Well-Posedness of the Einstein–Euler System in Asymptotically Flat Spacetimes: The Constraint Equations

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Abstract

This paper deals with the construction of initial data for the coupled Einstein-Euler system. We use the condition that the energy density might vanish or tends to zero at infinity and that the pressure is a fractional power of the energy density, conditions which are used to describe simplified stellar models. In order to achieve our goals we are enforced, by the complexity of the problem, to deal with these equations in a new type of weighted Sobolev spaces of fractional order.

The common Lichnerowicz-York scaling method [14], [9], [34], for solving the constraint equations cannot be applied here directly, since it violates the relations between the matter variables and the initial data for the fluid. We show that if the matter variables are restricted to a certain region, then Einstein's constraints equations have a unique solution in the weighted Sobolev spaces of fractional order. The regularity depends upon the fractional power of the equation of state.

1 Introduction

This paper deals with the Einstein-Euler system describing a relativistic self-gravitating perfect fluid, whose density either has, compact support or falls off at infinity in an appropriate manner, that is, the density belongs to a certain weighted Sobolev space.

The evolution of the gravitational field is described by the Einstein equations

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi T_{\alpha\beta} \tag{1.1}$$

where $g_{\alpha\beta}$ is a semi Riemannian metric having a signature $(-,+,+,+)$, $R_{\alpha\beta}$ is the Ricci curvature tensor, these are functions of $g_{\alpha\beta}$ and its first and second order partial derivatives and R is the scalar curvature. The right hand side of (1.1) consists of the energy-momentum tensor of the matter, $T_{\alpha\beta}$ and in the case of a perfect fluid the latter takes the form

$$T^{\alpha\beta} = (\epsilon + p)u^\alpha u^\beta + pg^{\alpha\beta}, \tag{1.2}$$

where ϵ is the energy density, p is the pressure and u^α is the four-velocity vector. The vector u^α is a unit timelike vector, which means that it is required to satisfy the normalization condition

$$g_{\alpha\beta}u^\alpha u^\beta = -1. \tag{1.3}$$

The Euler equations describing the evolution of the fluid take the form

$$\nabla_\alpha T^{\alpha\beta} = 0, \tag{1.4}$$

where ∇ denotes the covariant derivative associated to the metric $g_{\alpha\beta}$. Equations (1.1) and (1.4) are not sufficient to determinate the structure uniquely, a functional relation between the pressure p and the energy density ϵ (equation of state) is also necessary. We choose an equation of state that has been used in astrophysical problems. It is the analogue of the well known polytropic equation of state in the non-relativistic theory, given by

$$p = f(\epsilon) = K\epsilon^\gamma, \qquad K, \gamma \in \mathbb{R}^+, \qquad 1 < \gamma. \tag{1.5}$$

The sound velocity is denoted by

$$\sigma^2 = \frac{\partial p}{\partial \epsilon}. \tag{1.6}$$

The unknowns of these equations are the semi Riemannian metric $g_{\alpha\beta}$, the velocity vector u^α and the energy density ϵ. These are functions of t and x^a where x^a $(a = 1, 2, 3)$ are the Cartesian coordinates on $\mathbb{R}^3$. The alternative notation $x^0 = t$ will also be used and Greek indices will take the values $0, 1, 2, 3$ in the following.

The common method to solve the Cauchy problem for the Einstein equations consists usually of two steps. Unlike ordinary initial value problems, initial data must satisfy constraint equations which are intrinsic to the initial hypersurface. Therefore, the first step is to construct solutions of these constraints. The second step is to solve the evolution equations with these initial data, in the present case these are first order symmetric hyperbolic systems. As we describe later in detail, the complexity of our problems forces us to consider an additional third step, that is, after solving the constraint equations, we have to construct the initial data for the equations of the fluid.

The nature of this Einstein-Euler system (1.1), (1.4) and (1.5) forces us to treat both the constraint and the evolution equations in the same type of functional spaces. Under the above consideration, we have established the well posedness of this Einstein-Euler system in a weighted Sobolev space of fractional order.

We will briefly resume the situation in the mathematical theory of self gravitation perfect fluids describing compact bodies, such as stars: For the Euler-Poisson system Makino proved a local existence theorem in the case the density has compact support and it vanishes at the boundary, [23]. Since the Euler equations are singular when the density ρ is zero, Makino had to regularize the system by introducing a new matter variable ($w = M(\rho)$). His solution however, has some disadvantages such as the fact they do not contain static solutions and moreover, the connection between the physical density and the new matter density remains obscure.

Rendall generalized Makino's result to the relativistic case of the Einstein–Euler equations, [28]. His result however suffers from the same disadvantages as Makino's result and moreover it has two essential restrictions: 1. Rendall assumed time symmetry, that means that the extrinsic curvature of the initial manifold is zero and therefore the Einstein's constraint equations are reduced to a single scalar equation; 2. Both the data and solutions are C_0^∞ functions. This regularity condition implies a severe restriction on the equation of state $p = K\epsilon^\gamma$, namely $\gamma \in \mathbb{N}$.

Similarly to Makino and Rendall, we have also used the Makino variable

$$w = M(\epsilon) = \epsilon^{\frac{\gamma-1}{2}}. \tag{1.7}$$

We are now encountering the compatibility problem of the initial data for the fluid and the gravitational fields. There are three types of initial data for the Einstein-Euler system:

- The gravitational data is a triple (M, h, K), where M is space-like manifold, $h = h_{ab}$ is a proper Riemannian metric on M and $K = K_{ab}$ is a second fundamental form on M (extrinsic curvature). The pair (h, K) must satisfy the constrain equations

$$\left\{ \begin{array}{rcl} R(h) - K_{ab}K^{ab} + (h^{ab}K_{ab})^2 & = & 16\pi z, \\ {}^{(3)}\nabla_b K^{ab} - {}^{(3)}\nabla^b(h^{bc}K_{bc}) & = & -8\pi j^a, \end{array} \right. \tag{1.8}$$

  where $R(h) = h^{ab}R_{ab}$ is the scalar curvature with respect to the metric h.

- The matter variables, consisting of the energy density z and the momentum density j^a, appear in the right hand side of the constraints (1.8).

- The initial data for Makino's variable w and the velocity vector u^α of the perfect fluid.

The projection of the velocity vector u^α, $\bar{u}^\alpha$, on the tangent space of the initial manifold M leads to the following relations

$$\left\{ \begin{array}{rcl} z & = & \epsilon + (\epsilon + p)h_{ab}\bar{u}^a\bar{u}^b \\ j^\alpha & = & (\epsilon + p)\bar{u}^a\sqrt{1 + h_{ab}\bar{u}^a\bar{u}^b} \end{array} \right. \tag{1.9}$$

between the matters variable (z, j^a) and $(w, \bar{u}^a)$. We cannot give w, $\bar{u}^b$ and by relations (1.5) and (1.9) solve for z and j^α, since this is incompatible with the conformal scaling (see Section 3.1). In order to overcome this obstacle, we let $z = y^{\frac{2}{\gamma-1}}$ and $j^\alpha = v^\alpha/z$, then (1.9) is equivalent to (3.10) and the last one is invertible. But now we need to estimate z by y in the corresponding norm of the function spaces, and this leads to in an algebraic relation between the order of the functional space k and the coefficient γ of the equation of state (1.5) of the form

$$1 < \gamma \leq \frac{2+k}{k}. \tag{1.10}$$

This relation can be easily derived by considering $\|D^\alpha y\|_{L_2}$, $|\alpha| \leq k$. Moreover, it can be interpreted either as a restriction on γ or on k. Thus, unlike typical elliptic and hyperbolic systems where often the regularity parameter is bounded from below, here we have both lower and upper bounds for differentiability conditions of the sort $k \leq \frac{2}{\gamma-1}$. A similar phenomenon for Euler-Poisson equations was noticed by Gamblin [17].

We want to interpret (1.10) as a restriction on k rather than on γ. Therefore, instead of imposing conditions on the equation of state and in order to sharpen the regularity conditions for existence theorems, we are lead to the conclusion of considering function spaces of fractional order, and in addition, the Einstein equations consist of quasi linear hyperbolic and elliptic equations. The only function spaces which are known to be useful for existence theorems of the constraint equations in the asymptotically flat case, are the weighted Sobolev spaces $H_{k,\delta}$, $k \in \mathbb{N}$, $\delta \in \mathbb{R}$, which were introduced by Nirenberg and Walker, [27] and Cantor [8], and they are the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm

$$(\|u\|_{k,\delta})^2 = \sum_{|\alpha|\leq k} \int \left((1+|x|)^{\delta+|\alpha|}|\partial^\alpha u|\right)^2 dx. \tag{1.11}$$

Hence we are forced to consider new function spaces $H_{s,\delta}$, $s \in \mathbb{R}$ which generalize $H_{k,\delta}$ to fractional order. The well posedness of the Einstein-Euler system is obtained in these spaces and to achieve this, we have to solve both the constraint and the evolution equations in the $H_{s,\delta}$ spaces.

This paper deals with the construction of the initial data and solution to the constraint equations. The solution of the evolution equations will be treated in a different publication and is available as an electronic preprint [7].

The paper is organized as follows: In Section 2 we define the weighted Sobolev spaces of fractional order $H_{s,\delta}$ and present our main results. These include a solution of the compatibility problem, the construction of initial data and a solution to the evolution equations in the $H_{s,\delta}$ spaces. The announcement of the main results has been published in [5].

Section 3 deals with the constructions of the initial data. The common Lichnerowicz-York scaling method for solving the constraint equations cannot be applied here directly [14], [9], [34], since it violates the relations (1.9). We need to invert of (1.9) in order construct the initial data and there are two conditions which guarantee it: the dominate energy condition $h_{ab}j^a j^b \leq z^2$, this is invariant under scaling; and the causality condition, which states that the speed of sound (1.6) has to be less than the speed of light. Unfortunately the last condition is not invariant under scaling. It is also necessary to restrict the matter variables (z, j^a) to a certain region. We show the inversion of (1.9) exists provided that (z, j^a) belong to a certain region. This fact enables us to construct initial data for the evolution equations.

In Section 4 we study elliptic theory in $H_{s,\delta}$ which is essential for the solution of the constraint equations. We will extend earlier results in weighted Sobolev spaces of integer order which were obtained by Cantor [9], Choquet-Bruhat and Christodoulou [12], Choquet-Bruhat, Isenberg and York [13], and Christodoulou and O'Murchadha [15] to the fractional ordered spaces. The central tool is a priori estimate for elliptic systems in the $H_{s,\delta}$ spaces (4.21). Its proof requires first the establishment of analogous a priori estimate in Bessel potential spaces H^s. Our approach is based on the techniques of pseudodifferential operators which have symbols with limited regularity and in order to achieve that we are adopting ideas being presented in Taylor's books [30]and [31]. A different method was derived recently by Maxwell [24] who also showed existence of solutions to Einstein constraint equations in vacuum in $H_{s,\delta}$ with the best possible regularity condition, namely $s > \frac{3}{2}$. The semi-linear elliptic equation is solved by following Cantor's homotopy argument [9] and generalize it in $H_{s,\delta}$ spaces.

