Numerical Approximations for Average Cost Markov Decision Processes

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Numerical Approximations for Average Cost MDPs

1. Introduction
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3. Approximation of the control model
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Statement of the problem

- We are interested in approximating the optimal average cost and an optimal policy of a discrete-time Markov control process.
- We consider a control model with general state and action spaces.
- Most of the approximation results in the literature are concerned with MDPs with discrete state and action spaces.

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Lipschitz-continuous control models
Approximation of the control model
An application

Our approach

- We propose procedures to discretize the state and action spaces.
- Discretization of the state space is based on sampling an underlying probability measure.
- Discretization of the action space is made by selecting actions that are "dense" in the Hausdorff metric.
- We show that our approximation error converges in probability to zero at an exponential speed.

Dufour and Prieto-Rumeau Approximation of average MDPs

Dynamics of the control model

It is a stochastic controlled dynamic system.

- The system is in state x_0.
- The controller takes an action a_0 and incurs a cost $c(x_0, a_0)$.
- The system makes a transition $x_1 \sim Q(\cdot|x_0, a_0)$.
- The system is in state x_1. Etc.

On an infinite horizon we have:

- a state process: $\{x_t\}_{t\geq 0}$;
- an action process: $\{a_t\}_{t\geq 0}$;
- a cost process: $\{c(x_t, a_t)\}_{t\geq 0}$.

Definition of the control model

The control model $\mathcal{M}$

Consider a control model $\big(X, A, \{A(x) : x \in X\}, Q, c\big)$ where

- The state space X is a Borel space, with metric ρ_X.
- The action space A is a Borel space, with metric ρ_A.
- $A(x)$ is the measurable set of available actions in state $x \in X$.
- $Q \equiv Q(B|x, a)$ is a stochastic kernel on X given $\mathbb{K}$, where

$$\mathbb{K} = \{(x, a) \in X \times A : a \in A(x)\}.$$

- $c : \mathbb{K} \to \mathbb{R}$ is a measurable cost function.

Definition of the control model

- Let Π the family of randomized history-dependent policies.
- Let $\mathbb{F}$ be the family of **deterministic stationary** policies, i.e., the class of $f : X \to A$ such that $f(x) \in A(x)$ for $x \in X$.

Optimality criteria

Given $\pi \in \Pi$ and an initial state $x \in X$, the total expected α-discounted cost $(0 < \alpha < 1)$ and the long-run average cost are

$$
\begin{aligned}
V_\alpha(x,\pi) &= E^{\pi,x}\left[\sum_{t=0}^{\infty} \alpha^t c(x_t,a_t)\right] \\
J(x,\pi) &= \limsup_{t\to\infty} E^{\pi,x}\left[\frac{1}{t}\sum_{k=0}^{t-1} c(x_t,a_t)\right].
\end{aligned}
$$

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Definition of the control model

Optimality criteria

- The optimal discounted cost is

$$V^*_\alpha(x) = \inf_{\pi\in\Pi} V_\alpha(x,\pi).$$

- The optimal average cost is

$$J^*(x) = \inf_{\pi\in\Pi} J(x,\pi).$$

- A policy $\pi^* \in \Pi$ is average optimal if

$$J(x,\pi^*) = J^*(x) \quad \text{for all } x \in X.$$

Discretizing the state space

Main idea

- We suppose that there exists a probability measure μ on X and a nonnegative measurable function $q(\cdot|\cdot,\cdot)$ on $X \times \mathbb{K}$ such that
$$Q(B|x,a) = \int_B q(y|x,a)\mu(dy)$$
for all measurable $B \subseteq X$ and every $(x,a) \in \mathbb{K}$.
- On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we take a sample of n i.i.d. random observations $\{Y_k\}_{1 \le k \le n}$ with distribution μ and we consider the empirical probability measure
$$\mu_n(B) = \frac{1}{n}\sum_{k=1}^{n} \mathbf{I}\{Y_k \in B\}.$$

