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HEAVY TAILS AND SKEWNESS OF THE SPANISH STOCK RETURNS

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1.INTRODUCTION

Most of the concepts in theoretical and empirical finance that have been developed over the past decades rest upon the assumption that asset returns follow a normal distribution.

By now, there is, however, ample empirical evidence that many – if not most – financial return series are heavy-tailed and, possibly, skewed.

This is not only of concern to financial theorists, but also to practitioners who are, in view of the frequency of sharp market downturns, troubled by the "compelling evidence that something is rotten in the foundation of the statistical edifice…" (Richard Hoppe)

The work builds upon Benoit Mandelbrot's fundamental work in the 1960s, which strongly rejected normality as a distributional model for asset returns. Examining various time series on commodity returns and interest rates, he conjectured that financial return processes behave like non-Gaussian processes.

Mandelbrot's contributions give rise to a new probabilistic foundation for financial theory and empirics; and they are of similar importance as the fundamental contributions of Louis Bachelier (1900) and Paul Samuelson (1955). His early investigations on asset returns were carried further by Eugene Fama l1965), among others.

Partly in response to these empirical "inconsistencies", various alternatives have been proposed in the literature. Among the candidates considered were the fat-tailed distributions.

Our work investigates the consequences of relaxing the normality assumption and developing generalizations of prevalent concepts in modern theoretical and empirical finance that can accommodate heavy-tailed returns. Under the mathematical idea of Lévy processes.

The asset price models we are going to test are some of the most famous ones representing some new approach into the late literature.
In Section 3 we review all these models. Section 4 compares different models by different criteria. Finally we briefly summarize the results.

2.BASIC CONCEPTS

Lévy Process:

An adapted real-valued process X_t with $X_0 = 0$, is called a Lévy process if:

- It has independent increments; that is, for any choice of $n \geq 1$ and $0 \leq t_0 < t_1 < ... < t_n$
- It is time-homogeneous; that is, the distribution of $\{X_{t+s} - X_s ; t \geq 0\}$ does not depend upon s.
- It is stochastically continuous; that is, for any $\varepsilon > 0$, $\Pr\{|X_{s+t} - X_s| > \varepsilon\} \to 0$ as $t \to 0$
- As a function of t, it is right-continuous with left limits.

Processes that satisfy (i) and (ii) are called processes with stationary (or time-homogeneous) independent increments (PIIS). Some authors (e.g. Bertoin) simply define a Levy process to be a PIIS process with $X_0 = 0$. Such processes can be thought of as analogs of random walks in continuous time.

One important implication of the independence and stationarity of increments is that the distributions of a Lévy process are completely determined by their one dimensional distribution. In other words

$$IE\left[e^{iuX_t}\right] = (IE\left[e^{iuX_1}\right])^t$$

Alternatively, if we denote the characteristic triplet of a Lévy process X_t by (m_t, σ_t^2, L_t) and the characteristic triplet of its one-dimensional distribution at $t = 1$ with (m, σ^2, L) we have that $m_t = tm$, $\sigma_t^2 = t\sigma^2$ and $L_t = tL$. See Nielsen-Shephard (2001)

The Lévy-Khintchine Representation

The Lévy Kintchine theorem characterizes all infinitely divisible random variables in terms of their characteristic function.

Let X_t be a Levy process. Then the characteristic function can be written as

$$\phi_{X_t}(u) = IE\left[e^{iuX_t}\right] = \exp\left[mitu - \frac{1}{2}\sigma^2 tu^2 + t\int_{\{IR-0\}}\left(e^{iux} - 1\right)L(dx)\right]$$

Where the drift rate $m \in IR,$ and the diffusion coefficient $\sigma \geq 0$. The Levy measure L must satisfy

$$\int_{IR-0} x^2 1_{(|x|<1)} L(dx) < \infty$$

Loosely speaking the Lévy measure $L(x) = l(x)dx$ specifies the arrival rate of jumps of size $(x, x + dx)$ .Hence it must be nonnegative and no measure is assigned to the origen. So well the process has well defined quadratic variation. See Feller (1971).

If $\phi_t(z)$, $a < \operatorname{Im} z < b$, is an infinitely divisible characteristic function, then it has the next Levy-Khintchine representation.

The Levy-Khintchine representation has the form $\phi_{X_t}(u) = \exp[-t\psi(u)]$, where $[\psi(u)]$ is called the characteristic exponent.

Like all characteristic functions, they are Fourier transforms of a density and typically have an analytic extension

3. FRAMEWORK

We will consider a market place in which a stock or security price $S_t \geq 0$ follows an exponential Lévy process X_t on two continuous-time probability space. $(\Omega, \Lambda, (\Lambda_t)_{t \in IR+}, P)$
Under P, the stock price at $t = T$ evolves as $S_T = S_0 \exp(\mu T + X_T)$

To prevent an arbitrage opportunity, the stock price (net of the cost of carry) must be a local martingale under Q. That is, $IE[S_t] = S_0 e^{\mu t}$ satisfying $IE(\exp(X_{LEVY,t})) = 1$

For those Lévy processes with $IE(\exp(X_{LEVY,t})) \leq +\infty$, this normalization can be achieved by the drift adjustment ω. So we will consider:

Therefore we will consider a relatively simple class of continuous-time process where all non-overlapping increments $s_t = \log(S_T / S_0)$ are independent random variables with stationary distributions. These set of processes s_t are Lévy processes with stationary independent increments.

The statistical stock price dynamics are assumed to be given by

$$S_t = S_0 \exp\left((m + \omega_s)t + X_t\right) \quad \text{with } t \geq 0$$

and the characteristic function for the statistical log asset price process $s_t = \{\log(S_t / S_0) = (m + \omega_s)t + X_t\}$ process is:

$$\phi^s_{s_t,t}(u) = IE\left[e^{ius_t}\right] = IE\left[e^{iu(m+\omega_s)t}\right] IE\left[e^{iuX_t}\right]$$

therefore

$$\phi^s_{s_t,t}(u) = \exp(iu(m + \omega_s)t))\phi_{X_t}(u)$$

$$\phi^s_{X_t,}(u) = IE\left[\exp(iuX_t)\right] = \exp(-t\psi(u))$$

Where $\psi(u)$ is the characteristic exponent, so to organize the forward price at $S_0 \exp(\mu t)$, we take the value of ω_s as defined by $\phi_t(-i)$

4. ASSET MODELS

The Classical Lognormal Model.

This model proposed by Samuelson (1955) and Osborne (1959), and later it was used by Black-Scholes (1973).

Considering the stochastic, non-negative, and stationary independent increments process $T = \{T_{t,} t \geq 0\}$ also called the *intrinsic time process* or stochastic internal time, and the stochastic process $W = \{W_t, t \geq 0\}$ which represents the noise process. The process subordinated to the standard Wiener process W by the independent intrinsic process T is denoted by $X_{CL,t} = \{W_{T_t}, t \geq 0\}$ is usually called driving process, is another stochastic process with stationary independent increments.

Let $T = \{\ t, t \geq 0\ \}$, i.e. the intrinsic time process is deterministic and identical to physical time, then at this particular case $X_{CL,t} = \{W_t, t \geq 0\}$. Hence the Lévy measure for $\Delta\ X_{CL,t} \approx N(0, \sigma^2 t)$ or we can say that jumps only come from a normal distribution, then

$$\log(\phi_{s_t,t}(u)) = \log IE\left[e^{ius_t}\right] = itu(\mu + \omega_s) - \frac{1}{2}\sigma^2 tu^2$$

The Merton Statistical Jump-Diffusion Model

In Merton's model (1976), as we have seen at introduction formula (1), the stock price is driven by a (1 dimensional) Brownian motion W_t and independent pure jump process.

