Inference for non-linear, non-Gaussian state-space models

Universidad Complutense de Madrid
Seminarios de Investigación del Departamento de Economía Cuantitativa

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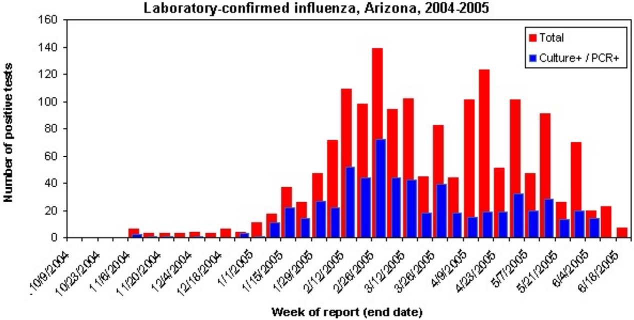
Example of dynamic systems: financial crisis

Another dynamic system: Infectious Disease Epidemics



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8. Modeling Disease Dynamics: Cholera as a Case Study

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Edward L. Ionides[1], Carles Bretó[2] and Aaron A. King[3]

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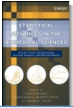
Book Title



Statistical Advances in the Biomedical Sciences: Clinical Trials, Epidemiology, Survival Analysis, and Bioinformatics

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On idiosyncratic stochasticity of financial leverage effects

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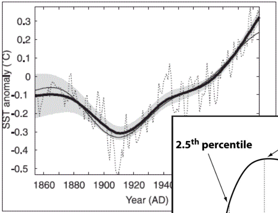
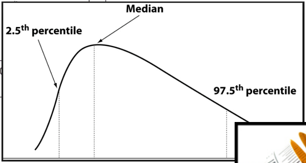
ABSTRACT

We model leverage as stochastic but independent of return shocks and of volatility and perform likelihood-based inference via the recently developed iterated filtering algorithm using S&P500 data, contributing new evidence to the still slim empirical support for random leverage variation.

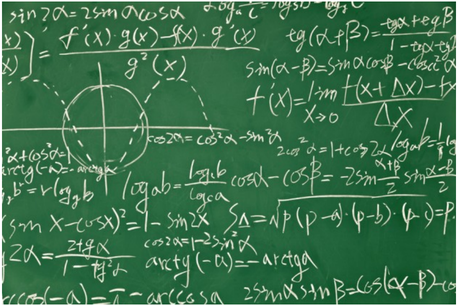
1. Introduction

Problem: realistic models are likely non-linear, non-Gaussian & partially observed

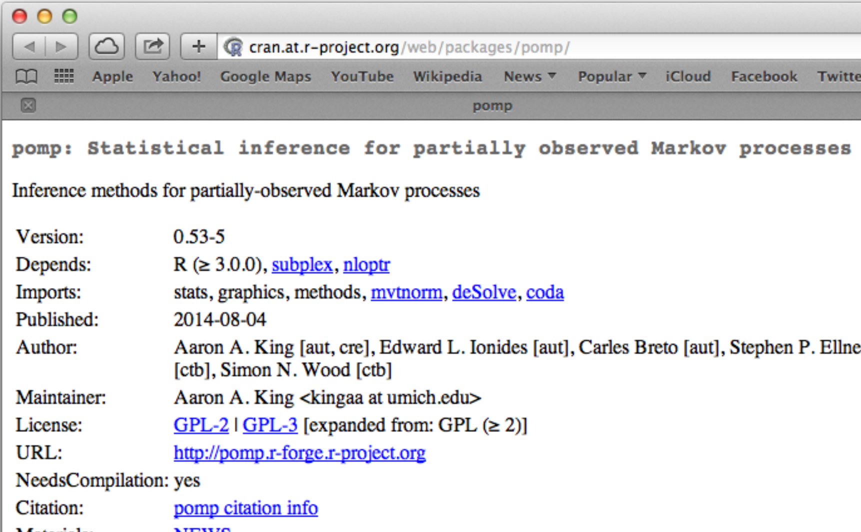




Hence: their statistical analysis is complicated & slows down scientific exploration

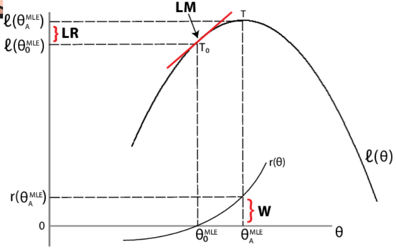


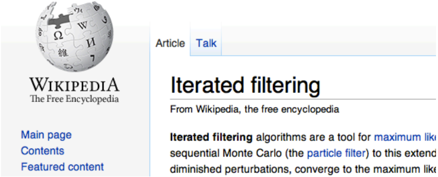
Contribution (I): R package POMP (Frequentist particle filter, PMCMC, ABC, etc)



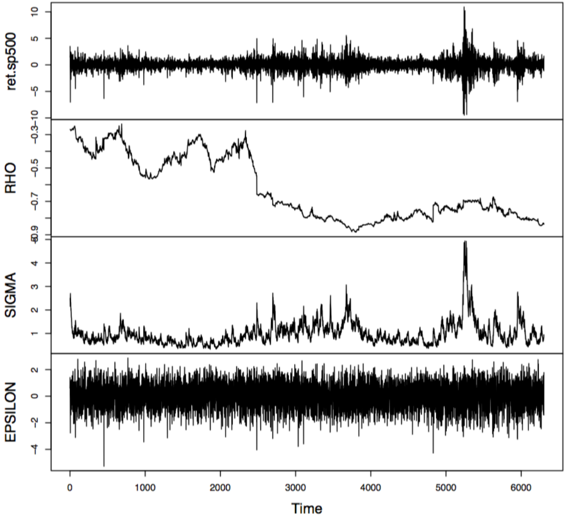
Contribution (II): iterated filtering algorithm

Plug-and-play likelihood-based inference on POMPs



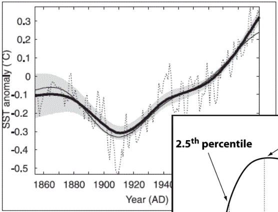


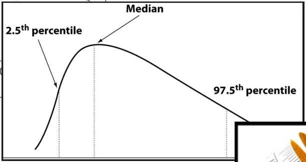
Contribution (III): Example from financial econometrics
Stochastic volatility with stochastic leverage



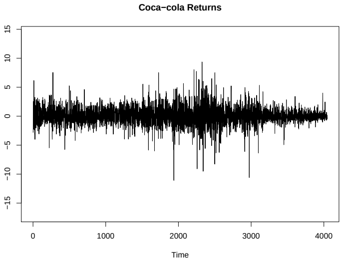
Take-home message: straightforward, likelihood-based inference is possible for general dynamic systems

Problem: realistic models are likely non-linear, non-Gaussian & partially observed



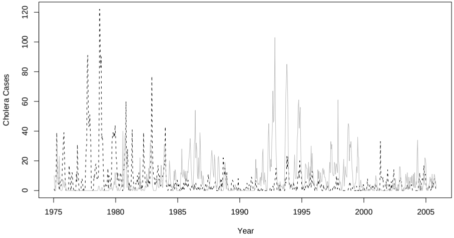


State-space: unobservable variables/mechanisms



- Stochastic volatility:

$$
\begin{aligned}
y_t &= \sigma_t \epsilon_t \\
\sigma_t^2 &= \exp(h_t) \\
h_t &= \mu(1-\phi) + \phi h_{t-1} + \eta_t^*
\end{aligned}
$$



- SIR-type compartment models:

$$
\begin{aligned}
C_t &= \rho I_t + \sigma \epsilon_t \\
I_t &= \exp(\tilde{I}_t) = (\beta S_{t-1}) I_{t-1}^{\alpha} \eta_t^* \\
\tilde{I}_t &= (\tilde{\beta} + \tilde{S}_{t-1}) + \alpha \tilde{I}_{t-1} + \tilde{\eta}_t^*
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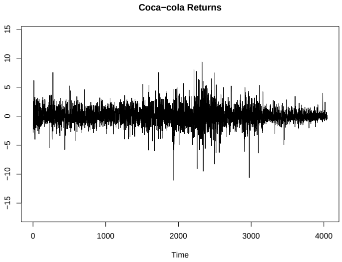
State-space: unobservable variables/mechanisms



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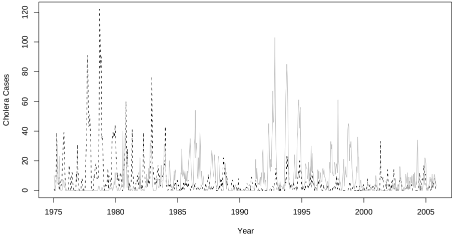

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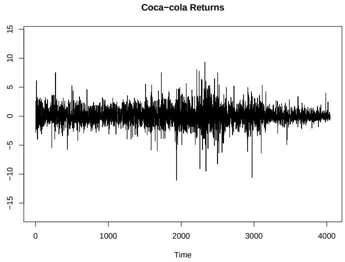

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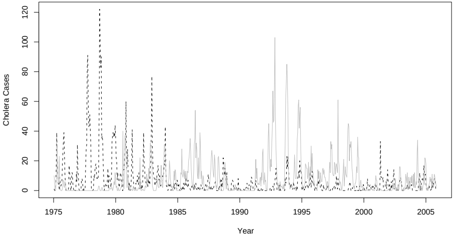

