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SUBVARIEDADES PRINCIPALES DE UNA VARIEDAD ALGEBRAICA

PRIMALS OF AN ALGEBRAIC VARIETY

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Introduction

Let P be an irreducible algebraic variety of dimension r over an arbitrary constant's field, k, with infinitely many elements. Let $(\xi_0, \ldots, \xi_n)$ be the homogeneous coordinates of a general point of P and set $\mathrm{P}=k[\xi_0, \ldots, \xi_n]$; $\Omega=k(\xi_0, \ldots, \xi_n)$; $\xi'_i=\frac{\xi_i}{\xi_0}$, $i=1, \ldots, n$; $\mathfrak{o}=k[\xi'_1, \ldots, \xi'_n]$, $\Sigma=k(\xi'_1, \ldots, \xi'_n)$. P is a homogeneous ring and Ω is a homogeneous field; i.e. they admit the isomorphisms τ: $\xi_i \to \lambda\xi_i$, $i=0, \ldots, n$, over k; where λ is an indeterminate over Ω. Since an irreducible variety is determined by one of its general points, there are not any inconvenient for the use of the same letter for the representation of a variety and of its polynomial ring. So, when we speek of the variety P we are refering to the variety with the general point $(\xi_0, \ldots, \xi_n)$, whose coordinates are the elements of the base of P. Analogously, the variety $\mathfrak{o}$ has as general point $(\xi'_1, \ldots, \xi'_n)$. Hence $\mathfrak{o}$ is the affin variety obtained cuting P by the hyperplane $x_0=0$. We shall represent the subvarieties of P, or of $\mathfrak{o}$, by the same letter that its corresponding ideal in P, or in $\mathfrak{o}$, respectively.

We call primal of P to every irreducible subvariety of P obtained as intersection of P with an hypersurface. The main goal of this paper is to give a constructive proof of the existence of primals that contain a finite number of arbitrarily given subvarieties, whose dimensions are $\leqslant r-2$. This study is made in the § 2. On the § 3 are given some consequences of the above results; specially two new caracterizations of the simple subvarieties of an algebraic variety and a proof of the resolutions of the singularities of an algebraic curve. In the § I is given some properties of the birational correspondences $\mathrm{P} \to \mathrm{P}'$ between the varieties P and P′ such that the ring P is contained in the ring P′. We shall call such correspondences antiprojections of P on P′.

The numbers in brackets are references to the papers quoted at the end.

§ 1. Antiprojections

Let $\mathfrak{o}^*$ be a finite overring of $\mathfrak{o}$, with Σ as its quotients field. Let

$$(1) \quad \mathfrak{o}^* = \mathfrak{o}\left[\frac{g_1(\xi')}{f_1(\xi')}, \ldots, \frac{g_s(\xi')}{f_s(\xi')}\right], \quad g_i, f_i \in \mathfrak{o}, \quad i = 1, \ldots, s.$$

We shall denote by $F_i(\xi)$ and $G_i(\xi)$, $i=1, \ldots, s$, to the homogeneous polynomials obtained multiplying numerator and denominator of the above fractions by a potence of ξ_0 of exponent equal to the greatest of the degrees the two polynomials f_i and g_i. Then

$$\frac{g_i(\xi')}{f_i(\xi')} = \frac{G_i(\xi)}{F_i(\xi)}; \; G_i, F_i \in P; \; \text{degr } G_i = \text{degr } F_i; \; i=1, \ldots, s.$$

If we multiply all the elements of the base of $\mathfrak{o}^*$ by ξ_0, we obtain

$$(2) \quad \overline{P} = k\left[\xi_0, \ldots, \xi_n, \frac{G_1(\xi)\xi_0}{F_1(\xi)}, \ldots, \frac{G_s(\xi)\xi_0}{F_s(\xi)}\right] = P\left[\frac{G_1(\xi)\xi_0}{F_1(\xi)}, \ldots, \frac{G_s(\xi)\xi_0}{F_s(\xi)}\right],$$

i. e. $\overline{P}$ is an overring of P and too an homogeneous ring. If we multiply every element of the base of $\overline{P}$ by $F(\xi) = \prod_{j=1}^{s} F_i(\xi)$ and if we put $F'_i = \prod_{j \neq i} F_j(\xi)$, it is obtained

$$(3) \quad P^* = k[\xi_0 F, \ldots, \xi_n F, \; G_1(\xi)F'_1\xi_0, \ldots, G_s(\xi)F'_s\xi_0].$$

$$(4) \quad (\bar{\xi}_0, \ldots, \bar{\xi}_n, \ldots, \bar{\xi}_{n+s}); \quad \bar{\xi}_i = \xi_i\zeta, \quad i=0, \ldots n; \quad \bar{\xi}_{n+j} = \frac{G_j\xi_0}{F_j(\xi)}\zeta,$$

$j = 1, \ldots, s$; ζ an indeterminate,

and

$$(5) \quad (\eta_0, \ldots, \eta_n, \ldots, \eta_{n+s}); \quad \eta_i = \xi_i F\zeta, \quad i=0, \ldots, n; \quad \eta_{n+j} = G_j F'_j \xi_0 \zeta, \quad j=1, \ldots, s,$$

ζ an indeterminate,

are general points of the same variety, that we shall denote by $\overline{P}$ or P^* indistinctely.

The birational correspondence, T, between P and P^* is defined by the following bihomogeneous ring

$$R = P.P^* = k[\xi_0, \ldots, \xi_n;\ \eta_0, \ldots, \eta_{n+s}],$$

Being the equations of T:

$$(6)\quad \begin{cases} \eta_0 \xi_1 - \eta_1 \xi_0 = 0 \\ \cdots\cdots\cdots\cdots\cdots \\ \eta_0 \xi_n - \eta_n \xi_0 = 0 \\ \eta_0 G_i F'_i - \eta_{n+i} F = 0, \quad i = 1, \ldots, s \\ \varphi_1(\xi) = \ldots = \varphi_t(\xi) = 0, \quad \varphi_1(\eta') = \ldots = \varphi_t(\eta') = 0; \end{cases}$$

where the equations on the last line are those of the variety P, write firts with the variables (ξ) and later with the (η). With the notation $\varphi_i(\eta')$, that we shall use too in the following pages, we will denote that in the polynomial φ_i occour only the variables $\eta_0, \ldots, \eta_n$ and d'ont occour the $\eta_{n+1}, \ldots, \eta_{n+s}$.

We shall denote by (Φ) the divisor of Ω corresponding to the element Φ.

Let

$$(7)\quad (\xi_0 F) = \mathfrak{M}\mathfrak{N}_0, \ldots, (\xi_n F) = \mathfrak{M}\mathfrak{N}_n, \quad (G_1 F'_1 \xi_0) = \mathfrak{M}\mathfrak{N}_{n+1}, \ldots, \\ (G_s F'_s \xi_0) = \mathfrak{M}\mathfrak{N}_{n+s},$$

where $\mathfrak{M}$ is the divisor h.c.d. of all the divisors on the left-hand side of (7). We shall intended as center of a prime divisor over a ring, or over a variety, the center of its corresponding valuation. If we represent by $\mathfrak{A}_i$ the ideal of P which is intersection of all ideals corresponding to the centers of the prime divisors of $\mathfrak{N}_i$, the ideal

$$\mathfrak{F} = \text{rad.}(\mathfrak{A}_0, \ldots, \mathfrak{A}_{n+s})$$

represent the fundamental subvariety of P in T. Every divisor, $\mathfrak{N}$, that divide the ideal $P(F, \xi_0)$ divide too all the divisors of the left-hand side of (7), and hence it divide $\mathfrak{M}$. Therefore, if $\mathfrak{N}$ is a prime divisor, its center on P divide the center of a primer divisor of $\mathfrak{M}$. Conversely, if $\mathfrak{M}_1, \ldots, \mathfrak{M}_a$ are prime divisors of (F) such that any one of them d'ont divide $\mathfrak{M}$, and if $\mathfrak{P}_{1j}$, $j=1, \ldots, a$, are the centers of $\mathfrak{M}_j$, $j=1, \ldots, a$, over P, these ideals shall be m.p.d. (m.p.d. signify: minimal prime divisor) of PF. Not every $G_i F'_i$, $i=1, \ldots, s$, can be contained in one of the ideal $\mathfrak{P}_{1j}$, because, if $G_i F'_i \equiv 0 (\mathfrak{P}_{1j})$, $i=1, \ldots, s$, it must $\mathfrak{M}_j | \mathfrak{M}\mathfrak{N}_i$, $i=0, \ldots, n+s$, and since $\mathfrak{M}_j$ is a prime divisor that d'ont divide $\mathfrak{M}$, it should be $\mathfrak{M}_j | \mathfrak{N}_i$, $i=0, \ldots, n+s$, and the center of $\mathfrak{M}_j$ on P should divide $\mathfrak{F}$ and its dimension should be $\leqslant r-2$, in contradiction with the hypothesis. Therefore, $G_i F'_i \not\equiv 0 (\mathfrak{P}_{1j})$, for at last one i, and, by the equations (6), it follows $\eta_0 \equiv 0 (\mathfrak{P}^*)$, where $\mathfrak{P}^*$ is any m.p.d. of R $\mathfrak{P}_{1j}$ that liess over $\mathfrak{P}_{1j}$. Since $\xi_0 \not\equiv 0 (\mathfrak{P}^*)$ it follows too that $\eta_1, \ldots, \eta_n \equiv 0 (\mathfrak{P}^*)$.