According to this specification, the arrival of normal information leads to price changes as a lognormal diffusion, while the arrival of abnormal information, is model as a Poisson process and give rise to jumps size $(e^{Y_t} - 1)$, in the security return.

For this process the Lévy measure has the form $L(x) = \mu_{MERTON}(x)dx = \lambda f(x)dx$ and $f(x) \approx N(\varsigma, \xi^2)$, with Poisson intensity λ and finite activity, that means $\int_{IR} \lambda f(x)dx = \lambda \int_{IR} f(x)dx < \infty$ the Lévy-Khintchine representation admits the next decomposition, Sato (1999, Theorem 8.1)

$$\log(\phi_{s_t,t}(u)) = \log IE\left[e^{ius_t}\right] = itu(\mu + \omega_s) - \frac{1}{2}\sigma^2 tu^2 + t\int_{R-\{0\}}\left[e^{iux} - 1\right]\mu_{MERTON}(x)dx$$

And $\mu_{MERTON}(x) = \lambda f(x)$ with $f(x) \approx N(\varsigma, \xi^2)$

And hence

$$\phi_{s_t,t}(u) = \exp(itu(\mu + \omega_s) - \frac{1}{2}\sigma^2 tu^2 + t\lambda(\phi_{f(x)}(u) - 1))$$

And therefore the characteristic function for the statistical process log (S_t) is:

$$\phi^{s}_{\log S_{t},t}\ (u) = \exp(iu \log S_0)\exp(itu(\mu + \omega_s) - \frac{1}{2}\sigma^2 tu^2 + t\lambda(\phi_{f(x)}(u) - 1))$$

The Double Exponential Jump Diffusion, and the Eraker's model

The double exponential jump diffusion Kou (1999) is one of the models that have been proposed to incorporate the leptokurtic feature, that is high peaks and heavy tails in asset returns and volatility smirk. It was based on the importance of the memoryless property of the exponential distribution.

In this model the price of the underlying asset is modelled by two parts, a continuous part driven by Brownian motion, and a jump part with the logarithm of jump sizes having a double exponential distribution and the jump times corresponding to the event of a Poisson process the double exponential jump diffusion model is a special case of Lévy processes.

We are going to obtain the characteristic function for $s_t = \log(S_T / S_0)$ double just like we did for Merton model. Here, the Lévy measure has the form $L(x) = \mu_{KOU}(x)dx = \lambda f(x)dx$ and $f(x) \approx \exp(-|x-\kappa|/\eta)/2\eta,$ 0<η <1, where κ and η are two real parameters. Hence:

$$\int_{IR} \lambda f(x)dx = \lambda \int_{IR} f(x)dx < \infty$$

λ = Poisson rate

And $\mu_{KOU}(x) = \lambda f(x)$

With $f(x) \approx \exp(-|x-\kappa|/\eta)/2\eta,$ 0<η <1, where κ and η are two real parameters

And hence $\phi^{s}_{s_t,t}(u) = \exp(itu(\mu + \omega_s) - \frac{1}{2}\sigma^2 tu^2 + t\lambda(\phi_{f(x)}(u) - 1))$

And the characteristic function for the double exponential density $\phi_{f(x)}(u) = \exp(iu\kappa)(1-\eta^2)/(1+u^2\eta^2),$

Another example, of finite jumps activity is Eraker, Johannes, and Polson (2000) and Eraker (2001), incorporate compound Poisson jumps into the process assuming that the jump size is controlled by a one-sided exponential density.

The Lévy measure has the form $L(x) = \mu_{ERAKER}(x)dx = \lambda f(x)dx$ and $f(x) \approx \exp(-x/\eta)/\eta,$ $x > 0$.

C.G.M.Y.

The model is called the CGMY model, after the authors of this model. Carr, Geman, Madan and Yor (2000).

The jump component of such processes is completely characterized by *L(x)* this Levy density.

The CGMY Levy density with parameters *C,G,M,Y, for this* process is, $L(x)=l_{CGMY}(x)dx$ and $l_{CGMY}(x)$ is given by

$$l_{CGMY}(x)= \quad C\frac{\exp(-G|x|)}{|x|^{1+Y}} \quad \text{for } x<0 \qquad (4)$$

$$C\frac{\exp(-M|x|)}{|x|^{1+Y}} \text{ for } x>0$$

Where *C >0;G ¸M > 0;Y < 2:* The condition $Y<2$ is induced by the requirement that Levy densities integrate x^2 in the neighbourhood of 0: We denote by $X_{CGMY,t}$ the infinitely divisible process of independent increments with Levy density given by (4).
The parameter *C* may be viewed as a measure of the overall level of activity.

For example, later to construct a model with a stochastic aggregate activity rate, then we could model *C* as an independent positive process, possibly following a square root law of its own.

The parameters *G* and *M* respectively control the rate of exponential decay on the right and left of the Levy density, leading to skewed distributions when they are unequal. For $G<M$; the left tail of the distribution for $X_{CGMY,t}$ is heavier than the right tail.

The exponential factor in the numerator of the Levy density leads to the finiteness of all moments for the process $X_{CGMY,t}$ as we typically construct a process at the return level, it is reasonable to enforce finiteness of the moments at this level.

The parameter Y is particularly useful in characterizing the fine structure of the stochastic process. For example, one may ask whether the up jumps and down jumps of the process have a completely monotone Levy density, and whether the process has finite or infinite activity, or variation, for example [1] the paths have infinitely many jumps in any finite interval if $Y\in[0,2)$, and they have infinite variation if $Y\in[1,2)$.

Following Carr, Geman, Yor, Madan (2000) Theorem 1, the characteristic function for the infinitely divisible process with independent increments and the CGYM Levy density (4) is derived from the Levy Khintchine theorem, by

$$\phi_{X_{CGMY},t}(u)=\exp\left(t\int_{-\infty}^{\infty}\left(e^{iux}-1\right)k_{CGMY}(x)dx\right)$$

The integral in the exponent may be written as the sum of two integrals of the form

$$\int_0^{\infty}\left(e^{iux}-1\right)C\frac{\exp(-\beta x)}{x^{1+Y}}dx$$

1 Carr,Geman,Yor,Madan(2000) for further detai about this issue

For β equal to G and M respectively, with *iu* replaced by - *iu* for $\beta = G$. This integration may be performed as follows.

$$\int_0^\infty Cx^{-Y-1}\left[\exp\left(-(\beta - iu)x\right) - \exp\left(-\beta x\right)\right]dx = C\int_0^\infty (\beta - iu)^Y w^{-Y-1}\exp(w)dw - C\int_0^\infty \beta^Y w^{-Y-1}\exp(-w)dw$$

$$= C\Gamma(-Y)\left[(\beta - iu)^Y - \beta^Y\right]$$

The result follows on substituting *M* and *G* for β and evaluating the case $\beta = G$ at *-iu*

$$\phi_{X_{CGMY},t}(u) = \exp\left(tC\Gamma(-Y)\left\{(M - iu)^Y - M^Y + (G + iu)^Y - G^Y\right\}\right)$$

Carr, Geman, Madan and Yor (2000) extend the model to include an orthogonal diffusion component. Define the extended CGMY process as the process

$$X_{CGMYe,t,\eta} = X_{CGMY,t} + \eta W_t$$

The characteristic function for the logarithm of the stock price in this diffusion extended CGMY model is given by

$$\phi^s_{CGMY,t}(u) = \exp(iut(\mu + \omega_s - \eta^2/2))\phi_{X_{CGMY},t}(u)\exp(-\eta^2 u^2/2)$$

The Variance Gamma Model.

The VG process, is a generalization of the two parameter stochastic process studied in Madan, Carr, Chang (1998).

