Preliminary version

OPTIMAL TIME-CONSISTENT FISCAL POLICY IN AN ENDOGENOUS GROWTH ECONOMY WITH PUBLIC CONSUMPTION AND CAPITAL

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ABSTRACT

In an endogenous growth model with public consumption and investment, we explore the time-consistent optimal choice for two policy instruments: an income tax rate and the split of government spending between consumption and investment. We compare the Markovian optimal policy with the Ramsey policy as well as with the solution to the planner's problem under lump-sum taxation. For empirically plausible parameter values we find that the Markov-perfect policy implies a higher tax rate and a larger proportion of government spending allocated to consumption than those chosen under a commitment constraint. As a result, economic growth is slightly lower under the Markov-perfect policy than under the Ramsey policy, with growth under lump-sum taxes being highest.

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1. Introduction

The relevance of time consistent policies stems from the fact that the government has no incentive to change its policy once private agents have made their decisions conditional on the announced policy. Unfortunately, the difficulty in solving for this optimality policy problem has led academic research into the characterization of the more limited Ramsey optimal policies. The same difficulty also explains that most research on time-consistent optimal policies has been done in exogenous growth environments. Ortigueira (2006) and Klein, Krusell and Rios-Rull (2008) consider the same stylized exogenous growth model with leisure and public consumption in the utility function to characterize the optimal time-consistent tax policy. Klein, Krusell and Rios-Rull (2008) consider a game in which the government is a dominant player that takes the optimal reaction of private agents as given when deciding the optimal policy. Ortigueira (2006) compares the results obtained under such game structure with an alternative design of the game in which the government and private agents make their respective decisions simultaneously, and characterizes the behavior of the economy along the transition to the optimal steady-state. These authors consider alternative fiscal structures with a single instrument: either a single tax levied on total income, a single tax on capital income or a single tax on labor income. Martin (2010) follows the same game structure as Klein, Krusell and Rios-Rull (2008) and considers simultaneously different tax rates for capital and labor income, to solve for the optimal time consistent choice for both fiscal instruments.

Park and Philippopoulos (2004) consider a one-sector endogenous growth economy with inelastic labour supply, where the government raises tax revenues on the stock of private capital and uses the proceeds to finance public investment and the public provision of consumption goods. In that setup, the authors characterize the optimal fiscal policy under commitment, conditional on a given distribution of public resources between public consumption and investment. Using a graphic representation of the analytical conditions for existence of equilibrium, they show that there may be either zero, one or two second-best steady-states, providing examples of plausible points in the parameter space for which either situation arises. Their analysis suggests the need of getting rid of the commitment restriction so as to characterize the time consistent optimal policy.[1]

In this paper we show that it is possible to avoid the two limitations mentioned above, by describing how to characterize the optimal time consistent fiscal policy in endogenous growth models. This is of central importance for the literature on optimal taxation, since it allows us to escape from the approximation involved in limiting the analysis to the Ramsey solution. Moreover, considering endogenous growth economies is essential, not only as a more plausible representation of actual economies, but also for explicitly taking into account the effect of fiscal policy on the rate of growth.

Our model considers an economic environment similar to that in Park and Philippopoulos (2004), with productive public capital, public consumption in the utility function of private agents, and a single income tax. We incorporate endogenously time-varying government expenditures, a general utility function with constant relative risk aversion on private and public consumption and an incomplete depreciation of capital, and the analysis is carried out under both inelastic and elastic labor supply assumptions. For standard parameterizations of our model, we show that a reasonable time consistent optimal policy exists described by the optimal choice of both, the income tax and the split of public spending between consumption and production activities.

When comparing the optimal Markov-perfect and Ramsey policies, we find that: *i*) the income tax rate is higher under the time consistent policy, since the Markov government cannot internalize the distortionary effects of the current tax on the investment in previous periods (see Ortigueira, 2006, in a neoclassical growth framework), *ii*) the proportion of public resources devoted to consumption is higher under the Markov government than under the Ramsey government, since the former only takes into account the current period effects, thereby giving priority to current consumption, with an immediate effect on utility, rather than to investment, whose effects on production and welfare will mainly take place in future periods, and *iii*) as a result, economic growth is slightly lower under the Markov-perfect policy than under the Ramsey policy, with the growth rate under lump-sum taxes being the highest.

[1] Malley et al. (2002) partially overcomes this weakness by obtaining the Markov tax policy in a model economy with logarithmic utility and complete depreciation of capital for an exogenously given split of government spending.

The implication is that a government that is aware that society knows its inability to pledge future policy decisions, should impose a slightly higher tax rate and devote a higher share of public resources to consumption, with a relatively lower rate of growth.

Finally, we introduce leisure as an argument in the utility function and characterize both the optimal time-consistent fiscal policy (Markov) and the one obtained under commitment (Ramsey policy). The results are qualitatively similar to those obtained under an inelastic labor supply. For the particular case of logarithmic preferences and complete depreciation of capital, the optimal time-consistent tax rate and the split of government spending are equal to the ones for the inelastic labor supply.

A natural extension should consider different tax rates for capital and labor rents. Another relevant extension would numerically characterize the time-consistent policy in an endogenous growth model with public debt and a non-trivial transitional dynamics to the balanced growth path.

In section 2 we describe the model economy and analyze the competitive equilibrium conditions. In section 3 we characterize the time-consistent optimal policy, while the optimal policy under commitment is determined in section 4. The time-consistent optimal policy and the optimal policy under commitment are compared in section 5. In section 6 we compute the welfare loss under the time-consistent optimal policy relative to the planner's allocation under lump-sum taxes, and the paper closes with some conclusions. In section 7 we extend the model economy by introducing leisure in the utility function and we characterize the Markov and Ramsey policies in this environment. Finally, the paper closes with some conclusions.

2. The model economy

Firms maximize profits subject to a technology that produces the single consumption commodity. The aggregate production function uses as inputs the stocks of private and public capital, K_t and $K_{p,t}$, together with labor, L_t, in a technology: $Y_t = BK_t^{\alpha}(L_t K_{p,t})^{1-\alpha}$. Firms pay rents on the use of private capital and labor to consumers, and solve each period the static optimization problem:

$$\underset{\{K_t, L_t\}}{Max}\, \Pi_t = BK_t^{\alpha}(L_t K_{p,t})^{1-\alpha} - r_t K_t - w_t L_t .$$

We assume in what follows that population does not grow and that it is equal to the labor supply, which we normalize to $1 : L_t = 1, \forall t$. As a consequence, the variables of the model can be regarded both as per capita or aggregate terms, and will be denoted by lower case letters.

Households maximize their life-time aggregate utility, defined over private and public consumption, subject to a flat tax on total income. Income is earned from lending physical capital to the firms and from working in the production of the single consumption commodity. We do not consider leisure as an argument in the utility function. So, households solve the intertemporal optimization problem,

$$\begin{aligned}
&\underset{\{c_t, k_{t+1}\}}{Max} \sum_{t=0}^{\infty} \rho^t U(c_t, g_t) \\
&s.t.: \quad k_{t+1} - (1-\delta)k_t + c_t = (1-\tau_t)\left(r_t k_t + w_t\right) \\
&\qquad\quad \text{given } k_0 \ .
\end{aligned}$$

The government uses the proceeds from income taxes in two ways: to finance public consumption, that enters as an argument into the utility function of consumers, and to accumulate public capital. We denote by η_t the proportion of revenues used at time t to purchase public consumption, the remaining public resources being used for capital accumulation. So, the government budget constraint is:

$$\tau_t\left(r_t k_t + w_t\right) = g_t + k_{p,t}, \quad \text{where} \quad \begin{cases} g_t = \eta_t \tau_t\left(r_t k_t + w_t\right), \\ k_{p,t} = (1-\eta_t)\tau_t\left(r_t k_t + w_t\right). \end{cases}$$

Notice that it is public investment (in line with Barro, 1990, or Cazzavillan, 1996) which is productive, since the same variable enters into the production function as an argument and into the government budget constraint as a use for tax revenues. Alternatively, we could think of public capital as fully depreciating each period.

2.1. The competitive equilibrium allocation

From the government budget expenditure rules and the first order optimality conditions for the competitive firms, we get,

$$\begin{aligned}
g_t &= \eta_t \tau_t\left(r_t k_t + w_t\right) = \eta_t \tau_t y_t = \eta_t \tau_t B k_t^{\alpha} k_{p,t}{}^{1-\alpha}, \\
k_{p,t} &= (1-\eta_t)\tau_t\left(r_t k_t + w_t\right) = (1-\eta_t)\tau_t y_t = (1-\eta_t)\tau_t B k_t^{\alpha} k_{p,t}{}^{1-\alpha},
\end{aligned}$$

so that,

$$k_{p,t} = \left[(1-\eta_t)\tau_t\right]^{1/\alpha} B^{1/\alpha} k_t \ , \tag{1}$$

$$g_t = \frac{\eta_t \tau_t}{1-\tau_t} \Omega(\tau_t, \eta_t) k_t \ , \tag{2}$$

$$y_t = \left[(1-\eta_t)\tau_t\right]^{(1-\alpha)/\alpha} B^{1/\alpha} k_t \ , \tag{3}$$

while the constraint of resources can be written as follows,

$$k_{t+1} = (1-\delta)k_t + \Omega(\tau_t, \eta_t) k_t - c_t , \tag{4}$$

where $\Omega(\tau_t, \eta_t) \equiv (1-\tau_t)\left[(1-\eta_t)\tau_t\right]^{(1-\alpha)/\alpha} B^{1/\alpha}$.

Equations (1) and (3) imply that in the competitive equilibrium allocation the ratio of public capital to output is equal to $(1-\eta)\tau$, an extension of the result in Barro (1990). From (3), the ratio of private capital to output in the competitive equilibrium allocation is a function of $(1-\eta)\tau$ and structural parameters α and B.

In competitive equilibrium, households maximize their time aggregate utility subject to their budget constraint, taking taxes and the composition of public spending as given. Under less than full depreciation of private capital, the Euler equation characterizing the dynamics of the competitive equilibrium allocation of resources is[2]:

$$U_{c_t} = \rho U_{c_{t+1}} \left[1-\delta + \alpha\, \Omega(\tau_{t+1}, \eta_{t+1})\right]. \tag{5}$$

Equations (1)-(5) characterize the competitive equilibrium, given paths for taxes and for the composition of government spending. Equation (5) suggests that the rate of growth of the economy will generally depend on policy choices.

As is typical in the Barro (1990) family of AK models, the proportionality between public and private capital in the competitive equilibrium together with the assumption of constant returns to scale in the cumulative factors, is the source of endogenous growth in our model economy.

3. The time-consistent optimal policy

We use the same equilibrium concept as in Klein, Krusell and Ríos-Rull (2008) and Ortigueira (2006).[3] We consider a dynamic game played by a sequence of governments, each one of them choosing current period policies on the basis of the state of the economy in

[2] Along the paper we denote partial derivatives by $F_v \equiv \frac{\partial F}{\partial v}$.

[3] The same equilibrium concept is also used in Krusell and Ríos-Rull (1999), Krusell, Quadrini and Ríos-Rull (1996).

the current period, the aggregate stock of private capital. So, the government chooses the current tax rate τ and the proportion of revenues used to purchase public consumption, η. Hence, the problem of the government is:

$$V(k) = \underset{\{\tau,\eta\}}{Max}\left[U\left(\mathcal{C}(k,\tau,\eta),\mathcal{G}(k,\tau,\eta)\right)+\beta V(k')\right]$$

where:

$$k' = (1-\delta)k + \Omega(\tau,\eta)\,k - \mathcal{C}(k,\tau,\eta), \qquad \text{[P1]}$$

$$\mathcal{G}(k,\tau,\eta) = \frac{\eta\tau}{1-\tau}\Omega(\tau,\eta)\,k$$

given k_0,

where *k'* denotes the future stock of capital, and the $\mathcal{G}(k_t,\tau_t,\eta_t)$ function is taken from (2).

Proposition 1. *The time consistent policy corresponding to the Markov equilibrium is the solution to the set of Generalized Euler Equations (GEE):*

$$\frac{U_c\mathcal{C}_\tau + U_g\mathcal{G}_\tau}{\mathcal{C}_\tau + \Lambda(\tau)\,\Omega(\tau,\eta)\,k} = \frac{U_c\mathcal{C}_\eta + U_g\mathcal{G}_\eta}{\mathcal{C}_\eta + \frac{1-\alpha}{\alpha}\frac{1}{(1-\eta)}\Omega(\tau,\eta)\,k}, \qquad (6)$$

and

$$\frac{U_c\mathcal{C}_\tau + U_g\mathcal{G}_\tau}{\mathcal{C}_\tau + \Lambda(\tau)\,\Omega(\tau,\eta)\,k} = \rho\left[U_c^{'}\mathcal{C}_k^{'} + U_g^{'}\mathcal{G}_k^{'} + \frac{U_c^{'}\mathcal{C}_\tau^{'} + U_g^{'}\mathcal{G}_\tau^{'}}{\mathcal{C}_\tau^{'} + \Lambda(\tau)\,\Omega(\tau',\eta')\,k'}\left(1-\delta+\Omega(\tau',\eta')-\mathcal{C}_k^{'}\right)\right], \qquad (7)$$

W*here:*

$$\Lambda(\tau) \equiv \frac{\tau-(1-\alpha)}{\alpha\tau(1-\tau)},$$

$$\Omega(\tau,\eta) \equiv (1-\tau)\left[(1-\eta)\tau\right]^{(1-\alpha)/\alpha}B^{1/\alpha}.$$

Proof.- *The result is obtained by taking derivatives in the Lagrangian with respect to the three policy controls* k_{t+1},τ_t,η_t, *and eliminating the Lagrange multiplier. See Appendix* 1.□

Equation (6) shows the condition for an optimal choice of policy instruments at a given point in time, while equation (7) characterizes the optimal intertemporal choice of income tax rates.