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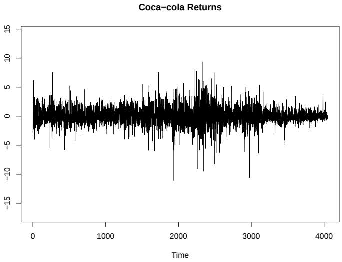
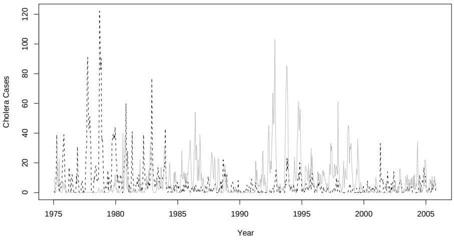
Linearity and Gaussianity: unusual but convenient



- Stochastic volatility:

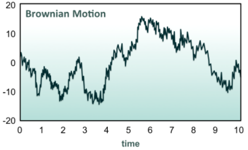
$$\begin{aligned} \log(y_t) &= \log(\sigma_t) + \log(\epsilon_t) \\ \sigma_t^2 &= \exp(h_t) \\ h_t &= \mu(1-\phi) + \phi h_{t-1} + \eta_t^* \end{aligned}$$

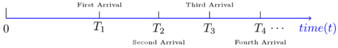


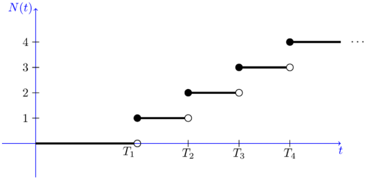
- SIR-type compartment models:

$$\begin{aligned} C_t &\sim \text{Pois}(\rho I_t) \\ I_t &\sim \text{bin}\left(S_{t-1}, e^{-\beta I_{t-1}^{\alpha}}\right) - \\ &- \text{bin}\left(I_{t-1}, e^{-\gamma}\right) \end{aligned}$$

Linearity and Gaussianity: unusual but convenient







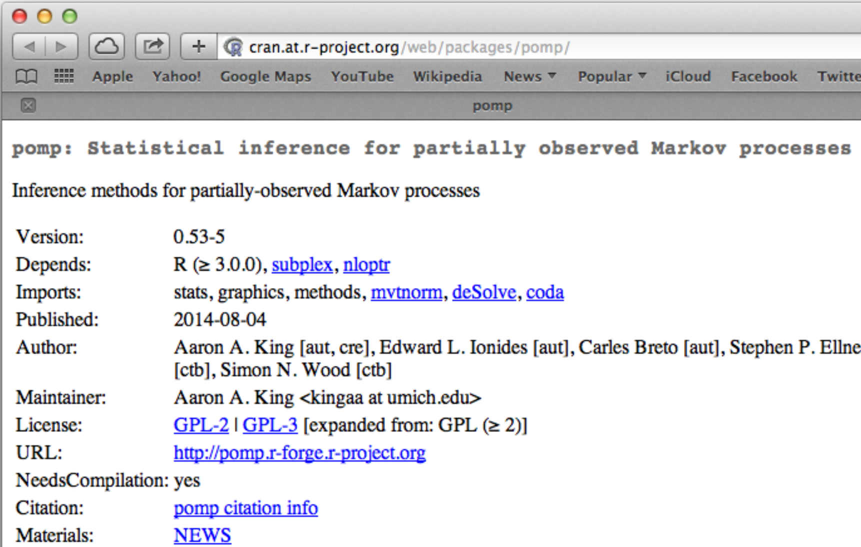
- Stochastic volatility:

$$
\begin{aligned}
dy^*(t) &= \mu + \beta\sigma^2(t)dt + \sigma(t)dB(t) \\
d\sigma^2(t) &= -\lambda\sigma^2(t)dt + dz(\lambda t)
\end{aligned}
$$

- SIR-type compartment models:

$$
\begin{aligned}
P(S \to S) &= 1 - \beta s i^\alpha h + o(h) \\
P(S \to I) &= \beta s i^\alpha h + o(h)
\end{aligned}
$$

Contribution (I): R package POMP (Frequentist particle filter, PMCMC, ABC, etc)



POMP object: notation

- Formal:
  - Markov unobservables: $\left(X_1(t), \ldots, X_{K_X}(t)\right)$
  - Unobservable time: either continuous $t \in \mathbb{R}_0^+$ or discrete $t \in \mathbb{N}_0$
  - Conditionally independent measurements: $\left(Y_1(t_n), \ldots, Y_{K_Y}(t_n)\right)$
  - Measurement time: discrete $t_1, \ldots, t_N$
  - Parameters θ
- Algorithmic (POMP code):
  - rprocess: a draw from $f_{\boldsymbol{X}(t_n)|\boldsymbol{X}(t_{n-1})}(\boldsymbol{x}(t_n)|\boldsymbol{x}(t_{n-1}), \boldsymbol{\theta})$
  - dprocess: evaluate $f_{\boldsymbol{X}(t_n)|\boldsymbol{X}(t_{n-1})}(\boldsymbol{x}(t_n)|\boldsymbol{x}(t_{n-1}), \boldsymbol{\theta})$
  - rmeasure: a draw from $f_{\boldsymbol{Y}(t_n)|\boldsymbol{X}(t_n)}(\boldsymbol{y}(t_n)|\boldsymbol{x}(t_n), \boldsymbol{\theta})$
  - dmeasure: evaluate $f_{\boldsymbol{Y}(t_n)|\boldsymbol{X}(t_n)}(\boldsymbol{y}(t_n)|\boldsymbol{x}(t_n), \boldsymbol{\theta})$
- Nuisances: Initial value parameters

POMP inference: difficult parameter estimation

- Alternative model-based inference approaches:
  - MoM (need to check moments, not full information)
  - MQLE (need to check Gaussian approx.)
  - Bayesian MCMC (Jaquier et al., 1994) (need to check priors)
  - EMM (Gallant & Tauchen, 1996) (need to check auxiliary model)
  - MCL (Sandmann and Koopman, 1998) (need to check approx.)
  - EIS (Lesenfeld and Richard, 2003) (need to check Imp. Sampler)
- Reasonable estimates on average across different samples (error compensation)
- We have only one (long) sample: importance of efficiency
- What about “plug-and-play” modelling?

POMP: Plug-and-play inference

- Plug-and-play algorithm:
  - rprocess but not dprocess
  - code simulating sample paths is “plugged” into inference software
- Not plug-and-play:
  - EM algorithm (dprocess)
  - MCMC (dprocess+dmeasure)
- Bayesian plug-and-play:
  - Artificial parameter evolution (Liu & West, 2001: posterior correction, rprocess+dmeasure)
  - ABC (Beaumont et al., 2002: sufficient statistics, rprocess+rmeasure)
  - PMCMC (Andrieu et al., 2010: SMC + MCMC, rprocess+dmeasure)
- Non-Bayesian plug-and-play:
  - Iterated filtering (Ionides et al., 2006: likelihood-based inference, rprocess+dmeasure)

Other settings: plug-and-play

- Optimization: Methods requiring only evaluation of the objective function to be optimized are sometimes called gradient-free. This is the same concept as plug-and-play: the code to evaluate the objective function can be plugged into the optimizer

- Complex systems: Methods to study the behavior of large numerical simulations (e.g., molecular models for phase transitions) that only employ the underlying code as a “black box” to generate simulations are called equation-free (Kevrekidis et al., 2003, 2004)

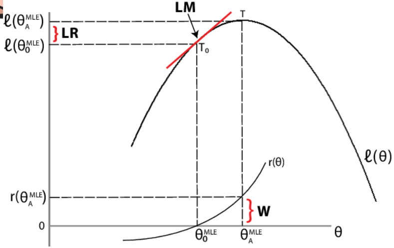
- ABC and PMCMC: Plug-and-play methods have recently been called likelihood-free. In this terminology, iterated filtering does likelihood-free likelihood-based inference

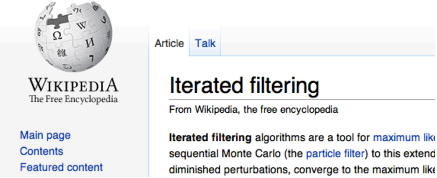
Cost: plug-and-play

- Efficiency: Approximate Bayesian methods and simulated moment methods lead to a loss of statistical efficiency
- Iterated filtering: enables (almost) exact likelihood-based inference
- Improvements: numerical efficiency may be possible when analytic properties are available (at the expense of plug-and-play). But many interesting dynamic models are analytically intractable—for example, it is standard to investigate systems of ordinary differential equations numerically