Finally, in the Appendix we deal with of the construction, properties and tools for PDEs in the weighted Sobolev spaces of fractional order $H_{s,\delta}$. Triebel extended the $H_{k,\delta}$ spaces given by the norm (1.11) to a fractional order [32], [33]. We present three equivalent norms, one of which is a combination of the norm (1.11) and the norm of Lipschitz-Sobolevskij spaces [29]. This definition is essential for the understanding of the relations between the integer and the fractional order spaces (see (5.3)). However the double integral makes it almost impossible to establish any property needed for PDEs. Throughout the effort to solve this problem, we were looking for an equivalent definition of the norm: we let $\{\psi_j\}_{j=0}^{\infty}$ be a dyadic resolution of unity in $\mathbb{R}^3$ and set

$$\left(\|u\|_{H_{s,\delta}}\right)^2 = \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j u)_{2^j}\|_{H^s}^2, \tag{1.12}$$

where $(f)_\epsilon(x) = f(\epsilon x)$. When s is an integer, then the norms (1.11) and (1.12) are equivalent. Our guiding philosophy is to apply the known properties of the Bessel potential spaces H^s term-wise to each of the norms in the infinite sum (1.12) and in that way to extend them to the $H_{s,\delta}$ spaces. Of course, this requires a careful treatment and a sound consideration of the additional parameter δ. Among the properties which we have extended to the $H_{s,\delta}$ are algebra, Moser type estimates, fractional power, compact embedding and embedding to the space of continuous functions.

2 New Function Spaces and the Principle Results

Our principle results concern the solution of the compatibility of the initial data for the equations of the fluid and the gravitational field (1.9) and the solutions to the Einstein constraint equations (1.8). The conformal scaling method reduces the constraint equations to an elliptic system, and the presence of an equation of state (1.5) compels us to treat these equations in the weighted Sobolev spaces of fractional order. The solution to the evolution equations (1.1) and (1.3) appears in our preprint [7] and it is also available as an electronic preprint in [6].

We first define the *weighted fractional Sobolev spaces*. We make a dyadic resolution of the unity in $\mathbb{R}^3$ as follows. Let $K_j = \{x : 2^{j-3} \leq |x| \leq 2^{j+2}\}$, $(j = 1, 2, ...)$ and $K_0 = \{x : |x| \leq 4\}$. Let $\{\psi_j\}_{j=0}^{\infty}$ be a sequence of $C_0^\infty(\mathbb{R}^3)$ such that $\psi_j(x) = 1$ on K_j, $\mathrm{supp}(\psi_j) \subset \cup_{l=j-4}^{j+3} K_l$, for $j \geq 1$ and $\mathrm{supp}(\psi_0) \subset K_0 \cup K_1$.

We denote by H^s the Bessel potential spaces with the norm $(p = 2)$

$$\|u\|_{H^s}^2 = c \int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi,$$

where $\hat{u}$ is the Fourier transform of u. Also, for a function f, $f_\varepsilon(x) = f(\varepsilon x)$.

Definition 2.1 ***(Weighted fractional Sobolev spaces: infinite sum of semi norms)*** *For $s \geq 0$ and $-\infty < \delta < \infty$,*

$$\left(\|u\|_{H_{s,\delta}}\right)^2 = \sum_j 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j u)_{(2^j)}\|_{H^s}^2. \tag{2.1}$$

The space $H_{s,\delta}$ is the set of all temperate distributions with a finite norm given by (2.1).

2.1 The principle results

2.1.1 The compatibility of the initial data for the fluid and the gravitational field

The matter data (non-gravitational) (z, j) which appear in the right hand side of (1.8) are coupled to the initial data of the perfect fluid (1.2) via the relations (1.9). Thus, an indispensable condition for obtaining a solution of the Einstein-Euler system is the inversion of (1.9). This system is not invertible for all $(z, j^a) \in \mathbb{R}_+ \times \mathbb{R}^3$, but the inverse does exist in a certain region.

Theorem 2.2 ***(Reconstruction theorem for the initial data)*** *There is a real function $S : [0, 1) \to \mathbb{R}$ such that if*

$$0 \leq z < S(\sqrt{h_{ab} j^a j^b}/z), \tag{2.2}$$

then system (1.9) has a unique inverse. Moreover, the inverse mapping is continuous in $H_{s,\delta}$ norm.

Remark 2.3 *The matter initial data (z, j^a) for the Einstein-Euler system with the equation of state (1.5) cannot be arbitrary . They must satisfy condition (2.2). This condition includes the inequality*

$$z^2 \geq h_{ab} j^a j^b, \tag{2.3}$$

which is known as the dominate energy condition.

2.1.2 Solution to the constraint equations

The gravitational data is a triple (M, h, K), where M is a space-like asymptotically flat manifold, $h = h_{ab}$ is a proper Riemannian metric on M, and $K = K_{ab}$ is the second fundamental form on M (extrinsic curvature). The metric h_{ab} and the extrinsic curvature K must satisfy Einstein's constraint equations (1.8). The free initial data is a set $(\bar{h}_{ab}, \bar{A}_{ab}, \hat{y}, \hat{v}^a)$, where $\bar{h}_{ab}$ is a Riemannian metric, $\bar{A}_{ab}$ is divergence and trace free form, $\hat{y}$ is a scalar function and $\hat{v}^a$ is a vector.

Theorem 2.4 ***(Solution of the constraint equations)*** *Given free data $(\bar{h}_{ab}, \bar{A}_{ab}, \hat{y}, \hat{v}^a)$ such that $(\bar{h}_{ab} - I) \in H_{s,\delta}$, $\bar{A}_{ab} \in H_{s-1,\delta+1}$, $(\hat{y}, \hat{v}^a) \in H_{s-1,\delta+2}$, $\frac{5}{2} < s < \frac{2}{\gamma-1} + \frac{3}{2}$ and $-\frac{3}{2} < \delta < -\frac{1}{2}$.*

(i) Then there exists two positive functions α and ϕ such that $(\alpha - 1), (\phi - 1) \in H_{s,\delta}$, a vector field $W \in H_{s,\delta}$ such that the gravitational data

$$h_{ab} = (\phi\alpha)^4 \bar{h}_{ab} \qquad \text{and} \qquad K_{ab} = (\phi\alpha)^{-2} \bar{A}_{ab} + \phi^{-2} \hat{\mathcal{L}}(W) \tag{2.4}$$

satisfy the constraint equations (1.8) with $z = \phi^{-8} \hat{y}^{\frac{2}{\gamma-1}}$ and $j^b = \phi^{-10} \hat{y}^{\frac{2}{\gamma-1}} \hat{v}^b$ as the right hand side, here $\hat{\mathcal{L}}$ is the Killing vector field operator. In addition, the $H_{s,\delta} \times H_{s-1,\delta+1}$ norms of $(h_{ab} - I, K_{ab})$ depend continuously on the $H_{s,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}$ norms of $(\bar{h}_{ab} - I, \bar{A}_{ab}, \hat{y}, \hat{j}^a)$.

(ii) Let $\hat{h}_{ab} = \alpha^4 \bar{h}_{ab}$, $\hat{z} = \hat{y}^{\frac{2}{\gamma-1}}$, $j^a = \hat{y}^{\frac{2}{\gamma-1}} \hat{v}^a$ and Ω^{-1} denote the inverse of relations (1.9). If $(\hat{h}_{ab}, \hat{z}, \hat{j}^a)$ satisfies (2.2), then the data for the four velocity vector and Makino variable are given by: $z = \phi^{-8} \hat{z}$, $j^a = \phi^{-10} \hat{j}^a$,

$$(w, \bar{u}^a) := \Omega^{-1}(z, j^a) \qquad \text{and} \qquad \bar{u}^0 = 1 + h_{ab} \bar{u}^a \bar{u}^b \tag{2.5}$$

and they satisfy the compatibility conditions (1.9). In addition, the $H_{s-1,\delta+2}$ norms of $(w, \bar{u}^a, u^0 - 1)$ depend continuously on the $H_{s,\delta} \times H_{s-1,\delta+2}$ norms of $(\bar{h}_{ab} - I, \hat{y}, \hat{j}^a)$.

3 The Initial Data

The Cauchy problem for the Einstein field equations (1.1) coupled with the Euler equations (1.4) consists of solving a coupled hyperbolic system with given initial data. There are two types of data for equations (1.1), the gravitational and the matter data.

The gravitational data is a triple (M, h, K), where M is a space-like manifold, $h = h_{ab}$ is a proper Riemannian metric on M and $K = K_{ab}$ is the second fundamental form on M (extrinsic curvature).

Let n be the unit normal to the hypersurface M, $\delta^\alpha_\beta + n^\alpha n_\beta$ be the projection on M and define

$$z = T_{\alpha\beta} n^\alpha n^\beta, \tag{3.1}$$
$$j^\alpha = (\delta^\alpha_\gamma + n^\alpha n_\gamma) T^{\gamma\beta} n_\beta. \tag{3.2}$$

The scalar z is the energy density and the vector j^α is the momentum density. These quantities are called matter variables and they appear as sources in the constraint equations (3.7) and (3.8) below.

In conjunction with these we must supply initial data for the velocity vector u^α. So we apply the projection to u^α and set $\bar{u}^\alpha = ({\delta^\alpha}_\beta + n^\alpha n_\beta) u^\beta$. Then from the relation of the perfect fluid (1.2) , (3.1), and (3.2) we see that

$$z = (\epsilon + p)(n_\beta u^\beta)^2 - p, \tag{3.3}$$
$$j^\alpha = (\epsilon + p)\bar{u}^\alpha (n_\beta u^\beta). \tag{3.4}$$

The vectors j^α and $\bar{u}^\alpha$ are tangent to the initial surface and so they can be identified with vectors j^a and $\bar{u}^a$ intrinsic to this surface. Recalling the normalization condition (1.3), we have $-1 = -(n_\beta u^\beta)^2 + h_{ab}\bar{u}^a \bar{u}^b$. Thus the matter data (z, j^a) can be identified with the initial data for the velocity vector as follow:

$$z = \epsilon + (\epsilon + p) h_{ab} \bar{u}^a \bar{u}^b, \tag{3.5}$$
$$j^\alpha = (\epsilon + p)\bar{u}^a \sqrt{1 + h_{ab}\bar{u}^a \bar{u}^b}. \tag{3.6}$$

These two types of data cannot be given freely, because the hypersurface (M, h) is a sub-manifold of (V, g), therefore the Gauss Codazzi equations lead to Einstein constraint equations

$$R(h) - K_{ab}K^{ab} + (h^{ab}K_{ab})^2 = 16\pi z \quad \text{(Hamiltonian constaint)} \tag{3.7}$$
$${}^{(3)}\nabla_b K^{ab} - {}^{(3)}\nabla^b (h^{bc}K_{bc}) = -8\pi j^a \quad \text{(momentum constraint).} \tag{3.8}$$

Here $R(h) = h^{ab}R_{ab}$ is the scalar curvature with respect to the metric h.