From the budget constraint: $k_{t+1}-k_t = -\delta k_t + \Omega(\tau_t,\eta_t)\,k_t - \mathcal{C}(k_t,\tau_t,\eta_t)$, the reduction in time *t* investment from an increase in taxes is: $\partial(k_{t+1}-k_t)/\partial\tau_t = \mathcal{C}_{\tau_t} + \Lambda(\tau_t)\,\Omega(\tau_t,\eta_t)\,k_t$. Hence, the left hand side at (6) gives the change in utility produced by the tax increase, per unit of crowded-out investment. This is what Ortigueira (2006) calls today's marginal value of taxation. By a similar argument, the right hand side at (6) is the change in utility from an increase in the share of resources devoted to public consumption, per unit of crowded-out investment. This equation captures the optimal choices of the two policy instruments at any given time period.

A unit less of capital at t+1 changes utility through private and public consumption by $U_{c_{t+1}}\mathcal{C}_{k_{t+1}}+U_{g_{t+1}}\mathcal{G}_{k_{t+1}}$. Lower taxes at t+1 stimulate investment, and the total effect from both tax changes on time t+2 capital stock is, $\frac{\partial k_{t+2}}{\partial k_{t+1}}=1-\delta+\Omega(\tau_{t+1},\eta_{t+1})-\mathcal{C}_{k_{t+1}}$. The change in utility per unit of additional investment is $\frac{U_{c_{t+1}}\mathcal{C}_{\tau_{t+1}}+U_{g_{t+1}}\mathcal{G}_{\tau_{t+1}}}{\mathcal{C}_{\tau_{t+1}}+\Lambda\left(\tau_{t+1}\right)\Omega(\tau_{t+1},\eta_{t+1})\,k_{t+1}}$. Equation (7) shows that the change in utility at time t is equal to the discounted change in utility at t+1.

Definition. A Markov-Perfect equilibrium is a set of functions $\mathcal{C}(k_t,\tau_t,\eta_t),\mathcal{G}(k_t,\tau_t,\eta_t),\,k_p(k_t,\tau_t,\eta_t),\tau(k_t),$ and $\eta(k_t)$ such that:

i) given $\mathcal{G}(k_t,\tau_t,\eta_t),\,k_p(k_t,\tau_t,\eta_t),\tau(k_t),$ and $\eta(k_t)$, then $\mathcal{C}(k_t,\tau_t,\eta_t)$ satisfies the constraint of resources (4) and the Euler equation (5) for the competitive equilibrium, and

ii) $\mathcal{G}(k_t,\tau_t,\eta_t),\,k_p(k_t,\tau_t,\eta_t),\tau(k_t),$ and $\eta(k_t)$ satisfy the conditions in the optimization problem of the government, *i.e.*, equations (1), (2), (4) and the Generalized Euler Equations (6) and (7).

3.1. An analytical solution: logarithmic utility and full depreciation of private capital

We focus in this section on the case of logarithmic preferences that are separable in private and public consumption,

$$U\left(\left\{c_t,k_{t+1}\right\}_{t=0}^{\infty}\right)=\sum_{t=0}^{\infty}\rho^t\left(\ln c_t+\theta\ln g_t\right),$$

together with full depreciation of private capital. The two assumptions together allow us to obtain an analytical characterization of the time consistent optimal fiscal policy that we can compare with the Ramsey solution as well as with the allocation that would be obtained under lump-sum taxes.

Under this utility function, the competitive equilibrium allocation is characterized by the system:

$$\left.\begin{aligned} k_{t+1} &= \Omega(\tau_t,\eta_t)k_t - c_t, \\ \frac{c_{t+1}}{c_t} &= \rho\left[\alpha\,\Omega(\tau_{t+1},\eta_{t+1})\right] \end{aligned}\right\} \qquad [S]$$

Note that we initially allow for future policies (taxes and government spending split) to be functions of the state variable at that point in time. However, we prove below that this economy is always on the balanced growth path, implying that bounded variables (like τ and η) remain constant for all *t*.

Proposition 2. *The competitive equilibrium allocations are given by:*

$$k_{t+1} = \rho\alpha\,\Omega(\tau_t,\eta_t)\,k_t, \tag{8}$$

$$c_t = (1-\rho\alpha)\,\Omega(\tau_t,\eta_t)\,k_t. \tag{9}$$

Proof. *Plugging in the previous system [S] a guess for the functional form for the competitive equilibrium allocation as:* $k_{t+1} = A\Omega_t k_t$, *it is easy to show that* $A=\rho\alpha$. □

Expressions (8) and (9) for k_{t+1}, c_t allow us to compute the partial derivatives that enter into the system (6)-(7) characterizing the time consistent optimal policy.

Proposition 3. *Under full depreciation of private capital and a logarithmic utility function, separable in private and public consumption, the optimal time-consistent fiscal policy is:*

$$\tau_t^M = \tau^M = 1 - \frac{\alpha(1+\rho\theta)}{1+\theta}, \quad \forall t$$

$$\eta_t^M = \eta^M = \frac{\alpha\theta(1-\rho)}{1-\alpha+\theta(1-\alpha\rho)}, \quad \forall t.$$

Proof. The problem solved by the government is:

$$\underset{\{\tau_t,\eta_t,k_{t+1}\}}{\text{Max}} \sum_{t=0}^{\infty} \rho^t \left[\ln \mathcal{C}(k_t,\tau_t,\eta_t) + \theta \ln \mathcal{G}(k_t,\tau_t,\eta_t)\right]$$

$$\text{subject to} \quad k_{t+1} = \Omega_t k_t - \mathcal{C}(k_t,\tau_t,\eta_t)$$

$$k_0 \text{ given}$$

where $\mathcal{C}(k_t,\tau_t,\eta_t)$ is given by (9) and $\mathcal{G}_t(k_t,\tau_t,\eta_t)$ is given by (2).

The first order conditions for this problem are:

$$\frac{\partial\mathcal{L}}{\partial\tau_t} = 0 \quad\to\quad \frac{(1-\tau_t)(1+\theta)-\alpha}{\tau_t-(1-\alpha)} = \alpha\rho\lambda_t\Omega_t k_t, \tag{10}$$

$$\frac{\partial\mathcal{L}}{\partial\eta_t} = 0 \quad\to\quad \frac{\theta(\alpha-\eta_t)}{\eta_t(1-\alpha)} - 1 = \alpha\rho\lambda_t\Omega_t k_t, \tag{11}$$

$$\frac{\partial\mathcal{L}}{\partial k_{t+1}} = 0 \quad\to\quad k_{t+1}\lambda_t = \rho(1+\theta) + \rho^2\alpha\lambda_{t+1}\Omega_{t+1}k_{t+1}. \tag{12}$$

From (10) and (11) we obtain a relationship between the optimal values of the tax rate and the government spending split in the Markov-perfect equilibrium:

$$\tau_t = \frac{1-\alpha}{1-\eta_t}, \tag{13}$$

where we can see that the optimal tax rate will be larger than $1-\alpha$.

Using (8), from (11) and (12) we obtain the dynamic equation:

$$\tilde{\eta}_{t+1} - \frac{1}{\rho}\tilde{\eta}_t + \frac{1+\rho\theta}{\rho} = 0, \tag{14}$$

where $\tilde{\eta}_t \equiv \frac{\theta(\alpha-\eta_t)}{\eta_t(1-\alpha)}$. The solution to the difference equation (14) is unstable, since $1/\rho > 1$; hence $\tilde{\eta}_t$ must stay constant over time, and the same applies to η_t, that is, $\eta_t = \eta, \forall t$. From (14), we obtain the value of η:

$$\eta^M = \frac{\alpha\theta(1-\rho)}{1-\alpha+\theta(1-\alpha\rho)}, \tag{15}$$

and using (13), we obtain the Markov perfect optimal tax rate:[4]

$$\tau^M = 1 - \frac{\alpha(1+\rho\theta)}{1+\theta}. \square \tag{16}$$

Notice that the optimal split of resources between public consumption and investment is well defined, taking values between 0 and 1 for any set of values for the structural parameters, while the optimal income tax rate is less than one. The fact that η_t and τ_t remain constant from the initial period implies that the Markov solution lacks transitional dynamics. In particular, the Markov-perfect optimal choice of tax rates and the split of public revenues does not depend on the state variable k_t, a fact that we will take into account when we analyze the general case in the next section.

***Corollary* 1.** *Under the Markov-perfect optimal policy, the economy is always on its balanced growth path, with constant values of the growth rate,* $\gamma_t \equiv k_{t+1}/k_t$, *and the ratios of private and public consumption to capital,* $\chi_t \equiv c_t/k_t, \phi_t \equiv g_t/k_t$. *The optimal allocation of resources is given by*:

[4] Malley et al. (2002) obtain a similar expression for the Markov perfect tax rate.

$$\gamma_t^M \equiv \left(\frac{k_{t+1}}{k_t}\right)^M = \rho\alpha\,\Omega(\tau^M, \eta^M) = \gamma^M \quad \forall t,$$

$$\chi_t^M \equiv \left(\frac{c_t}{k_t}\right)^M = (1-\rho\alpha)\,\Omega(\tau^M, \eta^M) = \chi^M \quad \forall t,$$

$$\phi_t^M \equiv \left(\frac{g_t}{k_t}\right)^M = \frac{\eta^M \tau^M}{1-\tau^M}\Omega(\tau^M, \eta^M) = \phi^M \quad \forall t.$$

Proof. It is straightforward.□

The three following corollaries can be readily shown from (15) and (16):

***Corollary* 2.** *When public consumption does not enter as an argument into the utility function (θ=0), the Markov-perfect optimal tax rate coincides with that in Barro* (1990)*:* $\tau = 1-\alpha$. *In that situation, public resources are fully devoted to investment.*

***Corollary* 3.** *The Markov-perfect optimal tax rate converges to the Barro tax as the discount rate approaches 1, with public resources again being fully devoted to public investment.*

***Corollary* 4.** *i*) *The proportion of public resources devoted to public consumption increases with θ and α, and it decreases with ρ,*

ii) the optimal time consistent income tax increases with θ, and it decreases with α and with ρ.

As expected, the proportion of public resources devoted to consumption increases with the relative importance of public consumption in the utility function. It also increases with the output elasticity of private capital. A more productive private capital, relative to public capital, allows for a higher share of public resources being consumed, rather than invested. Turning the argument around, the more productive is public capital relative to private capital, the more interesting is to allocate to resources to productive activities rather than to consumption. The share of public resources dedicated to consumption decreases for a larger ρ. We then value future consumption almost as much as current consumption, and it becomes interesting to increase investment and defer consumption for the future.

As public consumption is more appreciated by consumers for higher values of θ and lower values of ρ, it is appropriate to raise higher tax revenues to finance that component of public spending. On the contrary, for a high elasticity of private capital, α, it is preferable that the private sector takes a more important role in investment, and taxes can be lower.

3.2. Optimal fiscal policy under a constant-relative risk aversion utility function and less than perfect depreciation

Using the definitions for the ratios to private capital introduced above: $\chi_t \equiv c_t / k_t$, $\phi_t \equiv g_t / k_t$, the global constraint of resources allows us to express the gross rate of growth of private capital stock, $\gamma_t \equiv k_{t+1} / k_t$, as:

$$\gamma_t = 1 - \delta + \Omega(\tau_t, \eta_t) - \chi_t, \tag{17}$$

while from the government budget expenditure rule (2) we get:

$$\phi_t = \frac{\eta_t \tau_t}{1 - \tau_t} \Omega(\tau_t, \eta_t). \tag{18}$$

Let the instantaneous utility function be $U(c_t, g_t) = \dfrac{\left(c_t g_t^{\theta}\right)^{1-\sigma} - 1}{1 - \sigma}$. It is well known that the competitive, Pareto and Ramsey solutions to the Barro (1990) model lack any transitional dynamics. Furthermore, we have just shown in the previous section that the Markov-perfect equilibrium under a logarithmic utility and full depreciation of private capital also lacks transitional dynamics.