Contribution (II): iterated filtering algorithm

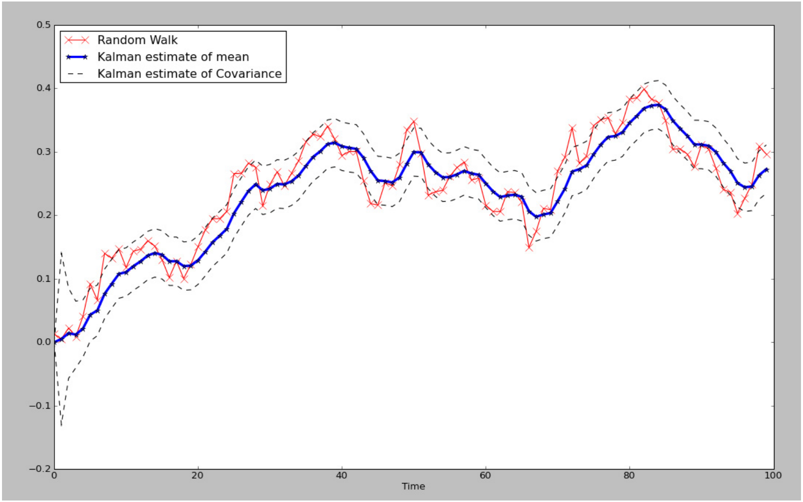
Plug-and-play likelihood-based inference on POMPs



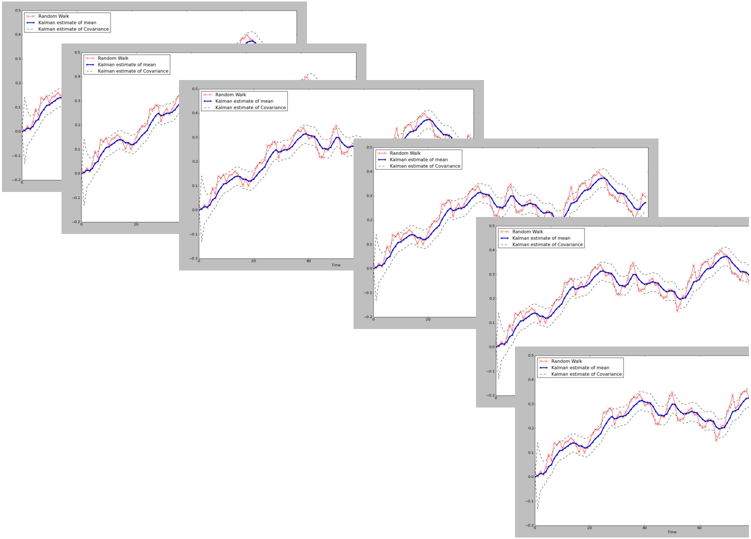


Iterated filtering. *Filtering Probl.*: extensively-studied

Cond. distr. of state $x(t_n)$ given obs. $y(t_1), \ldots, y(t_n)$

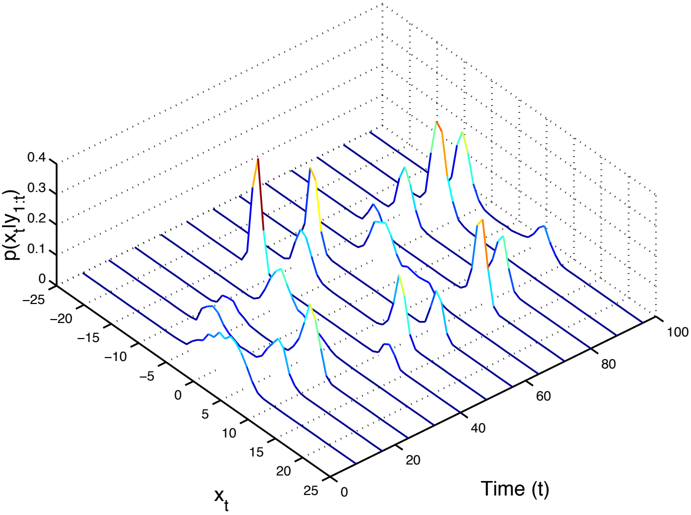


Iterated filtering: sequence of solutions to filtering to maximize likelihood over unknown parameters



Iterated filtering. Sequential Monte Carlo

Provides plug-and-play filter (for P&P IF and PMCMC)



Likelihood-based inference via iterated filtering

- MLE: asymptotically smallest estimator variance across different samples
- The properties of likelihood-based inference have been extensively studied:
  1. Invariant estimators
  2. Nested and non-nested hypothesis testing (via LR and AIC) (meaningful differences in the criterion)
  3. Computationally cheap standard errors (via FI)
  4. Likelihood profiles: robustness to identifiability issues

Toy example: AR(1) with noisy measurement

$$\begin{aligned} X_t &= \phi X_{t-1} + \epsilon_{xt} \\ Y_t &= X_t + \epsilon_{yt} \end{aligned}$$

$$\begin{pmatrix} \epsilon_{xt} \\ \epsilon_{yt} \end{pmatrix} = N\left(0, \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix}\right)$$

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Extend the model: Random-walk time-varying parameters

$$
\begin{aligned}
X_t &= \text{logit}(\phi_t) X_{t-1} + \epsilon_{xt} \\
Y_t &= X_t + \epsilon_{yt} \\
\phi_t &= \phi_{t-1} + \epsilon_{\phi t}
\end{aligned}
$$

$$
\begin{pmatrix} \epsilon_{xt} \\ \epsilon_{yt} \\ \epsilon_{\phi t} \end{pmatrix} = \mathrm{N} \left(0, \begin{pmatrix} \sigma_x^2 & 0 & 0 \\ 0 & \sigma_y^2 & 0 \\ 0 & 0 & \sigma_\phi^2 \end{pmatrix} \right)
$$

Take a limit as $\sigma_\phi \downarrow 0$

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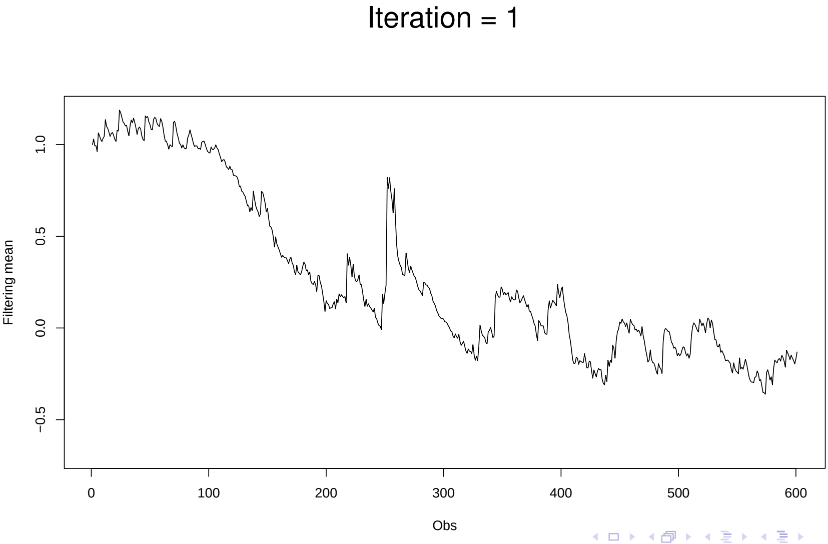
$$

Take a limit as $\sigma_\phi \downarrow 0$

$E[\phi_t|y_{1:t}]$: Local estimates of fixed AR parameter ϕ
$V[\phi_t|y_{1:t}]$: weighted by local uncertainty



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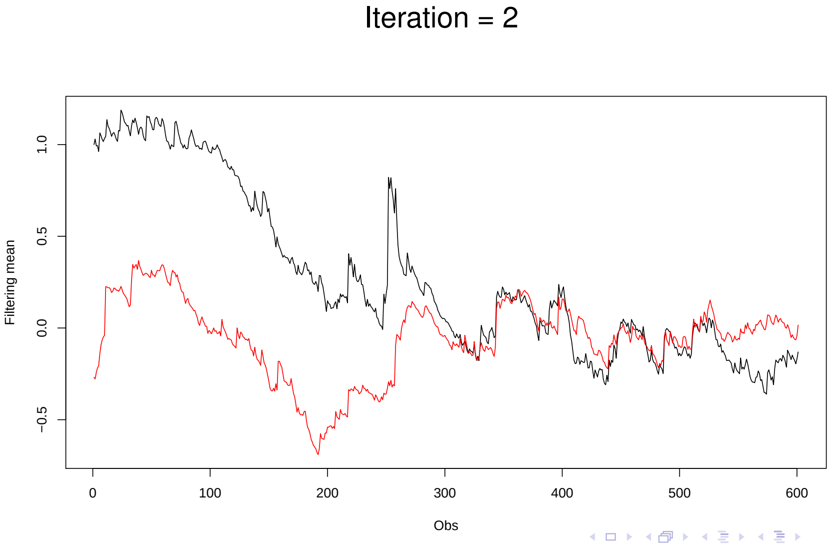