We turn now to the conformal method which allows us to construct the solutions of the constraint equations (3.7) and (3.8). Before entering into details we have to discuss the relations between the initial data for the system of Einstein gravitational fields (1.1) and Euler equations (1.4) which are given by (3.5) and (3.6). As it turns out this relations is by no means trivial, and indeed they will force us to modify the conformal method.

3.1 The compatibility problem of the initial data for the fluid and the gravitational fields

On the one hand, the initial data for the Euler equations are $w(\epsilon)$ and u^α. On the other hand $z = F(w(\epsilon), \bar{u}^a)$ and $j^a = H(w(\epsilon), \bar{u}^a)$, which are given by (3.5) and (3.6) respectively, appear as sources in the constraint equations (3.7) and (3.8). There we have the possibility of either to consider w and u^α as the fundamental quantities and construct then z and j^a or, vice verse, to consider z and j^a as the fundamental quantities and construct then w and u^α.

The first possibility does not work because the geometric quantities which occur on the left hand side of the constraint equations are supposed to scale with some power of a scalar function ϕ. So z and j^a, which are the source terms in the constraint equations, must also scale with a certain power of ϕ. If ϵ is scaled with a certain power of ϕ, then p would be scaled, according to the equation of state (1.5), to a different power. Hence, by (3.5) z is a sum of different powers. Thus, the power which ϵ and p are scaled would have to be zero and they would be left unchanged by the rescaling. Similarly it can be seen that $\bar{u}^a$ would remain unchanged. So in fact z would be unchanged and this is inconsistent with the scalding used in the conformal method.

Instead of constructing $(w, \bar{u}^a)$ from (z, j^a) it is more useful to introduce some auxiliary quantities. Beside the Makino variable $w = \epsilon^{\frac{\gamma-1}{2}}$, we set

$$y = z^{\frac{\gamma-1}{2}} \qquad \text{and} \qquad v^a = \frac{j^a}{z}. \tag{3.9}$$

Now we consider the following map

$$\Phi \begin{pmatrix} w \\ \bar{u}^a \end{pmatrix} = \begin{pmatrix} w[1 + (1+Kw^2)h_{ab}\bar{u}^a\bar{u}^b)]^{\frac{\gamma-1}{2}} \\ \frac{(1+Kw^2)\bar{u}^a\sqrt{1+h_{bc}\bar{u}^b\bar{u}^c}}{1+(1+Kw^2)h_{bc}\bar{u}^b\bar{u}^c} \end{pmatrix} = \begin{pmatrix} y \\ v^a \end{pmatrix}, \tag{3.10}$$

which is equivalent to the equations (3.5) and (3.6). The initial data $(w, \bar{u}^a)$ for the fluid are reconstructed through the inversion of Φ above.

Theorem 3.1 ***(Reconstruction theorem for the initial data)*** *There is a function $s : [0, 1) \to \mathbb{R}$ such that the map Φ defined by (3.10) is a diffeormophism from $[0, (\sqrt{\gamma K})^{-\frac{1}{2}}) \times \mathbb{R}^3$ to Ω, where*

$$\Omega = \{(y, v^a) : 0 \le y < s\left(\sqrt{h_{ab}v^a v^b}\right), h_{ab}v^a v^b < 1\}. \tag{3.11}$$

Proof (of theorem 3.1) Let $\rho = \sqrt{h_{ab}\bar{u}^a\bar{u}^b}$, $\bar{u}_0$ be a unit vector and $R_{\bar{u}^a}$ be the rotation with respect to the metric h_{ab} such that $\bar{u}^a = \rho R_{\bar{u}^a}\bar{u}_0$. Then

$$\Phi \begin{pmatrix} w \\ \bar{u}^a \end{pmatrix} = \Phi \begin{pmatrix} w \\ \rho R_{\bar{u}^a}\bar{u}_0 \end{pmatrix} = \begin{pmatrix} w[1 + (1+Kw^2)\rho^2]^{\frac{\gamma-1}{2}} \\ \frac{(1+Kw^2)R_{\bar{u}^a}\bar{u}_0\rho\sqrt{1+\rho^2}}{1+(1+Kw^2)\rho^2} \end{pmatrix}. \tag{3.12}$$

Therefore, we can first invert the two dimensional map

$$\Theta\begin{pmatrix} w \\ \rho \end{pmatrix} := \begin{pmatrix} w[1+(1+Kw^2)\rho^2]^{\frac{\gamma-1}{2}} \\ \frac{(1+Kw^2)\rho\sqrt{1+\rho^2}}{1+(1+Kw^2)\rho^2} \end{pmatrix} \tag{3.13}$$

for $\rho \geq 0$ and then apply again the rotation. For $w > 0$, we decompose Θ of (3.13) as follows:

$$\begin{pmatrix} w \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} \epsilon \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} \epsilon + (\epsilon + p(\epsilon))\rho^2 \\ (\epsilon + p(\epsilon))\rho\sqrt{1+\rho^2} \end{pmatrix} =: \begin{pmatrix} z \\ r \end{pmatrix} \mapsto \begin{pmatrix} z^{\frac{\gamma-1}{2}} \\ \frac{r}{z} \end{pmatrix}. \tag{3.14}$$

In order to show that this is a one to one map, we need to show that the Jacobian of $G(\epsilon, \rho) := (\epsilon + (\epsilon + p(\epsilon))\rho^2, (\epsilon + p(\epsilon))\rho\sqrt{1+\rho^2})$ does not vanish. This computation results with

$$\det\begin{pmatrix} 1+(1+p')\rho^2 & (1+p')\rho\sqrt{1+\rho^2} \\ (\epsilon+p)2\rho & (\epsilon+p)\frac{1+2\rho^2}{\sqrt{1+\rho^2}} \end{pmatrix} = \frac{(\epsilon+p)}{\sqrt{1+\rho^2}}\left(1+\rho^2(1-p')\right). \tag{3.15}$$

Recall that is the speed of sound is given by (1.6), therefore the causality condition $\sigma^2 < c^2 = 1$ imposes the below restriction of the domain of definition of the map Θ:

$$\sigma^2 = p' = \frac{\partial p}{\partial \epsilon} = \frac{\partial}{\partial \epsilon}(K\epsilon^\gamma) = \gamma K \epsilon^{\gamma-1} = \gamma K w^2 < 1. \tag{3.16}$$

Let S be the strip $\{0 \leq w < (\sqrt{\gamma K})^{-\frac{1}{2}}, 0 \leq \rho < \infty\}$. We now want to show that $\Theta : S \to \Theta(S)$ is a bijection. Clearly, $\Theta(0,\rho) = (0, \frac{\rho}{\sqrt{1+\rho^2}})$ maps $\{0\} \times [0,\infty)$ to $\{0\} \times [0,1)$ in a one to one manner, and $\Theta(w, 0) = (w, 0)$ is of course a bijection. The line $(\sqrt{\gamma K})^{-\frac{1}{2}}, \rho)$ is mapped to the curve

$$\begin{pmatrix} y(\rho) \\ x(\rho) \end{pmatrix} = \begin{pmatrix} (\sqrt{\gamma K})^{-\frac{1}{2}}(1+2\rho^2)^{\frac{\gamma-1}{2}} \\ \frac{2\rho\sqrt{1+\rho^2}}{1+2\rho^2} \end{pmatrix}. \tag{3.17}$$

Since $\frac{dx}{d\rho} > 0$, there exists a function $s : [0, 1) \to \mathbb{R}$ such that the curve (3.17) is given by the graph of s and the image of Θ is the set below this graph, that is,

$$\Theta(S) = \{(y, x) : y < s(x), 0 \leq x < 1\}. \tag{3.18}$$

By (3.14), (3.15) and (3.16) we conclude that the Jacobian of the map Θ does not vanish in the interior of S, hence $\Theta : S \to \Theta(S)$ is locally one to one map. It is well known that a locally one to one map between two simply connected sets is a bijective map. ■

3.2 Cantor's conformal method for solving the constraint equations

In principle there are two possibilities for solving the constraint equation for an asymptotically flat manifold:

- Either to adapt directly the method of York et all, but then one is forced to impose certain relations between $R(\bar{h})$ and the second fundamental form (see Choquet-Bruhat and York [14] for details).
- These undesirable conditions can be substituted by a method developed by Cantor and we will describe it in the following. (This method has been discussed in detail in the literature, see for example [2], [14], [9] [15] and reference therein.)

In this method parts of the data are chosen (the so-called free data), and the remaining parts are determined by the constraint equations (3.7) and (3.8). The free initial data are $\left(\bar{h}_{ab}, \bar{A}_{ab}, \bar{z}, \bar{j}\right)$, where A_{ab} is a divergence and trace free 2-tensor. The main idea is to consider two conformal scaling functions, α and ϕ.

1. We start with $\hat{h}_{ab} = \alpha^4 \bar{h}_{ab}$. If α is a positive solution to (3.24), then $R(\hat{h}) = 0$. The Brill-Cantor condition (see Definition 3.5) is necessary and sufficient for the existence of positive solutions. Having solved equation (3.24), we now adjust the given data to the new metric: $\hat{A}^{ab} = \alpha^{-10}\bar{A}^{ab}$, $\hat{z} = \alpha^{-8}\bar{z}$ and $\hat{j}^a = \alpha^{-10}\bar{j}^a$.
2. The second step here is solve the Lichnerowicz Laplacian (3.28) and set
$$\hat{K}^{ab} = (\mathcal{L}(W))^{ab} + \alpha^{-10} A^{ab}, \tag{3.19}$$
where $(\mathcal{L}(W))^{ab}$ is the Killing operator giving by (3.26).
3. The third step is: If ϕ is a solution to the Lichnerowicz equation (3.29), then it follows from (3.30) that the data $h_{ab} = \phi^4 \hat{h}_{ab}$, $K^{ab} = \phi^{-10}\hat{K}^{ab}$, $z = \phi^{-8}\hat{z}$ and $j^a = \phi^{-10}\hat{j}^a$ satisfy the constraint equations (3.7) and (3.8).

