The deterministic control problem in continuous time

Let us consider the dynamic optimization problem,

$$\underset{v_t}{Max}\int_0^T f(x_t, v_t, t)dt$$

subject to the constraint,

$$\dot{x}_t = h(x_t, v_t, t)$$

and given x_0

where v_t is known as the control variable, x_t being the state variable. Both could be vectors, in which case there could several restrictions similar to the one above. We write the Hamiltonian for this problema as

$$H(x_t, v_t, \lambda_t, t) = f(x_t, v_t, t) + \lambda_t h(x_t, v_t, t),$$

where λ_t are co-state variables, one for each restriction, having the interpretation of shadow prices, or marginal value of the corresponding so-state variable at time t. The differential equation defining the constraint is the law of motion of the system, and we need as many of these equations as state variables. It indicates how will be the change in a state variable, as a function of its current level, and the actions taken in that period.

Pontryagin's principle indicates that maximization of the Hamiltonian by choice of the v_t sequence leads to optimality conditions:

1) $$\frac{\partial H}{\partial v_t} = 0 \rightarrow \frac{\partial f}{\partial v_t} + \lambda_t \frac{\partial h}{\partial v_t} = 0$$

which is the state equation, also known as order condition, and

2) $$\dot{\lambda}_t \equiv \frac{\partial H}{\partial x_t} = -\frac{\partial f}{\partial x_t} - \lambda_t \frac{\partial h}{\partial x_t},$$

known as the "co-state equation" (one for each state variable).

If the control variable v_t is restricted in sign, $v_t \geq 0$, the state equation becomes,

$$\frac{\partial H}{\partial v_t} \equiv \frac{\partial f}{\partial v_t} + \lambda_t \frac{\partial h}{\partial v_t} \leq 0 \text{ together with } v_t \frac{\partial H}{\partial v_t} = 0.$$

Transversality condition

If the terminal value of the state variable, $x(T) \equiv x_T$ is restricted in sign, as it is usually the case in economic applications of the maximum principle, where a typical state variable is the stock of productive capital in the economy, we have as transversality condition,

$$x_T \geq 0, \; x_T \lambda_T = 0,$$

which implies that either $x_T = 0$, or else, $\lambda_T = 0$. If, on the contrary, x_T is not restricted in value or sign, then we must have $\lambda_T = 0$.

If the planning problem has an infinite horizon, the transversality condition becomes,

$$\lim_{T\to\infty} x_T \geq 0, \quad \lim_{T\to\infty} x_T \lambda_T = 0$$

when x_T is not restricted in sign, and

$$\lim_{T\to\infty} \lambda_T \geq 0,$$

when x_T is restricted in sign or value.

The discounted problem

Let us now assume that, as it is the case in many economic applications and, specifically, in growth problems, the global intertemporal objective is the result to aggregate over time a given single-period objective function, subject to some time discount. That is, function $f(x_t, v_t, t)$ is of the form,

$$f(x_t, v_t, t) = e^{-rt} g(x_t, v_t, t)$$

where the time discount makes the contribution of a given level of the objective function to be lower the farther it occurs into the future. If, for simplicity, we assume that $f(x_t, v_t, t), g(x_t, v_t, t)$ or $h(x_t, v_t, t)$ do not change with time $t,$ we will have the control problem,

$$\underset{v_t}{Max} \int_0^T e^{-rt} g(x_t, v_t) dt$$

subject to the constraint,

$$\dot{x}_t = h(x_t, v_t)$$

and given x_0

with Hamiltonian,

$$H(x_t, v_t, \lambda_t) = e^{-rt} g(x_t, v_t) + \lambda_t h(x_t, v_t)$$

state equation or order condition,

$$\frac{\partial H}{\partial v_t} \equiv \frac{\partial f}{\partial v_t} + \lambda_t \frac{\partial h}{\partial v_t} = e^{-rt} \frac{\partial g}{\partial v_t} + \lambda_t \frac{\partial h}{\partial v_t} = 0$$

co-state equation,

$$\dot{\lambda}_t \equiv -\frac{\partial H}{\partial x_t} = -\frac{\partial f}{\partial x_t} - \lambda_t \frac{\partial h}{\partial x_t} = -e^{-rt} \frac{\partial g}{\partial x_t} - \lambda_t \frac{\partial h}{\partial x_t} \qquad (1)$$

and the same transversality condition as before.

All variables are discounted as of time $t = 0.$ In particular, the multiplier λ_t converts the contribution of the state variable x_t to the Hamiltonian in units valid at the initial period, providing us with the value of the state variable at time t, discounted at time $t = 0$. However, it is more useful to write the problem in terms of their time t values, i.e., their current values. Besides, unless we do so, the differential equations defining the optimality conditions would depend on the variable t, due to the presence of the e^{-rt}-factor.

To work in current values, we rewrite the Hamiltonian,

$$H(x_t, v_t, \lambda_t) = e^{-rt}\left[g(x_t, v_t) + \lambda_t e^{rt} h(x_t, v_t)\right]$$

and define the *current-value multiplier*,

$$\mu_t = \lambda_t e^{rt}$$

which provides us with the shadow price or marginal value of the state variable at time t, in terms of that same period. We also define the current value *Hamiltonian*, H^*,

$$H^* = e^{rt} H = g(x_t, v_t) + \lambda_t e^{rt} h(x_t, v_t)$$

where we can see that the solution to the problem of maximizing H^* is the same as the solution to the problem of maximizing H.

Furthermore, using (1), we get,

$$\dot{\mu}_t = r\mu_t + e^{rt}\dot{\mu}_t = r\mu_t - e^{rt}\frac{\partial H}{\partial x_t} = r\mu_t - \frac{\partial g}{\partial x_t} - \mu_t\frac{\partial h}{\partial x_t} = r\mu_t - \frac{\partial H^*}{\partial x_t}$$

and, since,

$$\frac{\partial H}{\partial v_t} = \frac{\partial\left(e^{-rt}H^*\right)}{\partial v_t} = e^{-rt}\frac{\partial H^*}{\partial v_t}$$

we get,

$$\frac{\partial H}{\partial v_t} = 0 \Leftrightarrow \frac{\partial H^*}{\partial v_t}$$

so the state and co-state equation can also be written,

$$\frac{\partial H^*}{\partial v_t} = \frac{\partial g}{\partial v_t} + \mu_t\frac{\partial h}{\partial v_t}0$$

co-state equation,

$$\dot{\mu}_t \equiv r\mu_t - \frac{\partial H^*}{\partial x_t} = r\mu_t - \frac{\partial g}{\partial x_t} - \mu_t\frac{\partial h}{\partial x_t} \quad (2)$$

We can perform a similar transformation in the transversality conditions, to have for the infinite horizon case, in terms of the current-value multiplier,

$$\lim_{T\to\infty} x_T \geq 0, \quad \lim_{T\to\infty} e^{-rT} x_T \mu_T = 0 \quad (3)$$

when x_T is restricted in sign, and

$$\lim_{T\to\infty} e^{-rT}\mu_T \geq 0, \quad (4)$$

when x_T is not restricted in sign or value.

Summarizing, when there is a discount factor in the objective function, the Hamiltonian can be written in two alternative forms, depending on the way the multipliers are defined, and we need to be careful with using the appropriate expression for the optimality conditions. In economic models with time discount, expressions (3) and (4) are often used.