Discretizing the state space

Main idea

- In the transition kernel, we replace μ with μ_n
$$Q(B|x,a) = \int_B q(y|x,a)\mu(dy) \rightsquigarrow \int_B q(y|x,a)\mu_n(dy)$$
- We have "discretized" the state space: from X to $\{Y_k\}_{1 \le k \le n}$. Integration is discretized: from μ to μ_n.
- We must be able to compute the estimation error
$$\left| \int_X g(y)\mu(dy) - \int_X g(y)\mu_n(dy) \right|.$$
- We need a **convergence** $\mu_n \to \mu$ allowing to measure such estimation errors for a **certain class** of functions g.

Convergence of probability measures on Polish spaces

Metrics

- *Total variation.*
  The metric $d(\lambda,\mu) = \sup_{B\in\mathcal{B}(X)} |\lambda(B)-\mu(B)|$ corresponds to

  $$d(\lambda,\mu) = \frac{1}{2}\sup_f \left| \int_X f d\lambda - \int_X f d\mu \right|$$

  for continuous $f : X \to [-1,1]$.
- *In our case...*
  We do not have $d(\mu_n,\mu) \to 0$.

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Metrics

- *Weak convergence.* The (Lévy-Prokhorov) metric $d(\lambda,\mu)$ is

  $$\inf_{\delta>0} \Big\{ \mu(A) \le \lambda(N(A,\delta)) + \delta, \lambda(A) \le \mu(N(A,\delta)) + \delta, \ \forall A \Big\},$$

  and corresponds to the convergence of sequences: $\lambda_n \to \lambda$ iff

  $$\int_X f d\lambda_n \to \int_X f d\lambda \quad \text{for bounded Lipschitz-cont. } f : X \to \mathbb{R}.$$
- *In our case...* There is no explicit relation between

  $$d(\lambda,\mu) \quad \text{and} \quad \sup_f \left| \int f d\mu - \int f d\lambda \right|.$$

Convergence of probability measures on Polish spaces

Lipschitz-continuous functions

- $f : A \to \mathbb{R}$ (for $A \subseteq \mathbb{R}$) is L-Lipschitz-continuous, for some $L > 0$, if

$$|f(x) - f(y)| \leq L \cdot |x - y| \quad \text{for all } x, y \in A.$$

- Roughly: functions with bounded derivative, e.g., $ax + b$, $\cos x$, e^{-x} on $[0, \infty)$.
- Not Lipschitz-continuous: e^{-x} on $\mathbb{R}$, $\sqrt{x}$ on $[0, \infty)$.
- This definition is extended for functions $f : Z_1 \to Z_2$, with Z_1 and Z_2 with metrics d_1 and d_2:

$$d_2(f(x), f(y)) \leq L \cdot d_1(x, y) \quad \text{for all } x, y \in Z_1.$$

Convergence of probability measures on Polish spaces

Metrics

- 1-*Wasserstein metric.* For probability measures in $\mathcal{P}_1(X)$ with finite first moment: $\int_X \rho_X(x, x_0)\mu(dx) < \infty$:

$$W_1(\lambda, \mu) = \inf_{\{\nu : \nu_1 = \lambda, \nu_2 = \mu\}} \int_{X \times X} \rho_X(x_1, x_2)\nu(dx_1, dx_2).$$

- N.B.: The p-Wasserstein metric uses $\big(\rho_X(x_1, x_2)\big)^p$.
- The dual Kantorovich-Rubinstein characterization gives

$$W_1(\lambda, \mu) = \sup_{f \in \mathbb{L}_1(X)} \left| \int f d\mu - \int f d\lambda \right|$$

for all 1-Lipschitz continuous functions.

Convergence of probability measures on Polish spaces

- The 1-Wasserstein metric is equivalent to weak convergence plus convergence of absolute first moments.
- For distribution functions F_1 and F_2 on $\mathbb{R}$: $W_1(\mu_1,\mu_2)=\int_{\mathbb{R}}|F_1(x)-F_2(x)|dx$.