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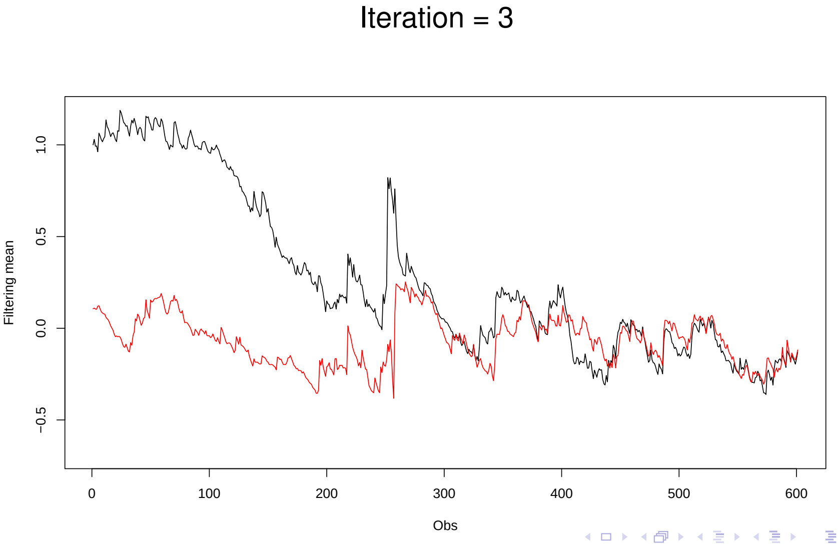

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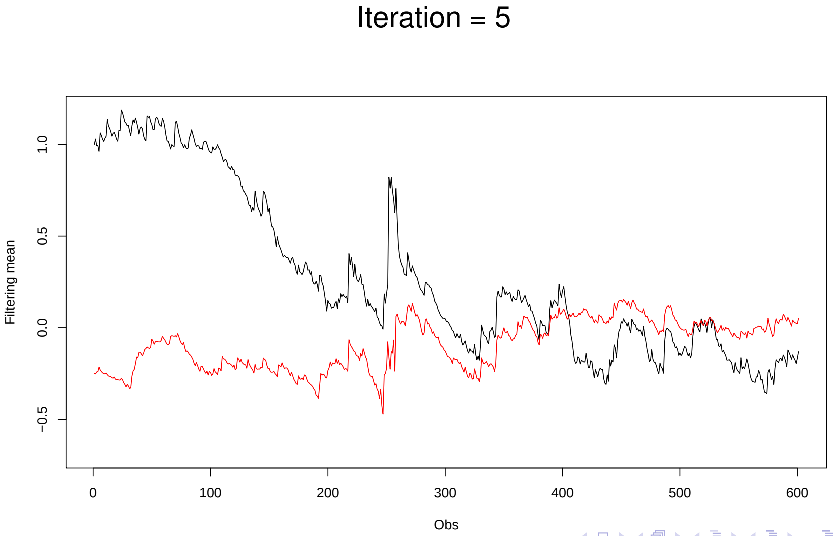

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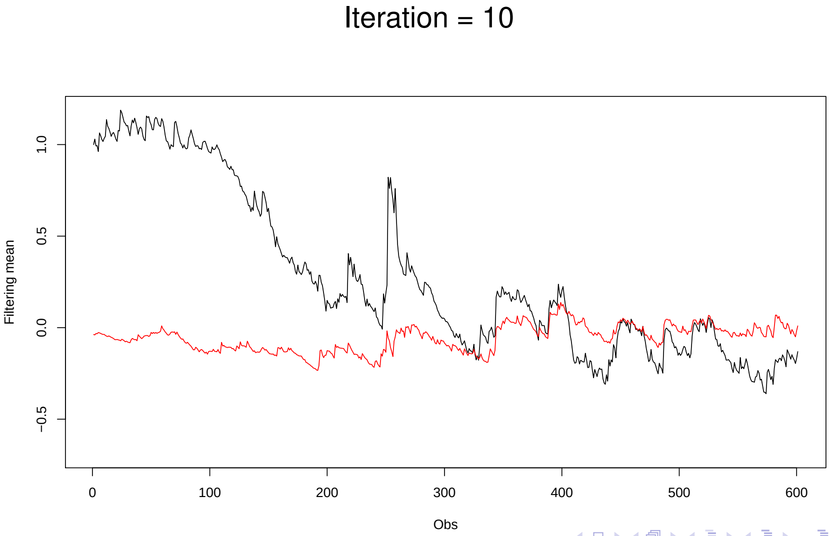

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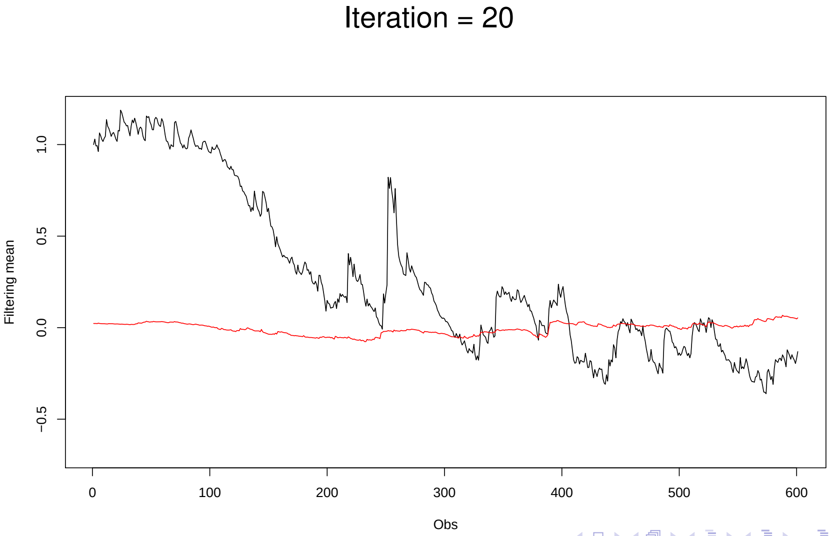


Easy to implement: Use POMP package!

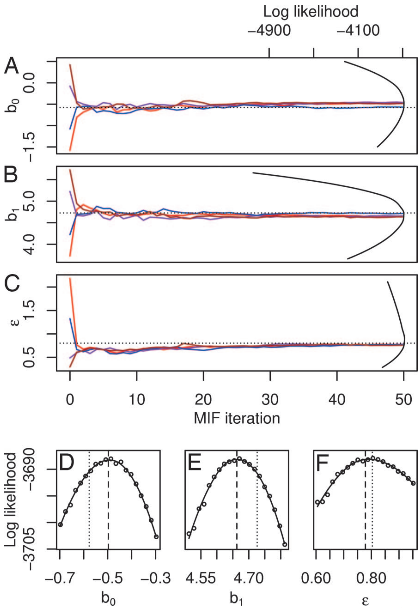
- SMC Filtering input:
  - $f(x_t|x_{t-1})$ or *rprocess*: rnorm(ϕx_{t-1}, σ_x^2)
  - $f(y_t|x_t)$ or *dmeasure*: dnorm(x_t, σ_y^2)

$$\begin{aligned} X_t &= \phi X_{t-1} + \epsilon_{xt} \\ Y_t &= X_t + \epsilon_{yt} \end{aligned}$$

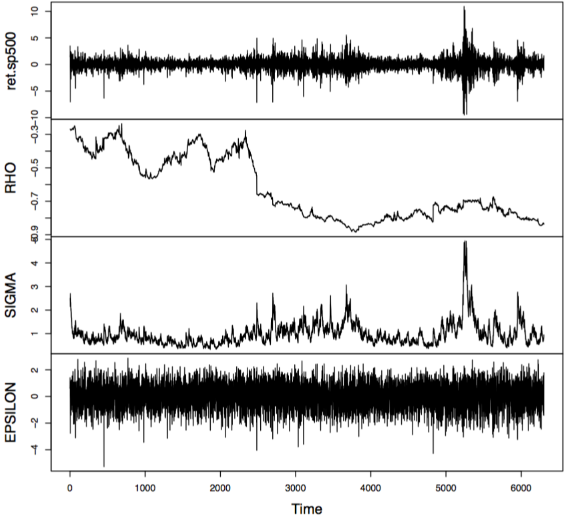
$$\begin{pmatrix} \epsilon_{xt} \\ \epsilon_{yt} \end{pmatrix} = N \left(0, \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix} \right)$$

- Algorithmic input:
  - Extended model variance (σ_ϕ^2): how big should the random walk variances in the extended model be
  - Speed of convergence ($\sigma_\phi^2 \to 0$): how fast should the extended model converge to the true model ($\propto$ number of iterations)

Accurate: Maximizing the likelihood



Contribution (III): Example from financial econometrics Stochastic volatility with stochastic leverage



Leverage equations: Harvey & Shephard, 1996

- Stochastic volatility with leverage (Harvey & Shephard, 1996)

$$
\begin{aligned}
y_t &= \sigma_t^2 \epsilon_t = \exp\left\{h_t/2\right\}\epsilon_t \\
h_t &= \mu_h(1-\phi) + \phi h_{t-1} + \sigma_\eta\sqrt{1-\phi^2}\left(\rho\epsilon_{t-1} + \omega_t\sqrt{1-\rho^2}\right)
\end{aligned}
$$

- Modifications to POMP code:
  - rprocess:
    $f(x_t|x_{t-1}) = \text{rnorm}(\phi x_{t-1}, \sigma_x^2) \longrightarrow \text{rnorm}(\mu_{x_t|x_{t-1}}, \sigma^2_{x_t|x_{t-1}})$
    $\mu_{x_t|x_{t-1}} = \mu(1-\phi) + \phi h_{t-1} + \sigma_\eta\sqrt{1-\phi^2}\rho\epsilon_{t-1}$
    $\sigma^2_{x_t|x_{t-1}} = \sigma_\eta\sqrt{1-\phi^2}\sqrt{1-\rho^2}$
  - *dmeasure*:
    $f(y_t|x_t) = \text{dnorm}(x_t, \sigma_y^2) \longrightarrow \text{dnorm}\ (0,\sigma_t^2)$

Idiosyncratic leverage: AR and RW equations

- Stochastic volatility with stochastic leverage (Bretó, 2013)

$$y_t = \sigma_t^2 \epsilon_t = \exp\left\{h_t/2\right\}\epsilon_t$$

$$h_t = \mu_h(1-\phi) + \phi h_{t-1} + \sigma_\eta \sqrt{1-\phi^2}\left(\rho_t \epsilon_{t-1} + \omega_t \sqrt{1-\rho^2}\right)$$

- Fisher-transformed correlation: $\rho_t = \frac{e^{2f_t}-1}{e^{2f_t}+1} \in (-1, 1)$
- AR(1) leverage: $f_t = \mu_f(1-\psi) + \psi f_{t-1} + \nu_t \sigma_\nu \sqrt{1-\psi^2}$
- RW leverage: $f_t = f_{t-1} + \nu_t \sigma_\nu$
- Stochastic volatility with random walk leverage: highly non-Gaussian, non-linear state-space model

Data analysis: S&P500 1988-2012 (25 y., 6302 obs.)