Conversely, if $\mathfrak{P}^* = P^*(\eta_0, \ldots, \eta_n, \ldots)$, taking into account the equations (6), it follows that $\eta_{n+i} F \equiv 0 (R\, \mathfrak{P}^*)$, $i=1, \ldots, s$; hence if $\overline{\mathfrak{P}}$ is a m.p.d. of R $\mathfrak{P}^*$ which d'ont liess over the irrelevant of P*, it will be $F \equiv 0 (\overline{\mathfrak{P}})$. Therefore we have obtained the folloving:

LEMME 1. *Every irreducible subvariety,* $\mathfrak{P}^* = P^*(\eta_0, \ldots, \eta_n, \ldots)$ *of* P^* *that contain all the elements* $\eta_0, \ldots, \eta_n$, *has as total transform [5] by* T^{-1} *a variety contained in the subvariety corresponding to the prime ideals of PF. For that the ideals of all components of the transform in T of a subvariety* $\mathfrak{P}$ *of P must divide to* $P^*(\eta_0, \ldots, \eta_n)$, *it is sufficient that* $\mathfrak{P}$ *divide the center of a prime divisor of (F) that d'on divide* $\mathfrak{M}$ *in (7).*

We shall denote by $T[\mathfrak{P}]$ the transform [5] of $\mathfrak{P}$ by T and by $T\{\mathfrak{P}\}$ its total transform. We shall denote too by $[\overline{P}\mathfrak{P}]$ the intersection of all m.p.d. of $\overline{P}\mathfrak{P}$ that liess over $\mathfrak{P}$; by $\{\overline{P}\mathfrak{P}\}$ the intersection of all m.p.d. of $\overline{P}\mathfrak{P}$ that d'ont liess over the irrelevant of P; and by $\{\overline{P}\mathfrak{P}]$ the intersection of all primary components of a normal decomposition of $\overline{P}\mathfrak{P}$ whose corresponding prime ideals d'ont contain all the elements $\overline{\xi}_0, \ldots, \overline{\xi}_n$. Similar notations shall be employed for the ideals $R\mathfrak{P}$. Between the rings P* and $\overline{P}$ there are the isomorphism τ: $\eta_i \rightarrow \overline{\xi}_i$, $i=0, \ldots, n+s$. If $\mathfrak{P}^*$ is any ideal of P*, we shall put $\tau(\mathfrak{P}^*) = \overline{\mathfrak{P}} \in \overline{P}$.

THEOREM 1. *If $\mathfrak{P}$ is an irreducible subvariety of P, it is verified that*

$$\tau(\{R\mathfrak{P}] \cap P^*) = \{\overline{P}\mathfrak{P}\ .$$

If $\mathfrak{P}^*$ is an irreducible subvariety of P^*, such that $(\eta_0, \ldots, \eta_n) \notin \mathfrak{P}^*$, it follows that

$$T^{-1}[\mathfrak{P}^*] = \overline{\mathfrak{P}} \cap P.$$

PROOF. Let $f(\eta)$ be an arbitrary form of $\{R\mathfrak{P}]\cap P^*$. There are not any inconvenient for suppose that the elements ξ_α and ξ_β d'ont belong to any of the m.p.d. of $\{R\mathfrak{P}\}$ and that $\overline{\xi}_\alpha$ and $\overline{\xi}_\beta$ d'ont belong to any of the m.p.d. of $\{\overline{P}\mathfrak{P}]$. Since the other m.p.d. of $R\mathfrak{P}$ that d'ont occour in $\{R\mathfrak{P}]$, if there are any one, must contain ξ_α or η_β (it must be observed that α, $\beta \leqslant n$), it follows that

$$\xi_\alpha^a\, \eta_\beta^b\, f(\eta) = \sum_{i=1}^{q} \varphi_i(\xi;\eta)\,\psi_i(\xi), \quad \varphi_i \in R, \quad \psi_i \in \mathfrak{P}, \quad i = 1, \ldots, q.$$

Multiplying this equation by η_β^a, and regarding (6), it result,

$$\eta_\alpha^a\, \eta_\beta^b\, f(\eta) = \sum_{i=1}^{q} \varphi_i(\eta';\ \eta)\,\psi_i(\eta'),$$

and applying the isomorphism τ,

$$\bar{\xi}_\alpha^a\, \bar{\xi}_\beta^b\, f(\bar{\xi}) = \sum_{i=1}^{q} \varphi_i(\bar{\xi}';\ \bar{\xi})\,\psi_i(\bar{\xi}'),$$

but, since $\psi_i(\bar{\xi}') = \psi_i(\xi_i) \in \mathfrak{P}$ and $\bar{\xi}_\alpha$ and $\bar{\xi}_\beta$ d'ont belong to any one of the m.p.d. $\{\overline{P}\mathfrak{P}]$, it follows

$$f(\bar{\xi}) \equiv 0\ (\{\overline{P}\mathfrak{P}]).$$

Conversely, if $f(\bar{\xi}) \equiv 0(\{\bar{P}\mathfrak{P}])$, it follows

$$\bar{\xi}_\beta^c f(\bar{\xi}) = \sum_{i=1}^{q} \varphi_i(\bar{\xi}'; \bar{\xi}) \psi_i(\bar{\xi}'), \quad \psi_i(\bar{\xi}') \in \mathfrak{P}, \quad i = 1, \ldots, q,$$

where $\bar{\xi}_\beta$ d'ont belong to any one of the m.p.d. of $\{\bar{P}\mathfrak{P}]$, and hence, one can take $\beta \leqslant n$. Following the inverse way of the above case, it is obtained

$$f(\eta) \equiv 0(\{R\mathfrak{P}] \cap P^*),$$

and hence

$$\tau(\{R\mathfrak{P}] \cap P^*) = \{\bar{P}\mathfrak{P}].$$

Let $f(\xi)$ be an arbitrary from of $[\mathfrak{P}^*R] \cap P$; then it shall be verified that

$$(9) \qquad g(\eta) f^\nu(\xi) = \sum_{i=1}^{s} \varphi_i(\xi; \eta) \psi_i(\eta), \quad \psi_i(\eta) \equiv 0(\mathfrak{P}^*), \quad i = 1, \ldots, s,$$

where $g(\eta)$ d'on belong to any of the m.p.d. of $[\mathfrak{P}^*R]$. If μ is the degree of (9) with respect to the (ξ) multiplying by η_α^μ, being $\alpha \leqslant n$ and $\eta_\alpha \not\equiv 0(\mathfrak{P}^*)$ and taking into account (6), it follows

$$g(\eta) f^\nu(\eta') = \sum_{i=1}^{s} \varphi_i(\eta'; \eta) \psi_i(\eta),$$

and applying the isomorphism τ:

$$g(\bar{\xi}) f^\nu(\bar{\xi}') = \sum_{i=1}^{s} \varphi_i(\bar{\xi}'; \bar{\xi}) \psi_i(\bar{\xi}) \equiv 0(\bar{\mathfrak{P}}),$$

and since $g(\bar{\xi}) \not\equiv 0(\bar{\mathfrak{P}})$ (then if were $g(\bar{\xi}) \equiv 0(\bar{\mathfrak{P}})$, by τ it should be $g(\eta) \equiv 0(\mathfrak{P}^*)$) it follows

$$f(\bar{\xi}') \equiv 0(\bar{\mathfrak{P}} \cap P).$$

Conversely, let $f(\bar{\xi}) \equiv 0\,(\bar{\mathfrak{P}} \cap \mathrm{P})$. Then, by τ^{-1}, it follows $f(\eta') \equiv 0(\mathfrak{P}^*)$ and since it is possible to determine an α such that η_α d'ont belong to any of the m.p.d. of $[\mathrm{R}\,\mathfrak{P}^*]$, being $\alpha \leqslant n$; if μ is the degree of f it will be

$$\xi_\alpha^\mu f(\eta') \equiv 0\,(\mathrm{R}\ \mathfrak{P}^*),$$

and by (6)

$$\eta_\alpha^\mu f(\xi) \equiv 0\,(\mathrm{R}\ \mathfrak{P}^*)$$

and, a fortiori,

$$\eta_\alpha^\mu f(\xi) \equiv 0\,([\mathrm{R}\,\mathfrak{P}^*]);$$

hence, since η_α d'ont belong to any m.p.d. of $[\mathrm{R}\,\mathfrak{P}^*]$, it follows

$$f(\xi) \equiv 0\,([\mathrm{R}\ \mathfrak{P}^*] \cap \mathrm{P}),$$

therefore

$$\tau([\mathfrak{P}^*\mathrm{R}] \cap \mathrm{P}) = \bar{\mathfrak{P}} \cap \mathrm{P}$$

and since

$$\mathrm{T}^{-1}[\mathfrak{P}^*] = [\mathfrak{P}^*\mathrm{R}] \cap \mathrm{P}$$

it follows the last part of the theorem.

§ 2. Primals containing other given subvarieties

We have called *primal* of P to an irreducible subvariety obtained as intersection of P with an hypersurface. We shall employ in this number the hypothesis and notations previously established. We will give a constructive proof of the following

THEOREM 2. *If* $\mathfrak{P}_1, \ldots, \mathfrak{P}_k$ *are prime ideals in P, whose dimensions are all* $\leqslant r-2$, *it is possible to find prime and principal ideals that are multiples of every* $\mathfrak{P}_i$, $i=1, \ldots, k$.

Let $\lambda_1, \ldots, \lambda_N$ be indeterminates over Ω, where N is a number that we shall fix a posteriori. Set

$$\mathrm{P}[\lambda] = \mathrm{P}[\lambda_1, \ldots, \lambda_N], \quad \Omega[\lambda] = \Omega[\lambda_1, \ldots, \lambda_N],$$
$$\Omega(\lambda) = \Omega(\lambda_1, \ldots, \lambda_N).$$

The ring $P[\lambda]$ is a $N+1$—homogeneous ring; i.e. it admit the isomorphims:

$$\xi_i \to \tau_1 \xi_i, \quad i = 0, \ldots, n, \quad \lambda_1 \to \tau_2 \lambda_1, \ldots; \quad \lambda_N \to \tau_{N+1} \lambda_N,$$

where $\tau_1, \ldots, \tau_{N+1}$ are indeterminates over $\Omega(\lambda)$. But we will consider it as a bihomogeneous one, with respect to the two series of variables $(\xi_0, \ldots, \xi_n)$ and $(\lambda_1, \ldots, \lambda_N)$, respectively.

It is convenient to observe, since we shall made a tacit use of it, that in the homogeneous or polyhomogeneous rings the only unities are the elements of the field of coefficients. Hence, a homogeneous polynomial contained in P, or in $P[\lambda]$, is irreducible if and only if there are not two polynomials of degree distinc of zero and last that its degree, whose product is equal to the given polynomial.

For the proof the Th. 2 it is possible suppose that not one of the ideals $\mathfrak{P}_1, \ldots, \mathfrak{P}_k$ is a proper multiple of other of them.