There are two representations for the variance gamma process, which are both useful in different contexts. In the first representation, which gave rise to the name, the variance gamma process is interpreted as a Brownian motion with drift, time changed by a gamma process. Let W_t be a standard Brownian motion with drift θ. and let $T_t = \{G_t^{1,\nu}, t \geq 0\}$ be an independent gamma process with mean rate unity, and variance rate v: The density of the gamma process at time t is given by

$$f_{T_t,t}(x) = \frac{x^{t/\nu - 1}\exp\left(-\frac{x}{\nu}\right)}{\nu^{t/\nu}\Gamma\left(\frac{t}{\nu}\right)} \qquad x > 0$$

While the $T_t = \{G_t^{1,\nu}, t \geq 0\}$ characteristic function is given by

$$\phi_{G_{1t}^{1,\nu}}(u) = E\left[\exp\left(iuG_t^{1,\nu}\right)\right] = \left(\frac{1}{1 - i\nu u}\right)^{t/\nu}$$

The variance gamma process has three parameters, σ, v and θ and the process $X_{VG,t} \approx N(\theta G_t^{1,v}, \sigma^2 W_{G_t^{1,v}})$ is given by:

$$X_{VG,t} = \theta G_t^{v} + \sigma W_{G_t^{v}}$$

Consequently, the distribution of $X_{VG,t}$ is a variance-mean mixture of normals, with a gamma distribution as mixing distribution.

$$f_{VG,1}(x) = \int_0^{\infty} N(x;\theta w, \sigma^2 w) d_{G_t^{1,v}}(w; v, \frac{1}{v}) dw$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_0^{+\infty} \frac{1}{\sqrt{u}} \exp\left\{-\frac{(x-\theta w)^2}{2\sigma^2 w}\right\} f_{G_{1t}^{1,v}}(w) dw . \qquad (x \in IR.)$$

The three parameters provide control over the skewness and kurtosis of the return distribution.
σ the volatility of the Brownian motion, v the variance rate of the gamma time change. These parameters measure the degree by which VG models fail to reproduce market prices. θ is consider the drift in the Brownian motion with drift.

The process therefore provides two dimensions of control on the distribution over and above that of the aggregate volatility. We observe below that control is attained over the skew via θ and over kurtosis with v.

In terms of (σ, v, θ) one may write the Lévy measure of the variance gamma as

$$k_{VG}(x)dx = \frac{\exp\left(\theta x / \sigma^2\right)}{v|x|} \exp\left(-\frac{\sqrt{\frac{2}{v} + \frac{\theta^2}{\sigma^2}}}{\sigma}|x|\right)dx \qquad (8)$$

The special case of $\theta = 0$ in (8) yields a Lévy measure that is symmetric about zero. It can be also observe from (8) that when $\theta < 0$ negative values of *x* receive a higher relative probability than the corresponding positive value. Hence, negative values of θ give rise to a negative skewness. We note further that large values of v, lower the exponential decay rate of the Lévy measure symmetrically around zero, and hence raise the likelihood of large jumps, thereby raising tail probabilities and kurtosis.
The distribution of the log returns, that is, the variable $s_t = \log(S_T / S_0)$ therefore:

$$\phi^{s}{}_{s_t}(u) = IE\left[e^{ius_t}\right] = \exp(iu(\mu + \omega_s)t)\ \phi_{X_{VG,t}}(u)$$

Generalized Hyperbolic Distributions

This class of distributions was introduced by Barndorff-Nielsen(1977), and firstly was used in various scientific fields, as the turbulence theory.
An important aspect is, that GH distributions embrace many special cases, respectively limiting distributions, of hyperbolic, normal inverse gaussian(NIG), and some others.
We consider X_{GH} the generalized hyperbolic Lévy-process.
The Lévy measure for the generalized hyperbolic distribution, $L(x) = ghyp(x)dx$

$$ghyp(x) = \left\{ \quad \frac{e^{\beta x}}{|x|} \left(\int_0^\infty \frac{\exp\left(-\sqrt{2y+\alpha^2}\,|x|\right)}{\pi^2 y \left(J_\lambda^2\left(\delta\sqrt{2y}\right) + Y_\lambda^2\left(\delta\sqrt{2y}\right)\right)} dy + \lambda e^{-\alpha|x|} \right) \qquad \lambda \geq 0 \right.$$

$$ghyp(x) = \left\{ \quad \frac{e^{\beta x}}{|x|} \left(\int_0^\infty \frac{\exp\left(-\sqrt{2y+\alpha^2}\,|x|\right)}{\pi^2 y \left(J_{-\lambda}^2\left(\delta\sqrt{2y}\right) + Y_{-\lambda}^2\left(\delta\sqrt{2y}\right)\right)} dy + \lambda e^{-\alpha|x|} \right) \qquad \lambda < 0 \right.$$

However, numerical evaluation of these expressions is cumbersome, especially for small values of $|x|$, where the decay of the exponential term in the numerator becomes very slow, because of the two Bessel functions J_λ and Y_λ appearing in the denominator .

For $\lambda \geq 0$ $\int_R L(x)$ is infinite, so Poisson intensity can not exist.

Generalized Hyperbolic distributions are infinitely divisible. This fact was shown by proving the infinite divisibility of the GIG. Are then infinitely divisible by Barndorff-Nielsen (1977).

We might get the characteristic function for the Lévy process X_{GH} by the Lévy-Khintchine representation of GH distribution. For an arbitrary λ that is rather unpleasant, but we are mainly interesting for $\lambda > 0$ cases, wich version is simpler.

Let the Lévy- measure $L(x) = ghyp_{\lambda \geq 0}(x)dx$

The Lévy-Khintchine representation of the natural logarithm of the characteristic function can be written as [2]

$$\log(\phi_{GH,t}(u)) = \log(-t\psi(u)) = \log \mathrm{E}\left[e^{iuX_{GH,t}}\right] = -t\int_{\{IR-0\}} \left(1 - e^{iux}\right) L(x)$$

It was pointed out by Barndorff-Nielsen (1977) that the hyperbolic distribution can be represented as a normal variance mean mixture where the mixing distribution is generalized inverse Gaussian GIG, with density

$$d_{GIG}(x) = \frac{(\psi/\gamma)^{\lambda/2}}{2K_\lambda\left(\sqrt{\psi\gamma}\right)} x^{\lambda-1} \exp\left(-\frac{1}{2}\left(\gamma x^{-1} + \psi x\right)\right) \qquad (x > 0) \qquad (10)$$

Let $\gamma = \delta^2$ and $\varphi = \alpha^2 - \beta^2$. This fact was shown by proving the infinite divisibility of the GIG. by Barndorff-Nielsen(1977)

Let, W_t a Wiener process with mean zero, and $T_t \approx GIG(\lambda,,\psi,\gamma)$, the increment of the intrinsec time process with density

$$d_{GIG}(x) = \frac{(\psi/\gamma)^{\lambda/2}}{2K_\lambda\left(\sqrt{\psi\gamma}\right)} x^{\lambda-1} \exp\left(-\frac{1}{2}\left(\gamma x^{-1} + \psi x\right)\right) \qquad (x > 0) \qquad (11)$$

Let $\gamma = \delta^2$ and $\psi = \alpha^2 - \beta^2$

by $X_{GH,t} = \{W_{T_t}, t \geq 0\}$ driving process

K_λ and are the modified Bessel functions of the third kind with orders λ .