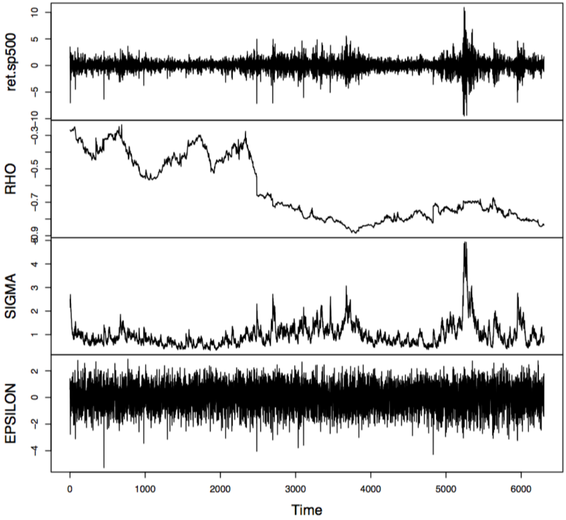
- Estimates for the volatility equation: usual & equal

Model	μ_h	ϕ	σ_η
Fixed leverage	-0.2506	0.9805	0.9003
	(0.0710)	(0.0017)	(0.0375)
Random-walk leverage	-0.2610	0.9818	0.9222
	(0.0776)	(0.0015)	(0.0406)

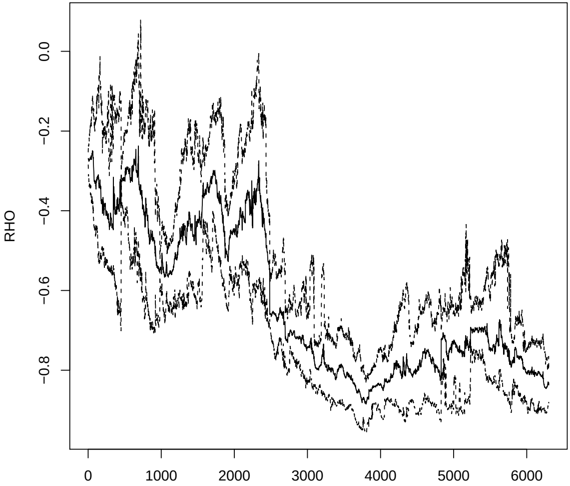
- ρ and likelihood (6.78 log-lik. units $\approx$ 7 parameters)

Model	ρ	σ_ν	ℓ
Fixed leverage	-0.6579	–	-8416.44
	(0.0599)	–	(0.0410)
Random-walk leverage	–	0.0086	-8409.06
	–	(0.0013)	(0.1333)

Random walk leverage (1988-2012, 25 y., 6302 obs.)



Random walk leverage (1988-2012, 25 y., 6302 obs.)



Flexibility: Extending stochastic volatility models

- Yu 2012

$$h_{t+1} = \varphi h_t + \gamma \sum_{i=1}^{m+1} (\rho_i \epsilon_t + \sqrt{1-\rho_i^2} w_t) \mathbf{1}(\tau_{i-1} \geq \epsilon_t > \tau_i)$$

- Bandi & Renó 2012

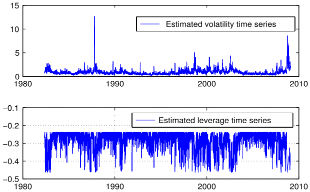
$$\begin{pmatrix} d \log p_t \\ d\xi(\sigma_t^2) \end{pmatrix} = \begin{pmatrix} \mu_t \\ m_t \end{pmatrix} dt + \begin{pmatrix} \sigma_t & 0 \\ 0 & \Lambda(\sigma_t^2) \end{pmatrix}$$
$$\times \begin{pmatrix} \rho(\sigma_t^2) & \sqrt{1-\rho^2(\sigma_t^2)} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix} + \begin{pmatrix} dJ_t^r \\ dJ_t^\sigma \end{pmatrix}$$

- Veraart & Veraart 2012

$$dY_t = \left(\mu + b\sigma_t^2\right) dt + \sigma_t dX_t,$$
$$dX_t = \rho_t dW_t + \sqrt{1-\rho_t^2} d\widetilde{W}_t,$$
$$d\sigma_t^2 = \alpha(\beta - \sigma_t^2)dt + \gamma \sigma_t dW_t,$$

$$d\rho_t = ((2\zeta - \eta) - \eta\rho_t)\, dt + \theta\sqrt{(1+\rho_t)(1-\rho_t)} dW_t^V$$

Empirical evidence: Bandi and Renó (2012)



- Model: driven by mean-reverting spot volatility $\rho_t = \rho(\sigma_t^2)$
- Period: 1982-2008 (long: 27 years)
- Estimation: non-parametric & high-frequency data
- Results: $\rho_t \in (-0.45, -0.25) \pm$ s.e.
- Implication: $E[\rho_t] \approx -0.30$ (?)

Empirical evidence: Yu (2012)

- Model: white noise driven by return noise ϵ_{t-1}

- $\rho_t = \rho(\epsilon_{t-1})$:

$$\rho_t = \begin{cases} \rho_1 & \text{if } \tau_1 < \epsilon_{t-1} \leq \tau_0 \\ \vdots & \\ \rho_{m+1} & \text{if } \tau_{m+1} < \epsilon_{t-1} \leq \tau_m \end{cases}$$

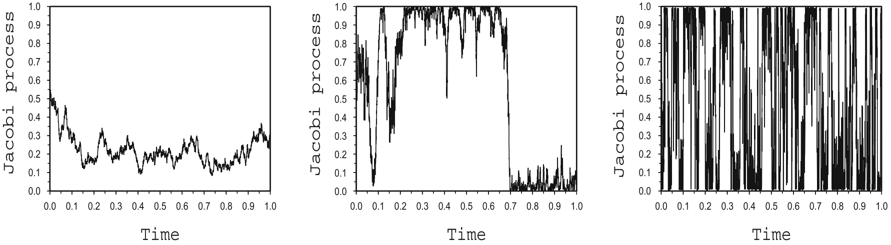
- $m = 1$ & $\tau_1 = 0$: simply two leverages

Empirical evidence: Yu (2012)

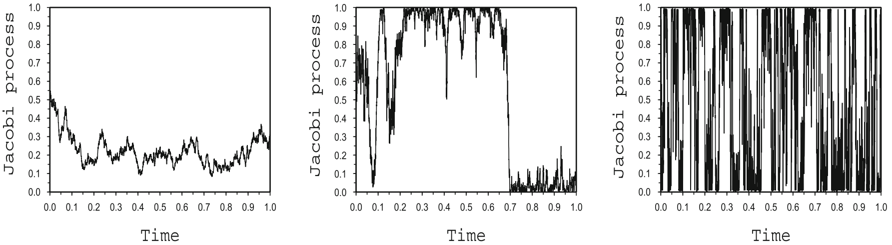
Data	Model	-Log MargLik	σ	φ	γ	ρ_1	ρ_2	ρ_3
S&P500	Basic	1730.26	0.9302	0.9426	0.2621			
			(0.067)	(0.019)	(0.042)			
			[0.001]	[0.0008]	[0.0021]			
	Leverage	1720.13	0.9336	0.9212	0.3066	−0.3691		
			(0.055)	(0.024)	(0.049)	(0.087)		
			[0.0014]	[0.0012]	[0.0030]	[0.0041]		
	Spline1	1700.40	2.077	0.9135	0.3689	−0.8386	0.1435	
			(0.3961)	(0.019)	(0.058)	(0.090)	(0.137)	
			[0.015]	[0.0008]	[0.0026]	[0.0133]	[0.005]	
	Spline2	1704.38	1.874	0.9157	0.3458	−0.8446	0.2059	0.1429
			(0.3537)	(0.019)	(0.051)	(0.1079)	(0.3484)	(0.1672)
			[0.014]	[0.0007]	[0.0022]	[0.0046]	[0.011]	[0.0066]