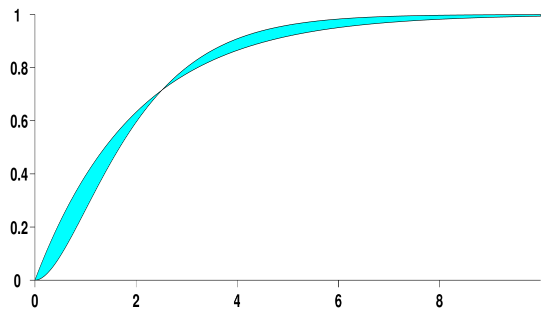


Figure: 1-Wasserstein distance between $\gamma(1/2,1)$ and $\gamma(1,2)$.

The transportation problem

- Given two probability measures λ and μ on X, transport the mass with distribution λ so as to obtain a mass with distribution μ, with cost function $c(x_1,x_2)\geq 0$.
- Find a function $T:X\to X$ minimizing

$$\int_X c(x_1,T(x_1))\lambda(dx_1)\quad \text{such that } \mu=\lambda\circ T^{-1}.$$

- The Kantorovich formulation is to find a probability measure ν on $X\times X$ with marginals λ and μ attaining

$$\inf_{\{\nu:\nu_1=\lambda,\nu_2=\mu\}}\int_{X\times X}c(x_1,x_2)\nu(dx_1,dx_2).$$

Convergence of empirical probability measures

Theorem (Boissard, 2011)

If $\mu \in \mathcal{P}_1(X)$ satisfies the modified transport inequality:

$$W_1(\mu, \lambda) \le C\Big(H(\lambda|\mu) + \sqrt{H(\lambda|\mu)}\Big)$$

for some $C > 0$ and all $\lambda \in \mathcal{P}_1(X)$ then there exists γ_0 such that for all $0 < \gamma \le \gamma_0$ there exist $C_1, C_2 > 0$ with

$$\mathbb{P}\{W_1(\mu_n, \mu) > \gamma\} \le C_1 \exp\{-C_2 n\} \quad \text{for all } n \ge 1.$$

Here, $H(\lambda|\mu)$ is the entropy $H(\lambda|\mu) = \int \log \frac{d\lambda}{d\mu} d\lambda$. A sufficient condition is the existence of $a > 0$ and $x_0 \in X$ such that

$$\int_X \exp\{a \cdot \rho_X(x, x_0)\}\mu(dx) < \infty.$$

Our setting

If f is L_f-Lipschitz-continuous

$$\Big| \int f(y)\mu_n(dy) - \int f(y)\mu(dy) \Big| \le L_f W_1(\mu_n, \mu)$$

and the probability that

$$\Big| \int f(y)\mu_n(dy) - \int f(y)\mu(dy) \Big| > \gamma$$

goes to zero at an exponential rate. So, we will place ourselves in the “Lipschitz continuity” setting.

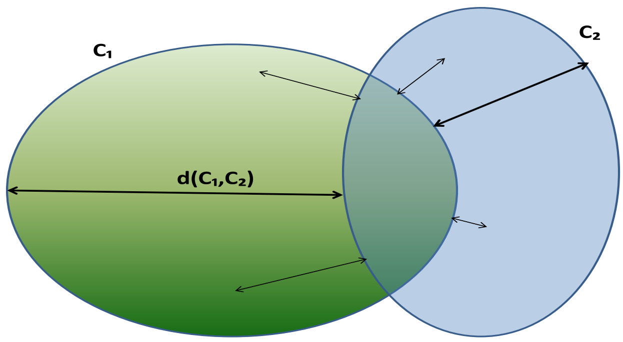
- The elements of the control model will be supposed to be Lipschitz-continuous.
- The action space will be discretized in a “Lipschitz-continuous” way.