Taking these results as a baseline reference, we assume that the time consistent solution to this more general version of our model economy, also lacks any transitional dynamics. This amounts to assuming that the variables that do not exhibit growth along the balanced growth path do not depend on the state variable k_t, so that: $\dfrac{\partial \tau_t}{\partial k_t} = \dfrac{\partial \eta_t}{\partial k_t} = \dfrac{\partial (c_t / k_t)}{\partial k_t} = \dfrac{\partial (g_t / k_t)}{\partial k_t} = 0$. Furthermore, since $c_t = \chi_t k_t$, $g_t = \phi_t k_t$, we have expressions for the partial derivatives that appear in the Generalized Euler conditions:

$$c_{k_t} = \frac{\partial \chi_t}{\partial k_t} k_t + \chi_t = \chi_t, \text{ since we assume that } \frac{\partial \chi_t}{\partial k_t} = 0,$$

$$c_{\tau_t} = \frac{\partial \chi_t}{\partial \tau_t} k_t = \chi_{\tau_t} k_t,$$

$$g_{k_{t+1}} = \frac{\eta_{t+1}\tau_{t+1}}{1-\tau_{t+1}} \Omega(\tau_{t+1},\eta_{t+1}), \text{ since we assume that } \frac{\partial \tau_{t+1}}{\partial k_{t+1}} = \frac{\partial \eta_{t+1}}{\partial k_{t+1}} = 0,$$

$$g_{\tau_{t+1}} = \frac{1}{\alpha}\frac{\eta_{t+1}}{1-\tau_{t+1}} \Omega(\tau_{t+1},\eta_{t+1})\, k_{t+1},$$

$$U_{c_{t+1}} = \chi_{t+1}^{-\sigma} \left[\frac{\eta_{t+1}\tau_{t+1}}{1-\tau_{t+1}} \Omega(\tau_{t+1},\eta_{t+1})\right]^{\theta(1-\alpha)} k_{t+1}^{(1+\theta)(1-\sigma)-1},$$

$$U_{g_{t+1}} = \theta\chi_{t+1}^{1-\sigma} \left[\frac{\eta_{t+1}\tau_{t+1}}{1-\tau_{t+1}} \Omega(\tau_{t+1},\eta_{t+1})\right]^{\theta(1-\alpha)-1} k_{t+1}^{(1+\theta)(1-\sigma)-1}.$$

The Euler equation for the competitive equilibrium (5) then becomes:

$$\left(\frac{\chi_{t+1}}{\chi_t}\right)^{\sigma} \gamma_t^{\sigma-\theta(1-\sigma)} = \rho\left[\frac{\phi_{t+1}}{\phi_t}\right]^{-\theta(1-\sigma)} \left[1-\delta+\alpha\,\Omega(\tau_{t+1},\eta_{t+1})\right], \tag{19}$$

and the two Markov equilibrium conditions (6) and (7) become:

$$\frac{\chi_{\tau_t} + \frac{\theta}{\alpha}\frac{1}{\tau_t}\chi_t}{\chi_{\eta_t} + \theta\chi_t\,\Lambda(\eta_t)} = \frac{\chi_{\tau_t} + \Lambda(\tau_t)\,\Omega(\tau_t,\eta_t)}{\chi_{\eta_t} + \frac{1-\alpha}{\alpha}\frac{1}{(1-\eta_t)}\Omega(\tau_t,\eta_t)}, \tag{20}$$

and

$$\begin{aligned}\frac{\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\left(\chi_{\tau_t} + \chi_t\frac{\theta}{\alpha}\frac{1}{\tau_t}\right)}{\chi_{\tau_t} + \Lambda(\tau_t)\,\Omega(\tau_t,\eta_t)} &= \rho\chi_{t+1}^{-\sigma}\phi_{t+1}^{\theta(1-\sigma)}\gamma_t^{(1+\theta)(1-\sigma)-1}\times \\ &\left[(1+\theta)\chi_{t+1} + \frac{\chi_{\tau_{t+1}} + \chi_{t+1}\frac{\theta}{\alpha}\frac{1}{\tau_{t+1}}}{\chi_{\tau_{t+1}} + \Lambda(\tau_{t+1})\,\Omega(\tau_{t+1},\eta_{t+1})}\gamma_{t+1}\right],\end{aligned} \tag{21}$$

where $\Lambda(\eta_t) \equiv \frac{\eta_t-(1-\alpha)}{\alpha\eta_t(1-\eta_t)}$.

Under our maintained assumption that the policy variables do not depend on k_t, we have arrived at the previous system that allows us to obtain the equilibrium values for $\{\gamma_t,\chi_t,\phi_t,\tau_t,\eta_t\}$ without the intervention of any state variable. As a consequence, in the absence of local indeterminacy of equilibrium, the economy is always placed on the balanced growth path, and hence, $\gamma_t=\gamma, \chi_t=\chi, \phi_t=\phi, \tau_t=\tau, \eta_t=\eta,\ \forall t$.

Imposing this property on the system above, we obtain the consumption to capital ratio along the balanced growth path:

$$\chi = 1-\delta+\Omega(\tau,\eta)-\left[\rho\left(1-\delta+\alpha\,\Omega(\tau,\eta)\right)\right]^{1/(\sigma-\theta(1-\sigma))}. \tag{22}$$

Since (22) characterizes χ as a function of τ and η alone, it can be considered as a policy function itself, and we can compute partial derivatives χ_τ and χ_η:

$$\chi_\tau \equiv \frac{\partial\chi}{\partial\tau} = \Lambda(\tau)\,\Omega(\tau,\eta)\left(\frac{\rho\alpha\gamma^{1-\sigma+\theta(1-\sigma)}}{\sigma-\theta(1-\sigma)}-1\right), \tag{23}$$

$$\chi_\eta \equiv \frac{\partial\chi}{\partial\eta} = \frac{1-\alpha}{\alpha(1-\eta)}\Omega(\tau,\eta)\left(\frac{\rho\alpha\gamma^{1-\sigma+\theta(1-\sigma)}}{\sigma-\theta(1-\sigma)}-1\right). \tag{24}$$

Thus, using (23) and (24) in the system (17)-(21) we obtain the time-consistent optimality conditions along the balanced growth path: [5]

$$\gamma = 1-\delta+\Omega(\tau,\eta)-\chi, \tag{25}$$

$$\phi = \frac{\eta\tau}{1-\tau}\Omega(\tau,\eta), \tag{26}$$

$$\gamma = \left\{\rho\left[1-\delta+\alpha\,\Omega(\tau,\eta)\right]\right\}^{\frac{1}{\sigma-\theta(1-\sigma)}}, \tag{27}$$

$$\frac{\chi_\tau+\frac{\theta}{\alpha}\frac{1}{\tau}\chi}{\chi_\eta+\theta\chi\,\Lambda(\eta)} = \frac{\chi_\tau+\Lambda(\tau)\,\Omega(\tau,\eta)}{\chi_\eta+\frac{1-\alpha}{\alpha}\frac{1}{(1-\eta)}\Omega(\tau,\eta)}, \tag{28}$$

$$\frac{\left(\chi_\tau+\chi\frac{\theta}{\alpha}\frac{1}{\tau}\right)\gamma^{\sigma-\theta(1-\sigma)}}{\chi_\tau+\Lambda(\tau)\,\Omega(\tau,\eta)} = \rho\left[(1+\theta)\chi+\frac{\chi_\tau+\chi\frac{\theta}{\alpha}\frac{1}{\tau}}{\chi_\tau+\Lambda(\tau)\,\Omega(\tau,\eta)}\gamma\right]. \tag{29}$$

This system of 5 equations in $\{\gamma,\chi,\phi,\eta,\tau\}$ characterizes the balanced growth path under the Markov-perfect optimal fiscal policy (τ^M,η^M) as well as the implied allocation of resources, described by (γ^M,χ^M,ϕ^M). The system can only be solved numerically and a later section is devoted to analyze its properties under several parameterizations of our model economy. We could also particularize the equations characterizing the balanced growth path under the optimal time consistent policy to the case of a logarithmic utility

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[5] Remember that $\Omega(\tau,\eta)=(1-\tau)\left[(1-\eta)\tau\right]^{(1-\alpha)/\alpha}B^{1/\alpha}$, $\Lambda(\tau)=\dfrac{\tau-(1-\alpha)}{\alpha\tau(1-\tau)}$, $\Lambda(\eta)=\dfrac{\eta-(1-\alpha)}{\alpha\eta(1-\eta)}$.

function (σ=1) and full depreciation (δ=1), to obtain the same expressions as in the previous section.

4. The Ramsey policy

As usual, we define the benchmark "Ramsey equilibrium" as the solution to an optimal-policy problem where the government can commit to future policies. The Ramsey optimal policy is then the solution to the utility maximization problem, subject to the equilibrium conditions (1) - (5) as constraints. The Ramsey policy takes into account the optimal reactions of private agents. However, it is time inconsistent, since once private agents adjust their decisions to the announced economic policy, it will be optimal for the government to change policy.

Therefore, the Ramsey policy is obtained by solving the problem:

$$\begin{aligned}
&\underset{\{\tau_t,\eta_t\}}{Max}\, U\left(\left\{c_t, k_{t+1}\right\}_{t=0}^{\infty}\right) \\
&\text{subject to:} \\
&k_{t+1} = (1-\delta)k_t + \Omega(\tau_t,\eta_t)k_t - c_t \qquad\qquad [P2] \\
&U_{c_t} = \rho U_{c_{t+1}}\left[1-\delta+\alpha\,\Omega\left(\tau_{t+1},\eta_{t+1}\right)\right] \\
&g_t = \eta_t\tau_t\left[(1-\eta_t)\tau_t\right]^{\frac{1-\alpha}{\alpha}} B^{\frac{1}{\alpha}} K_t \ .
\end{aligned}$$

As shown in Appendix 2, optimization problem [P2] leads to the system of equations characterizing the optimal Ramsey policy in stationary ratios:

$$\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)} = \tilde{\mu}_{1t} - \sigma\chi_t^{-\sigma-1}\phi_t^{\theta(1-\sigma)}\left\{\tilde{\mu}_{3t} - \tilde{\mu}_{3t-1}\frac{1}{\gamma_t}\left[\left(1-\delta+\alpha\,\Omega(\tau_t,\eta_t)\right)\right]\right\}, \qquad (30)$$

$$\theta\chi_t^{1-\sigma}\phi_t^{\theta(1-\sigma)-1} = \tilde{\mu}_{2t} - (1-\sigma)\theta\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)-1}\left\{\tilde{\mu}_{3t} - \tilde{\mu}_{3t-1}\frac{1}{\gamma_t}\left[\left(1-\delta+\alpha\,\Omega(\tau_t,\eta_t)\right)\right]\right\}, \qquad (31)$$

$$\tilde{\mu}_{1t} = \rho\gamma_{t+1}^{\theta(1-\sigma)-\sigma}\left[\tilde{\mu}_{1t+1}\left(1-\delta+\Omega(\tau_{t+1},\eta_{t+1})\right) + \tilde{\mu}_{2t+1}B^{1/\alpha}\tau_{t+1}^{1/\alpha}(1-\eta_{t+1})^{\frac{1-\alpha}{\alpha}}\right], \qquad (32)$$

$$\tilde{\mu}_{1t}\left(\frac{1-\alpha}{\alpha}\frac{1-\tau_t}{\tau_t} - 1\right) + \tilde{\mu}_{2t}\eta_t\frac{1}{\alpha} + \tilde{\mu}_{3t-1}\frac{1}{\gamma_t}\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}\alpha\left(\frac{1-\alpha}{\alpha}\frac{1-\tau_t}{\tau_t} - 1\right) = 0\,, \qquad (33)$$

$$-\tilde{\mu}_{1t}\frac{1-\alpha}{\alpha} + \tilde{\mu}_{2t}\frac{\tau_t}{1-\tau_t}\left(1-\frac{\eta_t}{\alpha}\right) - \tilde{\mu}_{3t-1}\chi_t^{-\sigma}\phi_t^{\theta(1-\sigma)}(1-\alpha)\frac{1}{\gamma_t} = 0\,, \qquad (34)$$

where $\tilde{\mu}_{1t}, \tilde{\mu}_{1t}$, and $\tilde{\mu}_{1t}$ are Lagrange multipliers associated to the three constraints in [P2], transformed as explained in Appendix 2. This system of dynamic equations characterizing the solution to the Ramsey problem is again made up only by control variables, with no participation of any state variable. So, again, in the absence of indeterminacy of equilibria, the only possible solution is that control variables stay on the balanced growth path (BGP) from the initial period.