2 For details see prause(1999)

For get symmetry $\beta = 0$ so now Let and $\psi = \alpha^2 \quad \gamma = \delta^2$

And the Laplace transform, at the symetric case, $\beta = 0$:

$$LAPLACE_{T_t}(s) = \int_0^{+\infty} e^{-sx} d_{GIG}(x)dx = \frac{\alpha^\lambda}{K_\lambda(\alpha\delta)} \frac{K_\lambda(\delta\sqrt{\alpha^2 + 2s})}{\left(\alpha^2 + 2s\right)^{\lambda/2}} \qquad (\mathrm{Re}(s) \geq 0)$$

For the general case [3]

$$LAPLACE_{T_t}(s) = . \frac{K_\lambda\left(\delta\sqrt{\alpha^2 - \beta^2 + 2s}\right)}{K_\lambda\left(\delta\sqrt{\alpha^2 - (\beta)^2}\right)} \left(\frac{\alpha^2 - \beta^2}{\alpha^2 - \beta^2 + 2s}\right)^{\lambda/2} \qquad (\mathrm{Re}(s) \geq 0)$$

And by Lema 1.37 Prause (1999) if we represent GH as a normal variance-mean mixture

$$f_{GH}(x) = \int_0^\infty N(x; \beta w, w) d_{GIG}(w; \lambda, \delta^2, \alpha^2 - \beta^2) dw$$

Where N is the normal density function with respect to mean and variance.

Therefore:

$$\phi_{GH}(u) = \int_{-\infty}^{+\infty} e^{iux} f_{GH}(x)dx = \int_{-\infty}^{+\infty} e^{iux} \int_0^\infty N(x; \beta w, w) d_{GIG}(w; \lambda, \delta^2, \alpha^2 - \beta^2) dwdx =$$

$$= \int_0^\infty \int_{-\infty}^{+\infty} e^{iux} N(x; \beta w, w) dx d_{GIG}(w; \lambda, \delta^2, \alpha^2 - \beta^2) dw =$$

$$\int_0^{+\infty} \exp(-(\frac{u^2}{2} - i\beta u)w) d_{GIG}(w)dw = e^{iu(\mu+\omega)} LAPLACE((\frac{u^2}{2} - i\beta u));$$

hence

$$\phi_{X_{GH},1}(u) = \frac{\left(\sqrt{\alpha^2 - \beta^2}\right)^\lambda}{K_\lambda\left(\delta\sqrt{\alpha^2 - \beta^2}\right)} \cdot \frac{K_\lambda\left(\delta\sqrt{\alpha^2 - (\beta + iu)^2}\right)}{\left(\sqrt{\alpha^2 - (\beta + iu)^2}\right)^\lambda}$$

The characteristic function of the time-t element of the convolution semigroup generated by

$$\phi_{X_{GH},t}(u) = \frac{\left(\sqrt{\alpha^2 - \beta^2}\right)^{\lambda t}}{K_\lambda\left(\delta\sqrt{\alpha^2 - \beta^2}\right)^t} \cdot \frac{K_\lambda\left(\delta\sqrt{\alpha^2 - (\beta + iu)^2}\right)^t}{\left(\sqrt{\alpha^2 - (\beta + iu)^2}\right)^{\lambda t}}$$

Note that we have be careful when taking the “t-th power” of the characteristic function; the main branch of the t-th power function, applied to the complex number $\phi_{GH,1}$, in general does *not* yield the desired characteristic function.

The domain of variation of the parameters is as follows. $\lambda \in IR, \alpha > 0, \beta \in (-\alpha, \alpha), \delta > 0$ and $\mu, \omega \in IR$. The functions K_λ and $K_{\lambda-1/2}$ are the modified Bessel functions of the third kind with orders λ and $\lambda - 1/2$, respectively.

[3] For details see prause(1999) lema 1.39

Different scale and location-invariant parametrizations of the generalized hyperbolic distributions have been proposed in the literature.

Subclasses:

Hyperbolic

Setting the first parameter $\lambda = 1$ yields the four-parameter class of hyperbolic distributions.

The log density of a hyperbolic distribution is a hyperbola, which is the origin of the name *hyperbolic distributio*n. Recall that for normal distributions the log-density is a parabola, so one can expect to obtain a reasonable alternative for heavy tail distributions.

Normal Inverse Gaussian Distribution(N.I.G.)

Setting $\lambda = -1/2$ leads to the class of Normal Inverse Gaussian (NIG) distributions. The name of this class stems from the fact that a NIG distribution can be represented as a variance-mean mixture of normal distributions, where the mixing distribution is inverse Gaussian. (See e. g. Barndorff-Nielsen (1998), Section 2).

In contrast to the case of hyperbolic distributions, the characteristic function of NIG is expressible by elementary functions: The Bessel function $K_{-1/2}(z)$ is equal to $K_{1/2}(z)$ by Abramowitz and Stegun (1968), 9.6.6, which in turn can be reduced to elementary functions

$$\phi_{X_{NIG},t}(u) = \frac{\left(\delta\sqrt{\alpha^2-\beta^2}\right)^{-(1/2)t}}{K_{-1/2}\left(\delta\sqrt{\alpha^2-\beta^2}\right)^{t}} \cdot \frac{K_{-1/2}\left(\delta\sqrt{\alpha^2-(\beta+iu)^2}\right)^{t}}{\left(\delta\sqrt{\alpha^2-(\beta+iu)^2}\right)^{-(1/2)t}}$$

$$= \exp(t\delta\sqrt{\alpha^2-\beta^2})\exp(-t\delta\sqrt{\alpha^2-(\beta+iu)^2})$$

Student-t Model

Praetz (1972) and Blattberg & Gonedes (1974) proposed for the asset price process to be driven by a process with increments that are Student t distributed. This occurs in a Generalized Hyperbolic, when $\lambda = \frac{-1}{2}\nu$, $\delta = \sqrt{\nu}$ and $\beta = \alpha = 0$,ie $\lambda < 0$.

Then we consider W_t the Wiener process $W_t \approx IN(0,\sigma^2 t)$, and we form the increment of the intrinsic time as $T_{t+1} - T_t \approx GIG(-\frac{\nu}{2},\nu,0) \approx \frac{\nu}{\chi^2(\nu)}$ $t \geq 0$.

Where $\chi^2(\nu)$ denotes the chisquare distribution with ν degrees of freedom. And we have the driving process $X_{STUDENT,t} = \{W_{T_t}, t \geq 0\}$. So that the subordinated process has t-Student increments.

Following Hurst (1995), $X_{STUDENT,t}$ as c.f.

$$\phi_{X_{STUDENT},t}(u) = \left(\frac{K_{\frac{1}{2}\nu}(\sigma\sqrt{\nu}|u|)(\sigma\sqrt{\nu}|u|)^{\frac{1}{2}\nu}}{\Gamma(\frac{1}{2}\nu)2^{\frac{1}{2}\nu-1}} \right)^t \quad u \in IR.$$

F.M.L.S.

The stable generalization of the familiar standard Brownian motion is often called the Lévy α-stable motion and is the subject of two recent monographs by Samorodnitsky and Taqqu (1994).

However, in the nondegenerate case of a $\alpha < 2$, any finite interval almost surely contains an infinite number of discontinuities, and unfortunately, a symmetric Lévy α-stable motion for log prices with $\alpha < 2$ has infinite expected arithmetic return.

Wu and Carr (2000) use a Lévy α-stable process, $X_{FMLS,t}$, with jumps and finite moments, finite but maximum negative skewness , which decreases like the reciprocal of the square-root of maturity, and finite kurtosis, which decreases with the reciprocal of maturity (see Konikov and Madan (2000) for a proof).
The use of a Lévy α-stable process retains the key advantages of a non-normal return distribution and self-similarity. The imposition of maximum negative skewness is required to deliver finite conditional moments of all orders for the stock price,.
Under the FMLS model, $X_{L_{\alpha,\beta=-1}}$ the structure of jumps is controlled by a maximum negatively skewed α-stable Lévy process. The arrival rate of jumps of size x is given by its Lévy measure

$$L(x) = l_{L_\alpha,-1}(x)dx$$

$$l_{L,\alpha,-1}(x) = c \pm / |x|^{\alpha+1}$$

Where $c_+ = 0$ for $x \in (0,\infty)$ and $c_- = 1/\Gamma(-\alpha)$ for $x \in (-\infty,0]$.