Hypotheses

For each $x \in X$, the set $A(x)$ is compact, and $x \mapsto A(x)$ is Lipschitz continuous with respect to the Hausdorff metric, i.e.,

$$d_H(A(x), A(y)) \le L\rho_X(x, y) \quad \text{for all } x, y \in X,$$

with $d_H(C_1, C_2) = \max\{\sup_{x_1 \in C_1} \rho_X(x_1, C_2), \sup_{x_2 \in C_2} \rho_X(x_2, C_1)\}$.



Hypotheses

There exists a Lipschitz-continuous function $w : X \to [1, \infty)$ such that for all $(x, a) \in \mathbb{K}$

- The cost function c is Lipschitz-continuous and

$$|c(x, a)| \le \overline{c}w(x).$$

The density function $q(y|x, a)$ verifies

- $q(y|x, a) \le \overline{q}w(x)$.
- It is Lipschitz-continuous in y (resp., (x, a)) uniformly in (x, a) (resp., y).
- $y \mapsto w(y)q(y|x, a)$ is $Lw(x)$-Lipschitz-continuous.

Hypotheses

- $Qw(x_0, a_0)$ is finite for some $(x_0, a_0) \in \mathbb{K}$ and there is some $0 < d < 1$ such that

$$\int_X w(y)|Q(dy|x,a) - Q(dy|x',a')| \leq 2d(w(x) + w(x')) \quad (1)$$

  for all (x, a) and (x', a') in $\mathbb{K}$.
- As a consequence of (1), there exists $b \geq 0$ such that

$$Qw(x,a) \leq dw(x) + b \quad \text{for all } (x,a) \in \mathbb{K}.$$

This is the usual "contracting" condition for average cost MDPs. We impose (1) because it implies a uniform geometric ergodicity condition under which we can use the vanishing discount approach to average optimality.

Dynamic programming equation

Notation

We say that $u : X \to \mathbb{R}$ is in $\mathbb{L}_w(X)$ if u is Lipschitz-continuous and there exists $M > 0$ with $|u(x)| \leq Mw(x)$ for all $x \in X$.

Theorem (Discounted cost)

Given a discount factor $0 < \alpha < 1$, the optimal discounted cost $V^*_\alpha \in \mathbb{L}_w(X)$ and it satisfies the α-DCOE

$$V^*_\alpha(x) = \min_{a \in A(x)} \left\{ c(x,a) + \alpha \int_X V^*_\alpha(y) Q(dy|x,a) \right\} \quad \textit{for } x \in X.$$

$x \mapsto V_\alpha(x, \pi)$ might not be continuous, but $x \mapsto \inf_{\pi \in \Pi} V_\alpha(x, \pi)$ is continuous!

Dynamic programming equation

Theorem (Average cost)

- *There exist $g \in \mathbb{R}$ and $h \in \mathbb{L}_w(X)$ that are a solution to the ACOE*

$$g + h(x) = \min_{a \in A(x)} \left\{ c(x,a) + \int_X h(y) Q(dy|x,a) \right\} \quad \text{for } x \in X.$$

- *We have $g = J^*(x) = \inf_{\pi \in \Pi} J(x,\pi)$ for all $x \in X$.*
- *If $f \in \mathbb{F}$ attains the minimum in the ACOE, then it is average optimal.*

Sketch of the proof: Define $h_\alpha(x) = V_\alpha^*(x) - V_\alpha^*(x_0)$. Show that $\{h_\alpha\}$ is equicontinuous, and that its Lipschitz constant does not depend on α. Let $\alpha \to 1$.

Approximation of the control model

Discretization of the action space

For all $\mathfrak{d} > 0$ there exists a family $A_\mathfrak{d}(x)$, for $x \in X$, of subsets of A satisfying:

- $A_\mathfrak{d}(x)$ is a nonempty closed subset of $A(x)$, for $x \in X$.
- For every $x \in X$,

$$d_H(A(x), A_\mathfrak{d}(x)) \le \mathfrak{d} w(x).$$

- The multifunction $x \mapsto A_\mathfrak{d}(x)$ is $L_\mathfrak{d}$-Lipschitz continuous with respect to the Hausdorff metric, with $\sup_{\mathfrak{d}>0} L_\mathfrak{d} < \infty$.