For the Einstein-Euler system with the equation of state (1.5) it is essential that the initial data will satisfy condition (3.11) of Theorem 3.1. Therefore in this case it is necessary to adjust this method.

Here the free initial data are:
$$\left(\bar{h}_{ab}, \bar{A}_{ab}, \hat{y}, \hat{v}^b\right). \tag{3.20}$$
where $\bar{A}_{ab}$ is trace and divergence free, that is, $\bar{D}_a \bar{A}^{ab} = \bar{h}_{ab}\bar{A}^{ab} = 0$, where $\bar{D}_a$ is the covariant derivative with respect to the metric $\bar{h}_{ab}$. We require that the matter data $(\hat{y}, \hat{v}^a)$, will satisfy the condition
$$0 \leq \hat{y} < s\left(\sqrt{\hat{h}_{ab}\hat{v}^a\hat{v}^b}\right), \tag{3.21}$$

where $s(\cdot)$ is given by (3.18). The remaining initial data are determined by the constraint equations (3.7) and (3.8), relations (3.9) and Theorem 3.1.

Remark 3.2 *The distinction between the gravitational data $(\bar{h}_{ab}, \bar{A}_{ab})$ and the matter data $(\hat{z}, \hat{j}^b)$ is caused by condition (3.11). For if we make the scaling $\hat{h}_{ab} = \phi^4 \bar{h}_{ab}$, $\hat{z} = \phi^{-8}\bar{z}$, and $\hat{j}^b = \phi^{-10}\bar{j}^b$, then $\hat{v}^b = \phi^{-2}\bar{v}^b$, $\hat{y} = \phi^{-4(\gamma-1)}\bar{y}$ and $\hat{h}_{ab}\hat{v}^a\hat{v}^b = \bar{h}_{ab}\bar{v}^a\bar{v}^b$. Thus under this conformal transformation, the argument of s in (3.11) is invariant, while the left hand side will be affected. Therefore the free initial data are partially invariant under conformal transformations.*

Now, if we perform the conformal transformation

$$\hat{h}_{ab} = \alpha^4 \bar{h}_{ab}, \tag{3.22}$$

then the scalar curvature with respect to the metric $\hat{h}_{ab}$, $R(\hat{h})$, satisfies

$$-8\Delta_{\bar{h}}\alpha + R(\bar{h})\alpha = R(\hat{h})\alpha^5. \tag{3.23}$$

Therefore, if there exists a nonnegative solution to the equation

$$-\Delta_{\bar{h}}\alpha + \frac{1}{8}R(\bar{h})\alpha = 0, \tag{3.24}$$

then the metric $\hat{h}_{ab}$ given by (3.22) will have zero scalar curvature. We continue the construction as follow. Let $\hat{A}^{ab} = \alpha^{-10}\bar{A}^{ab}$, $\hat{D}_a$ denotes the covariant derivative with respect to the metric $\hat{h}_{ab}$, since $\hat{D}_a\hat{A}^{ab} = \alpha^{-10}\bar{D}_a\bar{A}^{ab}$, $\hat{A}^{ab}$ is a divergence and trace free 2 tensor.

Assume $\hat{K}$ is a symmetric covariant 2-tensor which satisfies the maximal slice condition, that is $\hat{h}_{ab}\hat{K}^{ab} = 0$. Then we split $\hat{K}$ by writing it for some vector W:

$$\hat{K}^{ab} = \hat{A}^{ab} + \hat{\mathcal{L}}^{ab}(W), \tag{3.25}$$

where $\hat{\mathcal{L}}$ is the Killing field operator

$$\left(\hat{\mathcal{L}}(W)\right)^{ab} = \left(\hat{\pounds}_W \hat{h}\right)^{ab} - \frac{1}{3}\hat{h}^{ab}\mathrm{Tr}\hat{\pounds}_W\hat{h} = \hat{D}_a W^b + \hat{D}_b W^a - \frac{1}{3}\hat{h}^{ab}\mathrm{Tr}\hat{\pounds}_W\hat{h}, \tag{3.26}$$

and $\hat{\pounds}_W\hat{h}$ is the Lie derivative. The momentum constraint (3.8) is now equivalent to

$$\hat{D}_a\hat{K}^{ab} = \hat{D}_a\left(\hat{\mathcal{L}}(W)\right)^{ab} = -8\pi\hat{j}^b, \tag{3.27}$$

that is, W is a solution to the Lichnerowicz Laplacian system

$$\left(\Delta_{L_{\hat{h}}}W\right)^b := \hat{D}_a\left(\hat{\mathcal{L}}(W)\right)^{ab} = \Delta_{\hat{h}}W + \frac{1}{3}\hat{D}^b\left(\hat{D}_a W^a\right) + \hat{R}^b_a W^a = -8\pi\hat{j}^b, \tag{3.28}$$

here $\hat{R}_a^b$ is the Ricci curvature tensor with respect to the metric $\hat{h}_{ab}$.

Having solved the Lichnerowicz Laplacian (3.28), we consider the Lichnerowicz equation

$$-\Delta_{\hat{h}}\phi = 2\pi\hat{z}\phi^{-3} + \frac{1}{8}\hat{K}_a^b\hat{K}_b^a\phi^{-7}. \tag{3.29}$$

Now we put $h_{ab} = \phi^4\hat{h}_{ab}$, $K_{ab} = \phi^{-2}\hat{K}_{ab}$, $z = \phi^{-8}\hat{z}$ and $j^b = \phi^{-10}\hat{j}^b$. Since

$$-\Delta_{\hat{h}}\phi = \phi^5\frac{1}{8}R(h) = \phi^5\left(2\pi z + \frac{1}{8}K_a^bK_b^a\right) = \phi^5\left(2\pi\hat{z}\phi^{-8} + \frac{1}{8}\hat{K}_a^b\hat{K}_b^a\phi^{-12}\right), \tag{3.30}$$

the Hamiltonian constraint (3.7) is valid, and since

$$D_aK^{ab} = \phi^{-10}\hat{D}_a\hat{K}^{ab} = -8\pi\phi^{-10}\hat{j}^b = -8\pi j^b \tag{3.31}$$

the data (h_{ab}, K_{ab}, z, j^b) satisfy the constraint equations (3.7) and (3.8). In order that the matter variables and (z, j) satisfy the compatibility conditions (3.5) and (3.6) it is necessary to check that $y = z^{\frac{\gamma-1}{2}} = (\phi^{-8}\hat{z})^{\frac{\gamma-1}{2}} = \phi^{-4(\gamma-1)}\hat{y}$ and $v^b = \frac{j^b}{z} = \phi^{-2}\hat{v}^b$ satisfy condition (3.11). Indeed,

$$0 \leq y < s\left(\sqrt{h_{ab}v^av^b}\right) \Leftrightarrow 0 \leq \phi^{-4(\gamma-1)}\hat{y} < s\left(\sqrt{\hat{h}_{ab}\hat{v}^a\hat{v}^b}\right), \tag{3.32}$$

but since $\phi \geq 1$, $\phi^{-4(\gamma-1)}\hat{y} \leq \hat{y}$ and thus assumption (3.21) assures condition (3.11).

Theorem 3.3 ***(Construction of the gravitational data)*** *Given the free data $(\bar{h}_{ab}, \bar{A}_{ab}, \hat{y}, \hat{v}^b)$ such that $(\bar{h}_{ab} - I) \in H_{s,\delta}$, $\bar{A}_{ab} \in H_{s-1,\delta+1}$, $(\hat{y}, \hat{v}^b) \in H_{s-1,\delta+2}$, $\frac{5}{2} < s < \frac{2}{\gamma-1} + \frac{3}{2}$ and $-\frac{3}{2} < \delta < -\frac{1}{2}$. Then the gravitational data:*

$$h_{ab} = (\phi\alpha)^4\bar{h}_{ab} \qquad \textit{and} \qquad K_{ab} = (\phi\alpha)^{-2}\bar{A}_{ab} + \phi^{-2}\hat{\mathcal{L}}(W)$$

satisfy the constraint equations (3.7) and (3.8) with $z = \phi^{-8}\hat{y}^{\frac{2}{\gamma-1}}$ and $j^b = \phi^{-10}\hat{y}^{\frac{2}{\gamma-1}}\hat{v}^b$ as the right hand side. In addition, $(h_{ab} - I) \in H_{s,\delta}$ and $K_{ab} \in H_{s-1,\delta+1}$ and therefore if $\frac{7}{2} < s < \frac{2}{\gamma-1} + \frac{3}{2}$, then these data have the needed regularity so they can serve as initial data for the evolution equations of the Einstein's Gravitational Field Equations (1.1).

Proof (of Theorem 3.3)

- The free data are $(\bar{h}_{ab}, \bar{A}_{ab}, \hat{y}, \hat{v}^b)$, where $(\bar{h}_{ab} - I) \in H_{s,\delta}$, $\bar{A}_{ab} \in H_{s-1,\delta+1}$ a divergence a trace free 2-tensor and $(\hat{y}, \hat{v}^b) \in H_{s-1,\delta+2}$.

- The function α satisfies equation (3.24), so by Theorem 3.6 $\alpha > 0$ and $(\alpha - 1) \in H_{s,\delta}$ provided that $s \geq 2$ and $\delta > -\frac{3}{2}$. Since α is continuous and $\lim_{|x|\to\infty}\alpha(x) - 1 = 0$, there is a compact set D of $\mathbb{R}^3$ such that $\alpha(x) \geq \frac{1}{2}$ for $x \notin D$ and $\min_D \alpha(x) \geq t_0 > 0$.

- The function $F(t) := \frac{1-t}{t}$ has bounded derivatives in $[\min\{t_0, \frac{1}{2}\}, \infty)$, so by Moser type estimate, Theorem 6.10, $\alpha^{-1} - 1 = \frac{1-\alpha}{\alpha} \in H_{s,\delta}$.

- Now, by algebra (Proposition 6.7), $(\hat{h}_{ab} - I) = (\alpha^4 \bar{h}_{ab} - I) \in H_{s,\delta}$ and $\hat{A}^{ab} = \alpha^{-10} A^{ab} \in H_{s-1,\delta+1}$.

- The matter variables $(\hat{z}, \hat{j}^b)$ are given by $\hat{z} = \hat{y}^{\frac{2}{\gamma-1}}$, $\hat{j}^b = \hat{z}\hat{v}^b$. Therefore Proposition 6.8 implies that $\hat{z} \in H_{s-1,\delta+2}$, provided that $\frac{3}{2} < s-1 < \frac{2}{\gamma-1} + \frac{1}{2}$ and also $j^b \in H_{s-1,\delta+2}$ by the Proposition 6.7.