- Model: white noise driven by return noise ϵ_{t-1}
- Period: 1986-1989 (shorter: 4 years)
- Estimation: non-parametric & MCMC
- Results (m=2): $\rho_t \in (-1, -0.65) \cup (-0.13, 0.40)$
- Implication: $E[\rho_t] \approx -0.35$ ($\approx \rho$)

Idiosyncratic leverage: Veraart & Veraart (2012)



Idiosyncratic leverage: Veraart & Veraart (2012)



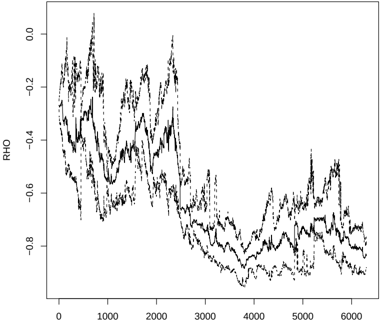
- **Model**: driven by idiosyncratic, independent Jacobi process V_t

$$dV_t = (\zeta - V_t)dt + \theta\sqrt{V_t(1 - V_t)}dW_t$$

$$\rho_t = 2V(t) - 1 \in [-1, 1]$$

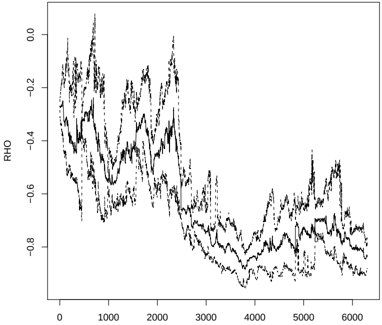
- **Period**, **Estimation**, **Results**: –
- **Implication**: $E[\rho_t] = 2\zeta - 1$
- Good candidate for new evidence!

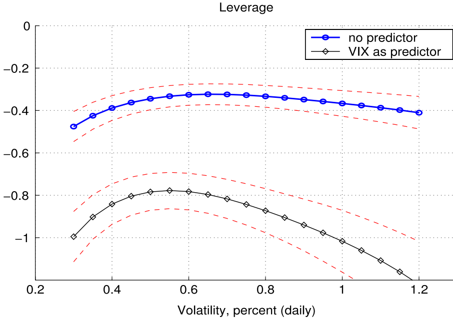
AR / RW evidence: consistent with Yu (2012)?



- Period 1988-1999:
  - RW: $\rho_t \in (-0.55, -0.10)$
  - Yu (2012): $\rho_1 \in (-0.5431, -0.1951)$

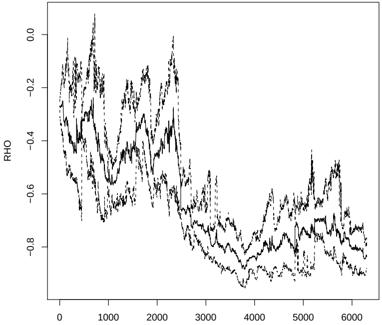
AR / RW evidence: consistent with Bandi (2012)?

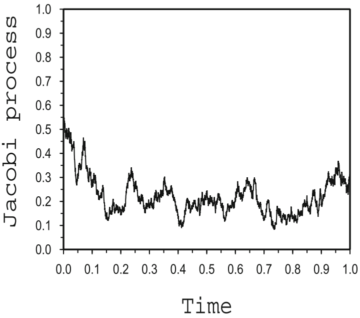




- Period 2003-2012:
  - RW: $\rho_t \in (-0.9, -0.6)$
  - Bandi (2012): ρ_t “more negative than those generally found in the literature” when using VIX as an instrumental variable for daily variance estimation

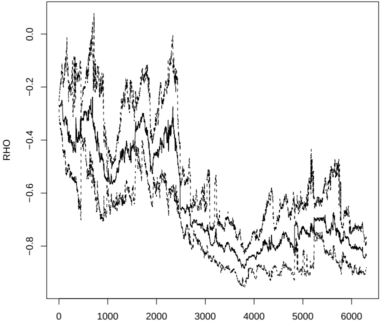
AR / RW evidence: consistent with Veraart (2012)?





- RW decade 1988-1997: $E[\rho_t] \approx -0.4$
- RW decade 2003-2012: $E[\rho_t] \approx -0.8$
- Fixed-leverage: $\rho \approx -0.6$

Idiosyncratic leverage: AR and RW evidence



- Model: driven by idiosyncratic AR/RW process
- Period: 1988-2012 (three sub-periods)
- Estimation: parametric & likelihood maximization
- Results: $\rho_t \in (-0.8, -0.3) \pm$ s.e. (consistent with Bandi, Yu, and Veraart?)
- Implication: $E[\rho_t] = ?$ $\quad E[\rho_t] \approx -0.6$ ($\approx \rho$) ?

Theory behind iterated filtering

- Update: $\hat{\theta}_0^{(n+1)} = V_{1,n}\Big(\sum_{t=1}^{T-1}(V_{t,n}^{-1} - V_{t+1,n}^{-1})\hat{\theta}_t^{(n)} + V_{T,n}^{-1}\hat{\theta}_T^{(n)}\Big)$
- Equivalently $\hat{\theta}^{(n+1)} = \hat{\theta}^{(n)} + V_{1,n}\sum_{t=1}^{T} V_{t,n}^{-1}(\hat{\theta}_t^{(n)} - \hat{\theta}_{t-1}^{(n)})$
- Assuming sufficient regularity conditions, a Taylor expansion gives

$$\lim_{\sigma\to 0}\sum_{t=1}^{T} V_t^{-1}(\hat{\theta}_t - \hat{\theta}_{t-1}) = \nabla \log f(y_{1:T}|\theta, \sigma=0)$$

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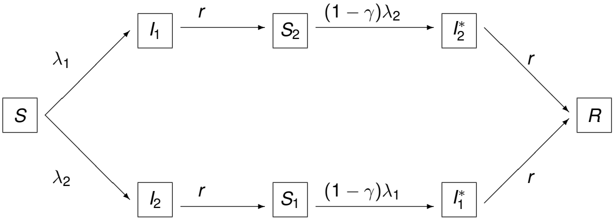
$$\lim_{\sigma\to 0} \sum_{t=1}^{T} V_t^{-1}(\hat{\theta}_t - \hat{\theta}_{t-1}) = \nabla \log f(y_{1:T}|\theta, \sigma{=}0)$$

A word of caution: Compartment models with random rates



$$\Delta N_{S_1 I_1^*}(t) = N_{S_1 I_1^*}(t+h) - N_{S_1 I_1^*}(t)$$

- Continuous Time Markov Chain (system of death processes)

$$\begin{aligned} P(\Delta N_{S_1 I_1^*}(t) = 0 | \boldsymbol{X}(t) = \boldsymbol{x}) &= 1 - (1-\gamma)\lambda_1 s_1 h + o(h) \\ P(\Delta N_{S_1 I_1^*}(t) = 1 | \boldsymbol{X}(t) = \boldsymbol{x}) &= (1-\gamma)\lambda_1 s_1 h + o(h) \\ P(\Delta N_{S_1 I_1^*}(t) > 1 | \boldsymbol{X}(t) = \boldsymbol{x}) &= o(h) \end{aligned}$$

Continuous-time & iterated filtering

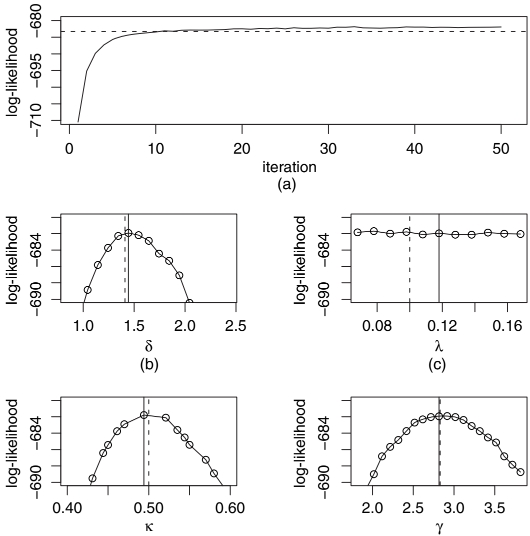
- IF has been applied to continuous-time stochastic volatility models
  - Comment to Andrieu et al., 2010, JRSSB: Particle MCMC and continuous-time Lévy-driven stochastic volatility model (Barndorff-Nielsen and Sheppard, 2001)

$$
\begin{aligned}
dy^*(t) &= \mu + \beta\sigma^2(t)dt + \sigma(t)dB(t) \\
d\sigma^2(t) &= -\lambda\sigma^2(t)dt + dz(\lambda t)
\end{aligned}
$$

  - Working paper - Bretó and Veiga: Forecasting performance and continuous-time log-linear one volatility factor model (Chernov et al., 2003)

$$
\begin{aligned}
dU_1(t) &= \alpha_{10}dt + \exp(\beta_{10} + \beta_{12}U_2(t))(\psi_{11}dW_1(t) + \psi_{12}dW_2(t)) \\
dU_2(t) &= \alpha_{22}U_2(t)dt + (1 + \beta_{22}U_2(t))dW_2(t).
\end{aligned}
$$