We shall suppose that we have select a basis of $\mathfrak{P}_i$ in such a way that the degree, μ_i, of the homogeneous polynomial of that basis with a greater degree is so small as possible; then we shall put $\mu = \max. \{\mu_1, \ldots, \mu_k\}$.

LEMME 2. *One can find bihomogeneous polynomials of $P[\lambda]$ such as*

$$F(\xi; \lambda) = \sum_{i=1}^{N} \lambda_i F_i, \quad F_i \in P, \quad i = 1, \ldots, N,$$

where degr $F_i = g \geqslant \mu$, $i = 1, \ldots, N$, *and that satisfy the following conditions:*

a) $F(\xi; \lambda) \equiv 0 \ (P[\lambda]\mathfrak{P}_i)$, $i = 1, \ldots, k$,

b) $F(\xi; \lambda)$ *is irreducible in* $P[\lambda]$.

Proof of the lemme 2.—Set dim * $(\mathfrak{P}_i) = d_i = r - s_i \leqslant r - 2$ and hence $s_i \geqslant 2$, $i = 1, \ldots, k$. Let g be a whole number $\geqslant \mu$. It is possible to select two forms of degree g in each one of the ideals $\mathfrak{P}_i$,

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* We denote as dimension of a homogeneous ideal the dimension of its corresponding projective model; i. e. its dimension as ideal diminished at one unity.

$i=1, \ldots, k$, in such a way that if $f_1^{(i)}$, $f_2^{(i)}$ are the forms corresponding to $\mathfrak{P}_i$, $i=1, \ldots, k$, it is verified that:

a) The ideals $(f_1^{(i)}, f_2^{(i)})$, $i=1, \ldots, k$, are all unmixed and of dimension $r-2$.

b) The ideals $(f_2^{(i)}, f_2^{(j)}$ $i \neq j$, $i, j=1, \ldots, k$, are all unmixed and of dimension $r-2$.

Let

$$\text{(1)} \qquad F_i = \prod_{\alpha=1}^{k} f_{j_\alpha}^{(\alpha)}, \quad j_\alpha = 1, 2, \quad i = 1, \ldots \, N = 2^k$$

and, in particular.

$$\text{(2)} \qquad \begin{cases} F_1 = f^{(1)} \ldots f_2^{(k-1)} \; f_2^{(k)} \\ F_2 = f_2^{(1)} \ldots f_2^{(k-1)} \; f_1^{(k)} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ F_{k+1} = f_1^{(1)} \ldots f_2^{(k-1)} \; f_2^{(k)} \end{cases}$$

where F_1 is the product of all forms with subindex equal 2, and the other forms are obtained from F_1 by succesive substitution of a $f_2^{(i)}$ by the corresponding $f_1^{(i)}$.

The set of forms $\{F_1, \ldots, F_{k+1}\}$ d'ond have any common divisor. Then if it where verified that

$$F_i = MG_i, \qquad i=1, \ldots, k+1.$$

it should be $F_i \equiv 0$ (PM), $i=1, \ldots, k+1$; and from

$$F_1 = f_2^{(1)} \ldots f_2^{(k)} \equiv 0 \, (\text{PM})$$

follows that if $\mathfrak{P}$ is a m.p.d. of PM one and only one of the $f_2^{(i)}$ must belong to $\mathfrak{P}$, then this ideal as m.p.d. of a principal ideal is of dimension $r-1$ and because *b*) it can not contain two of the $f_2^{(i)}$. If f.e. $f_2^{(k)} \equiv 0 \, (\mathfrak{P})$ it should be $f_2^{(1)}, \ldots, f_2^{(k-1)}, f_1^{(k)} \not\equiv 0 \, (\mathfrak{P})$ and this is in contradiction with the condition $F_2 \equiv 0 \, (\mathfrak{P})$.

The form

$$(3) \qquad F = \sum_{i=1}^{N} \lambda_i F_i$$

is an irreducible one in $P[\lambda]$, then as F is linear relatively to the (λ) if it were reducible, one of the factors should must be a form in the (ξ) alone, and such a factor should divide to all F_i, $i=1, \ldots, N$.

Q.e.d.

LEMME 3.

$$I = P[\lambda]F$$

is a prime ideal.

We ground the proof of this lemme on the following:

LEMME 4. *Let $N(\xi)$ be a form of P, $P[\lambda]N = \mathfrak{Q}_1 \cap \ldots \cap \mathfrak{Q}_r$ be a normal decomposition of $P[\lambda]N$ in primary ideals, $\mathfrak{P}_i$ be the prime ideal corresponding to $\mathfrak{Q}_i$, $i=1, \ldots, r$, $\mathfrak{P}_i \cap P = \mathfrak{p}_i$ and $\mathfrak{Q}_i \cap P = \mathfrak{q}_i$, $i=1, \ldots, r$, it follows that*

$$PN = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r$$

is an irreducible decomposition of PN in primary ideals, $\mathfrak{p}_i$ is the prime ideal corresponding to $\mathfrak{q}_i$, $i=1, \ldots, r$, and $\mathfrak{P}_i = P[\lambda]\mathfrak{p}_i$, $i=1, \ldots, r$.

Proof of the lemme 4. Since the λ_i are algebraically independents over P, it is verified that $P[\lambda]N \cap P = PN$ and therefore $PN = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r$ is a decomposition of PN in primary ideals and $\mathfrak{p}_i$ is the prime ideal corresponding to $\mathfrak{q}_i$, $i=1, \ldots, r$.

It remain only to prove the last part of the lemme and that the bove descomposition of PN is irreducible; but this is an inmediate consequence of that.

Actually $P[\lambda]\mathfrak{p}_i \subset \mathfrak{P}_i$. Let $A(\lambda\,;\,\xi)$ be an element of $\mathfrak{P}_i$ that d'ont belong to $P[\lambda]\mathfrak{p}_i$; then it can be assummed that no one of the coefficients of $A(\lambda\,;\,\xi)$, relatively to the (λ), is contained in $\mathfrak{p}_i$. We assume that all the ideals $\mathfrak{p}_i$, $i=1, \ldots, r$ are distinct. Then, since all them are m.p.d. of PN, no one of them is contained in other one ; therefore one can find an element b such that $b \equiv 0\ (\mathfrak{q}_j)$, $j=1, \ldots, i-1, i+1, \ldots, r$ and $b \not\equiv 0\ (\mathfrak{p}_i)$. Then it will be verified that $A^\rho(\lambda\,;\,\xi)\, b \equiv 0\ (P[\lambda]N)$ i.e.

$$(4) \qquad A^\rho(\lambda;\ \xi)\, b(\xi) = f(\lambda;\ \xi)\, N(\xi), \qquad f \in P[\lambda].$$

We shall call *first term* of a polynomial of $P[\lambda]$, to the term obtained by the following process : we take up all the terms of the polynomial for which λ_1 has the greatest exponent ; amongs these terms we take up those for which λ_2 has the greatest exponent ; and so succesively, till to proceed in the same fashion with λ_N. In this fashion is obtained one and only one term of the polynomial which we call the first ; if neccessary it must be add : relatively to the order $\lambda_1, \ldots, \lambda_N$ of the variables. The coefficient of the first term will be a polynomial of P. From this definition it folows inmediately : *a*) If two polynomials of $P[\lambda]$ are equal, its firts terms are too equal. *b*) The first term of a product is equal to the product of the first terms of the factors.

Let $a(\xi)$ and $f_0(\xi)$ be the coefficients of the first terms of A and *f*, respectively. From (4) follows that

$$a^\rho(\xi)\, b(\xi) = f_0(\xi)\, N(\xi),$$

that is

$$a^\rho(\xi)\, b(\xi) \equiv 0\ (\mathfrak{p}_i),$$

and since $b(\xi) \not\equiv 0\ (\mathfrak{p}_i)$, we obtain the contradiction $a(\xi) \equiv 0\ (\mathfrak{p}_i)$.

For complete the proof it remain only to prove that the ideals $\mathfrak{p}_i$, $i=1, \ldots, a$ are all distinct. Let us admit that $\mathfrak{p}_1 = \ldots = \mathfrak{p}_\alpha$, being the other ideals distinct of these ones. Then it should be possible find a polynomial, $B(\lambda\,;\,\xi)$ that should belong to $\mathfrak{P}_i$, $i=2, \ldots, \alpha$ and d'ont belong to $\mathfrak{P}_1$. We can suppose that no one of the coefficients of $B(\lambda\,;\,\xi)$, with respect to the (λ), belong to $\mathfrak{p}_1$. Let $b(\xi)$ be an element of P such that $b(\xi) \equiv 0\ (\mathfrak{q}_j)$ $j=\alpha+1, \ldots, r$; $b(\xi) \not\equiv 0\ (\mathfrak{p}_1)$. Then if we admit that $P[\lambda]\mathfrak{p}_1 \subset \mathfrak{P}_1$ it should exist a form $A(\lambda, \xi)$ of $\mathfrak{P}_1$ whose coefficients with respect to the (λ) d'ont belong, any one of them, to $\mathfrak{p}_1$. Then it would be

$$A^\rho(\xi;\ \lambda)\, B^\mu(\xi;\ \lambda)\, b = g(\lambda;\ \xi)\, N(\xi), \qquad g \in P[\lambda].$$

If $a(\xi)$, $b_0(\xi)$ and $g_0(\xi)$ are the coefficients of the first terms of A, B and g, respectively, it would result

$$a^{\rho}(\xi)\cdot b_0^{\mu}(\xi)\cdot b(\xi)=g_0(\xi)\,N(\xi)\equiv 0(\mathfrak{p}_1),$$

which is a contradiction; hence also in this case should be $P[\lambda]\mathfrak{p}_1=\mathfrak{P}_1$ and from this follows that $\mathfrak{P}_1=\ldots=\mathfrak{P}_\alpha$, which is too a contradiction.

Q.e.d.

Proof of the lemme 3.