- The vector W is a solution of the Lichnerowicz Laplacian (3.28), thus according to Theorem 3.8 below, $W \in H_{s,\delta}$ if $s \geq 2$. Hence $\hat{K}^{ab}$ given in (3.25) belongs to $H_{s-1,\delta+1}$. Again, by Proposition 6.7, $\hat{K}^b_a \hat{K}^a_b \in H_{s-2,\delta+2}$ if $s \geq 2$ and $\delta \geq -\frac{3}{2}$.

- Setting $u = \phi - 1$, then Lichnerowicz equation (3.30) becomes

$$-\Delta_{\hat{h}} u = 2\pi \hat{z}(u+1)^{-3} + \frac{1}{8}\hat{K}^b_a \hat{K}^a_b (u+1)^{-7}. \tag{3.33}$$

- So applying Theorem 4.12 with $s' = s$ and $\delta' = \delta$ results that $(\phi - 1) = u \in H_{s,\delta}$ and $(\phi - 1) = u \geq 0$.

■

Combining our results of Section 3.1 with Theorem 3.3 we obtain the following corollary:

Corollary 3.4 ***(Construction of the data for the fluid)*** *Given the free data $(\bar{h}_{ab}, \bar{A}_{ab}, \hat{y}, \hat{v}^b)$ such that $(\bar{h}_{ab} - I) \in H_{s,\delta}$, $\bar{A}_{ab} \in H_{s-1,\delta+1}$, $(\hat{y}, \hat{v}^b) \in H_{s-1,\delta+2}$, $\frac{5}{2} < s < \frac{2}{\gamma-1} + \frac{3}{2}$, $-\frac{3}{2} < \delta < -\frac{1}{2}$ and $(\hat{y}, \hat{v}^a) \in \Omega$, where Ω is given by (3.11). Then the data of the four velocity vector u^α and the Makino variable w are: $y = \phi^{-4(\gamma-1)}\hat{y}$, $v^b = \phi^{-2}\hat{v}^b$,*

$$(w, \bar{u}^a) := \Phi^{-1}(y, v^a) \qquad \text{and} \qquad \bar{u}^0 = 1 + h_{ab}\bar{u}^a \bar{u}^b$$

and the data for the energy and momentum densities are: $z = y^{\frac{2}{\gamma-1}}$, $j^a = zv^a$. These data satisfy the compatibility conditions (3.5) and (3.6). In addition, by Moser type estimate Theorem 6.10 and Proposition 6.7, $(w, \bar{u}^a) \in H_{s-1,\delta+2}$ and $\bar{u}^0 - 1 \in H_{s-1,\delta+2}$ and therefore if $\frac{7}{2} < s < \frac{2}{\gamma-1} + \frac{3}{2}$, then these data have the needed regularity so they can serve as initial data for Euler equations (1.4).

3.3 Solutions to the elliptic systems

This section is devoted to the solutions the linear elliptic systems (3.24) and (3.28). The assumption on the given metric $\bar{h}_{ab}$ is that $(\bar{h}_{ab} - I) \in H_{s,\delta}$. So according to Theorem 4.7 of Section 4, the operator $\Delta_{\bar{h}} : H_{s,\delta} \to H_{s-2,\delta+2}$ is semi Fredholm. In fact, it is an isomorphism, this can be shown in a similar manner to Step 1 of Section 4.3. We now consider the operator

$$L := -\Delta_{\bar{h}} + \frac{1}{8}R(\bar{h}) : H_{s,\delta} \to H_{s-2,\delta+2}, \tag{3.34}$$

this is also semi Fredholm. If $R(\bar{h}) \geq 0$, then L is injective. However, a weaker condition is that L does not have non-positive eigenvalues, the variational formulation of this property known as the *Brill-Cantor condition* [10].

Let us first introduce few notations. For a Riemannian metric $\bar{h}_{ab}$, we set $(Du, Dv)_{\bar{h}} = \bar{h}^{ab}\partial_a u \partial_b v$, $|Du|^2_{\bar{h}} = (Du, Du)_{\bar{h}}$ and $\mu_{\bar{h}}$ is the volume element with respect to the metric $\bar{h}_{ab}$.

Definition 3.5 ***(Brill-Cantor condition)*** *A metric $\bar{h}_{ab}$ satisfies the Brill-Cantor condition if*

$$\inf_{u\neq 0} \frac{\int \left(|Du|^2_{\bar{h}} + \frac{1}{8}R(\bar{h})u^2\right) d\mu_{\bar{h}}}{\int u^2 d\mu_{\bar{h}}} > 0, \tag{3.35}$$

where the infimum is taken over all $u \in C^1_0(\mathbb{R}^3)$.

This condition is invariant under conformal transformations, a fact that has been proved for example in [13]

Theorem 3.6 ***(Construction of a metric having zero scalar curvature)*** *Assume the given metric $\bar{h}_{ab}$ satisfies $(\bar{h}_{ab} - \delta_{ab}) \in H_{s,\delta}$, $s \geq 2$, $\delta > -\frac{3}{2}$ and $\bar{h}_{ab}$ satisfies the Brill-Cantor condition (3.35). Then there exists a scalar function α such that $\alpha - 1 \in H_{s,\delta}$, $\alpha(x) > 0$ and the metric $\hat{h}_{ab} = \alpha^4 \bar{h}_{ab}$ has a scalar curvature zero.*

Proof (of Theorem 3.6) The desired α is a solution to the elliptic equation (3.24). By setting $u = \alpha + 1$ this equation goes to

$$Lu = -\Delta_{\bar{h}} u + \frac{1}{8}R(\bar{h})u = -\frac{1}{8}R(\bar{h}). \tag{3.36}$$

We define for $\tau \in [0, 1]$, $L_\tau u = -\Delta_{\bar{h}} u + \frac{\tau}{8}R(\bar{h})u$. If $L_\tau u = 0$, then by Lemma 4.9, $u \in H_{s,-1}$ so

$$0 = (u, L_\tau u) = \int \left(|Du|^2_{\bar{h}} + \frac{\tau}{8}R(\bar{h})u^2\right) d\mu_{\bar{h}}. \tag{3.37}$$

Now, if $\int R(\bar{h})u^2 d\mu_{\bar{h}} \geq 0$, then obviously (3.37) implies that $u \equiv 0$. Otherwise $\int R(\bar{h})u^2 d\mu_{\bar{h}} < 0$, then there is sequence $\{u_n\} \subset C^\infty_0$ such that $u_n \to u$ in $H_{s,-1}$ - norm and

$$\int \left(|Du|^2_{\bar{h}} + \frac{1}{8}R(\bar{h})u^2\right) d\mu_{\bar{h}} = \lim_n \int \left(|Du_n|^2_{\bar{h}} + \frac{1}{8}R(\bar{h})u_n^2\right) d\mu_{\bar{h}} > 0 \tag{3.38}$$

by the Brill-Cantor condition (3.35). Substitution of (3.38) in (3.37) yields

$$0 = \int \left(|Du|^2_{\bar{h}} + \frac{1}{8}R(\bar{h})u^2\right) d\mu_{\hat{h}} + \frac{(\tau - 1)}{8} \int R(\bar{h})u^2 d\mu_{\bar{h}}, \tag{3.39}$$

this is certainly a contradiction. Thus L_τ is injective for each $\tau \in [0,1]$, $L_0 = -\Delta_{\bar{h}}$ is isomorphism, hence $L_1 = -\Delta_{\bar{h}} + \frac{1}{8}R(\bar{h})$ is isomorphism by Theorem 4.8. Having proved the existence, we now show that $\alpha = u + 1$ is nonnegative. The set $\{x : \alpha(x) < 0\}$ has compact support since $\lim_{x\to\infty} u(x) = 0$ by the embedding Theorem 6.13. So letting $w = -\min(\alpha, 0)$, we have $w \in H_0^1(\mathbb{R}^3)$ and if the set $\{x : \alpha(x) < 0\}$ is not empty, then $w \not\equiv 0$ and then the Brill-Cantor condition gives

$$\int_{\{\alpha<0\}} \left(|Dw|_{\bar{h}}^2 + \frac{1}{8}R(\bar{h})w^2 \right) d\mu_{\bar{h}} > 0. \tag{3.40}$$

On the other hand, according to Definition 4.10 of weak solutions,

$$0 = \int \left((D\alpha, Dw)_{\bar{h}} + \frac{1}{8}R(\bar{h})\alpha w \right) d\mu_{\bar{h}} = -\int_{\{\alpha<0\}} \left(|Dw|_{\bar{h}}^2 + \frac{1}{8}R(\bar{h})w^2 \right) d\mu_{\bar{h}}. \tag{3.41}$$

So we conclude that $\alpha \geq 0$. Since $\alpha \geq 0$, we have by Harnack's inequality

$$\sup_{B_r} \alpha \leq C \inf_{B_r} \alpha$$

provided that B_r is sufficiently small ball. Hence, the set $\{\alpha(x) = 0\}$ is both open and closed, which is impossible. Thus $\alpha(x) > 0$. ■

Remark 3.7 *The conditions for applying Harnack's inequality to a second order elliptic operator*

$$Lu = \partial_a \left(A_{ab}(x)\partial_b u \right) + C(x)u$$

are boundedness of the coefficients (see e. g. [18]; Section 8) However, following carefully the proofs we found it can be applied also when the zero order coefficient belongs to $L^q_{\text{loc}}(\mathbb{R}^3)$ with $q > \frac{3}{2}$. In local coordinates equation (3.24) takes the form

$$L\alpha = \partial_a \left(\sqrt{|\bar{h}|}\bar{h}^{ab}\partial_b\alpha \right) + \sqrt{|\bar{h}|}R(\bar{h})\alpha = 0.$$

For $s \geq 2$, $\sqrt{|\bar{h}|}\bar{h}^{ab}$ are bounded and non-degenerate, while $\sqrt{|\bar{h}|}R(\bar{h}) \in L^2_{\text{loc}}(\mathbb{R}^3)$.

We turn now the Lichnerowicz Laplacian system (3.28).