Denoting by:

$$\Psi = \frac{1}{\gamma}\left[1-\delta+\alpha\,\Omega(\tau,\eta)\right]-1,\; F = 1-\delta+\Omega(\tau,\eta), \text{ and } \Gamma = \frac{1-\rho\gamma^{\theta(1-\sigma)-\sigma}F}{\rho\gamma^{\theta(1-\sigma)-\sigma}\phi},$$

we characterize the balanced growth path by particularizing the system of equations for the Ramsey equilibrium:

$$\gamma = \left\{\rho\left[\left(1-\delta+\alpha\,\Omega(\tau,\eta)\right)\right]\right\}^{\frac{1}{\sigma-\theta(1-\sigma)}},$$

$$\chi = 1-\delta+\Omega(\tau,\eta)-\gamma,$$

$$\phi = B^{1/\alpha}\tau^{1/\alpha}\eta(1-\eta)^{\frac{1-\alpha}{\alpha}},$$

$$\tilde{\mu}_1 = \frac{1/\sigma}{\chi^{\sigma}\phi^{-\theta(1-\sigma)}\left[\frac{1}{\theta}\frac{\phi\Gamma}{\chi}+\frac{1-\sigma}{\sigma}\right]},$$

$$\tilde{\mu}_2 = \Gamma\tilde{\mu}_1,$$

$$\tilde{\mu}_3 = \frac{1-\chi^{\sigma}\phi^{-\theta(1-\sigma)}\tilde{\mu}_1}{\Psi\sigma/\chi},$$

$$\left[\tilde{\mu}_1+\tilde{\mu}_3\frac{1}{\gamma}\chi^{-\sigma}\phi^{\theta(1-\sigma)}\alpha\right]\left(\frac{1-\alpha}{\alpha}\frac{1-\tau}{\tau}-1\right)+\tilde{\mu}_2\eta\frac{1}{\alpha}=0,$$

$$-\tilde{\mu}_1\frac{1-\alpha}{\alpha}+\tilde{\mu}_2\frac{\tau}{1-\tau}\left(1-\frac{\eta}{\alpha}\right)-\tilde{\mu}_3\chi^{-\sigma}\phi^{\theta(1-\sigma)}(1-\alpha)\frac{1}{\gamma}=0,$$

a system of 8 equations in $\{\gamma,\chi,\phi,\eta,\tau,\tilde{\mu}_1,\tilde{\mu}_2,\tilde{\mu}_3\}$ that allows us to compute the balanced growth path for the Ramsey policy (τ^R,η^R) as well as the implied allocation of resources, characterized by (γ^R,χ^R,ϕ^R).

Given the complexity involved in characterizing optimal policy under lack of commitment, Ramsey optimal policies have usually been computed for growth economies in spite of their well-known limitations of assuming commitment on the part of the government. It is therefore important to evaluate to what extent the Markov-perfect fiscal policy differs from the Ramsey policy in our setup. We will perform such analysis in two ways: first, by comparing the analytical expressions for both policies under a logarithmic utility function and full depreciation of capital. Second, by comparing the numerical solutions obtained under the more general framework considered in this section, with CRRA utility and incomplete depreciation of private capital.

5. Comparing the Ramsey and Markov solutions

In this section we compare the Markov and Ramsey solutions between themselves, as well as with the allocation of resources that would be achieved under lump-sum taxes, which is characterized in Appendix 3. As explained in that Appendix, we can introduce a measure of the size of the public sector in the planner solution as: $\tau_t^P = \frac{g_t + k_{p,t}}{y_t}$, and for the composition of public expenditures as: $\eta_t^P = \frac{g_t}{g_t + k_{p,t}}$. Both of them will be used in the graphs we present below.

Due to the lack of an analytical solution in the case of a general utility function and less than perfect depreciation of private capital, we are forced to compute numerical solutions to the different systems of nonlinear equations characterizing each equilibrium concept in the previous sections. Even though the systems are written in terms of ratios to physical capital, they are readily transformed into ratios to output using the *y/k*-ratio. Obviously, non-growing variables do not need such transformation. Two reasons difficult the comparison of our numerical results with those in the literature: *i*) even though the theoretical discussion in previous work is often made under a generic utility function, numerical results are usually derived using a logarithmic, separable utility function, whereas our results correspond to general CRRA utility functions, *ii*) our consideration of endogenous growth.

5.1. Under logarithmic utility and full depreciation of private capital

The conditions considered in this section allow for an analytical solution to the policy problems. The following proposition shows that, for this special case, the Ramsey policy is the same as the Markov tax.

***Proposition* 4.** *Under a logarithmic utility function and full depreciation, the optimal Ramsey policy, becomes:*

$$\tau^R = 1 - \frac{\alpha(1+\rho\theta)}{1+\theta},$$

$$\eta^R = \frac{\alpha\theta(1-\rho)}{1-\alpha+\theta(1-\alpha\rho)}.$$

Proof: *See Appendix 4.* □

The Ramsey tax and the proportion of public resources devoted to public consumption under the Ramsey policy coincide with the values obtained under the time-consistent policy, so the properties analyzed in Corollaries 1 to 4 for the Markov-perfect optimal policy apply to the Ramsey policy as well.

5.2. The general case

Let us now examine the values taken by the main variables in the economy along the balanced growth path under three alternative fiscal policies: the planner's policy under lump-sum taxes, the Ramsey policy and the time-consistent policy under the more general setup, with a CRRA utility function and incomplete depreciation of private capital. Parameter values are standard in the literature for annual data except for θ, which is chosen so that the ratio of public consumption to private consumption for the Markov solution is in line with data for the postwar US economy (g/c=0.25).

We start by considering the benchmark case of full depreciation of private capital and logarithmic preferences. Figure 1 illustrates the solutions for the case of full depreciation of private capital under the parameterization: $\theta = 0.40, \alpha = 0.80, \rho = 0.98, \delta = 1.0, B = 2.5$, for values of the risk aversion parameter between 1 and 5. We are assuming a real interest rate around 2% (ρ=0.98). A relatively high value of the total productivity parameter, B, is needed in this case for a solution to exist. Over the whole range of values considered for the intertemporal elasticity of substitution of consumption, the income tax falls between 20% and 30%, being higher under the Markov-perfect policy than under the time-inconsistent Ramsey policy. The proportion of public resources devoted to consumption, relative to investment, is also higher under the Markov-perfect solution than under the Ramsey policy. As a proportion of output, private consumption is higher under the Ramsey policy while public consumption is higher under the Markov policy. In terms of specific values, private consumption never exceeds 30% of output under either policy, while public consumption remains below 8% of output, both observations below the levels observed in actual economies. A planner with access to lump-sum taxes under commitment would devote an even higher proportion of public resources to consumption than the Markov and Ramsey solutions, and the growth rate would also be higher.

That the income tax is higher under the Markov-perfect policy than under the Ramsey solution is consistent with the result obtained by Ortigueira (2006) in an exogenous growth economy under inelastic labor supply.[6] This result arises because the Markovian government cannot internalize the distortionary effects of current taxation on past investment, while in the Ramsey solution, the government takes fully into account the negative effect of the income tax on future investment. A similar argument explains that the Markov government devotes a higher proportion of public resources to consumption, which has a direct impact on current utility, to the expense of public investment, which would have a positive effect mainly in the future. That the growth rate is higher under the Ramsey than under the Markov solution is an implication from the fact that the income tax rate is lower and the share of investment in public expenditures is also higher under the former policy.

Under incomplete depreciation of private capital and parameter values:[7] $\theta = 0.4, \alpha = 0.80, \rho = 0.99, \delta = 0.10, B = 0.4555$, Figure 2 shows that the previous qualitative results stay the same for values of the risk aversion parameter, σ, above 1.0. For instance, for σ=2, we have that: *i*) a value θ=0.40 guarantees that the ratio public to private consumption is around 0.25 for the Markov solution; *ii*) a value of the total productivity parameter, B=0.4555, leads to an annual growth rate, γ=1.5%; *iii*) a value ρ=0.99 implies a real interest rate around 3% (since $1/\left(\rho\gamma^{(1-\sigma)(1+\theta)}\right) \approx 1.03$ with σ=2, θ=0.4 and γ=1.015). As before, the optimal tax rate increases with the risk aversion parameter, with values between 22% and 30%. The proportion of public resources devoted to consumption is also increasing in σ, staying between 6% and 32%. Steady state growth is slightly higher under the Ramsey policy. Growth rates are large for low values of the risk aversion parameter, but they become quite realistic for values of σ above 2.0. Under the Markov policy, public consumption increases to about 10% of output, while private consumption stays below 30% of output under both policies, below their values in actual economies. However, the public to private consumption ratio is around 25%, as in observed data.

Figure 3 presents results under the parameterization $\sigma = 2.0, \alpha = 0.80, \rho = 0.99$, $\delta = 0.10, B = 0.4555$, for values of the relative weight of public consumption in the utility function, θ, between 0.2 and 1.5. As expected, public consumption as a share of total public spending increases with θ. Qualitative results stay the same, with the Markov-perfect policy imposing a higher income tax than the Ramsey policy, and devoting a higher proportion of

[6] Even though the two results are not strictly comparable, since one of them refers to an exogenous growth economy and the other to an endogenous growth economy.

[7] We have reduced the value of B so as to have growth rates similar to those in actual economies.

public resources to consumption. The growth rate is again higher under the Ramsey than under the Markov policy.

Table 1 summarizes the results by displaying a single point from Figure 2 and Figure 3. Table 2 analyzes the effects of a change in α. The value of B has been chosen to guarantee positive growth rates under the Markov and Ramsey solutions.

Table 1. Values for the main variables under the three solution concepts. Effects of a change in θ

	$B = 0.4555$ $\sigma = 2.00$ $\theta = 0.40$ $\alpha = 0.80$ $\delta = 0.10$ $\rho = 0.99$			$B = 0.4555$ $\sigma = 2.00$ $\theta = 1.00$ $\alpha = 0.80$ $\delta = 0.10$ $\rho = 0.99$		
	Planner	*Markov*	*Ramsey*	*Planner*	*Markov*	*Ramsey*
η (%)	26.7	24.9	20.4	41.6	38.7	30.9
τ (%)	27.3	26.6	25.1	34.2	32.6	28.9
γ (%)	3.6	1.5	1.6	2.9	0.8	1.1
c/y(%)	18.3	27.4	28.4	14.3	24.2	26.9
g/y(%)	7.3	6.6	5.1	14.3	12.6	8.9
kp/y(%)	20.0	20.0	20.0	20.0	20.0	20.0
k/y	4.0	4.0	4.0	4.0	4.0	4.0

Table 2. Values for the main variables under the three solution concepts. Effects of a change in α

	$B = 0.658$ $\sigma = 2.00$ $\theta = 0.40$ $\alpha = 0.80$ $\delta = 0.10$ $\rho = 0.99$			$B =0.658$ $\sigma = 2.00$ $\theta = 0.40$ $\alpha = 0.70$ $\delta = 0.10$ $\rho = 0.99$		
	Planner	*Markov*	*Ramsey*	*Planner*	*Markov*	*Ramsey*
η (%)	32.9	32.1	28.9	19.2	16.9	13.1
τ (%)	29.8	29.4	28.1	37.1	36.1	34.5
γ (%)	8.1	4.5	4.7	4.8	1.5	1.6
c/y(%)	24.6	33.9	34.8	17.9	28.9	30.0
g/y(%)	9.8	9.5	8.1	7.1	6.1	4.5
kp/y(%)	20.0	20.0	20.0	30.0	30.0	30.0
k/y	2.5	2.5	2.5	3.0	3.0	3.0

Note to the tables: for the planner solution $\tau_t^P = \frac{g_t + k_{p,t}}{y_t}$ and $\eta_t^P = \frac{g_t}{g_t + k_{p,t}}$.

Since the resource allocations obtained under the three solution concepts satisfy the conditions for competitive equilibrium, the fact that the ratio of public capital to output is the same for the three solutions means that the product $(1-\eta)\tau$ is also the same for the three solution concepts. The value of $(1-\eta)\tau$ turns out to be equal to the elasticity of output with respect to public capital, again an extension of the result obtained by Barro (1990) in a model with just public capital.

Furthermore, since the product $(1-\eta)\tau$ is the same for the three solution concepts, the ratio of private capital to output is also the same for the three solutions under any parameterization.

The solution under lump-sum taxes leads to the largest public sector and devotes a lowest share of public resources to investment. Since taxes are nondistortionary under the planner's solution, a largest proportion of resources extracted by the public sector is compatible with a higher rate of growth.

The comparison between the two panels in Table 1 shows what happens as public consumption becomes more important in the utility function: while the ratios of both types of capital to output remain unchanged, the tax rate increases, as it does the proportion of public resources devoted to consumption. These two changes lead to a lower rate of growth.

Table 2 shows that an increase in the productivity of public capital (lower α) leads to larger tax rates, that is, the government detracts more aggregate resources and devotes a larger proportion of them to consumption. Because of the increase in the tax rate generated by a lower α parameter, the productivity of private capital decreases and, hence, also does the rate of growth.

6. Welfare

In this section we compute the level of welfare that would arise along the balanced growth path under the time consistent Markov policy and compare it with the level of welfare that would be obtained under lump-sum taxes.[8] As in Lucas (1987), what we compute is the consumption compensation (as a percentage of output) that would be needed under the Markov rule to achieve the same level of welfare than under the resource allocation of the planner with non-distortionary taxation.

Under a CRRA utility, welfare can be written,

$$W_i = \sum_{t=0}^{\infty} \rho^t \frac{c_{t,i}^{1-\sigma} g_{t,i}^{\theta(1-\sigma)} - 1}{1-\sigma} = \frac{1}{1-\sigma}\left[\frac{\chi_i^{1-\sigma}\phi_i^{1-\sigma}}{1-\rho\,\gamma_i^{(1-\sigma)(1+\theta)}} - \frac{1}{1-\rho}\right], i = Planner, Markov.$$

[8] We do not consider the level of welfare under the Ramsey solution because of its time-inconsistent nature.