Continuous-time SV



Sequential Monte Carlo: Particle Filter

$t = 0$ $t = 1$ $t = 2$ $t = 3$

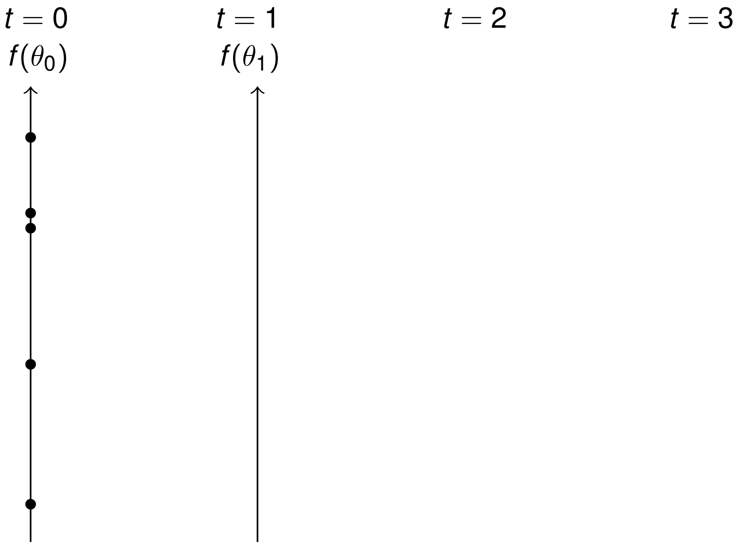
$f(\theta_0)$

Sequential Monte Carlo: Particle Filter

$t = 0$ $t = 1$ $t = 2$ $t = 3$

$f(\theta_0)$

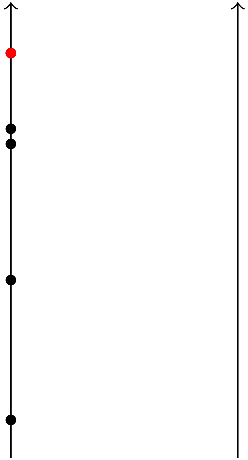
Sequential Monte Carlo: Particle Filter



Sequential Monte Carlo: Particle Filter

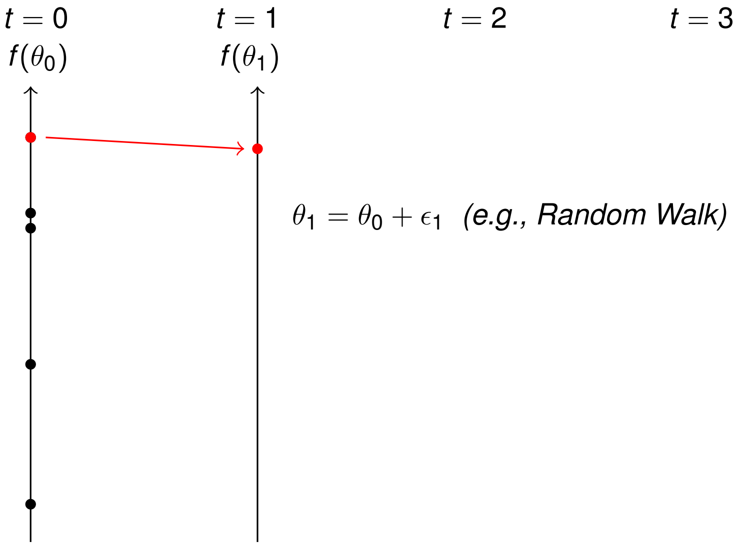
$t = 0$ $t = 1$ $t = 2$ $t = 3$

$f(\theta_0)$ $f(\theta_1)$



$\theta_1 = \theta_0 + \epsilon_1$ *(e.g., Random Walk)*

Sequential Monte Carlo: Particle Filter



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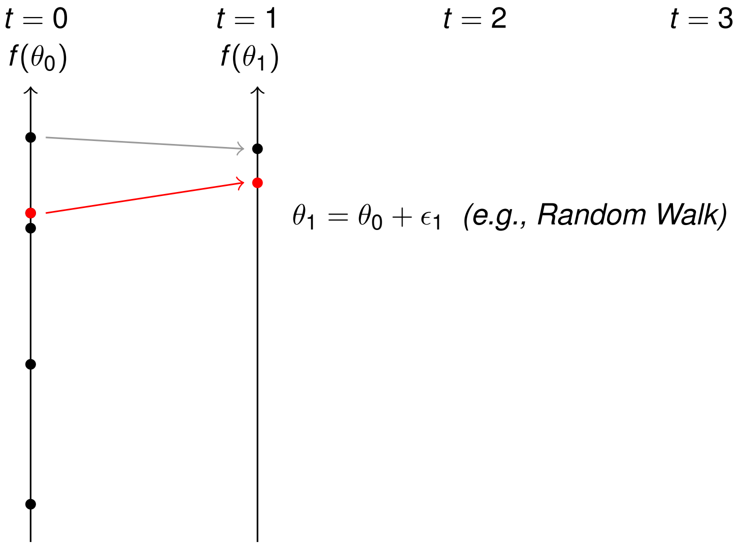
Sequential Monte Carlo: Particle Filter