Approximation of the control model

Definition

Given $n \geq 1$ and $\mathfrak{d} > 0$, the control model $\mathcal{M}_{n,\mathfrak{d}}$ is defined by the elements

$$\big(X, A, \{A_{\mathfrak{d}}(x) : x \in X\}, Q_n, c\big),$$

Recall that $Q(B|x,a) = \int_B q(y|x,a)\mu(dy)$. Here,

$$Q_n(B|x,a) = \frac{\int_B q(y|x,a)\mu_n(dy)}{\int_X q(y|x,a)\mu_n(dy)} = \frac{\sum_{k:Y_k \in B} q(Y_k|x,a)}{\sum_{k=1}^n q(Y_k|x,a)}.$$

Note that $Q_n(\cdot|x,a)$ has finite support, and it assigns probability proportional to $q(Y_k|x,a)$ to Y_k.

Properties of $\mathcal{M}_{n,\mathfrak{d}}$

If $v \in \mathbb{L}_w(X)$ —w-bounded and Lipschitz-continuous— we can compare Qv and $Q_n v$:

$$|Qv(x,a) - Q_n v(x,a)| \leq \mathcal{C}_v w(x) W_1(\mu, \mu_n),$$

but not when v is not Lipschitz-continuous.

We will use the notation:

- $\mathbb{K}_{\mathfrak{d}} = \{(x,a) \in X \times A \ : \ a \in A_{\mathfrak{d}}(x)\}$.
- $\Pi_{\mathfrak{d}}$ and $\mathbb{F}_{\mathfrak{d}}$ are the families of all policies and deterministic stationary policies for the control model $\mathcal{M}_{n,\mathfrak{d}}$.
- The expectation operator is $E^{\pi,x}_{n,\mathfrak{d}}$.
- Let

$$J^*_{n,\mathfrak{d}}(x) = \inf_{\pi \in \Pi_{\mathfrak{d}}} \limsup_{t \to \infty} E^{\pi,x}_{n,\mathfrak{d}}\left[\frac{1}{t}\sum_{k=0}^{t-1} c(x_t, a_t)\right].$$

Properties of $\mathcal{M}_{n,\mathfrak{d}}$

Define

$$\mathfrak{c} = \frac{1-d}{4(L_{wq} + L_q(1+4(d+b)))}$$

and suppose that $\omega \in \Omega$ is such that $W_1(\mu, \mu_n(\omega)) \leq \mathfrak{c}$. Then we have:

- $Q_n(X|x,a) = 1$ for all $(x,a) \in \mathbb{K}_{\mathfrak{d}}$.
- For all $(x,a) \in \mathbb{K}_{\mathfrak{d}}$,

$$Q_n w(x,a) \leq \frac{1+d}{2} w(x) + 2b.$$

- For all (x,a) and (x',a') in $\mathbb{K}_{\mathfrak{d}}$

$$\int_X w(y)|Q_n(dy|x,a) - Q_n(dy|x',a')| \leq (1+d) \cdot (w(x) + w(x'))$$

Properties of $\mathcal{M}_{n,\mathfrak{d}}$

Theorem

If $\omega \in \Omega$ is such that $W_1(\mu, \mu_n(\omega)) \leq \mathfrak{c}$ then