Theorem 3.8 ***(Solution of Lichnerowicz Laplacian)*** *Let $\hat{h}_{ab}$ be a Riemannian metric in $\mathbb{R}^3$ so that $(\hat{h} - I) \in H_{s,\delta}$. Let vector $\hat{j}^b \in H_{s-2,\delta+2}$, $s \geq 2$ and $\delta > -\frac{3}{2}$. Then equation (3.28) has a unique solution $W \in H_{s,\delta}$.*

Proof (of theorem 3.8) In order to verify condition (H1) of Section 4.2 we compute the principle symbol of $L_{\Delta_{\hat{h}}}$ in (3.28). For each $\xi \in T^*_x M$, the principle symbol $\left(\Delta_{L_{\hat{h}}}(\xi)\right)^b_a$ is a linear map from E_x to F_x, where E_x and F_x are a fibers in T_xM. In local coordinates $\Delta_{\hat{h}} = \hat{h}^{ab}\partial_a\partial_b +$ lower terms and $D_a = \partial_a + \Gamma(\hat{h}^{ab}, \partial\hat{h}_{ab})$, hence

$$\left(\Delta_{L_{\hat{h}}}(\xi)\right)_a^b = |\xi|_{\hat{h}}^2\delta_a^b + \frac{1}{3}\xi^b\xi_a. \tag{3.42}$$

Therefore

$$\left(\left(\Delta_{L_{\hat{h}}}(\xi)\right)\eta, \eta\right)_{\hat{h}} = \hat{h}^{bc}\left(L_{\Delta_{\hat{h}}}(\xi)\right)_a^b\eta^a\eta^c = |\xi|_{\hat{h}}^2|\eta|_{\hat{h}}^2 + \frac{1}{3}(\xi_a\eta^a)^2 \geq |\xi|_{\hat{h}}^2|\eta|_{\hat{h}}^2. \tag{3.43}$$

Thus $\left(\Delta_{L_{\hat{h}}}(\xi)\right)_a^b$ has positive eigenvalues and therefore $L_{\Delta_{\hat{h}}}$ is strongly elliptic. Furthermore, by Proposition 6.7 and Remark 6.11 we have that if $(\hat{h}_{ab} - I) \in H_{s,\delta}$, $s \geq 2$ and $\delta > -\frac{3}{2}$, then

$$\Delta_{L_{\hat{h}}} : H_{s,\delta} \to H_{s-2,\delta+2}.$$

Hence, we may apply Theorem 4.8 in order to obtain existence of the elliptic system (3.28). For the given metric $\hat{h}_{ab}$ we define one parameter family of metrics $h_t = (1-t)I + t\hat{h}$, $0 \leq t \leq 1$, and the following associated operators with respect to these metrics: $(D_a)_t$ the covariant derivative, $\mathcal{L}_t$ the Killing operator and $L_t = \Delta_{L_{h_t}} = (D)_t \cdot \mathcal{L}_t$ the Lichnerowicz Laplacian. We want to show that L_t is injective. We recall that $-2\mathcal{L}_t$ is the formal adjoint of D_t (see e. g. [3]), in addition, if $L_t(W) = 0$, then by Lemma 4.9 implies $W \in H_{s,-1}$. Thus we may use integration by parts and get

$$\begin{aligned} 0 = (W, L_tW)_{h_t} &= \int (h_t)_{ab} W^a L_t(W)^b d\mu_{h_t} = \int (h_t)_{ab} W^a (D_c)_t \cdot (\mathcal{L}_tW)^{cb} d\mu_{h_t} \\ &= -2\int (h_t)_{ab}(h_t)_{dc}(\mathcal{L}_tW)^{ad}(\mathcal{L}_tW)^{cb}\mu_{h_t} = -2\int |\mathcal{L}_tW)|_{h_t}^2\mu_{h_t} \end{aligned} \tag{3.44}$$

Now, if let $\widetilde{h} = |h_t|^{-\frac{1}{3}}h_t$, then

$$\pounds_W\widetilde{h} = |h_t|^{-\frac{1}{3}}\left(\pounds_W h_t - h_t\frac{2}{3}(D_a)_tW^a\right) = |h_t|^{-\frac{1}{3}}\mathcal{L}_t(W). \tag{3.45}$$

Thus $L_t(W) = \Delta_{L_{h_t}}(W) = 0$ implies $W \equiv 0$ if and only if there are no non-trivial Killing vector fields W in $H_{s,-1}$. This fact has been proved by G. Choquet and Y.Choquet-Bruhat [11] for $s > \frac{7}{2}$, D. Christodoulou and N. O'Murchadha for $s > 3 + \frac{3}{2}$ [15], and Bartnik for $s \geq 2$ [1] (See also Maxwell [24], where he obtained the minimum regularity $s > \frac{3}{2}$). Now $L_0 = \Delta_{L_I}$ is an operator with constant coefficients, so by Lemma 4.5 is an isomorphism. ■

4 Quasi Linear Elliptic Equations in $H_{s,\delta}$

In this section we will establish the elliptic theory in $H_{s,\delta}$ which is essential for the solution of the constraint equations. We will extend earlier results in weighted Sobolev spaces of integer order which were obtained by Cantor [9], Choquet-Bruhat and Christodoulou [12] and Christodoulou and O'Murchadha [15] to the fractional ordered spaces. The essential tool is the a priori estimate (4.18) and proving it requires first to establish an analogous a priori estimate in Bessel potential spaces. Our approach is based on the techniques of Pseudodifferential Operators which have symbols with limited regularity and we are adopting ideas being presented in Taylor's books [30]and [31]. A different method was derived recently by Maxwell [24].

4.1 A priori estimates for linear elliptic systems in H^s

In this section we consider a second order homogeneous elliptic system

$$(Lu)^i = \sum_{\alpha,\beta,j} a_{ij}^{\alpha\beta}(x)\partial_\alpha\partial_\beta u^j, \tag{4.1}$$

where the indexes $i, j = 1, , , N$ and $\alpha, \beta = 1, 2, 3$ (since only $\mathbb{R}^3$ is being discussed in this paper). We will use the convention

$$Lu = A(x)D^2u, \tag{4.2}$$

where $A(x) = a_{ij}^{\alpha\beta}(x)$, $(D^2u)^j_{\alpha\beta} = \partial_\alpha\partial_\beta u^j$ and $(A(x)D^2u)_i = a_{ij}^{\alpha\beta}(x)(D^2u)^j_{\alpha\beta}$. The symbol of (4.1) is $N \times N$ matrix $A(x,\xi)$, defined for all $\xi \in \mathbb{C}^3$ as follows:

$$A(x,\xi)_{ij} := -\sum_{\alpha,\beta} a_{ij}^{\alpha\beta}(x)i\xi_\alpha i\xi_\beta. \tag{4.3}$$

The following definitions are due to Morrey [26].

Definition 4.1

1. *The system (4.1) is* **elliptic** *provided that*

$$\det(A(x,\xi)) = \det\left(\sum_{\alpha,\beta} a_{ij}^{\alpha\beta}(x)\xi_\alpha\xi_\beta\right) \neq 0, \qquad \textit{for all} \qquad 0 \neq \xi \in \mathbb{R}^3; \tag{4.4}$$

2. *The system (4.1) is* **strongly elliptic** *provided that for some positive* λ

$$\langle A(x,\xi)\eta, \eta\rangle = \sum_{\alpha,\beta,i,j} a_{ij}^{\alpha\beta}(x)\xi_\alpha\xi_\beta\eta^i\eta^j \geq \lambda|\xi|^2|\eta|^2. \tag{4.5}$$

Our main task is to obtain a priori estimate in the Bessel potential spaces H^s for the operator (4.1) whose coefficients $a^{ij}_{\alpha\beta}$ belong to H^{s_2}. In case s and s_2 are integers, then one may prove (4.8) below by means of induction and the classical results of Douglas and Nirenberg [16], and Morrey [26]. We will employ techniques of Pseudodifferential calculus. If the coefficients of the matrix A belongs to H^{s_2}, then $A(x,\xi) \in H^{s_2}S^2_{1,0}$, that is, $\|\partial_\xi^\alpha A(\cdot,\xi)\|_{H^{s_2}} \le C_\alpha(1+|\xi|^2)^{(2-|\alpha|)/2)}$. We follow Taylor and decompose

$$A(x,\xi) = A^{\#}(x,\xi) + A^b(x,\xi) \tag{4.6}$$

in such way that a good parametrix can be constructed for $A^{\#}(x,\xi)$, while $A^b(x,\xi)$ will have order less than 2. According to Proposition 8.2 in [31], for $s_2 > \frac{3}{2}$ there is $0 < \delta < 1$ such that

$$A^{\#}(x,\xi) \in S^2_{1,\delta}, \qquad A^b(x,\xi) \in H^{s_2}S^{2-\sigma\delta}_{1,\delta}, \qquad \sigma = s_2 - \frac{3}{2}$$

where $A^{\#}(x,\xi) = \sum_{k=0}^{\infty} J_{\epsilon_k}A(x,\xi)\phi_k(\xi)$, $\epsilon_k = c2^{-k\delta}$. Here $\{\phi_k\}$ is the Littlewood-Paley partition of unity, that is, $\phi_0 \in C_0^\infty(\mathbb{R}^3)$, $\phi_0(0) = 1$, $\phi_k(\xi) = \phi_0(2^{-k}\xi) - \phi_0(2^{-k+1}\xi)$ and $\sum_{k=0}^{\infty}\phi_k(\xi) = 1$. The smoothing operator J_ϵ is defined as follows:

$$J_\epsilon f(x) = \phi_0(\epsilon D)f(x) = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \int \epsilon^{-3}\widehat{\phi}_0(\frac{y}{\epsilon})f(x-y)dy,$$

where $\widehat{\phi}_0$ is the inverse Fourier transform. In order that $A^{\#}$ will have a good parametrix we need to verify that it is a strongly elliptic. Since the original operator is strongly elliptic,

$$\begin{aligned}
&\sum_{\alpha,\beta,i,j} J_{\epsilon_k}a_{ij}^{\alpha\beta}(x)\phi_k(\xi)\xi_\alpha\xi_\beta\eta^i\eta^j \\
&= \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \int \left(\sum_{\alpha,\beta,i,j} \epsilon_k^{-3}\widehat{\phi}_0(\frac{y}{\epsilon_k})(y)a_{ij}^{\alpha\beta}(y-x)\phi_k(\xi)\xi_\alpha\xi_\beta\eta^i\eta^j\right)dy \\
&\ge \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \quad \lambda\phi_k(\xi)|\xi|^2|\eta|^2 \int \epsilon_k^{-3}\widehat{\phi}_0(\frac{y}{\epsilon_k})dy = \lambda\phi_k(\xi)|\xi|^2|\eta|^2\phi_0(0) \\
= &\quad \lambda\phi_k(\xi)|\xi|^2|\eta|^2
\end{aligned}$$

for each fixed k. Summing over the k we have,

$$\langle A^{\#}(x,\xi)\eta,\eta\rangle = \sum_{k=0}^{\infty}\sum_{\alpha,\beta,i,j}\left(J_{\epsilon_k}a_{ij}^{\alpha\beta}\right)(x)\phi_k(\xi)\xi_\alpha\xi_\beta\eta^i\eta^j \ge \sum_{k=0}^{\infty}\lambda\phi_k(\xi)|\xi|^2|\eta|^2 = \lambda|\xi|^2|\eta|^2,$$

thus (4.5) holds for $A^{\#}$. The last step assures that $\|A^{\#}(x,\xi)^{-1}\| \le \frac{1}{\lambda|\xi|^2}$ and then it follows from the identity $\partial(A^{-1}) = A^{-1}(\partial(A))A^{-1}$ that

$$\|\partial_x^\beta\partial_\xi^\alpha(A^{\#}(x,\xi))^{-1}\| \le C_{\alpha\beta}(1+|\xi|^2)^{(-2-|\alpha|+\delta|\beta|)/2},$$

that is, $(A^{\#}(x,\xi))^{-1} \in S^{-2}_{1,\delta}$. Hence, the operator $A^{\#}(x,D)$ has a parametrix $E^{\#}(x,D) \in OPS^{-2}_{1,\delta}$ satisfying

$$E^{\#}(x,D)A^{\#}(x,D) = I + S, \tag{4.7}$$

where $S \in OPS^{-\infty}$ (See e. g. [30] Section 0.4).