Although the random component of the process features only negative jumps, the predictable component of the process compensates by so much that the support of the distribution for any positive time is the entire real line.

$\int_{-\infty}^{\infty} l_{L_{\alpha,-1}}(x)dx = \infty$, so there are too many small jumps around the origin, and not poisson intensity can exist then.

For this type of Lévy process is easy to derive the characteristic function from the canoncal Lévy-Khintchine representation using the Lévy measure $L(x) = l_{L_{\alpha,-1}}(x)dx$, see Prause (1999).
In general, a stable random variable $x \approx X_{L_\alpha}(m,\sigma,\beta)$
The parameter α governs the thickness of tails. It takes values $\alpha \in$ (0,2] and in particular when α = 2 we get the Normal distribution. This is the most important parameter that characterises Lévy-Stable variables since it can be seen as a "departure" from the Gaussian case as α moves away from α = 2.

The parameter $\sigma \geq 0$ is the scale parameter. It cannot be interpreted as the standard deviation of the process since this is only true for the Gaussian case. However, the larger σ is, the 'wider' is the pdf of the random variable.

The parameter $\beta \in$ [-1,1] dictates the skewness of the density function. When β = 1 the distribution is "totally skewed" to the right and similarly, when β = -1 it is "totally skewed" to the left. When β =0 we have a symmetric pdf. The location parameter is $m \in IR$.

However it can be represented by its characteristic function:

$$\phi_{L_\alpha}(u) \equiv E[e^{iux}] = \quad \exp\left[ium - |u|^\alpha \sigma^\alpha \left(1 - i\beta(\operatorname{sgn} u) tan \frac{\pi\alpha}{2}\right)\right] \quad \text{if } \alpha \neq 1 \quad (14)$$

$$\exp\left[ium - |u|\sigma\left(1 + i\beta \frac{2}{\pi}(\operatorname{sgn} u)\log|u|\right)\right] \quad \text{if } \alpha = 1$$

As a result of this tail behavior, the Laplace transform of an α-stable variable $X_{L\alpha}(0, \sigma, \beta)$, is not finite unless $\beta = 1$. When $\beta = 1$, the Laplace transform is given by:

$$LAPLACE_{X_{L\alpha},t}(\lambda) \equiv E\left[e^{-\lambda x}\right] = \{ \quad \exp\left(-\lambda^\alpha \sigma^\alpha \sec \frac{\alpha\pi}{2}\right) \quad if\ \alpha \neq 1 \quad (Re((\lambda) \geq 0) \qquad (15)$$

$$LAPLACE_{X_{L\alpha},t}(\lambda) \equiv E\left[e^{-\lambda x}\right] = \{ \quad \exp\left(-\sigma \frac{2}{\pi} \lambda t \log \lambda\right) \quad if\ \alpha = 1$$

With $X_{L\alpha} \approx X_{L\alpha}(0, \sigma, -1)$ that is m=0 and $\beta = -1$.

The characteristic function of the log return $s_t = \log(S_t / S_0)$,that is

$$\phi_{s_t,t}(u) = IE\left[e^{ius_t}\right] = IE\left[e^{iu(\mu+\omega_s)}\right] IE\left[e^{iuX_{l_\alpha}}\right] = \exp((\mu + \omega_s)iut)\phi_{X_{L_\alpha,t}}(u) \quad (16)$$

That means, $s_t \approx X_{L_\alpha,t}(\mu + \omega_s, \sigma, -1)$ so we get the characteristic function from (16) given by:

$$\phi_{s_t,t}(u) = \exp((\mu + \omega_s)iut)\exp(-t(iu\sigma)^\alpha \sec \frac{\pi\alpha}{2}) \quad 1 < \alpha < 2.$$

Laplace Model

In Mittnik and Rachev (1993) the asset process is driven by the symmetric geometric-stable process is proposed. In particular the symmetric Laplace process belongs to this class of processes playing the role of the Wiener process in the geometric summation scheme. For the subordinated process $X_{LAPLACE,t}$ to become the symmetric Laplace process we require the intrinsic time process T to be a negative exponential process with increments.

$$T_{t+1} - T_t \approx Exp(\lambda) .$$

For all $t \geq 0$ and $\lambda > 0$ .The increment of T has probability density function $f_\lambda(x) = \lambda \exp(-\lambda x)$. for $x > 0$.

We consider the Brownian motion $W_t \approx N(0, \sigma^2 t\ )$
Then under subordination we have the driving Lévy process
$X_{LAPLACE,t} = \{W_{T_t}, t \geq 0\}$
that means, like always, the distribution of the process $X_{LAPLACE,t}$ will be defined as a normal variance-mean mixture where $Exp(\lambda)$ is the mixing distribution.

The Laplace transform is given by:

$$LAPLACE_{T,1}(u) = \frac{\lambda}{\lambda + u} \quad \text{for } u \geq 0.$$

The Fama Model

Mandelbrot (1963, 1967) and Fama (1963, 1965) proposed the log asset price process to be driven by a symmetric α-stable Lévy process. For the subordinated driving process $X_{FAMA,t} = \{ W_{T_t}, t \geq 0 \}$ to become a symmetric α-stable Lévy process it is required the intrinsic time process *T* to be a scaled maximally skewed α /2-stable Lévy process (see e.g. Mandelbrot & Taylor (1967)). This means we require T to have stationary non-negative independent increments

$$T_{t+s} - T_t \approx S_{\frac{\alpha}{2}}\left(2(\cos(\pi\alpha/4)s)^{\frac{2}{\alpha}}, 1, 0\right) \qquad \text{s,t} \geq 0$$

where $\alpha \in (0,2)$ and the notation is consistent with Samorodnitsky & Taqqu (1994).

The first parameter α /2 denotes the index of stability. The second parameter is a scale parameter and is chosen such that the increments are closed under convolutions. The third parameter represents the skewness and here obtains its maximal value of 1. Finally the fourth parameter is a location parameter that is set to zero. We note here that the increments are non-negative because a stable distribution with index of stability less than one, maximal skewness of one and zero location parameter is always non-negative (see e.g. Samorodnitsky & Taqqu (1994)).

The increment of T has c.f.

$$\phi_{T,s}(u) = \exp\left\{ -s|u|^{\frac{\alpha}{2}} 2^{\frac{\alpha}{2}} \cos(\pi\alpha/4)(1 - i\tan(\pi\alpha/4)\operatorname{sgn}(u) \right\} \qquad u \in IR$$

and Laplace transform

$$LAPLACE_{T_t}(\lambda) = \exp\left\{ -s(2\lambda)^{\frac{\alpha}{2}} \right\} \qquad \operatorname{Re}(\lambda) \geq 0 \qquad (19)$$

By using $\phi_{X_{FAMA,t}}(u) = LAPLACE_{T_t}(\frac{1}{2}\sigma^2 u^2)$ and (19) the c.f. of the increment of L is therefore

$$\phi_{X_{FAMA},t}(u) = \exp(-\sigma^\alpha t|u|^\alpha) \qquad u \in IR$$

. Peter K. Clark Model

Clark, P.K. (1973) proposed the process $X_{CLARK,t} = \{W_{T_t}, t \geq 0\}$ where W_t is a Wiener process $W_t \approx IN(0, \sigma^2 t)$, and T has increments

$$T_{t+1} - T_t \approx \log\left(0, e^{-\frac{1}{2}\varphi^2}, \varphi\right) \qquad t \geq 0 .$$