- *The control model $\mathcal{M}_{n,\mathfrak{d}}$ is uniformly geometrically ergodic and it verifies the "same" properties as $\mathcal{M}$.*
- *The optimal average cost $J^*_{n,\mathfrak{d}}(x) \equiv g^*_{n,\mathfrak{d}}$ is constant and it satisfies the ACOE: for all $x \in X$*

$$g^*_{n,\mathfrak{d}} + h(x) = \min_{a \in A_{\mathfrak{d}}(x)} \left\{ c(x,a) + \int_X h(y) Q_n(dy|x,a) \right\}$$

  for some $h \in \mathbb{B}_w(X)$.
- *Besides, h is unique up to additive constants.*

Convergence of the optimal average cost

Theorem

There exists $\varepsilon_0 > 0$ *such that for any* $0 < \varepsilon \leq \varepsilon_0$ *there exist* $\mathfrak{d} > 0$ *and constants* $\mathcal{S}, \mathcal{T} > 0$ *such that*

$$\mathbb{P}^*\{|g^*_{n,\mathfrak{d}} - g| > \varepsilon\} \leq \mathcal{S} \exp\{-\mathcal{T} n\}.$$

for all $n \geq 1$.

Sketch of the proof

- From the ACOE for $\mathcal{M}$ we have

$$g + h(x) \leq c(x, a) + Qh(x, a).$$

- Replace Q with Q_n and obtain

$$g + h(x) \leq c(x, a) + Q_n h(x, a) + Cw(x) W_1(\mu, \mu_n).$$

- Iterate this inequality t times, divide by t, and take the limit as $t \to \infty$ to obtain $g \leq g^*_{n,\mathfrak{d}} + CW_1(\mu, \mu_n)$.
- For an $\mathcal{M}$-canonical policy $f \in \mathbb{F}$

$$g + h(x) = c(x, f) + Qh(x, f).$$

- Take the "projection" $\tilde{f}$ of f on $\mathbb{F}_{\mathfrak{d}}$ and obtain

$$g + h(x) \geq c(x, \tilde{f}) + Qh(x, \tilde{f}) - C\mathfrak{d} w(x).$$

- Replace Q with Q_n and proceed as before.

Approximation of an optimal policy

Main idea

- Starting from the ACOE for $\mathcal{M}_{n,\mathfrak{d}}$

$$g^*_{n,\mathfrak{d}} + h(x) = \min_{a\in A_{\mathfrak{d}}(x)} \Big\{c(x,a) + \int_X h(y)Q_n(dy|x,a)\Big\},$$

  let $\widetilde{f}_{n,\mathfrak{d}} \in \mathbb{F}_{\mathfrak{d}}$ be a canonical policy.
- Since $\widetilde{f}_{n,\mathfrak{d}} \in \mathbb{F}$, "use it" in the control model $\mathcal{M}$ to obtain the expected average cost $J(x,\widetilde{f}_{n,\mathfrak{d}})$
- Compare $J(x,\widetilde{f}_{n,\mathfrak{d}})$ and g.

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Approximation of an optimal policy

Difficulties

- For a function v, we have that Qv is Lipschitz-continuous, but $Q_n v$ is locally Lipschitz-continuous.
- The function h in the ACOE for $\mathcal{M}_{n,\mathfrak{d}}$ is locally Lipschitz-continuous.
- We cannot directly compare Qh with $Q_n h$.
- There exists a Lipschitz-continuous $\tilde{h}$ with

$$||h - \tilde{h}||_w \leq CW_1(\mu, \mu_n).$$

- Use this $\tilde{h}$ to compare $Q\tilde{h}$ and $Q_n\tilde{h}$.

Approximation of an optimal policy

Theorem

There exists $\varepsilon_0 > 0$ *such that for any* $0 < \varepsilon \leq \varepsilon_0$ *there exist* $\mathfrak{d} > 0$ *and constants* $\mathcal{S}, \mathcal{T} > 0$ *such that*

$$\mathbb{P}^*\{J(\widetilde{f}_{n,\mathfrak{d}}, x) - g > \varepsilon\} \leq \mathcal{S} \exp\{-\mathcal{T} n\}.$$

for all $n \geq 1$ *and* $x \in X$.