$\Omega[\lambda]F$ is a prime ideal. If $A\,.\,B\equiv 0$ (I), $A, B\in P[\lambda]$, one of them, for example A, shall belong to $\Omega[\lambda]F$. Hence

$$\text{(5)}\qquad A(\xi;\,\lambda)=\frac{M(\lambda;\,\xi)}{N(\xi)}F(\lambda;\,\xi),\quad M\in P[\lambda],\quad N\in P.$$

From this follows

$$\text{(6)}\qquad M(\lambda\,;\,\xi)F(\lambda\,;\,\xi)\equiv 0\ (P[\lambda]N).$$

Employing for $P[\lambda]N$ the results of the L. 4, it follows that if $F\equiv 0\ (\mathfrak{P}_i)$, since $\mathfrak{P}_i=P[\lambda]\mathfrak{p}_i$, all polynomials F_i (L. 2) should belong to $\mathfrak{p}_i$, and therefore the dimensión of this ideal should be $\leqslant r-2$, which is in contradiction with the fact that $\mathfrak{p}_i$ is a m.p.d. of a principal ideal. Hence (6) imply

$$M(\lambda\,;\,\xi)\equiv 0\ (P[\lambda]N),$$

that is

$$\lambda_i^{\nu}\,\xi_j^{\tau}\,M(\lambda;\,\xi)=g(\lambda;\,\xi)\,N(\xi),\quad g(\lambda;\,\xi)\in P[\lambda],$$

and, by substitution in (5),

$$\lambda_i^{\nu}\,\xi_j^{\tau}\,A(\xi;\,\lambda)=g(\lambda;\,\xi)\,F(\lambda;\,\xi)\equiv 0\,(P[\lambda]F),$$

and since one can choice i and j so that λ_i and ξ_j d'ont belong to the eventuell m.p.d. of $P[\lambda]F$, it follows that

$$A(\xi\,;\,\lambda)\equiv 0\ (P[\lambda]F).$$

Q. e. d.

The indeterminates $(\lambda_1, \ldots, \lambda_N)$ can be considered as homogeneous coordinates of the general point of a projective space of dimension N−1. We shall put $Q=k[\lambda_1, \ldots, \lambda_N]$, and, following our agreement, we shall denote the projective space too by Q. If I is the ideal of the lemme 3, we put

$$(7) \qquad R=P[\lambda]/I.$$

Hence R is a bihomogeneous domain of integrity, that define an irreducible algebraic correspondence, $\mathfrak{T}$, between the varieties $Q'=Q/I$ and $P'=P/I$.

LEMME 5.—$P \backsimeq P'$, $Q \backsimeq Q'$.

Proof.—If $\varphi(\xi)\equiv 0$ (I), $\varphi \neq 0$, $\varphi \in P$, being φ a form, the preceding congruence imply that

$$(8) \quad g(\xi\,;\lambda)\varphi(\xi)=h(\xi\,;\lambda)F(\xi\,;\lambda), \qquad g,h \in P[\lambda], \qquad g \not\equiv 0\ (I).$$

Hence $h(\lambda\,;\xi)F(\lambda\,;\xi)\equiv 0\ (P[\lambda]\varphi)$ and, following the same way as in the proof of the L. 4, of this congruence follows that $h(\lambda\,;\xi)=l(\lambda\,;\xi)\varphi(\xi)$, which, by substitution in (8) give

$$g(\lambda;\,\xi) \equiv 0\,(I),$$

which is a contradiction. Hence from $\varphi(\xi)\equiv 0$ (I) is must follows $\varphi(\xi)=0$ and hence $P' \backsimeq P$.

We shall now admit that $\psi(\lambda)\equiv 0$ (I) ; $\psi(\lambda)\neq 0$, $\psi(\lambda)\in Q$, being $\psi(\lambda)$ a form. From the first congruence follows $\psi(\lambda)\equiv 0\ (\Omega[\lambda]F)$ i.e.

$$\lambda_i^{\nu}\,\psi(\lambda) = \frac{g(\xi;\,\lambda)}{h(\xi)}\,F(\xi;\,\lambda), \qquad h \in P, \qquad g \in P[\lambda].$$

And, as in the above cases, from this follows $g(\xi\,;\lambda)=l(\xi\,;\lambda)h(\xi)$ and by substitution in the above equality

$$(9) \qquad \lambda_i^{\nu}\psi(\lambda)=l(\xi\,;\,\lambda)F(\xi\,;\,\lambda)$$

If we consider the first terms of (9) obtain that a constant must be equal to a form of degree greater that one; which is a contradiction. Hence of $\psi(\lambda)\equiv 0$ (I), $\psi(\lambda)\in Q$ must follows $\psi(\lambda)=0$, and therefore $Q'\cong Q$.

Q.e.d.

Hence we can identify P′ and Q′ with P and Q, respectively; and so we can put

$$P\subset R, \quad Q\subset R.$$

We have proved in [1] that one can find $a+1$ elements of R, $a=r-1$, that we will denote by $\zeta_0, \ldots, \zeta_a$ such that they are algebraically independents over Q and that R is integrally dependent over $\bar{Q}=Q[\zeta_0, \ldots, \zeta_a]$. Let $Q^*=\bar{Q}[\xi_0, \ldots, \xi_n]$, $R\subset Q^*$. We shall distinguished two cases:

a) *$\bar{Q}$ and Q^* have the same quotients field.*

b) *$\bar{Q}$ and Q^* have distinct quotients field.*

a) Let $\mathfrak{p}_0$ be a point of Q. $\bar{Q}\mathfrak{p}_0$ is a prime ideal of dimension a. Since $\bar{Q}$ and Q^* have the same quotients field and Q^* is integrally dependent on $\bar{Q}$, the conductor, $\mathfrak{c}$, of $\bar{Q}$ relatively to Q^* is distinct of zero.

The following lemme was proved by Zariski for zero-dimensional ideals [4].

LEMME 6. *If $\bar{Q}\mathfrak{p}_0 \not\supset \mathfrak{c}$, $Q^*\mathfrak{p}_0$ is a prime ideal.*

PROOF. Since $\mathfrak{p}_0$ is homogeneous and zero-dimensional ideal and Q is a projective space, $Q/\mathfrak{p}_0\cong k[\lambda]$, where λ is an indeterminate over k. Since $\bar{Q}$ is a pure transcendental extension of Q, it shall be $\bar{Q}/\bar{Q}\mathfrak{p}_0\cong k[\lambda, \bar{\zeta}_0, \ldots, \bar{\zeta}_a]$, $\zeta_i\equiv\bar{\zeta}_i(\bar{Q}\mathfrak{p}_0)$, $i=0, \ldots, a$, and $k[\lambda, \bar{\zeta}_0, \ldots, \bar{\zeta}_a]$ is a polynomial ring with $a+2$ indeterminates. From the hypothesis follows that there are one element A, such that $A\equiv 0$ ($\mathfrak{c}$), $A\not\equiv 0$ ($\bar{Q}\mathfrak{p}_0$). Let $A\equiv\bar{A}(\bar{Q}\mathfrak{p}_0)$, then $\bar{A}\neq 0$. If A^* is an arbitrary element of Q^*, it shall be $A \, . \, A^*=B\in\bar{Q}$. Taking into account that since Q^* depend integrally of $\overline{Q}$, it is verified that $Q^*\mathfrak{p}_o \cap \overline{Q} = \overline{Q}\mathfrak{p}_o$, it follows that $\overline{Q}/Q^*\mathfrak{p}_o \approx \overline{Q}/\overline{Q}\mathfrak{p}_o$, hence one can put $A \equiv \overline{A}(Q^*\mathfrak{p}_o)$ and if $A^* \equiv \overline{A}^*(Q^*\mathfrak{p}_o)$, $B \equiv \overline{B}(Q^*\mathfrak{p}_o)$, it follows that $\overline{A}^* = \dfrac{\overline{B}}{\overline{A}} \in k(\lambda\,;\,\overline{\zeta}_o, \ldots, \overline{\zeta}_a)$, and therefore $Q^*\mathfrak{p}_o$ is prime.

Q.e.d.

The theorem 2 one can ennounce now moore precisely in the case *a*) in the following fashion:

THEOREM 3. *If* $\mathfrak{p}_0$ *is a regular point of* Q *relatively to the correspondence* $\mathfrak{T}$, *such that* $\overline{Q}\mathfrak{p}_0 \not\ni \mathfrak{c}$, *it is verified that* $\mathfrak{T}\{\mathfrak{p}_0\} = R\mathfrak{p}_0 \cap P$ *is a principal prime ideal, multiple of* $\mathfrak{P}_1, \ldots, \mathfrak{P}_k$.

PROOF. Let $\mathfrak{p}_0 = Q(\lambda_2 - a_2\lambda_1, \ldots, \lambda_N - a_N\lambda_1)$, $a_i \in k$, $i=2, \ldots, N$. Then. $R\mathfrak{p}_0 = R\left(\lambda_2 - a_2\lambda_1, \ldots, \lambda_N - a_N\lambda_1, \sum_{i=1}^{N} a_i F_i \lambda_1\right)$, $a_1 = 1$; and $\mathfrak{T}\{\mathfrak{p}_0\} = R\left(\lambda_2 - a_2\lambda_1, \ldots, \lambda_N - a_N\lambda_1, \sum_{i=1}^{N} a_i F_i\right) \cap P = P\left(\sum_{i=1}^{N} a_i F_i\right)$. Let $\mathfrak{a} = R\left(\lambda_2 - a_2\lambda_1, \ldots, \lambda_N - a_N\lambda_1, \sum_{i=1}^{N} a_i F_i\right)$. Since $\mathfrak{p}_0$ is a point, its total transform coincide with its transform: $\mathfrak{T}\{\mathfrak{p}_0\} = \mathfrak{T}[\mathfrak{p}_0]$. Therefore all the m.p.d. of $\mathfrak{a}$ lie on $\mathfrak{p}_0$. But, since $Q^*\mathfrak{p}_0$ is prime ideal and is regular in $\mathfrak{T}$, by the Th 1.11. [1], $\mathfrak{a}$ has only one m.p.d. and therefore $\mathfrak{a}$ will be prime or primary ideal. Since $\mathfrak{p}_0$ is regular, one can find the denominator, H, of the ς_i, $i=0, \ldots, a$ [1], in such a fashion that $H \not\equiv 0\ (R\mathfrak{p}_0)$. Then it is verified [2] that $\mathfrak{a} = R_H\mathfrak{p}_0 \cap R$ and since $R_H\mathfrak{p}_0 \cap R \supseteq Q^*\mathfrak{p}_0 \cap R$, it will be $\mathfrak{a} \supseteq Q^*\mathfrak{p}_0 \cap R$. But $\mathfrak{a} \subseteq Q^*\mathfrak{p}_0 \cap R$ and therefore $\mathfrak{a} = Q^*\mathfrak{p}_0 \cap R$ is a prime ideal and hence

$$\mathfrak{T}\{\mathfrak{p}_0\} = \mathfrak{a} \cap P = P\left(\sum_{i=1}^{N} a_i F_i\right)$$

is too a prime ideal.