Let $\{c_{t,i}, g_{t,i}\}$, $i=P,M$, be the optimal path for private and public consumption for the planner's solution and the Markov solution, respectively, that is:

$$c_{t,i} = \chi_i k_{t,i} = \chi_i k_0 \gamma_i^t \underbrace{=}_{k_0=1} \chi_i \gamma_i^t,$$
$$g_{t,i} = \phi_i k_{t,i} = \phi_i k_0 \gamma_i^t \underbrace{=}_{k_0=1} \phi_i \gamma_i^t, \quad i = P, M$$

where we have indicated the normalization $k_0=1$.

The consumption compensation λ needed for the Markov solution to achieve the same level of welfare as under the planner's allocation can be obtained by solving the following equation:

$$W_P = \sum_{t=0}^{\infty} \rho^t \frac{(1+\lambda)^{1-\sigma} c_{t,M}^{1-\sigma} g_{t,M}^{\theta(1-\sigma)} - 1}{1-\sigma},$$

that is,

$$\frac{1}{1-\sigma}\left[\frac{\chi_P^{1-\sigma}\phi_P^{1-\sigma}}{1-\rho\,\gamma_P^{(1-\sigma)(1+\theta)}} - \frac{1}{1-\rho}\right] = \frac{1}{1-\sigma}\left[\frac{(1+\lambda)^{1-\sigma}\chi_M^{1-\sigma}\phi_M^{1-\sigma}}{1-\rho\,\gamma_P^{(1-\sigma)(1+\theta)}} - \frac{1}{1-\rho}\right],$$

and finally,

$$1+\lambda = \left[\frac{1-\rho\,\gamma_M^{(1-\sigma)(1+\theta)}}{1-\rho\,\gamma_P^{(1-\sigma)(1+\theta)}}\right]^{\frac{1}{1-\sigma}} \frac{\chi_P}{\chi_M}\left(\frac{\phi_P}{\phi_M}\right)^{\theta}. \tag{35}$$

To translate this compensation into output units, we have to compute $100\lambda\frac{c_{t,M}}{y_{t,M}}$, which is the compensation shown in Figure 4.

As the risk aversion parameter changes between 1 and 5, the Markov consumption compensation falls from 45% to 3% of output**.** In particular, for $\sigma = 2$, the compensation that would be necessary to achieve the planner's welfare is around 8% of output. By and large, the decrease is due to the decline in the value of the first factor in (35).[9]

The consumption compensation increases with θ. For $\sigma=2$, the Markov consumption compensation increases from 6% to 23% of output. Again, this increase in the consumption compensation is mainly due to the first factor.[10] So, the difference in growth rates is the

[9] The first factor, which depends on growth rates, falls from 17.13 for σ=1.1, to 1.23 for σ=5. The second factor increases from 0.29 to 0.86, while the third factor initially increases from its starting value of 1.018 to 1.054, and it decreases after that to essentially the same initial level.

[10] The first factor increases from 1.72 to 3.02 as θ changes from 0.2 to 1.5. The second factor gradually decreases from 0.70 to 0.54, and the third factor shows a moderate increase, from 1.13 to 1.23

main determinant of the welfare loss of the Markov solution relative to the planner's solution, over and above the effects of differences in the ratios of private or public consumption to output.

7. The model with leisure in the utility function

In this section we incorporate leisure as an additional argument in the utility function. Our goal is to analyze the extent to which the distortions on labor supply produced by the choice of tax rates and the split of public resources between consumption and investment affect the characterization of optimal policy in our endogenous growth economy.

Let us assume that preferences can be represented by the utility function, $U(c_t, l_t, g_t) = \dfrac{\left(c_t(1-l_t)^{\upsilon} g_t^{\theta}\right)^{1-\sigma} - 1}{1-\sigma}$, while the production technology is, $y_t = Bk_t^{\alpha}\left(k_{p,t} l_t\right)^{1-\alpha}$. Consumers solve the time aggregate utility maximization problem subject to $k_{t+1} - (1-\delta)k_t + c_t = (1-\tau_t)\left(r_t k_t + w_t l_t\right)$, given k_0 , while firms maximize profits as in the model with inelastic labor supply. The government raises revenues imposing a global tax rate on total income, and distributes the proceeds between public consumption and investment. The government budget constraint is, $\tau_t\left(r_t k_t + w_t l_t\right) = g_t + k_{p,t}$.

The conditions for competitive equilibrium are:

$$\upsilon \mathcal{C}(k_t, \tau_t, \eta_t) \frac{\ell(k_t, \tau_t, \eta_t)}{1 - \ell(k_t, \tau_t, \eta_t)} = (1-\alpha)(1-\tau_t)\left[(1-\eta_t)\tau_t\right]^{\frac{1-\alpha}{\alpha}} B^{1/\alpha}\left[\ell(k_t, \tau_t, \eta_t)\right]^{\frac{1-\alpha}{\alpha}} k_t,$$

$$\begin{aligned}
&\left[\mathcal{C}(k_t, \tau_t, \eta_t)\right]^{-\sigma}\left[1 - \ell(k_t, \tau_t, \eta_t)\right]^{\upsilon(1-\sigma)}\left[\mathcal{G}(k_t, \tau_t, \eta_t)\right]^{\theta(1-\sigma)} = \\
&\quad \rho\left[\mathcal{C}(k_{t+1}, \tau_{t+1}, \eta_{t+1})\right]^{-\sigma}\left[1 - \ell(k_{t+1}, \tau_{t+1}, \eta_{t+1})\right]^{\upsilon(1-\sigma)}\left[\mathcal{G}(k_{t+1}, \tau_{t+1}, \eta_{t+1})\right]^{\theta(1-\sigma)} \times \\
&\quad \left[1 - \delta + \alpha(1-\tau_{t+1})[(1-\eta_{t+1})\tau_{t+1}]^{\frac{1-\alpha}{\alpha}} B^{1/\alpha}\left[\ell(k_{t+1}, \tau_{t+1}, \eta_{t+1})\right]^{\frac{1-\alpha}{\alpha}}\right]
\end{aligned}$$

The *Generalized Euler Equations (GEE)* become:

$$\frac{U_{c_t}\mathcal{C}_{\tau_t} + U_{l_t}\ell_{\tau_t} + U_{g_t}\mathcal{G}_{\tau_t}}{\mathcal{C}_{\tau_t} + \ell_t^{\frac{1-\alpha}{\alpha}}\left(\Lambda(\tau_t) - \dfrac{1-\alpha}{\alpha}\dfrac{\ell_{\tau_t}}{\ell_t}\right)\Omega(\tau_t, \eta_t) k_t} = \frac{U_{c_t}\mathcal{C}_{\eta_t} + U_{l_t}\ell_{\eta_t} + U_{g_t}\mathcal{G}_{\eta_t}}{\mathcal{C}_{\eta_t} + \dfrac{1-\alpha}{\alpha}\left(\dfrac{1}{1-\eta_t} - \dfrac{\ell_{\eta_t}}{\ell_t}\right)\Omega(\tau_t, \eta_t)\ell_t^{\frac{1-\alpha}{\alpha}} k_t},$$

and

$$\frac{U_{c_t}\mathcal{C}_{\tau_t}+U_{l_t}\ell_{\tau_t}+U_{g_t}\mathcal{G}_{\tau_t}}{\mathcal{C}_{\tau_t}+\ell_t^{\frac{1-\alpha}{\alpha}}\left(\Lambda(\tau_t)-\frac{1-\alpha}{\alpha}\frac{\ell_{\tau_t}}{\ell_t}\right)\Omega(\tau_t,\eta_t)k_t}=\rho\Big[U_{c_{t+1}}\mathcal{C}_{k_{t+1}}+U_{l_t}\ell_{k_{t+1}}+U_{g_{t+1}}\mathcal{G}_{k_{t+1}}+$$

$$\left.\frac{U_{c_{t+1}}\mathcal{C}_{\tau_{t+1}}+U_{l_{t+1}}\ell_{\tau_{t+1}}+U_{g_{t+1}}\mathcal{G}_{\tau_{t+1}}}{\mathcal{C}_{\tau_{t+1}}+\ell_{t+1}^{\frac{1-\alpha}{\alpha}}\left(\Lambda(\tau_{t+1})-\frac{1-\alpha}{\alpha}\frac{\ell_{\tau_{t+1}}}{\ell_{t+1}}\right)\Omega(\tau_{t+1},\eta_{t+1})k_{t+1}}\left(1-\delta+\Omega(\tau_{t+1},\eta_{t+1})\ell_{t+1}^{\frac{1-\alpha}{\alpha}}-\mathcal{C}_{k_{t+1}}\right)\right].$$

We follow the same steps as in the model with inelastic labor supply, transforming these four conditions as well as the global constraint of resources in ratios to private capital, and then particularizing them for the balanced growth path, we obtain the Markov-solution. The Ramsey solution for this model also follows the same steps as that for the model with inelastic labor supply.[11]

Table 3 displays values for the main variables for two parameterizations differing in the relative weight of public consumption in the utility function. We have chosen the same parameter values as in the inelastic labor supply case, except for B and υ. These have been chosen so that the rate of growth in the Markov solution is 1.5% for the benchmark parameterization, and the proportion of time devoted to work is around 1/3.

Table 3. Values for the main variables under the three solution concepts. Effects of a change in θ

	$B = 0.573$ $\sigma = 2.00$ $\theta = 0.40$ $\alpha = 0.80$ $\delta = 0.10$ $\rho = 0.99$ $\upsilon = 1.2$			$B = 0.573$ $\sigma = 2.00$ $\theta = 1.00$ $\alpha = 0.80$ $\delta = 0.10$ $\rho = 0.99$ $\upsilon = 1.2$		
	Planner	*Markov*	*Ramsey*	*Planner*	*Markov*	*Ramsey*
η (%)	28.4	26.3	20.5	44.3	41.0	31.0
τ (%)	27.9	27.1	25.2	35.9	33.9	29.0
l (%)	47.8	32.7	32.3	53.4	33.7	32.4
γ (%)	4.5	1.5	1.6	3.7	0.8	1.1
c/y(%)	19.9	27.2	28.5	15.9	23.6	27.0
g/y(%)	7.4	7.1	5.2	15.9	13.9	9.0
kp/y(%)	20.0	20.0	20.0	20.0	20.0	20.0
k/y	3.61	3.97	3.98	3.51	3.94	3.98

Note to the table: for the planner solution $\tau_t^P=\frac{g_t+k_{p,t}}{y_t}$ and $\eta_t^P=\frac{g_t}{g_t+k_{p,t}}$.

[11] The analytical details for both solutions can be checked in a Technical Appendix available from the authors upon request.

Qualitative results are similar to the inelastic labor supply case in that the Markov solution leads to a higher income tax rate and a lower proportion of public resources devoted to investment than the Ramsey solution. Additionally, we find that the amount of time devoted to work is slightly higher in the Markov than in the Ramsey solution in order to compensate a slightly lower *k/y* ratio.

The qualitative effects of an increase in the relative weight of public consumption in the utility function are again similar to the case of an inelastic labor supply: the tax rate increases in the three solutions, and a higher proportion of public expenditures is devoted to consumption.

7.1 An analytical solution: logarithmic utility and full depreciation of private capital.

Let us assume a logarithmic utility and full depreciation, $\sigma=\delta=1$. The competitive equilibrium conditions become:

$$\frac{\upsilon c_t l_t}{1-l_t}=(1-\alpha)(1-\tau_t)B^{1/\alpha}\left[(1-\eta_t)\tau_t l_t\right]^{\frac{1-\alpha}{\alpha}}k_t \tag{36}$$

$$k_{t+1}+c_t=(1-\tau_t)B^{1/\alpha}\left[(1-\eta_t)\tau_t l_t\right]^{\frac{1-\alpha}{\alpha}}k_t, \tag{37}$$

$$\frac{c_{t+1}}{c_t}=\rho\left[(1-\tau_{t+1})\alpha B^{1/\alpha}\left[(1-\eta_{t+1})\tau_{t+1}l_{t+1}\right]^{\frac{1-\alpha}{\alpha}}\right]. \tag{38}$$

Proposition 5. *The competitive equilibrium allocations are given by,*

$$k_{t+1}=\rho\alpha\left[\frac{1-\alpha}{\upsilon(1-\rho\alpha)+1-\alpha}\right]^{\frac{1-\alpha}{\alpha}}\Omega(\tau_t,\eta_t)k_t,$$

$$c_t=(1-\rho\alpha)\left[\frac{1-\alpha}{\upsilon(1-\rho\alpha)+1-\alpha}\right]^{\frac{1-\alpha}{\alpha}}\Omega(\tau_t,\eta_t)k_t,$$

$$l_t=\frac{1-\alpha}{\upsilon(1-\rho\alpha)+1-\alpha},$$

$$\textit{where}\quad \Omega(\tau_t,\eta_t)\equiv(1-\tau_t)B^{1/\alpha}\left[(1-\eta_t)\tau_t\right]^{\frac{1-\alpha}{\alpha}}.$$

Proof. *Plugging in the system made up by (36)-(38) a guess for the functional form:* $k_{t+1}=A\,\Omega(\tau_t,\eta_t)k_t$, $c_t=B\,\Omega(\tau_t,\eta_t)k_t$, $l_t=D$, *for the competitive equilibrium allocation, with A, B, and D being unknown constants, it is easy to show that,*

$$A = \rho\alpha\left[\frac{1-\alpha}{\upsilon(1-\rho\alpha)+1-\alpha}\right]^{\frac{1-\alpha}{\alpha}},\ B = (1-\rho\alpha)\left[\frac{1-\alpha}{\upsilon(1-\rho\alpha)+1-\alpha}\right]^{\frac{1-\alpha}{\alpha}} \text{ and } D = \frac{1-\alpha}{\upsilon(1-\rho\alpha)+1-\alpha}.$$

Proposition 5. *Under full depreciation of private capital and a logarithmic utility function, separable in private and public consumption, the optimal time-consistent fiscal policy is:*[12]

$$\tau_t^M = \tau^M = 1 - \frac{\alpha(1+\rho\theta)}{1+\theta}, \quad \forall t$$

$$\eta_t^M = \eta^M = \frac{\alpha\theta(1-\rho)}{1-\alpha+\theta(1-\alpha\rho)}, \quad \forall t.$$

Proof. The proof is straightforward by following the same steps of proposition 3.