Finite state and action approximations

- For applications, suppose that the sets $A_{\mathfrak{d}}(x)$ are finite.
- Take a sample $\Gamma_n = \{Y_k(\omega)\}$ of the probability measure μ.
- The control model $\mathcal{M}_{n,\mathfrak{d}}$ has finite state and action spaces.
- We need to determine its optimal average cost $g^*_{n,\mathfrak{d}}$.
- We need to solve the ACOE for $\mathcal{M}_{n,\mathfrak{d}}$ to find a canonical policy.

The linear programming approach

Primal linear programming problem P

$$\min \quad \sum_{x\in\Gamma_n}\sum_{a\in A_{\mathfrak{d}}(x)} c(x,a)z(x,a) \quad \text{subject to}$$

$$\sum_{a\in A_{\mathfrak{d}}(x)} z(x,a) = \sum_{x'\in\Gamma_n}\sum_{a'\in A_{\mathfrak{d}}(x')} z(x',a')Q_n(\{x\}|x',a')$$

$$\sum_{x\in\Gamma_n}\sum_{a\in A_{\mathfrak{d}}(x)} z(x,a) = 1 \quad \text{and} \quad z(x,a)\geq 0$$

It is known that $\min P = g^*_{n,\mathfrak{d}}$, the optimal average cost of the control model $\mathcal{M}_{n,\mathfrak{d}}$.

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The linear programming approach

Dual linear programming problem D

$$\max \quad g \quad \text{subject to}$$

$$g + h(x) \leq c(x,a) + \sum_{y\in\Gamma_n} Q_n(\{y\}|x,a)h(y)$$

$$g\in\mathbb{R} \quad \text{and} \quad h(x)\in\mathbb{R}.$$

Its optimal value is $g^*_{n,\mathfrak{d}}$ and, at optimality, we obtain a solution of

$$g^*_{n,\mathfrak{d}} + h(x) \leq \min_{a\in A_{\mathfrak{d}}(x)} \Big\{c(x,a) + \sum_{y\in\Gamma_n} Q_n(\{y\}|x,a)h(y)\Big\} \qquad (2)$$

but not necessarily of the ACOE.

Solving the ACOE by linear programming

Our approach to approximate an optimal policy is based on a canonical policy for $\mathcal{M}_{n,\mathfrak{d}}$. We need to solve the ACOE for $\mathcal{M}_{n,\mathfrak{d}}$.

Lemma (Maximal property)

Let $\{z^*(x,a)\}$ be an optimal solution of P, and fix arbitrary x^* with $z^*(x^*,a) > 0$.

Let h^* be the unique solution of the ACOE for $\mathcal{M}_{n,\mathfrak{d}}$ such that $h^*(x^*) = 0$, and let h, with $h(x^*) = 0$, verify the inequalities in (2).

Then we have $h \le h^*$.

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Solving the ACOE by linear programming

Modified dual linear programming problem D'

$$\max \quad \sum_{x\in\Gamma_n} h(x) \quad \text{subject to}$$

$$g^*_{n,\mathfrak{d}} + h(x) \le c(x,a) + \sum_{y\in\Gamma_n} Q_n(\{y\}|x,a)h(y)$$

$$h(x^*) = 0 \quad \text{and} \quad h(x) \in \mathbb{R}.$$

Theorem

Solving P and then D' yields a solution of the ACOE for $\mathcal{M}_{n,\mathfrak{d}}$.

An inventory management system

Consider the dynamics

$$x_{t+1} = \max\{x_t + a_t - \xi_t, 0\} \quad \text{for } t \in \mathbb{N}$$

where

- x_t is the stock level at the beginning of period t;
- a_t is the amount ordered at the beginning of period t;
- ξ_t is the random demand at the end of period t.

The capacity of the warehouse is $M > 0$. Therefore,

$$X = A = [0, M] \quad \text{and} \quad A(x) = [0, M - x].$$

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An inventory management system

The controller incurs:

- a buying cost of $b > 0$ for each unit;
- a holding cost $h > 0$ for each period and unit;
- and receives $p > 0$ for each unit that is sold.