Q.e.d.

b) Let $K=k(\lambda_1, \ldots, \lambda_N)$ be the quotients field of Q. One can find the linear forms $\pi_i=\sum_{j=0}^{n} a_{ij}\xi_j$, $a_{ij}\in Q$, $i=0, \ldots, a$; $j=0, \ldots, n$ in such a fashion that $K[\xi_0, \ldots, \xi_n]$ is integrally dependent of $K[\pi_0, \ldots, \pi_a]$:

$$\xi_i^{\rho_i}+\varphi_{i1}(\pi)\xi_i^{\rho_i-1}+\ldots+\varphi_{i\rho_i}(\pi)=0, \ \varphi_{ij}\in K[\pi_0, \ldots, \pi_a], \ i=0, \ldots, n, \ j=1, \ldots \rho_i;$$

or

(10)
$$\xi_i^{\rho_i}+\frac{\psi_{i1}(\pi)}{\psi(\lambda)}\xi_i^{\rho_i-1}+\ldots+\frac{\psi_{i\rho_i}(\pi)}{\psi(\lambda)}=0,$$
$$\psi_{ij}(\pi)\in Q[\pi_0, \ldots, \pi_a], \quad \psi(\lambda)\in Q.$$

if we put in (10)

$$\theta_i=\sum_{j=0}^{n} a_{ij}\,\psi^{-1}(\lambda)\,\xi_j=\frac{\pi_i}{\psi(\lambda)},$$

it is obtained

(11)
$$\xi_i^{\rho_i}+\psi_{i1}(\theta)\xi_i^{\rho_i-1}+\ldots+\psi_{i\rho_i}(\theta)=0,$$
$$\psi_{ij}(\theta)\in Q[\theta_0, \ldots, \theta_a], \quad i=0, \ldots, n; \quad j=1, \ldots, \rho_i.$$

Let $\overline{Q}=Q[\theta_0, \ldots, \theta_a]$, Λ be the quotients field of R and $K^*=K(\theta_0, \ldots, \theta_a)$ the quotients field of $\overline{Q}$. By the hypothesis $\Lambda\neq K^*$, but Λ is a finite algebraic extension of K^*. We shall admit that these extension is separable over K^* and that the same is truth for the forthcoming specializations of these fields. Let

(12) $\qquad \theta=c_0\xi_0+c_1\xi_1+\ldots+c_n\xi_n, \qquad c_i\in\overline{Q}, \qquad i=0, \ldots, n,$

be a primitiv element of Λ over K^*; $\Lambda=K^*(\theta)$. There are not any inconvenient for admit that the c_i, $i=0, \ldots, n$ are elements of $\overline{Q}$ and R. Then $\theta\in R$ and hence is integrally dependent on $\overline{Q}$. Since this ring is integrally closed, if

(13) $\qquad \theta^g+d_1\theta^{g-1}+\ldots+d_g=0,$

is the irreducible equation of θ over K^*, it shall be verified that

$$d_i \in \overline{Q}, \quad i=1, \ldots, g.$$

From $\Lambda = K^*(\theta)$ follows that

$$(14) \quad \xi_i = \frac{t_{1i}\,\theta^{g-1} + \ldots + t_{g_i i}}{H(\lambda;\, \theta_i)}, \quad i=0, \ldots, n; \quad H,\ t_{ji} \in \overline{Q}, \quad \begin{matrix} i=0, \ldots, n, \\ j=1, \ldots, g. \end{matrix}$$

Let $\lambda_i \to \mu_i$, $\mu_i \in k$, $i=1, \ldots, N$ be a spezialization of the (λ) over k, such that $\psi(\mu) \neq 0$ and $H(\mu\,;\, \overline{\theta}_i) \neq 0$. Let $\overline{\theta}_i$, $\overline{\theta}$, $i=0, \ldots, a$ be the values of θ_i, θ, $i=0, \ldots, a$ corresponding to the above spezialization. Then follows

$$(15) \qquad P_1 = k[\overline{\theta}_0, \ldots, \overline{\theta}_a, \overline{\theta}] \subset P$$

and P is integrally dependent on P_1. From (14) follows that P and P_1 have the same quotients field and the varieties P and P_1 are birationally equivalent.

Let

$$\overline{\mathfrak{P}}_i = \mathfrak{P}_i \cap P_1, \quad i=1, \ldots, k,$$

then [3] dim. $\overline{\mathfrak{P}}_i = \text{dim.}\,\mathfrak{P}_i = d_i \leqslant r-2$.

Lemme 7. *If* $P_1F \subset \overline{\mathfrak{P}}_i$, $i=1, \ldots, k$ *and if* P_1F *is a prime ideal. PF is too a prime ideal and* $PF \subset \mathfrak{P}_i$, $i=1, \ldots, k$.

Proof. Since P_1F is a principal prime ideal, it is simple and $\mathfrak{I} = P_{1_{P_1F}}$ is a valuation ring, R_v. Evidently, $P_1 \subset R_v$ and v is the only valuation with center P_1F. Since P is integrally dependent on P_1, it follows that $P \subset R_v$ and $PF \subseteq \mathfrak{p}_v \cap P$. Let ω be an arbitrary element of $\mathfrak{p}_v \cap P$, one can put $\omega = \frac{\omega_1}{\omega_2}$; $\omega_1, \omega_2 \in P_1$, $\omega_2 \notin P_1F$, $\omega_1 \in P_1F$. Therefore, $\omega_1 = \omega\omega_2 \in PF$, $\omega_2 \notin \mathfrak{p}_v \cap P$ and hence

$$PF = (\mathfrak{p}_v \cap P) \cap I. \quad (I \text{ an ideal}).$$

But, since PF is a principal ideal, all their m.p.d. are minimal ideals of P and since there are only one valuation with center P_1F, the m.p.d. of I should lie over proper divisors of P_1F, and this is not possible since P is integrally dependent over P_1 [3]. Therefore

$$PF = \mathfrak{p}_v \cap P,$$

and PF is a prime ideal. From $P_1 F \subset \overline{\mathfrak{P}}_i$ follows that $F \subset \overline{\mathfrak{P}}_i \subset \mathfrak{P}_i$, $i=1, \ldots, k$.

Q.e.d.

From this lemme follows that the proof of the Th. 2 in the case b is reduced to the proof of the hypothesis of the L. 7.

The variety P_1 is an hypersurface of the projective space of dimension $a+2$. Let $(x_0, \ldots, x_a, x_{a+1}) \to (\overline{\theta}_0, \ldots, \overline{\theta}_a, \overline{\theta})$ be an specialization of the general point of the projective space to the general point of P; and let $\mathbf{O} = k[\overline{\theta}_0, \ldots, \overline{\theta}_a]$, this ring is an homogeneous ring. One can admit that the cones $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$, obtained by projection of the varieties $\overline{\mathfrak{P}}_1, \ldots, \overline{\mathfrak{P}}_k$ from the point $x_0 = \ldots = x_a = 0$ are all irreducibles; i. e. the ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ are prime ideals of $\mathbf{O}$. Since $\mathbf{O}$ is of dimension r and $\mathfrak{p}_i$, $i=1, \ldots, k$ are of dimension $\leqslant r-2$ we can apply the L. 2 for determine the forms $F_i(\overline{\theta}_0, \ldots, \overline{\theta}_a)$, $i=1, \ldots, N$, which have not any common factor in $\mathbf{O}$, that they have the same degree μ, and that $F_i \equiv 0\ (\mathfrak{p}_j)$, $i=1, \ldots, N$; $j=1, \ldots, k$.

Making the specialization $\lambda_i \to \mu_i$ in (13) we obtain

$$\overline{\theta}^g + \overline{d}_1 \overline{\theta}^{g-1} + \ldots + \overline{d}_g = 0, \tag{16}$$

where $\overline{d}_i \in \mathbf{O}$, $i=1, \ldots, g$. Let $\varphi(\overline{\theta})$ be a form of $\mathbf{O}$ of degree ν that belong to all the ideals $\mathfrak{p}_i$, $i=1, \ldots, k$, where $\nu = \mu - g \geqslant 0$. Then, by the L. 2, the bihomogeneous form

$$F(\lambda) = \lambda_0 \varphi(\overline{\theta}_i)(\overline{\theta}^g + \overline{d}_1 \overline{\theta}^{g-1} + \ldots + \overline{d}_{g-2} \overline{\theta}^2) + \sum_{i=1}^{N} \lambda_i F_i \tag{17}$$

is irreducible in $P_1[\lambda_0, \ldots, \lambda_N]$ and, by the L. 3

$$I = P_1[\lambda]F$$

is a prime ideal. I is too a bihomogeneous ideal with respect to the (λ) and the $(\bar{\theta}_0, \ldots, \bar{\theta}_a, \bar{\theta})$ hence I define an algebraic correspondence between $Q'=Q/I$ and $P'_1=P_1/I$. If $R_1=P_1[\lambda]/I$, it follows from the lemme 5 that Q and P_1 can be considered as subrings of R_1.

$$\bar{Q}_1=Q[\bar{\theta}_0, \ldots, \bar{\theta}_a]$$

is a polynomial ring over k with $N+a+1$ indeterminates and therefore it is integrally closed. From (16) follows that

$$Q^*_1=\bar{Q}_1[\bar{\theta}]$$

is integrally dependent from $\bar{Q}_1$. We are going now to show that with relation to the correspondence R_1 we are in the case a). It rest to prove only that Q^*_1 and $\bar{Q}_1$ have the same quotient field. Actually, from the definition of R_1, from (16) and from (17) follows that

$$\lambda_0\varphi(\bar{\theta}_i)[-\bar{d}_{g-1}\,\bar{\theta}-\bar{d}_g]+\sum_{i=1}^{N}\lambda_i F_i(\bar{\theta}_i)=0,$$

and hence

$$\bar{\theta}=\frac{\sum_{i=1}^{N}\lambda_i\,F_i(\bar{\theta}_0, \ldots, \bar{\theta}_a)}{\bar{d}_{g-1}\,\lambda_0\,\varphi(\bar{\theta}_0, \ldots, \bar{\theta}_a)}-\frac{\bar{d}_g}{\bar{d}_{g-1}}\in k(\lambda;\,\bar{\theta}_0, \ldots, \bar{\theta}_a).$$

Q.e.d.