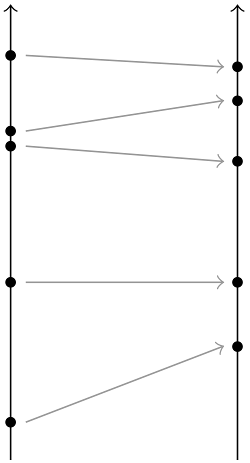


$\theta_1 = \theta_0 + \epsilon_1$ *(e.g., Random Walk)*

Sequential Monte Carlo: Particle Filter

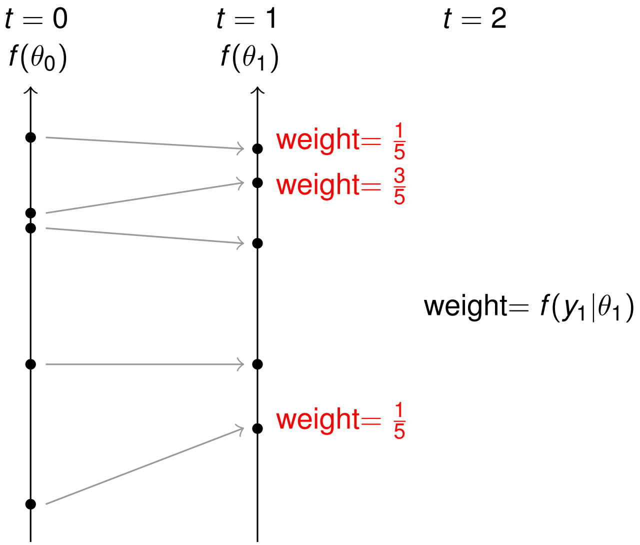
$t = 0$ $t = 1$ $t = 2$ $t = 3$

$f(\theta_0)$ $f(\theta_1)$

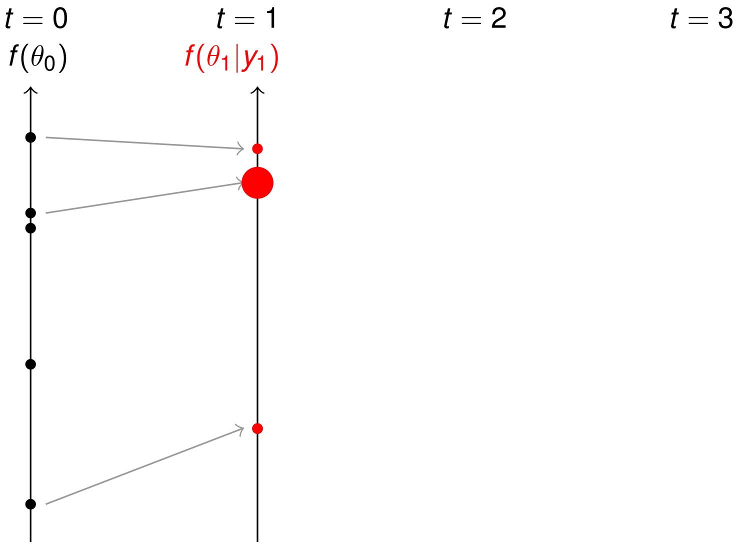


$\theta_1 = \theta_0 + \epsilon_1$ *(e.g., Random Walk)*

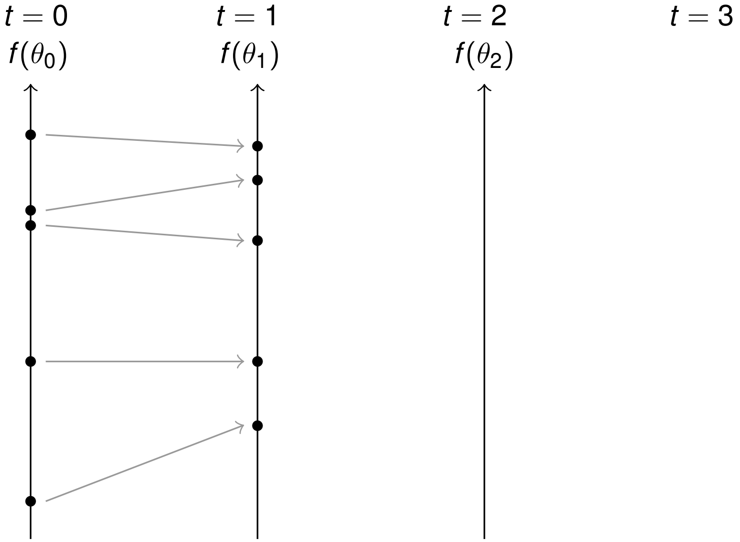
Sequential Monte Carlo: Particle Filter



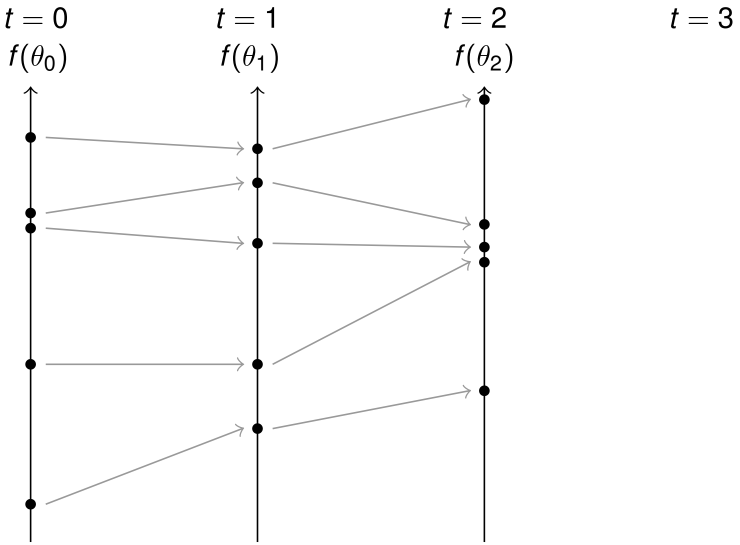
Sequential Monte Carlo: Particle Filter



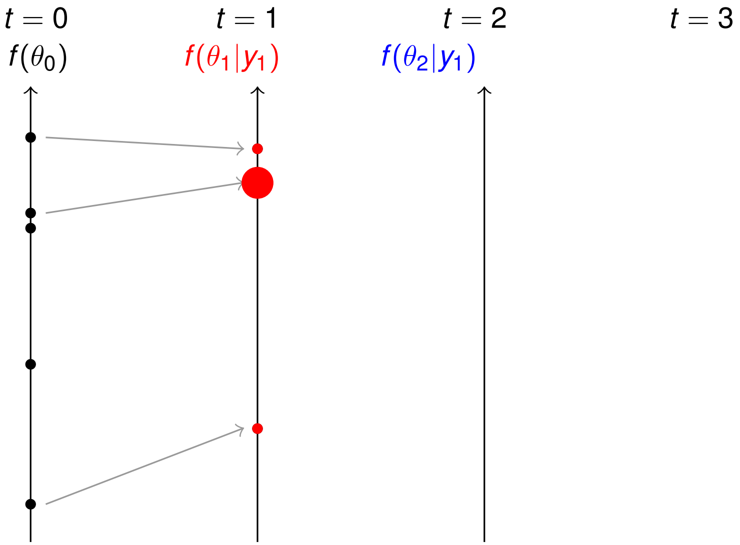
Sequential Monte Carlo: Particle Filter



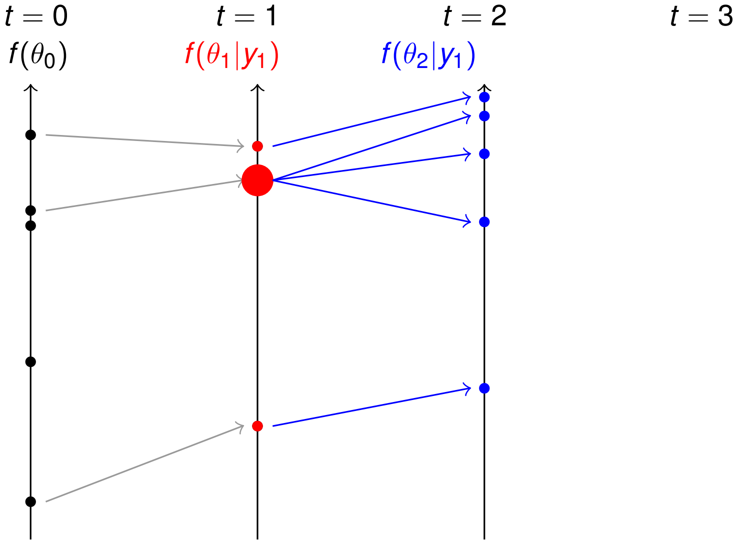
Sequential Monte Carlo: Particle Filter



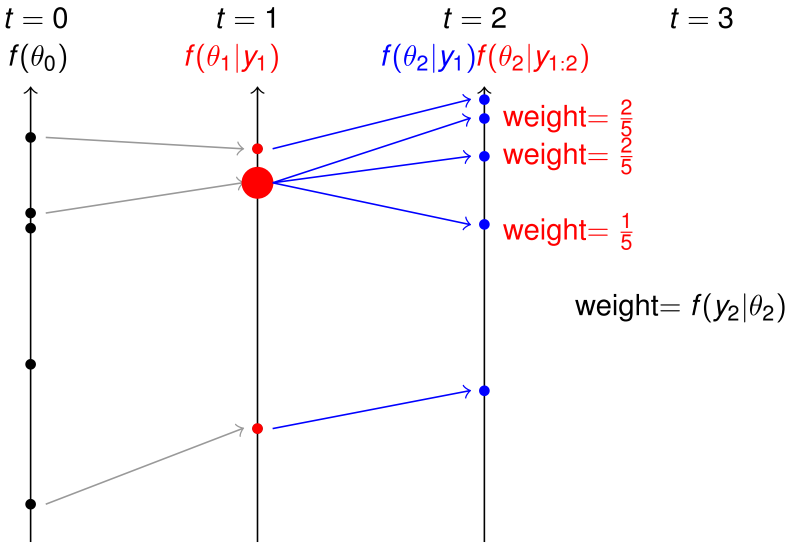
Sequential Monte Carlo: Particle Filter



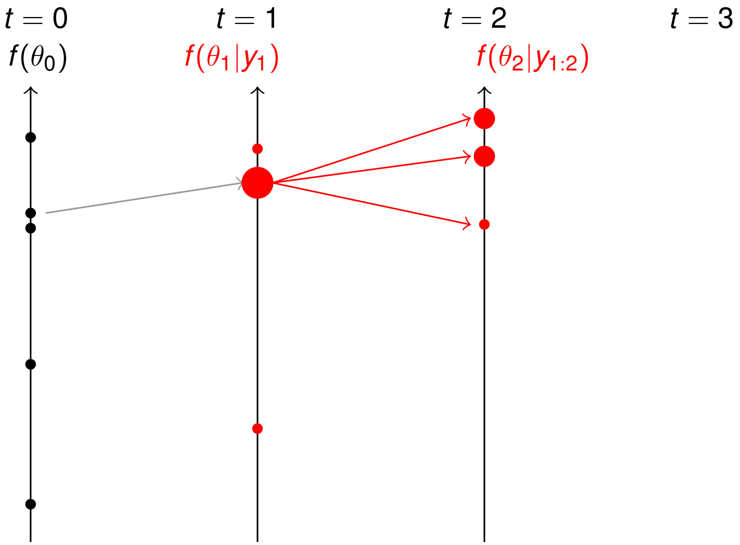
Sequential Monte Carlo: Particle Filter



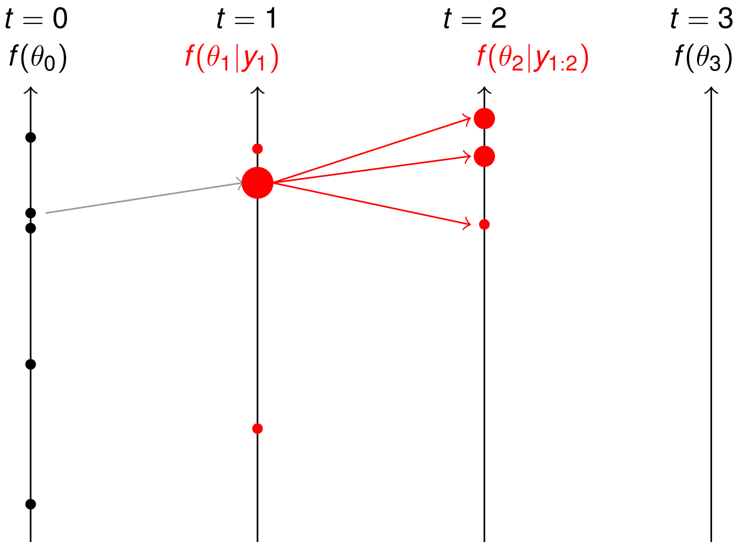
Sequential Monte Carlo: Particle Filter



Sequential Monte Carlo: Particle Filter



Sequential Monte Carlo: Particle Filter



Take-home message: straightforward, likelihood-based inference is possible for general dynamic systems

THANK YOU