8. Conclusions

We have characterized the optimal Markov-perfect fiscal policy in an endogenous growth economy with public consumption and capital, where the fiscal authority cannot commit to policy choices beyond the current period. We have considered two policy variables: a single tax on total income and the split of public resources between investment and consumption.

Under logarithmic preferences and full depreciation of capital, we obtain the analytical expressions for the two policy variables. With this particular specification, we show that the Markov-perfect policy coincides with the optimal Ramsey policy that would arise by imposing commitment. The optimal policy reduces to that of Barro if we assume away public consumption.

For the more general case of a CRRA utility function and less than perfect depreciation of private capital, there is no closed form solution, but we compute numerical values for the Markov-perfect and the Ramsey optimal policies under parameter values calibrated to the US economy. We also explore the sensitivity of the numerical solutions to the values of three parameters: the intertemporal elasticity of substitution of consumption, the relative weight of public consumption in agents' utility function and the elasticity of output with respect to private capital. For empirically plausible parameter values, the income tax is higher under the Markov policy than under the Ramsey solution, and a higher

[12] This result generalizes Malley et al. (2002) in a double direction: by considering an endogenous labor supply and by characterizing the optimal split of government spending in addition to the optimal income tax rate.

proportion of public resources are devoted to consumption. Consequently, the growth rate is lower under the Markov policy than under the Ramsey solution.

The welfare loss of the Markov solution relative to the planner's allocation is mainly determined by the differences in growth rates, more than by differences in the ratios of private or public consumption to output.

The implication is that if the private sector knows the government's inability to pledge future policy decisions, then the government should impose a slightly higher tax rate and devote a higher share of public resources to consumption, with a relatively low cost in terms of growth.

When we include leisure as an additional argument in the utility function we obtain similar results to those corresponding to the benchmark economy. Considering a more complex tax structure, as well as non-trivial transitional dynamics in an endogenous growth model with public debt, are left as future extensions of our work.

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Appendix 1: *Proof of Proposition* 1

First order optimality conditions for the government's problem are:

With respect to τ :

$$U_C\mathcal{C}_\tau + U_G\mathcal{G}_\tau + \beta V_k^{'}\left(\frac{\partial\Omega}{\partial\tau}k - C_\tau\right) = 0$$

where:

$$\frac{\partial\Omega}{\partial\tau} = -\left[(1-\eta)\tau\right]^{\frac{1-\alpha}{\alpha}} B^{1/\alpha} + (1-\tau)\left[(1-\eta)\tau\right]^{\frac{1-\alpha}{\alpha}} \frac{1-\alpha}{\alpha}\frac{1}{\tau}B^{1/\alpha} = -\Omega(\tau,\eta)\Lambda(\tau,\eta)$$

So that:

$$U_C C_\tau + U_G G_\tau - \beta V_k^{'}\left(\Omega(\tau,\eta)\Lambda(\tau,\eta)k + C_\tau\right) = 0$$

With respect to η :

$$U_C\mathcal{C}_\eta + U_G\mathcal{G}_\eta + \beta V_k^{'}\left(\frac{\partial\Omega}{\partial\eta}k - C_\eta\right) = 0$$

Where:

$$\frac{\partial\Omega}{\partial\eta} = -(1-\tau)\left[(1-\eta)\tau\right]^{\frac{1-\alpha}{\alpha}}\frac{1-\alpha}{\alpha}\frac{1}{1-\eta}B^{1/\alpha} = -\Omega(\tau,\eta)\frac{1-\alpha}{\alpha}\frac{1}{1-\eta}$$

So that:

$$U_C C_\eta + U_G G_\eta - \beta V_k^{'}\left(\Omega(\tau,\eta)\frac{1-\alpha}{\alpha}\frac{1}{1-\eta}k + C_\eta\right) = 0$$

The envelope condition is:

$$V_k = U_C\mathcal{C}_k + U_G\mathcal{G}_k + \beta V_k^{'}\left(1-\delta+\Omega(\tau',\eta') - C_k\right)$$

where

$$\tau' = \psi(k'); \quad \eta' = \mu(k')$$

From the optimality conditions above we get,

$$\beta V_k^{'} = \frac{U_C C_\tau + U_G G_\tau}{\Omega(\tau,\eta)\Lambda(\tau,\eta)k + C_\tau},$$

$$\beta V_k^{'} = \frac{U_C C_\eta + U_G G_\eta}{\Omega(\tau,\eta)\frac{1-\alpha}{\alpha}\frac{1}{1-\eta}k + C_\eta},$$

which leads to condition (6).

Plugging the first equation into the envelope condition we get,

$$V_k = U_C \mathcal{C}_k + U_G \mathcal{G}_k + \frac{U_C C_\tau + U_G G_\tau}{\Omega(\tau,\eta)\Lambda(\tau,\eta)k + C_\tau}\left(1 - \delta + \Omega(\tau',\eta') - C_k\right)$$

and, finally, we get equation (7):

$$\frac{U_C C_\tau + U_G G_\tau}{\Omega(\tau,\eta)\Lambda(\tau,\eta)k + C_\tau} = \beta\left[U_C^{'} \mathcal{C}_k^{'} + U_{\mathcal{G}}^{'} \mathcal{G}_k^{'} + \frac{U_C^{'} C_\tau^{'} + U_G^{'} G_\tau^{'}}{\Omega(\tau^{'},\eta^{'})\Lambda(\tau^{'},\eta^{'})k^{'} + C_\tau}\left(1 - \delta + \Omega\left((\tau^{'},\eta^{'})\right) - C_k^{'}\right)\right]$$

Appendix 2: *Optimal Ramsey policy under a CRRA utility and incomplete depreciation of private capital*

The Ramsey optimal policy is the solution to the utility maximization problem, subject to the equilibrium conditions as constraints. Under the CRRA utility function, the Lagrangian for the Ramsey problem becomes:

$$\begin{aligned} L = \sum_{t=0}^{\infty} \rho^t \frac{c_t^{1-\sigma} g_t^{\theta(1-\sigma)} - 1}{1-\sigma} &+ \rho^t \mu_{1t}\left[\left(1 - \delta + \Omega(\tau_t,\eta_t)\right)k_t - c_t - k_{t+1}\right] + \\ &\rho^t \mu_{2t}\left[\eta_t\left(1-\eta_t\right)^{\frac{1-\alpha}{\alpha}} \tau_t^{1/\alpha} B^{1/\alpha} k_t - g_t\right] + \\ &\rho^t \mu_{3t}\left[\rho c_{t+1}^{-\sigma} g_{t+1}^{\theta(1-\sigma)}\left(1 - \delta + \alpha\,\Omega(\tau_{t+1},\eta_{t+1})\right) - c_t^{-\sigma} g_t^{\theta(1-\sigma)}\right]. \end{aligned}$$

Taking the derivatives with respect to $c_t, g_t, k_{t+1}, \tau_t, \eta_t$ to be equal to zero, we obtain the optimality conditions for the Ramsey problem:

$$c_t^{-\sigma} g_t^{\theta(1-\sigma)} = \mu_{1t} - \mu_{3t}\sigma c_t^{-\sigma-1} g_t^{\theta(1-\sigma)} + \sigma\mu_{3,t-1} c_t^{-\sigma-1} g_t^{\theta(1-\sigma)}\left(1 - \delta + \alpha\,\Omega(\tau_t,\eta_t)\right), \tag{39}$$

$$\theta c_t^{-\sigma} g_t^{\theta(1-\sigma)} = \mu_{2t} - (1-\sigma)\theta c_t^{-\sigma-1} g_t^{\theta(1-\sigma)}\left[\mu_{3t} - \mu_{3,t-1}\left(1 - \delta + \alpha\,\Omega(\tau_t,\eta_t)\right)\right], \tag{40}$$

$$\mu_{1t} = \rho\left[\mu_{1,t+1}\left(1 - \delta + \Omega(\tau_{t+1},\eta_{t+1})\right) + \mu_{2,t+1} B^{1/\alpha} \tau_{t+1}^{1/\alpha} \eta_{t+1}(1-\eta_{t+1})^{\frac{1-\alpha}{\alpha}}\right], \tag{41}$$

$$\mu_{1t} k_t\left(\frac{1-\alpha}{\alpha}\frac{1-\tau_t}{\tau_t} - 1\right) + \mu_{2t}\eta_t k_t \frac{1}{\alpha} + \mu_{3t-1} c_t^{-\sigma} g_t^{\theta(1-\sigma)} \alpha\left(\frac{1-\alpha}{\alpha}\frac{1-\tau_t}{\tau_t} - 1\right) = 0, \tag{42}$$

$$-\mu_{1t}\frac{1-\alpha}{\alpha} k_t + \mu_{2t}\frac{\tau_t}{1-\tau_t} k_t\left(1 - \frac{\eta_t}{\alpha}\right) - \mu_{3t-1} c_t^{-\sigma} g_t^{\theta(1-\sigma)}\left(1-\alpha\right) = 0. \tag{43}$$

Transforming the multipliers by: $\tilde{\mu}_{1t} \equiv \mu_{1t} k^{\sigma-\theta(1-\sigma)}$, $\tilde{\mu}_{2t} \equiv \mu_{2t} k^{\sigma-\theta(1-\sigma)}$, $\tilde{\mu}_{3t} \equiv \frac{\mu_{3t}}{k_t}$, and defining the rate of growth $\gamma_{t+1} = \frac{k_{t+1}}{k_t}$, the consumption to capital ratio $\chi_t = \frac{c_t}{k_t}$, and the ratio between public and private capital: $\phi_t = \dfrac{g_t}{k_t}$, we can get a system of equations in stationary ratios. First, from the global constraint of resources, we get an expression for the growth rate:

$$\gamma_{t+1} = 1 - \delta + \Omega(\tau_t, \eta_t) - \chi_t \,.$$

Whereas from the government budget constraint, we can write the ratio of public to private capital:

$$\phi_t = B^{1/\alpha} \tau_t^{1/\alpha} \eta_t (1-\eta_t)^{\frac{1-\alpha}{\alpha}} \,.$$

From the Euler equation for the competitive equilibrium:

$$x_t^{-\sigma} \phi_t^{\theta(1-\sigma)} \gamma_{t+1}^{\sigma-\theta(1-\sigma)} = \rho x_{t+1}^{-\sigma} \phi_{t+1}^{\theta(1-\sigma)} \left[\left(1 - \delta + \alpha\, \Omega(\tau_{t+1}, \eta_{t+1})\right) \right],$$

and from the set of optimality conditions above, we finally get the system of equations characterizing the optimal Ramsey policy represented in stationary ratios:

$$x_t^{-\sigma} \phi_t^{\theta(1-\sigma)} = \tilde{\mu}_{1t} - \sigma x_t^{-\sigma-1} \phi_t^{\theta(1-\sigma)} \left\{ \tilde{\mu}_{3t} - \tilde{\mu}_{3t-1} \frac{1}{\gamma_t} \left[\left(1 - \delta + \alpha\, \Omega(\tau_t, \eta_t)\right) \right] \right\}, \tag{44}$$

$$\theta x_t^{1-\sigma} \phi_t^{\theta(1-\sigma)-1} = \tilde{\mu}_{2t} - (1-\sigma)\theta x_t^{-\sigma} \phi_t^{\theta(1-\sigma)-1} \left\{ \tilde{\mu}_{3t} - \tilde{\mu}_{3t-1} \frac{1}{\gamma_t} \left[\left(1 - \delta + \alpha\, \Omega(\tau_t, \eta_t)\right) \right] \right\}, \tag{45}$$

$$\tilde{\mu}_{1t} = \rho \gamma_{t+1}^{\theta(1-\sigma)-\sigma} \left[\tilde{\mu}_{1t+1} \left(1 - \delta + \Omega(\tau_{t+1}, \eta_{t+1})\right) + \tilde{\mu}_{2t+1} B^{1/\alpha} \tau_{t+1}^{1/\alpha} (1-\eta_{t+1})^{\frac{1-\alpha}{\alpha}} \right], \tag{46}$$

$$\tilde{\mu}_{1t} \left(\frac{1-\alpha}{\alpha} \frac{1-\tau_t}{\tau_t} - 1 \right) + \tilde{\mu}_{2t} \eta_t \frac{1}{\alpha} + \tilde{\mu}_{3t-1} \frac{1}{\gamma_t} x_t^{-\sigma} \phi_t^{\theta(1-\sigma)} \alpha \left(\frac{1-\alpha}{\alpha} \frac{1-\tau_t}{\tau_t} - 1 \right) = 0, \tag{47}$$

$$-\tilde{\mu}_{1t} \frac{1-\alpha}{\alpha} + \tilde{\mu}_{2t} \frac{\tau_t}{1-\tau_t} \left(1 - \frac{\eta_t}{\alpha} \right) - \tilde{\mu}_{3t-1} \chi_t^{-\sigma} \phi_t^{\theta(1-\sigma)} (1-\alpha) \frac{1}{\gamma_t} = 0. \tag{48}$$