§ 3. Simple subvarieties

If $\mathfrak{P}$ is an irreducible subvariety of P of dimension s, one can find $r-s-1$ forms $\Phi_1, \ldots, \Phi_{r-s-1}$, by repeat application of the Th. 2, such that all the ideals of the chain:

(1) $$P\Phi_1\subset P(\Phi_1, \Phi_2)\subset \ldots \subset P(\Phi_1, \ldots, \Phi_{r-s-1})\subset \mathfrak{P}$$

are distinct prime ideals. We shall say that $P(\Phi_1, \ldots, \Phi_{r-s-1})$

is a *canonical over-variety of* $\mathfrak{P}$, with respect to the chain (1) or with respect to the succession of the forms $\{\Phi_1, \ldots, \Phi_{r-s-1}\}$.

THEOREM 4. *If $\mathfrak{P}$ is a simple variety in one of its canonical over-varieties, it is a simple variety in P.*

PROOF. Let (1) be the chain of ideals corresponding to the over-variety $\mathfrak{P}_0$ of $\mathfrak{P}$ and let

$$\mathfrak{P}_i = P(\Phi_1, \ldots, \Phi_i), \quad \mathfrak{p}_i = \mathfrak{P}/\mathfrak{P}_i, \quad P_i = P/\mathfrak{P}_i,$$
$$\mathfrak{J}_i = P_{i_{\mathfrak{p}_i}}, \quad \mathfrak{p}^*_{i+1} = \mathfrak{P}_{i+1}/\mathfrak{P}_i; \quad i=1, \ldots, r-s-1.$$

It follows that

$$\mathfrak{J}_{i+1} \approx \mathfrak{J}_i/\mathfrak{p}^*_{i+1}\mathfrak{J}_i,$$

hence, if $\mathfrak{M}_i = \mathfrak{p}_i \mathfrak{J}_i$, $i=1, \ldots, r-s-1$, it will be

$$\mathfrak{M}_{i+1} \approx \mathfrak{M}_i/\mathfrak{p}^*_{i+1}\mathfrak{J}_i.$$

Therefore, if ω is an arbitrary element of $\mathfrak{M}_i$ and if

$$\omega \equiv \overline{\omega}(\mathfrak{p}^*_{i+1}\mathfrak{J}_i), \text{ and } \mathfrak{M}_{i+1} = \mathfrak{J}_{i+1}(\overline{\theta}_1, \ldots, \overline{\theta}_\lambda)$$

it will be

$$\overline{\omega} = \sum_{i=1}^{\lambda} \overline{\varphi}_i \overline{\theta}_i.$$

If θ_i and φ_i are elements of $\mathfrak{J}_i$ whose images in the homomorphism mod $\mathfrak{p}^*_{i+1}\mathfrak{J}_i$ are the elements $\overline{\theta}_i$, $\overline{\varphi}_i$, $i=1, \ldots, \lambda$, respectively, it will follows

$$\omega - \sum_{j=1}^{\lambda} \varphi_i \theta_i \equiv 0(\mathfrak{p}^*_{i+1}\mathfrak{J}_i),$$

but. since if $\Phi_j \equiv \overline{\Phi}_j(\mathfrak{P}_i)$, $j=1, \ldots, i+1$, it is $\overline{\Phi}_j = 0$, $j \leqslant i$, it follows

$$\mathfrak{p}^*_{i+1}\mathfrak{J}_i = \mathfrak{J}_i(\overline{\Phi}_{i+1})$$

and therefore

$$\omega = \sum_{j=1}^{\lambda} \varphi_i \theta_i + \varphi_{\lambda+1} \overline{\Phi}_{1+1}.$$

For $i=0$ is

$$\mathfrak{P}_0 = (0), \quad P_0 = P, \quad \mathfrak{p}_0 = \mathfrak{P}, \quad \mathfrak{J}_0 = \mathfrak{J} = P_{\mathfrak{P}},$$

$$\mathfrak{p}^*_1 = \mathfrak{P}_1 / \mathfrak{P}_0 = \mathfrak{P}_1, \quad \mathfrak{J}_1 \approx \mathfrak{J}/\mathfrak{P}_1\mathfrak{J}, \quad \mathfrak{M}_1 \approx \mathfrak{M}/\mathfrak{P}_1\mathfrak{J}.$$

Hence if the minimal base of $\mathfrak{M}_{r-s-1}$ are built with e elements, that of $\mathfrak{M}$ shall have $e+r-s-1$ elements. Therefore if $\mathfrak{P}$ is simple in $\mathfrak{P}_c$ it is too simple in P.

Q.e.d.

THEOREM 5. *If $\mathfrak{P}$ is simple in P, one can determine a canonical overvariety of $\mathfrak{P}$ in which $\mathfrak{P}$ is simple subvariety.*

PROOF. Let s be the dimension of $\mathfrak{P}$. For $s=r-1$ the pro-proposition is clear. Let $s \leqslant r-2$. Then there are [4] $r-s$ forms algebraic and linearly independents mod $\mathfrak{P}^2$. Taking these forms as the forms $f_i^{(1)}$ of the L. 2, one can built, in virtue of this lemme, a form Φ_1 such trat $\Phi_1 \equiv 0(\mathfrak{P})$, $\Phi_1 \not\equiv 0(\mathfrak{P}^2)$, and that $P\Phi_1$ is a prime ideal.

For convenience, we shall emploi non homogeous coordinates with the plane $x_0=0$ as plane at infinity. Let $\mathfrak{p}$ be the non homogeneous ideal of $\mathfrak{o}$ corresponding to the ideal $\mathfrak{P}$ of P and φ_1 the non homogeneous form corresponding to Φ_1. Let us suppose that we have find the polynomials φ_i, $i=1, \ldots, j$ such that $\varphi_i \not\equiv 0(\mathfrak{p}^2)$, $\varphi_i \equiv 0(\mathfrak{p})$ and that $\mathfrak{o}(\varphi_1, \ldots, \varphi_h)$, $h=1, \ldots, j$ are prime ideals. Let

$$\mathfrak{m} = \mathfrak{p}\, \mathfrak{o}_{\mathfrak{p}} = \mathfrak{p}\mathfrak{J}, \quad \mathfrak{J} = \mathfrak{o}_{\mathfrak{p}} \quad \text{and} \quad \mathfrak{m} = \mathfrak{J}(\theta_1, \ldots, \theta_{r-s});$$

where the base of $\mathfrak{m}$ have $r-s$ elements, since $\mathfrak{p}$ is simple. Let

$$_i\mathfrak{o} = \mathfrak{o}/\mathfrak{p}_i = \mathfrak{o}_{i-1}/\mathfrak{p}''_{i-1}, \quad \mathfrak{p}'_i = \mathfrak{p}/\mathfrak{p}_i, \quad \mathfrak{p}''_i = \mathfrak{p}_{i+1}/\mathfrak{p}_i, \quad i=1 \ldots j.$$

and

$$\mathfrak{J}_i = \mathfrak{o}_{i\mathfrak{p}'_i}, \quad \mathfrak{m}_i = \mathfrak{p}'_i \mathfrak{J}_i, \quad i=1, \ldots, j.$$

Then follows that

$$(2) \qquad \mathfrak{m}_i \approx \mathfrak{m}_{i-1}/\,\mathfrak{p}''_{i-1}\,\mathfrak{J}_{i-1}, \quad i = 1, \ldots, i,$$

and in particular

$$(3) \qquad \mathfrak{p}''_0 = \mathfrak{p}_1/_{\mathfrak{p}_0} = \mathfrak{p}_1/_{(0)} = \mathfrak{p}_1, \qquad \mathfrak{m}_1 \approx \mathfrak{m}/\mathfrak{p}_1\mathfrak{J}.$$

Since $\theta_1, \ldots, \theta_{r-s}$ are linearly independents mod $\mathfrak{p}^2$, $\varphi_1 \not\equiv 0\ (\mathfrak{p}^2)$, and since $\mathfrak{p}$ is simple, it follows that

$$\varphi_1 \equiv a_1\theta_1 + \ldots + a_{r-s}\theta_{r-s}(\mathfrak{p}^2), \ a_i \in k, \ i=1, \ldots, r-s.$$

Then, if f. e. $a_{r-s} \neq 0$, one can take $\theta_1, \ldots, \theta_{r-s-1}, \varphi_1$ as a base of $\mathfrak{m}$ i. e.: $\mathfrak{m} = \mathfrak{J}\,(\theta_1, \ldots, \theta_{r-s-1}, \varphi_1)$. Since $\varphi_1, \ldots, \varphi_j$ are linearly independents mod $\mathfrak{p}^2$ it follows inmediately by induction that

$$\mathfrak{m} = \mathfrak{J}\cdot(\theta_1, \ldots, \theta_{r-s-j}, \varphi_1, \ldots, \varphi_j).$$