That is, The increment $T_{t+1} - T_t$, where $t \geq 0$ has probability density function.

$$f_{T_1}(x) = \frac{1}{\varphi\sqrt{2\pi}}\, x^{-1} \exp\left\{-\frac{(\log x + \frac{1}{2}\varphi^2)^2}{2\varphi^2}\right\} \qquad x>0$$

where $\varphi \geq 0$. Here log denotes the lognormal distribution (consistent with Kotz et al. (1988)) where the first parameter is a location parameter that is set to zero. The second parameter is a scale parameter and is chosen such that the increments $T_{t+1} - T_t$, $t \geq 0$ have a mean of one. Finally, the third parameter φ is a shape parameter which reflects the tail thickness. The larger φ , the larger the tail thickness. Furthermore it can be shown that as $\varphi \to 0$ the intrinsic time process *T* asymptotically approaches physical time which leads us to the lognormal model. The Clark model has certain similarity to the stochastic model proposed by Hull & White (1987).
The Hull & White (1987) stochastic volatility model has the logarithmic instantaneous variance modelled by a Wiener process with drift.

By using the Levy-Kinntchine representation we have

$$\phi_{X_{CLARK,t_1}}(u) = \int_0^{+\infty} \exp\left\{-\frac{1}{2}\sigma^2 x u^2\right\} f_{T_1}(x)dx \qquad u \in IR \quad \text{therefore}$$

$$\phi_{X_{CLARK,t_1}}(u) = \frac{1}{\varphi\sqrt{2\pi}} \int_0^{+\infty} y^{-1} \exp\left\{u^2 y + \frac{(\log y + \frac{1}{2}\varphi^2 - \log\sigma^2)^2}{\varphi^2}\right\} dy \qquad u \in IR$$

The c.f. for $X_{CLARK,s} - X_{CLARK,0}$ for all s≥ 0 is then defined by

$\phi_{X_{CLARK,t_1}}(u)$ =($\phi_{X_{CLARK,t_1}}(u)$)^{s} since the increments are stationary and independent

.The Meixner Process

The Meixner process is a special type of Lévy process which originates from the theory of orthogonal polynomials. It is related to the Meixner-Pollaczek polynomials.

The Meixner process is very flexible, has a simple structure and leads to analytically and numerically tractable formulas. It was introduced in Schoutens,W.and Teugels,J.L.(1998) also and originates from the theory of orthogonal polynomials and was proposed to serve as a model of financial data in Schoutens(2002).This is another infinite jumps models, so it allows wider tails and prolonged skew.

Schoutens(2002) provide evidence that the Meixner model performs also significabtly better than the Black-Scholes model.
The Meixner process $\{M_t, t \geq 0\}$ is a stochastic process wich stars at zero and has independent and stationary increments.

$\{M_t, t \geq 0\}$ has no Brownian part and a pure jump part. That Meixner process has no Brownian part and a pure jump part governed by the Lévy measure $L(x) = m(x)dx$

$$L(x) = \frac{\exp(bx/a)}{xsinh(\pi x/a)} dx$$

The Lévy measure for this GZ-Process is given by

$$L_{GZ}(x) = \begin{cases} \dfrac{2d\exp(2\pi b_1 x/a)}{|x|(1-\exp(2\pi x/a))} dx & x < 0 \\ \dfrac{2d\exp(2\pi b_2 x/a)}{x(1-\exp(-2\pi x/a))} dx & x > 0 \end{cases}$$

For

$b_1 = \frac{1}{2} + \frac{b}{2\pi}$ and $b_2 = \frac{1}{2} - \frac{b}{2\pi}$

we obtain $L(x) = m(x)dx$

The Generalized z-distribution (GZ), Grigelionis, B. (2000) is defined through the characteristic function:

$$\phi_{GZ}(z; a, b_1, b_2, d, m) = \left(\frac{B\left(b_1 + \frac{i_{az}}{2\pi}, b_2 - \frac{i_{az}}{2\pi}\right)}{B(b_1, b_2)} \right)^{2d} \exp(imz)$$

Where $a, b_1, b_2, d > 0$ and $m \in R$.

This distribution is infinitely divisible and we can associate with it a Lévy process, such that its time t distribution has characteristic function $\phi_{GZ}(z; a, b_1, b_2, dt, mt)$.
Again setting

$b_1 = \frac{1}{2} + \frac{b}{2\pi}$ And $b_2 = \frac{1}{2} - \frac{b}{2\pi}$

The Meixner (a, b, d, m) c.f. is given by

$$\phi_{M,t}(u) = IE[\exp(iuM_1)] = \left(\frac{\cos(b/2)}{\cosh \frac{au - ib}{2}} \right)^{2d}$$

Where $a > 0$, $-\pi < b < \pi$, $d > 0$, and the distribution of an increment over $[s, s+t], s, t \geq 0$, i.e. $X_{t+s} - X_s$ has $(\phi(u))^t$ as characteristic function

The distribution of the log returns, that is, the variable $s_t = \log(S_t / S_0)$ therefore:

$$\phi^s{}_{s_t}(u) = IE\left[e^{ius_t}\right] = \exp(iu(\mu + \omega_s)t)\ \phi_{X_{M,t}}(u)$$

$$\phi^s_{X_{M,t}}(u) = IE\left[\exp(iuX_{M,t})\right] = \exp(-t\psi_M(u))$$

Where $\psi_M(u)$ is the characteristic exponent, so to organize the forward price at $S_0\exp(\mu t)$, we take the value of ω_s as defined by $\phi_{M,t}(-i)$

Truncated Lévy Processes

The infinite moments of Lévy-Stable random variables are due to the fact that the so called "fat-tails" do not allow finiteness of moments.

The truncated Lévy distribution is Lévy like in the central part of the distribution, but has a cut-off in the far tails that is faster than the Lévy power law tails.

The cut-off will ensure the variance of the truncated Lévy distribution is finite.

What is interesting is the existence of a characteristic timescale separating the Lévy and Gaussian regimes. This timescale can be arbitrarily long due to the stable nature of the Lévy distribution.

Mantegna and Stanley (1994) were the first to make the above observations

Inspired by these results, Koponen (1995),and later Cartea, A. (2002), Boyarchenko, S.I. and Levendorskii, S.Z. (2000) among others researchers derived an analytical form for the characteristic function of a truncated Lévy distribution with an exponential cut-off in the tails.

Truncation of the tails is one obvious choice to ensure finite moments. Mantegna and Stanley (1994) were the first to propose a truncation or cut-off of the tails at some arbitrary point. Koponen (1995) introduced a smooth exponential cut-off of the tails. For the sake of brevity, we consider only symmetric distributions of this family, with characteristic function defined by:

$$\phi_{X_{KOPONEN,t}}(u) = \exp\left[-\sigma^2\lambda^2\left[1 - ((u/\lambda)^2 + 1)^{\alpha/2}\cos(\alpha arctan(u/\lambda))\right]/\alpha(\alpha-1)\right]$$

Where $\sigma > 0, \lambda > 0$ and $\alpha \in (0,2], \alpha \neq 1$ are the model's parameters. We have here chosen a normalization s that the variance is independent of α and λ.