Lemma 4.2 ***(An a priori estimates in H^s)*** *Let $Lu = A(x)D^2u$ be a strongly elliptic system and assume $A \in H^{s_2}$, $s_2 > \frac{3}{2}$ and $0 \le s-2 \le s_2$. Then there is a constant C such that*

$$\|u\|_{H^s} \le C\left\{\|Lu\|_{H^{s-2}} + \|u\|_{H^{s-2}}\right\}. \tag{4.8}$$

Proof (of Lemma 4.2) We decompose $A(x,D)$ as in (4.2) and let $E^{\#}(x,D)$ be the above parametrix, then by (4.7)

$$E^{\#}(x,D)A(x,D)u = u + Su + E^{\#}(x,D)A^b(x,D)u. \tag{4.9}$$

Since $E^{\#}(x,D), S : H^{s-2} \to H^s$ are bounded,

$$\|E^{\#}(x,D)A(x,D)u\|_{H^s} = \|E^{\#}(x,D)Lu\|_{H^s} \le C\|Lu\|_{H^{s-2}} \tag{4.10}$$

and

$$\|Su\|_{H^s} \le C\|u\|_{H^{s-2}}. \tag{4.11}$$

According to [31] Proposition 8.1, (see also [30] Proposition 2.1.J)

$$A^b(x,D) : H^{s-\sigma\delta} \to H^{s-2}.$$

Hence,

$$\|E^{\#}(x,D)A^b(x,D)u\|_{H^s} \le C\|A^b(x,D)u\|_{H^{s-2}} \le C\|u\|_{H^{s-\sigma\delta}}. \tag{4.12}$$

Using the intermediate estimate $\|u\|_{H^{s-\sigma\delta}} \le \epsilon\|u\|_{H^s} + C(\epsilon)\|u\|_{H^{s-2}}$, and combining it with (4.9)-(4.11), we obtain the estimate (4.8). ∎

4.2 A priori estimates in $H_{s,\delta}$

Our main task here is to extend the a priori estimate (4.8) to $H_{s,\delta}$-spaces and for second order elliptic systems of the form:

$$\begin{aligned}(Lu)^i &= \sum_{\alpha,\beta,j} a_{ij}^{\alpha\beta}(x)\partial_\alpha\partial_\beta u^j + \sum_{\alpha,j} b_{ij}^{\alpha}(x)\partial_\alpha u^j + \sum_j c_{ij}(x)u^j \\ &= A(x)D^2u + B(x)(Du) + C(x)u.\end{aligned} \tag{4.13}$$

Here $A(x)$ is as in the previous subsection, $B(x) = b_{ij}^{\alpha}(x)$, $(Du)_\alpha^j = \partial_\alpha u^j$, $(B(x)Du)_i = b_{ij}^{\alpha}(x)\partial_\alpha u^j$ and $C(x) = c_{ij}(x)$ is $N \times N$. We introduce the following hypotheses:

Hypotheses (H)

(H1) $\sum a_{i,j}^{\alpha,\beta}(x)\eta^i\eta^j\xi_\alpha\xi\beta \ge \lambda|\eta|^2|\xi|^2$ (i.e. L is strongly elliptic);

(H2) $(A(\cdot) - A_\infty) \in H_{s_2,\delta_2}$, $B \in H_{s_1,\delta_1}$, $C \in H_{s_0,\delta_0}$
$s_i \ge s-2, i = 0,1,2$, $s_2 > \frac{3}{2}, s_1 > \frac{1}{2}, s_0 \ge 0$ and $\delta_i > \frac{1}{2} - i$, $i = 0,1,2$, the matrix A_∞ has constant coefficients and $A_\infty D^2u$ is an elliptic system, that is, $\det\left(\sum (a_\infty)_{ij}^{\alpha,\beta}\xi_\alpha\xi_\beta\right) \neq 0$.

We shall first derive an a priori estimate for a second order homogeneous operator

$$L_2u = A(x)D^2u.$$

Lemma 4.3 ***(An a priori estimate for homogeneous operator in*** $H_{s,\delta}$***)*** *Assume the operator* L_2 *satisfies hypotheses* (H) *and* $s \geq 2$*. Then*

$$\|u\|_{H_{s,\delta}} \leq C\left\{\|L_2u\|_{H_{s-2,\delta+2}} + \|u\|_{H_{s-2,\delta}}\right\}, \tag{4.14}$$

where the constant C *depends on* s, δ *and* $\|A - A_\infty\|_{H_{s_2,\delta_2}}$*.*

Proof (of Lemma 4.3) According to Corollary 5.6,

$$\sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j}\left\|\left(\psi_j^4 u\right)_{2^j}\right\|_{H^s}^2$$

is an equivalent norm in $H_{s,\delta}$. The main idea of the proof is to apply Lemma 4.2 to each term of the equivalent norm above. We use the convention (4.2) and compute

$$L_2(\psi^4 u) = \psi^4 L_2\left(D^2 u\right) + \psi A(x) R(u,\psi),$$

where

$$R(u,\psi)_{\alpha\beta} = 8\psi\left(D\psi\right)_\alpha\left(\psi Du\right)_\beta + 12\left(D\psi\right)_\alpha\left(D\psi\right)_\beta\left(\psi u\right) + 4\psi\left(D^2\psi\right)_{\alpha\beta}(\psi u).$$

Applying the a priori estimate (4.8), we have

$$\begin{aligned}
\|u\|_{H_{s,\delta}}^2 &\lesssim \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j}\left\|\left(\psi_j^4 u\right)_{2^j}\right\|_{H^s}^2 \\
&\lesssim \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j}\left\{\left\|L_2\left((\psi_j^4 u)\right)_{2^j}\right\|_{H^{s-2}}^2 + \left\|\left(\psi_j^4 u\right)_{2^j}\right\|_{H^{s-2}}^2\right\} \\
&\lesssim \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j}\left\{2^{4j}\left\|\left(\psi_j^4 L_2(u)\right)_{2^j}\right\|_{H^{s-2}}^2 + \left\|\left(\psi_j^4 u\right)_{2^j}\right\|_{H^{s-2}}^2\right\} \\
&+ \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta+2\right)2j}\left\|\left(\psi_j A R(u,\psi_j)\right)_{2^j}\right\|_{H^{s-2}}^2 \\
&\lesssim \|L_2(u)\|_{H_{s-2,\delta+2}}^2 + \|u\|_{H_{s-2,\delta}}^2 + \|AR\|_{H_{s-2,\delta+2}}^2.
\end{aligned} \tag{4.15}$$

The assumption on s_2 and δ_2 enable us to use Proposition 6.7 and get

$$\begin{aligned}
\|AR\|_{H_{s-2,\delta+2}} &\leq C\left(\|(A-A_\infty)R\|_{H_{s-2,\delta+2}} + \|A_\infty R\|_{H_{s-2,\delta+2}}\right) \\
&\leq C\left(\|(A-A_\infty)\|_{H_{s_2,\delta_2}} + \|A_\infty\|\right)\|R\|_{H_{s-2,\delta+2}}.
\end{aligned} \tag{4.16}$$

Property (5.4) of ψ_j and inequality (6.5) imply

$$\|(\psi_j R)_{2^j}\|_{H^{s-2}} \leq C\left(2^{-j}\|(\psi_j Du)_{2^j}\|_{H^{s-2}} + 2^{-2j}\|(\psi_j u)_{2^j}\|_{H^{s-2}}\right)$$

and hence $\|R\|_{H_{s-2,\delta+2}} \leq C\left(\|u\|_{H_{s-1,\delta}} + \|u\|_{H_{s-2,\delta}}\right)$. Thus, inequalities (4.15) and (4.16) yields

$$\|u\|_{H_{s,\delta}} \leq C\left\{\|L_2 u\|_{H_{s-2,\delta+2}} + \left(\|A - A_\infty\|_{H_{s_2,\delta_2}} + 1\right)\left(\|u\|_{Hs-1,\delta} + \|u\|_{H_{s-2,\delta}}\right)\right\}. \quad (4.17)$$

Invoking the intermediate estimate $\|u\|_{Hs-1,\delta} \leq \sqrt{2\epsilon}\|u\|_{Hs,\delta} + C(\epsilon)\|u\|_{Hs-2,\delta}$ (see Proposition 6.6) and taking ϵ so that $C\left(\|A - A_\infty\|_{H_{s_2,\delta_2}} + 1\right)\sqrt{2\epsilon} \leq \frac{1}{2}$, we obtain from (4.17) the desired estimate (4.14). ∎

Lemma 4.4 ***(An a priori estimate in*** $H_{s,\delta}$***)*** *Assume the operator L of the form (4.13) satisfies hypotheses (H) and $s \geq 2$. Then*

$$\|u\|_{H_{s,\delta}} \leq C\left\{\|Lu\|_{H_{s-2,\delta+2}} + \|u\|_{H_{s-2,\delta}}\right\}, \quad (4.18)$$

where the constant C depends on s, δ and the coefficients of L.

