (monthly risk-free interest rate). The x-axis shows a grid for the mean of monthly portfolio returns going from 0.5182% to 0.9553%, see Subsection 7.2 for more details. The y-axis displays one graph for a different m containing the RMSDI, given by (36) with $N_p = 1$, in percentage for each EF portfolio. The *right-hand graph* exhibits three lines but now the x-axis shows a grid for the monthly return thresholds such that $\tau \in [0\%, 0.39\%]$. Meanwhile, the y-axis contains the RMSDI with $N_p = 100$ in (36). Thus, the case of $\tau = 0.39\%$ in the left-hand graph corresponds to the last 3 points further on the right whatever the selected line (depending on m).

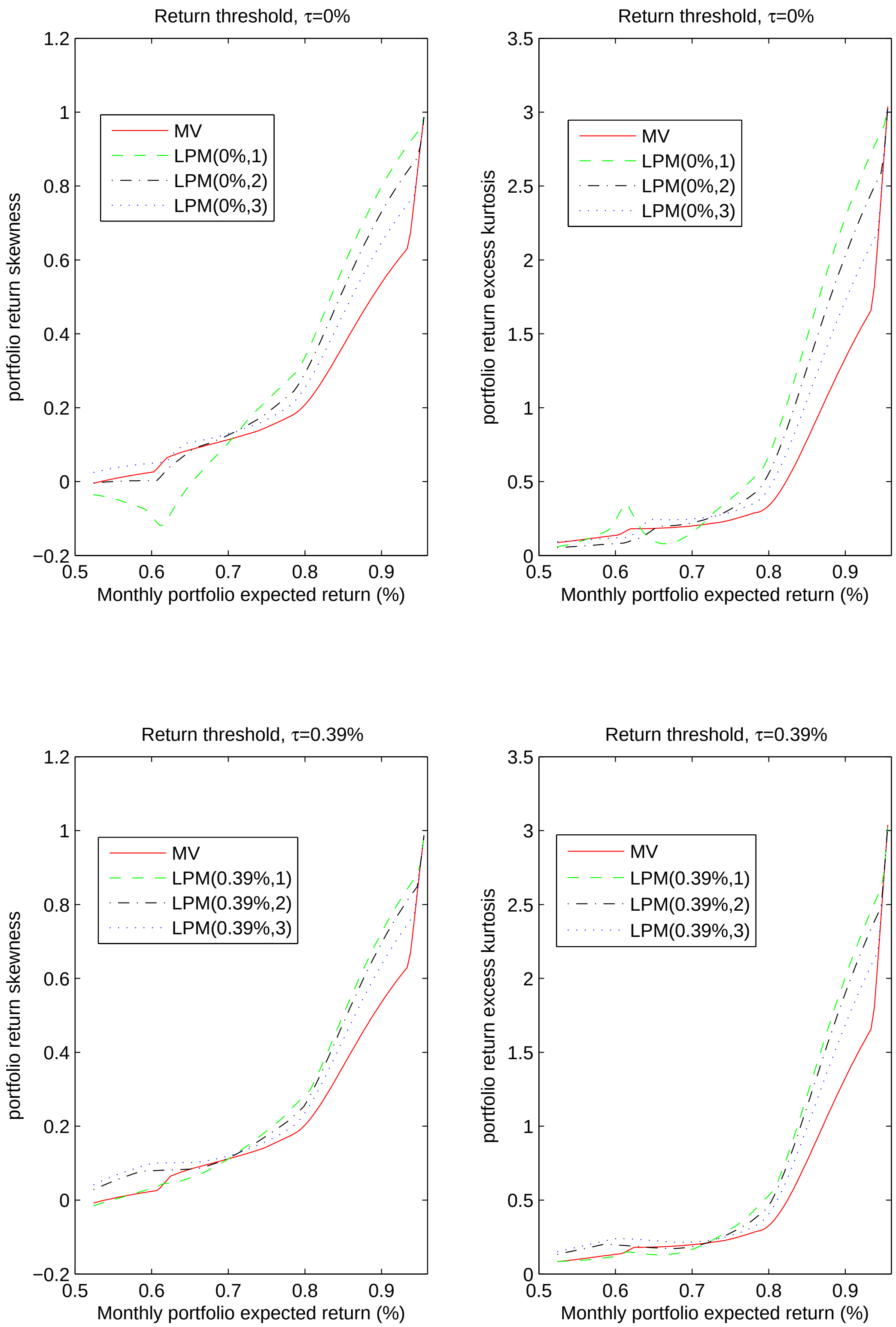

Figure 5: This figure provides the skewness (s_p) and kurtosis excess (ek_p) for the four series of EF portfolios as a function of the monthly portfolio expected return. The grid is the same as in Figure 4. The values of s_p and ek_p have been obtained by using, respectively, (41) and (42). Each graph includes four lines that show, respectively, the results under the MV framework and the LPM measures $LPM(\tau, m)$, $m = 1, 2, 3$, where τ denotes the monthly threshold return. The *upper* and *lower* graphs include the results for, respectively, $\tau = 0\%$ and $\tau = 0.39\%$. The co-skewness and co-kurtosis matrices, M_3 and M_4, are given in closed-form assuming a Gaussian copula with sample correlation matrix from hedge fund (HF) data in Appendix A. The parameters of the GC pdf for the marginal distribution of each HF monthly return are given in Table 3 in Appendix A.