Inside this family for α =2,we obtain $\phi_{X_{KOPONEN,t}}(u) = \exp(-\sigma^2 u^2/2)$ which means that is a gaussian distribution. In the limit $\lambda \to +0$, we recover a Lévy distribution with

$$\phi_{X_{\lambda\to+0,t}}(u) = \exp\left[-c|u|^\alpha \cos(\alpha\pi/2)/\alpha(\alpha-1)\right].$$

Under Koponen approach Cartea, A. (2002), consider the Lévy measure $L(x)$ of a Lévy-Stable random variable $x \approx X_{L_\alpha}\left(0,\sigma,\beta\right)$ described by

$$L(x) = l(x)dx$$

Where

$$l(x) = \begin{cases} C(1-p)\left|x\right|^{-1-\alpha} & x<0 \\ Cpx^{-1-\alpha} & x>0 \end{cases}$$

$C>0$ a scale constant, $1 \geq p \geq 0$ and α is the scale constant. Now, introduce an exponential cut-off to obtain the truncated Lévy measure $L_{trunc}(x) = l_{trunc}(x)dx$ for the truncated Lévy process $X_{TLP_\alpha}(0,\sigma,\beta,\lambda)$.

$$l_{trunc}(x) = \begin{cases} C(1-p)\left|x\right|^{-1-\alpha}\exp(-\lambda\left|x\right|) & x<0 \\ Cpx^{-1-\alpha}\exp(-\lambda x) & x>0 \end{cases}$$

When $\lambda \to 0$, we recover the usual Lévy-Stable measure.

Using the Lévy-Khintchine representation, the characteristic function for the Truncated Lévy distribution is

$$\phi_{X_{TLP}}(u) = \begin{cases} C\Gamma(-\alpha)\left\{p(\lambda-iu)^\alpha + (1-p)(\lambda+iu)^\alpha - \lambda^\alpha\right\} & 0<\alpha<1 \\ C\Gamma(-\alpha)\left\{p(\lambda-iu)^\alpha + (1-p)(\lambda+iu)^\alpha - \lambda^\alpha - iu\alpha\lambda^{\alpha-1}(1-2p)\right\} & 1<\alpha\leq 2 \end{cases}$$

S.I. Boyarchenko and S.Z. Levendorskii (2000) use a purely non-Gaussian Lévy-process, called Kobol process with Lévy measure $L(x) = l(x)dx$ with order $\alpha \in \left(0,2\right)$ and steepness parameters $\lambda_- < 0$ and $\lambda_+ > 0$.

$$l(x) = \begin{cases} C_+ {x_+}^{-1-\alpha}\exp(\lambda_- x) & x<0 \\ C_- {x_-}^{-1-\alpha}\exp(\lambda_+ x) & x>0 \end{cases}$$

Where $x_+ = max\left\{x,0\right\}$ and $C_{\substack{+\\-}} > 0$. Direct calculation show that if $\alpha \in \left(0,2\right), \alpha \neq 1,$ and $C_+ = C_- = C > 0,$ then the characteristic function $\phi_{X\,kobol}(u) = \exp(-t\psi_{kobol}(u))$ defined by characteristic exponent is of the form

$$\psi_{kobol}(u) = C\Gamma(-\alpha)\left\{{\lambda_+}^\alpha - (\lambda_+ + iu)^\alpha + (-\lambda_-)^\alpha - (-\lambda_- - iu)^\alpha\right\}$$

MODELS ESTIMATION

The p.d.f. is calculated by taking the Fourier transform of the characteristic function. Note that the characteristic function is by definition the inverse Fourier transform of the p.d.f.
Following Carr, Geman, Madan, Yor (2001),
The fast Fourier transform was used to invert the characteristic function once for each parameter setting. This method efficiently renders the level of the probability density at a prespecified set of values for returns. For integration spacing of .25; where *N* is a power of 2 used in the fast Fourier discrete transform. For *N = 4096*; the return spacing is too coarse at .00613592. We used instead *N = 16384*; and a return spacing of 0 .00153398.

With the density evaluated at these prespecified points, we binned the return series by counting the number of observations at each pre-specified return point, assigning data observations to the closest pre-specified return point. We then searched for parameter estimates that maximized the likelihood of this binned data.
In order to use maximun likelihood estimation procedure for the Clarck model, the c.f. involves an improper integral which we calculat by transforming the integral to a definite integral and then use the extended midpoint rule (see e.g. Press et al.(1992),Chapter 4.

THE DATA

The data comprise 3037 daily closing values of the future of IBEX-35.

We formed the time series of daily log price relatives, as:
Let n be the number of observations of the index and $S_j = S_{t_j}$ be the observation of the index on day t_j, *j*=0,1,2,..,n-1. Here t_0 is the first observation (January 7, 1990)and t_{n-1} is the last obsevation ,(December 30,2002). The sample of increments of the log index is defined by

$$R_j \approx \frac{S_j - S_{j-1}}{S_{j-1}} \qquad \textit{(j=1,2,…,n-1)} \quad \text{and } \log(1+x) = x + o(x) \text{ For } x \to 0$$

hence $R_j = \log(S_j / S_{j-1}) = \log(1 + \frac{S_j - S_{j-1}}{S_{j-1}}) \approx \frac{S_j - S_{j-1}}{S_{j-1}}$

Then

$$R_j = \log S_j - \log S_{j-1} \qquad \textit{(j=1,2,…,n-1)}$$

COMPARISON AND TESTS OF THE MODELS

The Likehood ratio

To answer the question, "Which market index model, in comparison to the classical lognormal model, is the best model to fit and explain the observed data?", one has to specify a criterion. Maximun Likelihood it may be viewed as an overall measure of goodnes of fit.

We choose the best model to be the model whose likelihood value, $LogL$ is significantly the largest, when compared to the likelihood value for the classical lognormal model. That is, the model

that is adding the most information to the classical lognormal model by allowing some extra parameter. We can check what we gain by allowing for extra parameters.
We define the likelihood ratio. [4]

$$\Lambda = \frac{LogL_{\log normal}}{LogL_{other}}$$

Where $LogL_{\log normal}$ is the likelihood value of the two parameter classical lognormal model and $LogL_{other}$ is the likelihood value of the other model we are testing.
The asymptotic distribution of $-2\log\Lambda$ is chisquare, with degrees of freedom equal to the difference in the number of parameters between the two models (see e.g. Hoel et al. (1971) §3.4).Under this criterion above, the model with the larger absolute likelihood value will be the best model.

Model	**log L**	**Parameter Estimates**
Classical Lognormal	**-8.6676e+003**	$\sigma = 0.0139$ $m = 0.0003$
Symmetric Laplace	-8.7864e+003	$\sigma = 0.0102$ $m = 0.0000$
Eraker	-8.7002e+003	$\sigma = 0.0132$ $\lambda = 0.0374$ $\eta = 0.0174$ $m = -0.0030$
Merton exponential	-8.7830e+003	$\sigma = 0.0119$ $\lambda = 0.0430$ $\varsigma = 0.0008$ $\xi = 0.0268$ $m = 0.0000$
Double exponential	-8.8106e+003	$\sigma = 0.0088$ $\lambda = 0.7904$ $\kappa = 0.\ 0003$ $\eta = 0.0085$ $m = 0.0000$
Hyperbolic $\lambda = 1$	-8.7891e+003	$\delta = 0.0001$ $\alpha = 98.3092$ $\beta = -0.0002$ $m = 0.0005$
Normal Inverse Gaussian $\lambda = 1/2$	-8.8147e+003	$\alpha = 75.9420$ $\beta = -4.9442$ $\delta = 0.0145$ $m = 0.0013$
FMLS	-8.7198e+003	$\alpha = 1.9545$ $\sigma = 0.0094$ $m = 0.0003$
Fama-Mandelbrot	-8.7966e+003	$\sigma = 0.0083$ $\alpha = 1.7546$ $m = 0.0005$
t-Student	-8.8150e+003	$\sigma = 0.0105$ $\nu = 1.7546$