From hear and from (2) and (3) follows that

$$\mathfrak{m}_j \approx \mathfrak{J}_i\left(\theta_1^{(i)}, \ldots, \theta_{r-s-j}^{(i)}, \varphi_{i+1}^{(i)}, \ldots, \varphi_j^{(i)}\right), \quad i=1, \ldots, j,$$

where $\theta_l^{(i)} \equiv \theta_l(\mathfrak{p}_i)$, $i=1, \ldots, j$; $l=1, \ldots, r-s-j$; $\varphi_k^{(i)} \equiv \varphi_k(\mathfrak{p}_i)$ $i=1, \ldots, j$, $k=i+1, \ldots, j$. This tell us that $\mathfrak{p}$ is a simple subvariety of all the varieties $\mathfrak{p}_i$, $i=1, \ldots, j$. This let us determine one element $\varphi_{j+1}^{(j)}$, of $\mathfrak{o}_j$ such that $\mathfrak{o}_j\varphi_{j+1}^{(j)}$ is prime, multiple of $\mathfrak{p}'_j$ and linearly independent of $\varphi_i^{(j)}$ mod $\mathfrak{p}'^2_j$, $i=1, \ldots, j$. Then if

$$\varphi_{j+1} \equiv \varphi_{j+1}^{(j)}(\mathfrak{p}_j), \quad \varphi_{j+1} \in \mathfrak{o},$$

it should be

$$\varphi_{j+1} \equiv 0\ (\mathfrak{p}), \qquad \varphi_{j+1} \not\equiv 0\ (\mathfrak{p}^2) \quad \text{and} \quad \varphi_{j+1} \quad \text{should be}$$

linearly independent of the φ_i, $i=1, \ldots, j$. mod $\mathfrak{p}^2$. Therefore, proceding in the same fashion as above, it shall be possible substitute one of the θ_i in the base of $\mathfrak{m}$, for example θ_{r-s-j}, by φ_{j+1} with which one can put

$$\mathfrak{m} = \mathfrak{J}\,(\theta_1, \ldots, \theta_{r-s-j-1}, \varphi_1, \ldots, \varphi_j, \varphi_{j+1}).$$

Since $\mathfrak{o}_j \varphi_{j+1}^{(j)}$ is prime, it follows that $\mathfrak{o}(\varphi_1, \ldots, \varphi_j, \varphi_{j+1})$ is too a prime ideal, and since

$$\mathfrak{m}_{j+1} \approx \mathfrak{J}_{j+1}(\theta_1^{(j+1)}, \ldots, \theta_{r-s-j-1}^{(j+1)}),$$

$\mathfrak{p}$ is too simple in $\mathfrak{o}_{j+1}$. Therefore, by induction, follows the theorem.

We shall call prime divisors to the valuations of Σ of maximal dimension ; and we shall mean as center of a prime divisor on $\mathfrak{o}$ the center on $\mathfrak{o}$ of the corresponding valuation.

THEOREM 6. *If a maximal irreducible subvariety, $\mathfrak{p}$, of $\mathfrak{o}$ is simple, there are one and only one prime divisor of Σ with center on $\mathfrak{p}$. This condition is sufficient when the minimal positiv value of the valuation is assumed by an element of $\mathfrak{o}$.*

PROOF. *The condition is necessary*. Since $\mathfrak{p}$ is simple it is verified that $\mathfrak{m} = \mathfrak{o}_{\mathfrak{p}}\, \mathfrak{p} = \mathfrak{o}_{\mathfrak{p}}(\theta)$, which imply that $\mathfrak{J} = \mathfrak{o}_{\mathfrak{p}}$ is a valuation ring. Any other valuation with center $\mathfrak{p}$ should contain $\mathfrak{J}$, therefore is should be composite with it [1] and their dimension should be less that $r-1$ and their center d'ont can be $\mathfrak{p}$, that hat a dimension equal to $r-1$.

The condition is sufficient. Let as before $\mathfrak{J} = \mathfrak{o}_{\mathfrak{p}}$, $\mathfrak{m} = \mathfrak{p}\mathfrak{J}$. Let $\overline{\mathfrak{J}}$ be the integral closure of $\mathfrak{J}$ and $\mathfrak{m}\overline{\mathfrak{J}} = \overline{\mathfrak{m}}_1^{\alpha_1}, \ldots, \overline{\mathfrak{m}}_s^{\alpha_s}$ the decomposition of $\mathfrak{m}\overline{\mathfrak{J}}$ in product of potences of prime ideals. It is verified that $\overline{\mathfrak{J}}_{\overline{\mathfrak{m}}_i}$, $i=1, \ldots, s$ are all the valuation rings with center on $\mathfrak{p}$, therefore, by the hypothesis it should be $s=1$. We are going to prove that $\alpha_1 = 1$. Let $R_v = \overline{\mathfrak{J}}_{\overline{\mathfrak{m}}}$, $\overline{\mathfrak{m}} = \overline{\mathfrak{m}}_1$. We are going to show that $\mathfrak{m} R_v$ is prime ideal. Actually if $A \cdot B \equiv 0(\mathfrak{m} R_v)$ it will be $v(A \cdot B) > 0$, hence the value of at least one of the factors must be greater than zero. Let $v(A) > 0$ and $A = \dfrac{a}{b}$, $a, b \in \mathfrak{J}$. It follows that $v(a) > v(b)$. But for the hypothesis there are one element c of $\mathfrak{o}$ with minimal positiv value ; therefore $v\left(\dfrac{a}{bc}\right) \geqslant 0$ and $A = \dfrac{a}{bc} \cdot c$ belong to $R_v \mathfrak{m}$.

If were $\mathfrak{m}\bar{\mathfrak{J}}=\bar{\mathfrak{m}}^{\alpha}$ it should be $\mathfrak{m}R_v=(\mathfrak{m}\bar{\mathfrak{J}})R_v=\bar{\mathfrak{m}}^{\alpha}R_v=\mathfrak{p}_v^{\alpha}$, contradiction; hence $\alpha=1$ and $\mathfrak{m}\bar{\mathfrak{J}}$ is prime ideal. But, since $\bar{\mathfrak{J}}$ is integrally dependent on $\mathfrak{J}$ and both rings have the same quotients field, the conductor of $\mathfrak{J}$ with respect to $\bar{\mathfrak{J}}$ d'ont vanish and, since $\mathfrak{J}$ is a local ring, $\mathfrak{m}$ should be divisor of the conductor besides that the conductor be the unit ideal. But since $\mathfrak{m}\bar{\mathfrak{J}}$ is prime, $\mathfrak{m}$ d'ont divide to the conductor and therefore the conductor coincide with $\mathfrak{J}$ and $\mathfrak{J}=\bar{\mathfrak{J}}$.

Q.e.d.

If $\mathfrak{P}$ is an s-dimensional irreducible subvariety of P and if $P(\Phi_1, \ldots, \Phi_{r-s-1})$ is a canonical overvariety of $\mathfrak{P}$ one can built a valuation of rang $r-s$ with center in $\mathfrak{P}$ and composite with valuations that have their centers on $P\Phi_1$, $P(\Phi_1, \Phi_2)$ ..., $P(\Phi_1, \ldots, \Phi_{r-s-1})$, respectively. A such valuation will be called a canonical valuation with center on M relative to the over variety $P(\Phi_1, \ldots, \Phi_{r-s-1})$. From the theorems 4, 5, 6 follows the following:

COROLLARY. *A necessary and sufficient condition for that the subvariety $\mathfrak{P}$ of P should be simple is that there exist a canonical overvariety of $\mathfrak{P}$ and a unique canonical valuation with center $\mathfrak{P}$ relative to that overvariety that assume the minimal positiv value on* $\mathfrak{o}$.

THEOREM 7. *If P is a curve, one can loss their singularities by means of an antiprojection.*

PROOF. Let $\mathfrak{P}_1, \ldots, \mathfrak{P}_s$ be all the singular points of P that we shall suppose at finity distance relatively to the plane of infinity $x_0=0$. Let $Q(\mathfrak{P}_i)=P_{\mathfrak{P}_i}$, $i=1, \ldots, s$; and $\bar{Q}(\mathfrak{P}_i)$ be the integral closure of $Q(\mathfrak{P}_i)$ Let $\mathfrak{P}_i\bar{Q}(\mathfrak{P}_i)=\mathfrak{P}_{i1}^{\alpha_{i1}}\ldots\mathfrak{P}_{it_i}^{\alpha_{it_i}}$ be the decomposition of the ideal of the left hand side in product of potences of prime ideals, and let

$$\mathfrak{J}_j(\mathfrak{P}_i)=\bar{Q}(\mathfrak{P}_i)_{\mathfrak{P}_{ij}}, \qquad \mathfrak{M}_{ij}=\mathfrak{P}_{ij}\mathfrak{J}_j(\mathfrak{P}_i).$$

$R_{v_{ij}} = \mathfrak{I}_j(\mathfrak{P}_i)$ is a valuation ring and $\mathfrak{P}_{v_{ij}} = \mathfrak{M}_{ij}$ their corresponding ideal of non unities. The valuations v_{ij}, $j=1, \ldots, t_i$ are the only ones that have center on $\mathfrak{P}_i$. Since P is an homogeneous ring, the rings $Q(\mathfrak{P}_i)$ and $\overline{Q}(\mathfrak{P}_i)$ are too homogeneous. Since $\mathfrak{P}_i$ is an homogeneous ideal, $Q(\mathfrak{P}_i)\mathfrak{P}_i$ and all the m.p.d. of $\overline{Q}(\mathfrak{P}_i)\mathfrak{P}_i$ are too homogeneous ideals; therefore, $\mathfrak{I}_j(\mathfrak{P}_i)$ and $\mathfrak{M}_{ij}$ are too homogeneous ones. Hence one can determine homogeneous elements of degree zero $\frac{f_{ij}}{g_{ij}}$, such that

$$\frac{f_{ij}}{g_{ij}} \equiv 0(\mathfrak{M}_{ij}), \quad \frac{f_{ij}}{g_{ij}} \not\equiv 0(\mathfrak{M}_{lk}), \quad i \neq l, \quad j \neq k, \quad i=1, \ldots, s, \quad j=1, \ldots, t_i.$$

We shall represent, as above, by $\left(\frac{f_{ij}}{g_{ij}}\right)$ the divisor corresponding to $\frac{f_{ij}}{g_{ij}}$. Let

$$(4) \qquad \left(\frac{f_{ij}}{g_{ij}}\right) = \mathfrak{M}_{ij}^{\alpha_{ij}} \mathfrak{N}_{1ij}^{\beta_{1ij}} \ldots \mathfrak{N}_{lij}^{\beta_{lij}} \mathfrak{V}_{1ij}^{-\gamma_{1ij}} \ldots \mathfrak{V}_{kij}^{-\gamma_{kij}}$$

be the decomposition of $\left(\frac{f_{ij}}{g_{ij}}\right)$ in product of potences of prime divisors.