The running cost function is

$$c(x, a) = ba + h(x + a) - pE[\min\{x + a, \xi\}].$$

Theorem

If the $\{\xi_t\}$ are i.i.d. with distribution function F, with $F(M) < 1$, and density function f, which is Lipschitz continuous on $[0, M]$ with $f(0) = 0$, then the inventory management system satisfies our assumptions.

An inventory management system

Fix $0 < \mathfrak{p} < 1$. The probability measure μ is

$$\mu\{0\} = \mathfrak{p} \quad \text{and} \quad \mu(B) = \frac{1-\mathfrak{p}}{M}\lambda(B) \quad \text{for measurable } B \subseteq (0, M],$$

The density function of the demand is

$$f(x) = \frac{1}{\lambda^2}\, x e^{-x/\lambda} \quad \text{for } x \geq 0.$$

The approximating action sets are

$$A_{\mathfrak{d}}(x) = \left\{ \frac{(M-x)j}{q_{\mathfrak{d}}-1} \ : \ j = 0, 1, \ldots, q_{\mathfrak{d}} - 1 \right\}.$$

An inventory management system

We take 500 samples of size n for the parameters

$$M = 10,\ b = 7,\ h = 3,\ p = 17,\ \mathfrak{p} = 1/10,\ \lambda = 5/2,\ q_{\mathfrak{d}} = 20.$$

	$n = 50$	$n = 150$	$n = 300$
Mean	-26.8755	-26.4380	-26.2817
Std. Dev.	2.2119	1.4578	1.0145
	$n = 500$	$n = 700$	$n = 1000$
Mean	-26.1717	-26.1553	-26.1659
Std. Dev.	0.8104	0.6662	0.5734

Table: Estimation of the optimal average cost g.

An inventory management system

We determine the canonical policy $\widetilde{f}_{n,\mathfrak{d}}$ for $\mathcal{M}_{n,\mathfrak{d}}$ and we evaluate it for $\mathcal{M}$.

	$n=50$	$n=150$	$n=300$
Mean	-25.6312	-25.8387	-25.9724
Std. Dev.	0.7648	0.5394	0.3954
	$n=500$	$n=700$	$n=1000$
Mean	-26.0406	-26.0497	-26.0833
Std. Dev.	0.3387	0.3276	0.3133

Table: Estimation of the average cost of the policy $\widetilde{f}_{n,\mathfrak{d}}$.

An inventory management system

We compute the relative error of $J(x, \widetilde{f}_{n,\mathfrak{d}})$ with respect to g.

$n=50$	$n=150$	$n=300$	$n=500$	$n=700$	$n=1000$
4.63%	2.27%	1.18%	0.50%	0.40%	0.32%

Table: Relative error.

An inventory management system

We display the approximation of an optimal policy for the control model $\mathcal{M}$.

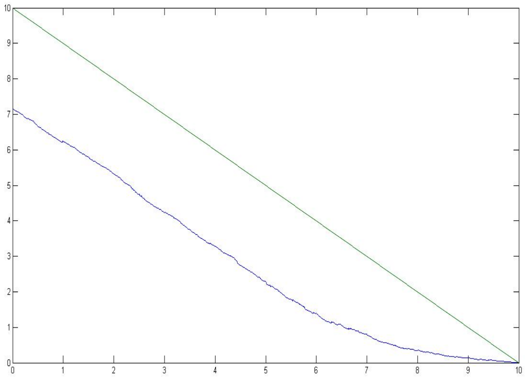

Figure: Estimation of an optimal policy

Conclusions

- We have proposed a general procedure to approximate a continuous state and action MDP.
- We can do this for a “Lipschitz-continuous” control model.
- We prove exponential rates of convergence (in probability).
- For applications, our method provides very good approximations.