Along the balanced growth path, the system of equations for the Ramsey equilibrium becomes:

$$\gamma^{\sigma-\theta(1-\sigma)} = \rho \left(1 - \delta + \alpha\, \Omega(\tau, \eta)\right), \tag{49}$$

$$\chi = 1 - \delta + \Omega(\tau, \eta) - \gamma, \tag{50}$$

$$\phi = B^{1/\alpha} \tau^{1/\alpha} \eta (1-\eta)^{\frac{1-\alpha}{\alpha}}, \tag{51}$$

$$1 = \tilde{\mu}_1 x^{\sigma} \phi^{-\theta(1-\sigma)} + \tilde{\mu}_3 \sigma \frac{1}{\chi} \left[\frac{1}{\gamma} \left(1 - \delta + \alpha\, \Omega(\tau, \eta)\right) - 1 \right], \tag{52}$$

$$1=\tilde{\mu}_2\frac{1}{\theta}x_t^{\sigma-1}\phi^{1-\theta(1-\sigma)}-\tilde{\mu}_3\frac{1-\sigma}{\chi}\left[\frac{1}{\gamma}\left(1-\delta+\alpha\,\Omega(\tau,\eta)\right)-1\right], \tag{53}$$

$$\tilde{\mu}_1\left[1-\rho\gamma^{\theta(1-\sigma)-\sigma}\left(1-\delta+\Omega(\tau,\eta)\right)\right]=\rho\gamma^{\theta(1-\sigma)-\sigma}\tilde{\mu}_2 B^{1/\alpha}\tau^{1/\alpha}\eta(1-\eta)^{\frac{1-\alpha}{\alpha}}, \tag{54}$$

$$\left(\frac{1-\alpha}{\alpha}\frac{1-\tau}{\tau}-1\right)+\tilde{\mu}_2\eta\frac{1}{\alpha}+\tilde{\mu}_3\frac{1}{\gamma}x^{-\sigma}\phi^{\theta(1-\sigma)}\alpha\left(\frac{1-\alpha}{\alpha}\frac{1-\tau}{\tau}-1\right)=0, \tag{55}$$

$$-\tilde{\mu}_1\frac{1-\alpha}{\alpha}+\tilde{\mu}_2\frac{\tau}{1-\tau}\left(1-\frac{\eta}{\alpha}\right)-\tilde{\mu}_3\chi^{-\sigma}\phi^{\theta(1-\sigma)}\left(1-\alpha\right)\frac{1}{\gamma}=0, \tag{56}$$

Appendix 3. *The planner's problem under lump-sum taxes*

A planner with access to lump-sum taxes would allocate resources so as to maximize time aggregate utility subject only to the global constraint of resources, thereby solving the problem,

$$\underset{\{c_t,k_{t+1},k_{pt},g_t\}}{Max}\sum_{t=0}^{\infty}\rho^t\,\frac{c_t^{1-\sigma}g_t^{\theta(1-\sigma)}-1}{1-\sigma}$$

subject to: $k_{t+1}-(1-\delta)k_t+c_t+g_t+k_{p,t}=Bk_t^{\alpha}k_{p,t}^{1-\alpha}$, leading to optimality conditions:

$$\frac{c_{t+1}}{c_t}=\left\{\rho\left[\alpha(1-\alpha)^{\frac{1-\alpha}{\alpha}}B^{1/\alpha}+(1-\delta)\right]\right\}^{\frac{1}{\sigma-\theta(1-\sigma)}},$$

that defines the rate of growth γ_P, and:

$$\frac{k_{t+1}}{k_t}-(1-\delta)+\chi_t+\theta\chi_t+\left[(1-\alpha)B\right]^{1/\alpha}=B^{1/\alpha}(1-\alpha)^{\frac{1-\alpha}{\alpha}},$$

that leads to the ratios of public investment and consumption to private capital:

$$\chi_P=\frac{1}{1+\theta}\left[\alpha B^{1/\alpha}(1-\alpha)^{\frac{1-\alpha}{\alpha}}+(1-\delta)-\gamma_P\right],$$

$$\phi_P=\frac{g_t}{k_t}=\theta\chi_P.$$

For the purpose of comparison with the Markov and Ramsey equilibria, we can introduce a measure of the size of the public sector, as $\tau_t^P=\frac{g_t+k_{p,t}}{y_t}$ and the composition of public expenditures, $\eta_t^P=\frac{g_t}{g_t+k_{p,t}}$.

Appendix 4: **Proof of proposition 4.** *Optimal Ramsey policy under logarithmic utility and full depreciation of private capital*

Particularizing the system of equations for the balanced growth path under the optimal Ramsey policy obtained in section 4 to the case of a logarithmic utility function (σ=1) and full depreciation (δ=1), we obtain:

$$\gamma = \rho B^{1/\alpha}(1-\tau)\alpha\left[\tau(1-\eta)\right]^{\frac{1-\alpha}{\alpha}} \Rightarrow \gamma = \rho\alpha F,$$

$$\chi = B^{1/\alpha}(1-\tau)\left[\tau(1-\eta)\right]^{\frac{1-\alpha}{\alpha}} - \gamma,$$

$$\phi = B^{1/\alpha}\tau^{1/\alpha}\eta(1-\eta)^{\frac{1-\alpha}{\alpha}},$$

$$\Omega = \frac{1}{\gamma}\left(B^{1/\alpha}(1-\tau)\alpha\left[\tau(1-\eta)\right]^{\frac{1-\alpha}{\alpha}}\right) - 1 = \frac{1}{\rho} - 1,$$

$$F = B^{1/\alpha}(1-\tau)\left[\tau(1-\eta)\right]^{\frac{1-\alpha}{\alpha}},$$

$$\Gamma = \frac{1-\rho\gamma^{-1}F}{\rho\gamma^{-1}\phi} = \frac{1-1/\alpha}{1-\rho\gamma^{-1}F},$$

$$\tilde{\mu}_1 = \frac{1}{\chi\frac{1}{\theta}\frac{\phi\Gamma}{\chi}} = \frac{\theta\rho\gamma^{-1}}{1-\rho\gamma^{-1}F} = -\frac{\theta}{B^{1/\alpha}(1-\tau)(1-\alpha)\left[\tau(1-\eta)\right]^{\frac{1-\alpha}{\alpha}}},$$

$$\tilde{\mu}_2 = \Gamma\tilde{\mu}_1 = \frac{\theta}{\phi} = -\frac{\theta}{B^{1/\alpha}\tau^{1/\alpha}\eta(1-\eta)^{\frac{1-\alpha}{\alpha}}},$$

$$\tilde{\mu}_3 = \frac{1-\chi\tilde{\mu}_1}{\left(\frac{1}{\rho}-1\right)\frac{1}{\chi}},$$

together with:

$$\left(\tilde{\mu}_1 + \tilde{\mu}_3\frac{1}{\gamma\chi}\alpha\right)\frac{1-\alpha-\tau}{\alpha\tau} + \tilde{\mu}_2\eta\frac{1}{\alpha} = 0,$$

$$-\tilde{\mu}_1\frac{1-\alpha}{\alpha} + \tilde{\mu}_2\frac{\tau}{1-\tau}\left(1-\frac{\eta}{\alpha}\right) - \tilde{\mu}_3\chi(1-\alpha)\frac{1}{\gamma} = 0.$$

Substituting the expressions for the Lagrange multipliers into the last two equations gives us:

$$\left(-\frac{\theta}{B^{1/\alpha}(1-\tau)(1-\alpha)\left[\tau(1-\eta)\right]^{\frac{1-\alpha}{\alpha}}}+\frac{1-\chi\tilde{\mu}_1}{\frac{1}{\rho}-1}\frac{1}{\gamma}\alpha\right)\frac{1-\alpha-\tau}{\alpha\tau}+\frac{\theta}{B^{1/\alpha}\tau^{1/\alpha}\eta(1-\eta)^{\frac{1-\alpha}{\alpha}}}\eta\frac{1}{\alpha}=0,$$

$$\frac{\theta}{B^{1/\alpha}(1-\tau)(1-\alpha)\left[\tau(1-\eta)\right]^{\frac{1-\alpha}{\alpha}}}\left(\frac{1-\alpha}{\alpha}-\frac{1-\frac{1}{\alpha}}{\rho\chi^{-1}\phi}\frac{\tau}{1-\tau}\left(1-\frac{\eta}{\alpha}\right)\right)-\frac{1-\chi\tilde{\mu}_1}{\left(\frac{1}{\rho}-1\right)\gamma}(1-\alpha)=0.$$

Finally leading to the system:

$$-\frac{\theta}{1-\alpha}+\frac{1+\frac{1-\rho\alpha}{1-\alpha}\theta}{1-\rho}(1-\alpha-\tau^R)+\theta(1-\tau^R)=0,$$

$$\eta^R=\frac{\theta\alpha(1-\tau^R)}{\frac{(1+\rho\theta)(1-\alpha)}{1-\rho}+(1-\tau^R)(1-\alpha)}.$$

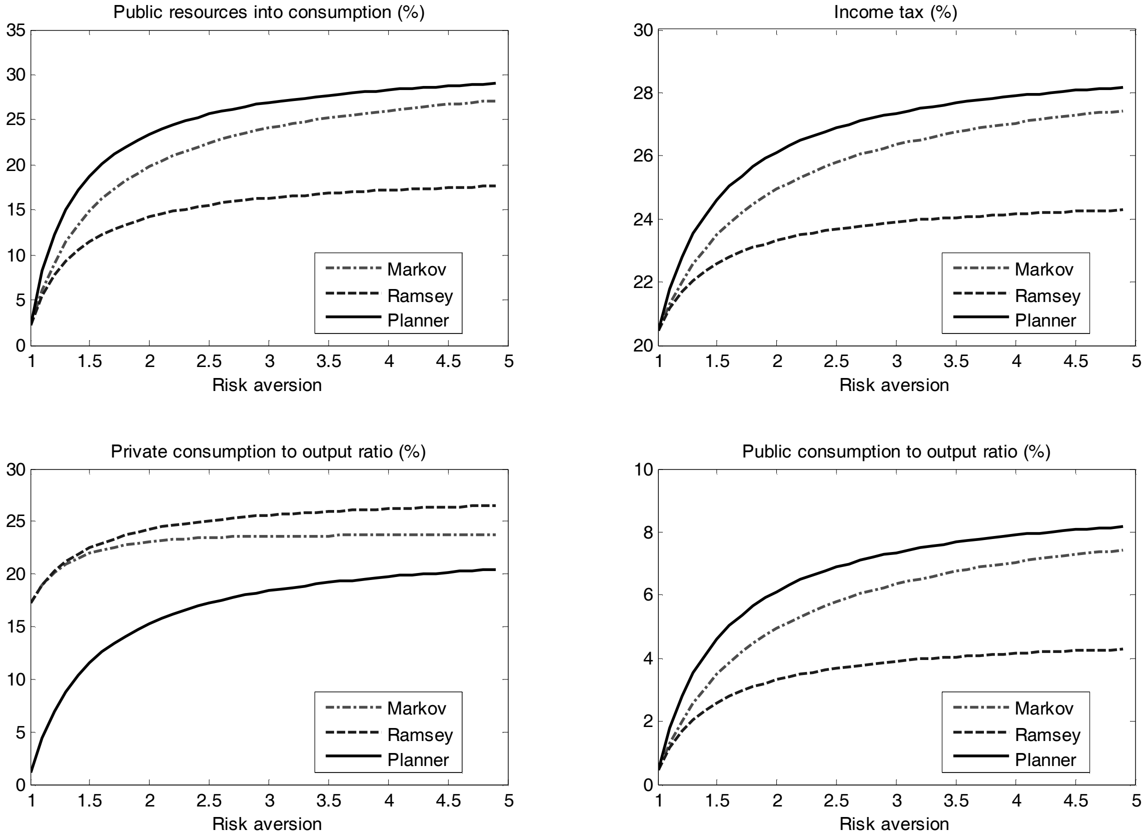
The first equation yields the Ramsey-optimal tax rate as a function of the structural parameters α, θ, ρ, while the second equation gives us the associated optimal split of public resources. It is easy to see that the solution to this system is given by