It is possible to choice $\frac{f_{ij}}{g_{ij}}$ in such a fashion that

$$\frac{f_{ij}}{g_{ij}} \in \overline{Q}(\mathfrak{P}_i), \quad \frac{f_{ij}}{g_{ij}} \equiv 0(\mathfrak{P}_{ij}), \quad \frac{f_{ij}}{g_{ij}} \not\equiv 0(\mathfrak{P}_{il})$$

for $l \neq j$. Also one can impose to f_{ij} and g_{ij} the condition that they d'ont belong to any one of the ideals $\mathfrak{P}_{ik}$ for $l \neq i$. Then among all the prime divisors $\mathfrak{M}_{ij}$ that lie on the ideal $\mathfrak{P}_i$, $i=1, \ldots, s$, only $\mathfrak{M}_{ij}$ can figure in the decomposition (4). One can choice too the elements $\frac{f_{ij}}{g_{ij}}$ with the condition $\alpha_{ij}=1$, i.e. $v_{ij}\left(\frac{f_{ij}}{g_{ij}}\right)=1$ and

$$(4') \qquad \left(\frac{f_{ij}}{g_{ij}}\right) = \mathfrak{M}_{ij} \mathfrak{N}_{1ij}^{\beta_{1ij}} \ldots \mathfrak{N}_{lij}^{\beta_{lij}} \mathfrak{V}_{1ij}^{-\gamma_{1ij}} \mathfrak{V}_{kij}^{-\gamma_{kij}}.$$

There are not any inconvenient for the asumption that not any one of the places of (f_{ij}) nor (g_{ij}) should be a zero of ξ_0. Let φ_{ij} be a form of P such that all the poles of (4′) are zeros of (φ_{ij}) and that the exponents of these are greater that the exponents of those; also will we assume that neither the places $\mathfrak{M}_{ij}$, $i=1, \ldots, s$, $j=1, \ldots, t_i$, nor the places of infinity occour among the places of (φ_{ij}). Then it will be

$$(5) \qquad \left(\frac{f_{ij}\,\varphi_{ij}}{g_{ij}\,\xi_0^{\mu_{ij}}}\right) = \mathfrak{M}_{ij}\,\mathfrak{D}_{ij}\,\mathfrak{L}^{-\mu_{ij}},$$

where $\mathfrak{D}$ is an integral divisor and $\mathfrak{L}$ is the divisor of infinity. Let

$$a_{ij} = \varphi_{ij} f_{ij}, \quad b_{ij} = g_{ij}\,\xi_0^{\mu_{ij}} \text{ and } \omega_{ij} = \frac{a_{ij}\,\xi_0}{b_{ij}}, \quad i=1, \ldots, s, \quad j=1, \ldots, t_i.$$

The ring

$$(6) \qquad P' = P[\omega_{ij}], \quad i = 1, \ldots, s; \quad j = 1, \ldots, t_i$$

is an homogeneous ring, too $P \subset P'$ and P and P′ have the same quotient field. Therefore P′ is an antiprojection of P. We will prove that P′ have not singularities.

From (5) follows that the elements $\dfrac{\omega_{ij}}{\xi_0}$ can take only a negative value for the places of infinity; hence in the correspondence $T : P \rightarrow P'$ the points at finite distance with relation to the plane of infinity $x_0=0$ are transformed in points at finite distance of P′ relatively to the plane of infinity $x_0=0$. This let us emploi for the points at finite distance the rings $\mathfrak{o}$ and

$$\mathfrak{o}' = \mathfrak{o}\left[\frac{a_{ij}}{b_{ij}}\right], \quad i = 1, \ldots, s, \quad j = 1, \ldots, t_i.$$

instead of P and P′, respectively.

Let $\mathfrak{p}'$ be an arbitrary point of $\mathfrak{o}'$ and $\mathfrak{p}' \cap \mathfrak{o} = \mathfrak{p}$. By the Th. 1 follows that $T^{-1}(\mathfrak{p}') = \mathfrak{p}$. From $\mathfrak{p}' \cap \mathfrak{o} = \mathfrak{p}$ follows that $\mathfrak{o}_{\mathfrak{p}} \subseteq \mathfrak{o}'_{\mathfrak{p}'}$. If $\mathfrak{p}$ is a simple point of $\mathfrak{o}$, $\mathfrak{o}_{\mathfrak{p}}$ is the only valuation ring with center on $\mathfrak{p}$, therefore $\mathfrak{o}_{\mathfrak{p}} \supseteq \mathfrak{o}'_{\mathfrak{p}'}$ and $\mathfrak{o}_{\mathfrak{p}} = \mathfrak{o}'_{\mathfrak{p}'}$. Hence $\mathfrak{p}'$ is a simple point of $\mathfrak{o}'$. If $\mathfrak{p}$ is a singular point of $\mathfrak{o}$, f.e. $\mathfrak{p}=\mathfrak{p}_1$ and if v is a valuation with center in $\mathfrak{p}$ and $\mathfrak{p}'$, v must coincide with one of the valuations v_{1i}; with v_{11}, for example. But $\mathfrak{p}_{v_{11}}$ contain $\frac{a_{11}}{b_{11}}$, therefore $\frac{a_{11}}{b_{11}} \in \mathfrak{p}'$ and since not any other place with center in $\mathfrak{p}_1$ contain to $\frac{a_{11}}{b_{11}}$, it follows that v_{11} is the only valuation with center in $\mathfrak{p}'$; and since $v_{11}\left(\frac{a_{11}}{b_{11}}\right)=1$, it follows by the corollary of the Th. 6 that $\mathfrak{p}'$ is a simple point.

We have see above that the points at finite distance of P are transformed by T in points at finite distance of P′, therefore, the points at infinity of P′ are transformed by T^{-1} in points of infinity of P. This observation let us show that the ideal $(\xi_0, \ldots, \xi_n)$ is irrelevant in P′. Let us assume that were this not the case. Then it should be a m.p.d., $\mathfrak{P}'$, of P′ $(\xi_0, \ldots, \xi_n)$ distinct of the irrelevant and we would can find a valuation, v, with center in $\mathfrak{P}'$ and such that $P' \subset R_v$. In this valuation should be $v(\xi_i) > 0$, $i=0, \ldots, n$. But since $\mathfrak{P}'$ is not the irrelevant of P′ it should have one element, f.e. $\frac{f_{ij}\,\varphi_{ij}\,\xi_0}{g_{ij}\,\xi_0^{\mu_{ij}}}$, of value zero. Hence

$$v\,(f_{ij}\,\varphi_{ij}) = v\,(g_{ij}) + (\mu_{ij} - 1)\,v\,(\xi_0), \qquad \mu_{ij} - 1 > 0.$$

But since the center of v in P must be a point at infinity, it follows that

$$v\,(\xi_0) > \min\,\{\, v\,(\xi_1), \;\ldots,\; v\,(\xi_n)\,\},$$

which involve that

$$v\,(f_{ij}\,\varphi_{ij}) > \nu \cdot v\,(l),$$

where ν is the degree of $f_{ij}\varphi_{ij}$ and l is a linear form of P of minimal value in v. Hence $f_{ij}\,\varphi_{ij}$ would have a zero at infitity, in contradiction with the construction of these forms. Therefore, by the Th. 1 follows that $T^{-1}[\mathfrak{P}'] = \mathfrak{P}' \cap P$ for every point, $\mathfrak{P}'$ at infinity of P′ and since $\mathfrak{P}' \cap P$ is a simple point also $\mathfrak{P}'$ is a simple point.

Q.e.d.

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Madrid, 16 september 1953
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ALGUNAS CORRECCIONES

El Prof. Krull (Z. B. f. Mathematik., vol. 44, pág. 354) ha señalado la posibilidad de que dada la, a su juicio, complicada notación que hemos empleado en nuestro trabajo: «Correspondencias algebraicas. II», publicado el año 1951 en esta Revista se hayan deslizado gran número de erratas de imprenta que dificultan su lectura. En carta particular (10-III-1953), y a nuestro requerimiento, ha tenido la amabilidad de señalarnos las siguientes erratas:

Pág. 163, línea 1.ª, dice: Ω on Σ y debe decir Ω^* on $\bar{\Sigma}$.

» 164, » 7.ª, » Hence $\mathfrak{p}_1^*$ » Hence $\tilde{\mathfrak{p}}_1^*$

Gustosamente las señalamos con la esperanza de que resulte menos ingrata la lectura del mencionado trabajo.

El Prof. Segre (Math. Rev., vol. 14, pág. 314), al hacer la recensión de nuestro trabajo: «Orientación de variedades algebraicas», publicado el año 1952 en esta Revista, pone a continuación del título: (Spanish. English summary). No creemos que la versión inglesa que publicamos de dicho trabajo pueda llamarse resumen, pues, como se advierte en la nota ** de la página 94, lo único que se ha suprimido en la versión inglesa han sido las demostraciones. Por este motivo nos extraña que al indicar en la mencionada recensión que el lema 2 no es correcto, no señale que tal lema fué suprimido de la versión inglesa (página 98, l.c.); ya que nos dimos cuenta que dicho lema no se había empleado en ningún momento en dicho trabajo, razón por la que tampoco nos preocupamos de su comprobación.

Señala también el Prof. Segre que el L. 5 es incorrecto, sin especificar que se trata de una simple errata de imprenta, pues como se dice antes de enunciarlo (pág. 88, l.c.) este lema es un caso particular del L. 1. Efectivamente, en lugar de

$$\begin{vmatrix} A_{11} M & A_{12} M & A_{1n} M \\ A_{21} & A_{22} & A_{2n} \\ \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & A_{nn} \end{vmatrix}$$

(pág. 88, l.c.) debe decir:

$$\begin{vmatrix} MA_{11} & MA_{12} & \dots & MA_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{vmatrix}$$

y ninguna de estas dos correcciones afectan lo más mínimo ni a la esencia ni al detalle de las demostraciones que figuran en el texto l.c.

PEDRO ABELLANAS.