4 A version of this ratio has been used by some authors before see Hoel et al (1971)

		$m = 0.0005$
Meixner	-8.8151e+003	$d = 0.4291$ $a = 0.0298$ $b = -0.1565$ $m = 0.0013$
Variance gamma	-8.8078e+003	$\sigma = 0.0137$ $\nu = 0.6089$ $\theta = -0.0009$ $m = 0.0012$
Clark	8.7876e+003	$\mu = -8.9548$ $\varphi = 1.3468$ $m = 0.0003$
CGMY	-8.8153e+003	C = 0.0036 G = 47.9424 M = 57.0024 Y = 1.0763 $\eta = -0.0010$ $m = 0.0012$
Boyarchenko	-8.8142e+003	$c = 0.0008$ $\alpha = 1.3231$ $\lambda_+ = 30.7690$ $\lambda_- = -39.7861$ $m = 0.0010$
Cartea	-8.7981e+003	$c = 0.0001$ $\alpha = 1.7150$ $\lambda = 1.6663$ $p = 0.3968$ $m = -0.0003$
Koponen	-8.8140e+003	$\sigma = 0.0139$ $\alpha = 1.1174$ $\lambda = 49.3473$ $m = 0.0005$

Table: The Likehood Ratio results

.Some Kurtosis

A more intuitive way of comparing the best models would be to use their kurtosis. The one day kurtosis for Meixner, t-Student, and CGMY are given below . These values can be compared to the sample one day measure of kurtosis of Ibex-35.

MODEL	**Kurtosis**
MEIXNER (4 parameters model)	5.3589
CGMY (6 parameters model)	6.3745
t-STUDENT(3 parameters model)	15.4275
IBEX-35	6.0571

Table: Some model's kurtosis

Some people may argue that the large negative increment caused by the stock market crashes like September 2001, is too much influencing in these results. Namely in favour of a model with very heavy tails opposed to one with less heavy tails. For principle reasons we do not like to exclude the extreme events such as stock market crashes from the sample because it is this feature (namely tail heaviness) we are explicitly emphasising to model in the simplest way.

The Kolmogorov distance

As a mesure for the goodness of the fit we also can use distances between the fitted data and the empirical cumulative density function (CDF): The Kolmogorov distance is defined as the supremum over the absolute differences between two cumulative density functions.

$$K = \max_{x \in IR} \left| F_{emp}(x) - F_{est}(x) \right|$$

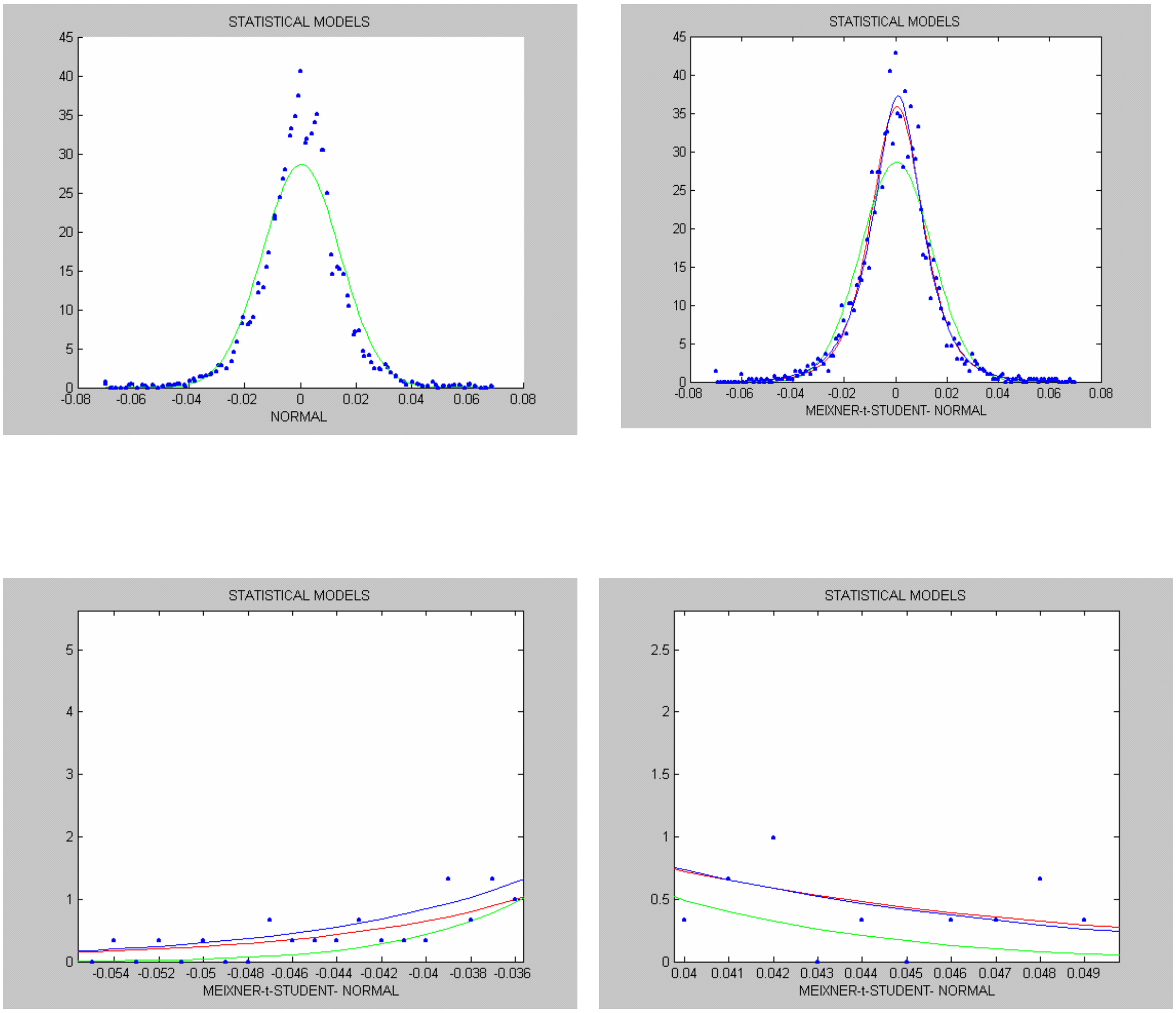
Model	**Kolmogorov-Statistic**
Classical Lognormal	**0.5079**
Symmetric Laplace	0.4961
Eraker	0.4977
Merton exponential	0.4881
Double exponential	0.4971
Hyperbolic $\lambda = 1$	0.4962
Normal Inverse Gaussian $\lambda = 1/2$	0.4957
FMLS	0.4799
Fama-Mandelbrot	0.4799
t-Student	0.4802
Meixner	0.4978
Variance gamma	0.4926
Clark	0.4973
CGMY	0.4925
Boyarchenko	0.4876
Cartea	0.4798
Koponen	0.4919

Table : Kolmogorov-Statistic results

2.7.3. Graphical Comparison

We provide plots of the best fitting models and some others interesting plots of models inside the same family or with very close likelihood function.
It is obvious that all models are leptokurtic, i.e. the peak in the center is higher and there is more mass in the tails than for the normal distribution.

Figures 2.10 (*probability density functions*)



.Conclusions

We tested a large number of asset models based on Lévy processes, whose returns have distributions with heavy tail properties. The parameters of interest were estimated by maximum-likelihood.

It is clear that there is not only a model by far better than the rest but, sophisticated models *as CGMY, NIG, Koponen, Boyarchenko, Kou, Meixner,* with four, five, and six, parameters fit reasonably better the returns of the Spanish index than models with less parameters . However, we should outstand the *Student- t* model, which reflects the tail probabilities of this market, just like these models above but only using one extra parameter to the celebrated *lognormal* model.

Our preliminary investigations suggested that stochastic time change might be profitably used to improve the fit of our models even further.

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