$$\tau^R=1-\frac{\alpha(1+\rho\theta)}{1+\theta},$$

$$\eta^R=\frac{\alpha\theta(1-\rho)}{1-\alpha+\theta(1-\alpha\rho)}.$$

Figure 1

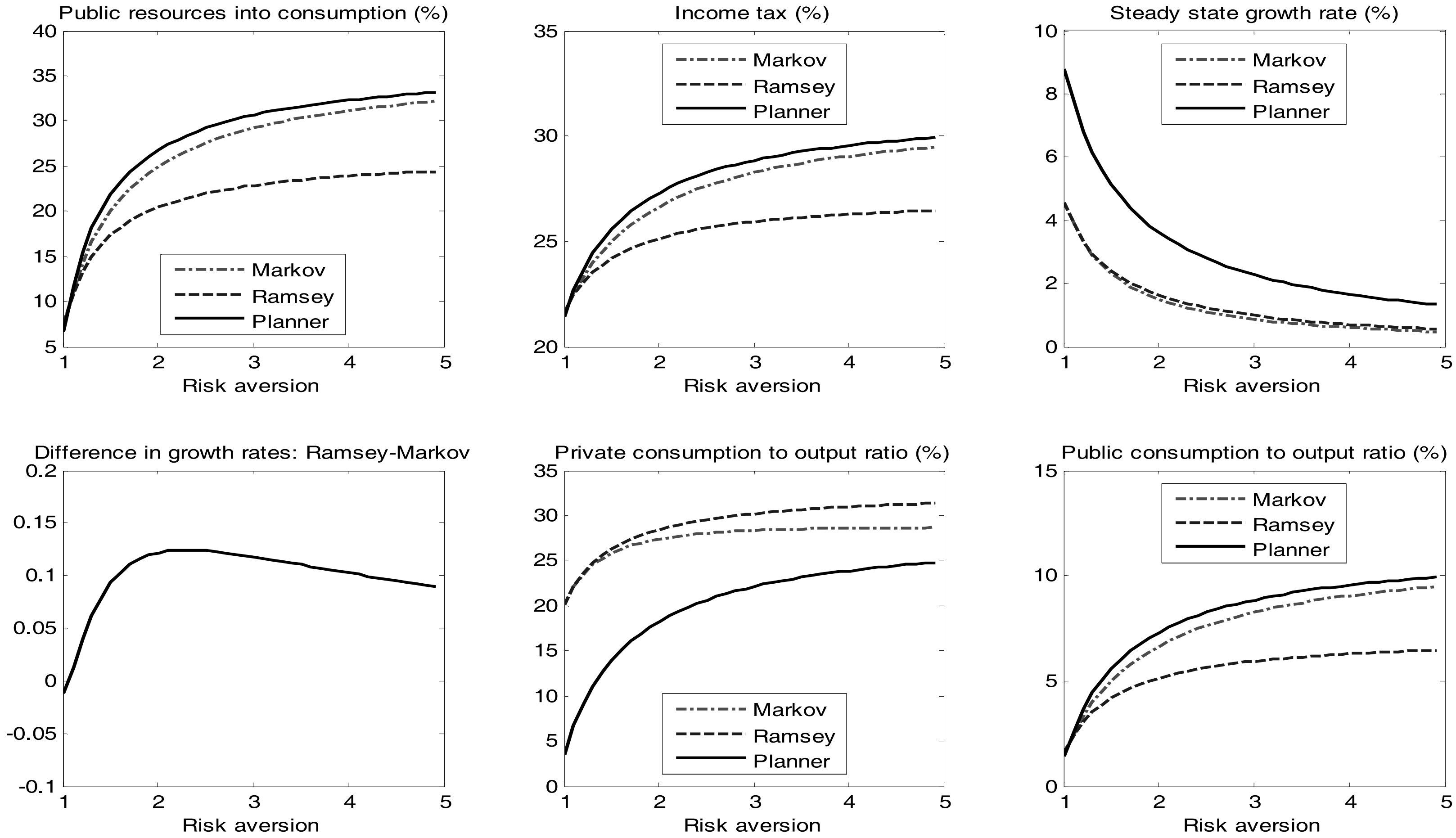
Values for the main variables in the economy under the three equilibrium concepts with full depreciation of private capital



From left to right and from above to below, the graphs display: the share of public resources devoted to public consumption, the optimal income tax rate, and the ratio of private and public consumption to output.

$\theta = 0.4$	Relative weight of public consumption in utility function
$\alpha = 0.8$	Elasticity of private capital in production function
$\rho = 0.98$	Discount rate
$\delta = 1.0$	Depreciation rate
$B = 2.5$	Productivity level

Figure 2
Values for the main variables in the economy under the three equilibrium concepts with incomplete depreciation of private capital, for different values of the risk aversion parameter

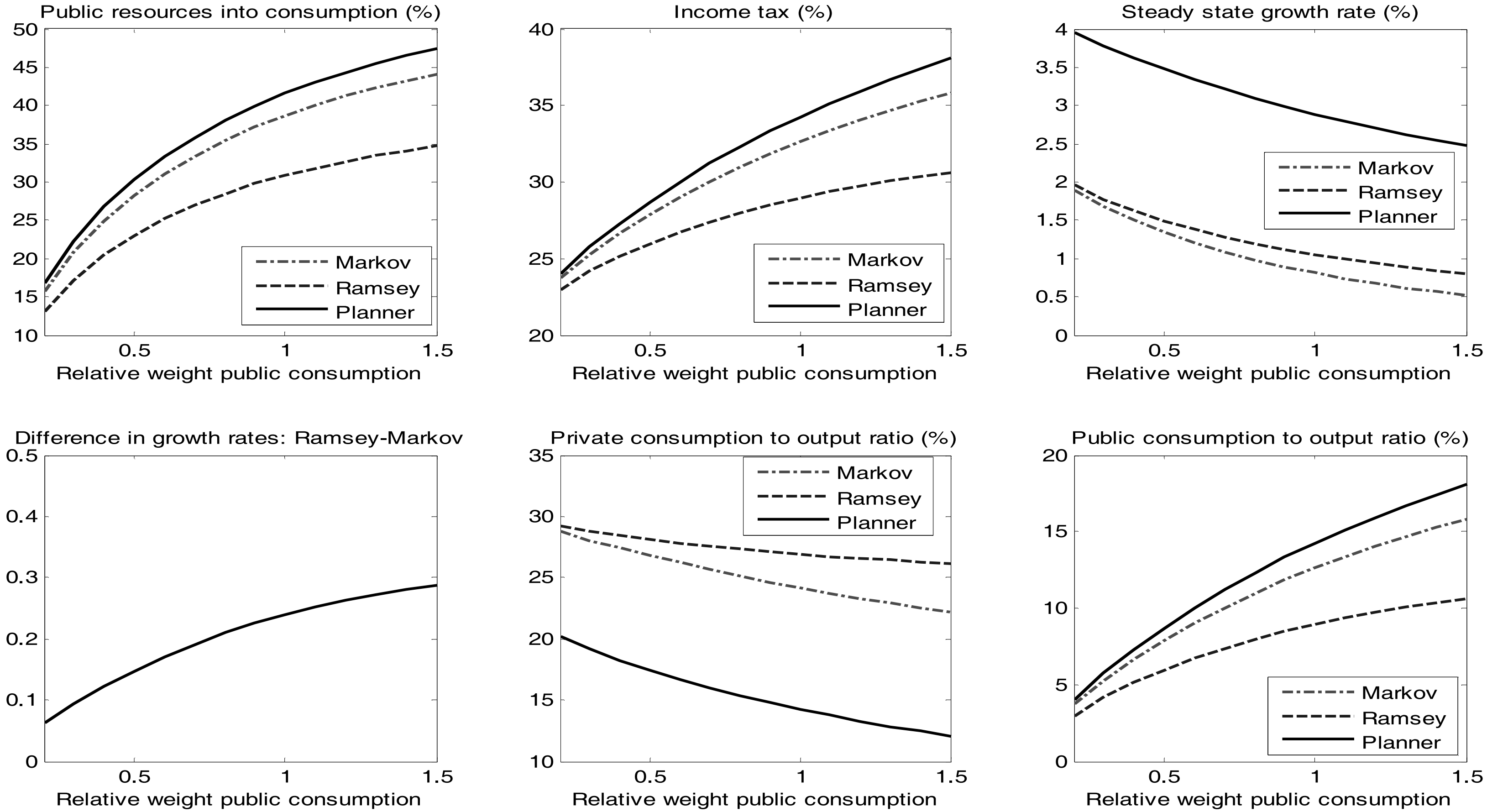


From left to right and from above to below, the graphs display: the share of public resources devoted to public consumption, the optimal income tax rate, the growth rate along the balanced path, the difference between the growth rates under the Ramsey and the Markov policies, and the ratios of private and public consumption to output.

$\theta = 0.40$	Relative weight of public consumption in utility function
$\alpha = 0.80$	Elasticity of private capital in production function
$\rho = 0.99$	Discount rate
$\delta = 0.10$	Depreciation rate
$B = 0.4555$	Productivity level

Figure 3

Values for the main variables in the economy under the three equilibrium concepts with incomplete depreciation of private capital, for different values of the relative weight of public consumption in the utility function

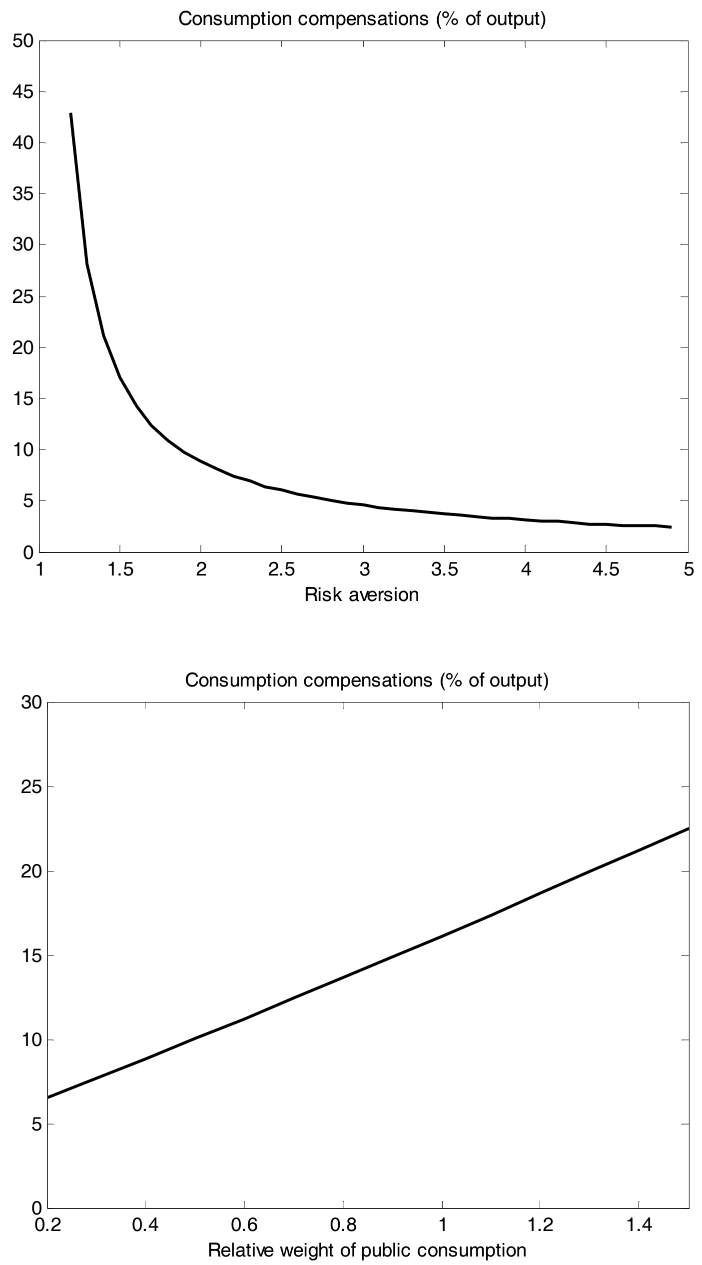


From left to right and from above to below, the graphs display: the share of public resources devoted to public consumption, the optimal income tax rate, the growth rate along the balanced path, the difference between the growth rates under the Ramsey and the Markov policies, and the ratios of private and public consumption to output.

$\sigma = 2.0$	Relative risk aversion
$\alpha = 0.80$	Elasticity of private capital in production function
$\rho = 0.99$	Discount rate
$\delta = 0.10$	Depreciation rate
$B = 0.4555$	Productivity level

Figure 4

Consumption compensations needed for the Markov policy to achieve the same level of welfare as the planner's allocation of resources



$\sigma = 2.00$	Relative risk aversion (in the second graph)
$\theta = 0.40$	Relative weight of public consumption in preferences (in the first graph)
$\alpha = 0.80$	Elasticity of private capital in production function
$\rho = 0.99$	Discount rate
$\delta = 0.10$	Depreciation rate
$B = 0.4555$	Productivity level