Proof (of Lemma 4.4) By Lemma 4.3,

$$\begin{aligned}\|u\|_{H_{s,\delta}} &\leq C\left\{\|L_2 u\|_{H_{s-2,\delta+2}} + \|u\|_{H_{s-2,\delta}}\right\} \\ &\leq C\left\{\|Lu\|_{H_{s-2,\delta+2}} + \|u\|_{H_{s-2,\delta}} + \|(L - L_2)u\|_{H_{s-2,\delta+2}}\right\},\end{aligned}$$

where $(L - L_2)u = B(x)(Du) + C(x)u$. Hypothesis (H2) together with Corollary 6.2 and Proposition 6.7 give

$$\|B(x)(Du)\|_{H_{s-2,\delta+2}} \lesssim \|B\|_{H_{s_1,\delta_1}}\|Du\|_{H_{s-2,\delta+1}} \lesssim \|B\|_{H_{s_1,\delta_1}}\|u\|_{H_{s-1,\delta}}, \quad (4.19)$$

and

$$\|C(x)u\|_{H_{s-2,\delta+2}} \lesssim \|C\|\|_{H_{s_0,\delta_0}}\|u\|_{H_{s-2,\delta}}.$$

Finally, we apply the intermediate estimate, Proposition 6.6, to the right hand side of (4.19) and by taking ϵ sufficiently small we obtain (4.18). ∎

Lemma 4.5 ***(Isomorphism of an operator with constant coefficients)*** *Let $A_\infty u := A_\infty D^2 u$ be a homogeneous elliptic system with constant coefficients. Then for any $s \geq 2$ and $-\frac{3}{2} < \delta < -\frac{1}{2}$, the operator $A_\infty : H_{s,\delta+2} \to H_{s-2,\delta}$ is isomorphism satisfying*

$$\|u\|_{H_{s,\delta}} \leq C\|A_\infty D^2 u\|_{H_{s-2,\delta+2}}. \quad (4.20)$$

Proof (of Lemma 4.5) Both statements are true when s is an integer and under the norm (5.1) (see e. g. [12], Theorem 5.1) and by Theorem 5.2 they hold also in the $H_{s,\delta}$ norm (2.1). For s between two integers m_0 and m_1, we have $s = s_\theta = \theta m_0 + (1-\theta)m_1$ and $s-2 = s_\theta - 2 = \theta(m_0-2) + (1-\theta)(m_1-2)$, where $0 < \theta < 1$. The interpolation Theorem 6.1 implies

$$H_{s,\delta} = [H_{m_0,\delta}, H_{m_1,\delta}]_\theta \quad \text{and} \quad H_{s-2,\delta} = [H_{m_0-2,\delta}, H_{m_1-2,\delta}]_\theta.$$

Since $A_\infty^{-1} : H_{m_i-2,\delta} \to H_{m_i,\delta+2}$, $i = 0,1$, is continuous, it follows from interpolation theory that $A_\infty^{-1} : H_{s_\theta-2,\delta} \to H_{s_\theta,\delta+2}$ is also continuous (see e. g. [33]). Hence (4.20) holds. ∎

The next lemma improves the a priori estimate (4.18).

Lemma 4.6 ***(Improved a priori estimate)*** *Let L be an elliptic operator of the form (4.13) which satisfies hypotheses (H). Assume $s \geq 2$ and $-\frac{3}{2} < \delta < -\frac{1}{2}$. Then for any δ' there is a constant C such that*

$$\|u\|_{H_{s,\delta}} \leq C\left\{\|Lu\|_{H_{s-2,\delta+2}} + \|u\|_{H_{s-1,\delta'}}\right\}. \tag{4.21}$$

The constant C depends on the H_{s_i,δ_i}-norm of the coefficients of L, s, δ and δ'.

Proof (of Lemma 4.6) Let $\chi_R \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function satisfying $\mathrm{supp}(\chi_R) \subset \{|x| \leq 2R\}$, $\chi_R(x) = 1$ for $|x| \leq R$, $0 \leq \chi_R(x) \leq 1$ and $\|\partial^\alpha \chi_R\|_\infty \leq C_\alpha R^{-|\alpha|}$. For $u \in H_{s,\delta}$ we write

$$u = (1-\chi_R)u + \chi_R u$$

and R will be determinate later on. We start with the estimation of $\|(1-\chi_R)u\|_{H_{s,\delta}}$ and for that purpose we use the convention (4.2) and compute

$$\begin{aligned} A_\infty(D^2(1-\chi_R)u) &= (1-\chi_R)A_\infty(D^2u) - 2A_\infty(D\chi_R)(Du) - A_\infty(D^2\chi_R)u \\ &= (1-\chi_R)(Lu) + E_1 + E_2, \end{aligned} \tag{4.22}$$

where

$$E_1 = -(1-\chi_R)\left\{(A - A_\infty)\left(D^2u\right) + B(x)(Du) + C(x)u\right\}$$

and

$$E_2 = -\left\{2A_\infty(D\chi_R)(Du) + A_\infty(D^2\chi_R)u\right\}$$

Applying inequality (4.20) of Lemma 4.5,

$$\begin{aligned} \|(1-\chi_R)u\|_{H_{s,\delta}} &\leq C\|A_\infty D^2((1-\chi_R)u)\|_{H_{s-2,\delta+2}} \\ &\leq C\left\{\|(1-\chi_R)Lu\|_{H_{s-2,\delta+2}} + \|E_1\|_{H_{s-2,\delta+2}} + \|E_2\|_{H_{s-2,\delta+2}}\right\}. \end{aligned} \tag{4.23}$$

Since $\|(1-\chi_R)Lu\|_{H_{s-2,\delta+2}} \lesssim \|Lu\|_{H_{s-2,\delta+2}}$ (see Proposition 6.4), it remains to estimate $\|E_1\|_{H_{s-2,\delta+2}}$ and $\|E_2\|_{H_{s-2,\delta+2}}$. We may choose δ_i' so that $\delta_i > \delta_i' > \frac{1}{2} - i$, $i = 0,1,2$ and then we put $\gamma = \min_{i=0,1,2}(\delta_i - \delta_i')$. Under these conditions the application of Corollary 6.2, Propositions 6.5 and 6.7 yield

$$\begin{aligned}
\|E_1\|_{H_{s-2,\delta+2}} &\leq C \left\|(1-\chi_R)\left\{(A-A_\infty)(D^2u) + B(Du) + Cu\right\}\right\|_{H_{s-2,\delta+2}} \\
&\leq C\{\|(1-\chi_R)(A-A_\infty)\|_{H_{s_2,\delta_2'}} \|D^2u\|_{H_{s-2,\delta+2}} \\
&+ \|(1-\chi_R)B\|_{H_{s_1,\delta_1'}} \|Du\|_{H_{s-1,\delta+1}} + \|(1-\chi_R)C\|_{H_{s_0,\delta'}} \|u\|_{H_{s,\delta}}\} \\
&\leq \frac{C_1}{R^\gamma}\left(\|(A-A_\infty)\|_{H_{s_2,\delta_2}} + \|B\|_{H_{s_1,\delta_1}} + \|C\|_{H_{s_0,\delta_0}}\right)\|u\|_{H_{s,\delta}} \\
&\leq \frac{C_1\Lambda}{R^\gamma}\|u\|_{H_{s,\delta}},
\end{aligned} \tag{4.24}$$

where $\Lambda = \left(\|A-A_\infty\|_{H_{s_2,\delta_2}} + \|B\|_{H_{s_1,\delta_2}} + \|C\|_{H_{s_0,\delta_0}}\right)$.

Next, since $D\chi_R$ has compact support, Remark 6.3 and Corollary 6.2 imply that

$$\begin{aligned}
\|E_2\|_{H_{s-2,\delta+2}} &\leq C(R)\left\{\|2A_\infty((D\chi_R)(Du))\|_{H_{s-2,\delta'+1}} + \|A_\infty((D^2\chi_R)u)\|_{H_{s-2,\delta'}}\right\} \\
&\leq C(R)\|A_\infty\|\left\{2\|Du\|_{H_{s-2,\delta'+1}} + \|u\|_{H_{s-2,\delta'}}\right\} \\
&\leq C(R)\|A_\infty\|\|u\|_{H_{s-1,\delta'}}.
\end{aligned} \tag{4.25}$$

We turn now to the estimation of $\|\chi_R u\|_{H_{s,\delta}}$. Noting that $(\chi_R u)$ has compact support, we have by Remark 6.3, (4.18) and Proposition 6.4 that

$$\|\chi_R u\|_{H_{s,\delta}} \leq C(R)\|\chi_R u\|_{H_{s,\delta'}} \leq C(R)\{\|L(\chi_R u)\|_{H_{s-2,\delta'+2}} + \|u\|_{H_{s-1,\delta'}}\}. \tag{4.26}$$

Similarly to (4.22) we compute

$$L(\chi_R u) = \chi_R L(u) + 2A(D\chi_R)(Du) + A(D^2\chi_R)u + B(D\chi_R)u. \tag{4.27}$$

We estimate each term of (4.27) separately. Once again, since $\chi_R Lu$ has compact support,

$$\|\chi_R(Lu)\|_{H_{s-2,\delta'+2}} \leq C(R)\|Lu\|_{H_{s-2,\delta+2}}. \tag{4.28}$$

Next, using the second assumption of (H), Proposition 6.7 and the compactness of $\mathrm{supp}(\chi_R)$ we get

$$\begin{aligned}
&\|2A(D\chi_R)(Du)\|_{H_{s-2,\delta'+2}} \\
&\leq 2\|(A-A_\infty)(D\chi_R)(Du)\|_{H_{s-2,\delta'+2}} + \|A_\infty(D\chi_R)(Du)\|_{H_{s-2,\delta'+2}} \\
&\leq C\left(\|(A-A_\infty)\|_{H_{s_2,\delta_2}} + \|A_\infty\|\right)\|(D\chi_R)(Du)\|_{H_{s-2,\delta'+2}} \\
&\leq C(R)\left(\|(A-A_\infty)\|_{H_{s_2,\delta_2}} + \|A_\infty\|\right)\|Du\|_{H_{s-2,\delta'+1}} \\
&\leq C(R)\left(\|(A-A_\infty)\|_{H_{s_2,\delta_2}} + \|A_\infty\|\right)\|u\|_{H_{s-1,\delta'}}.
\end{aligned} \tag{4.29}$$

In a similar manner we estimate the other terms and together with inequalities (4.23)-(4.26), (4.28) and (4.29) we have

$$\begin{aligned}\|u\|_{H_{s,\delta}} &\leq \|(1-\chi_R)u\|_{H_{s,\delta}} + \|\chi_R u\|_{H_{s,\delta}} \\ &\leq C\left\{\|Lu\|_{H_{s-2,\delta+2}} + C_2\|u\|_{H_{s-1,\delta'}} + \frac{C_1\Lambda}{R^\gamma}\|u\|_{H_{s,\delta}}\right\},\end{aligned} \tag{4.30}$$

where C_1 and C_2 depend on the norms of the coefficients of L and in addition C_2 depends in R. We now take R such that $\frac{C_1\Lambda}{R^\gamma} \leq \frac{1}{2}$, then (4.21) follows from (4.30). ∎

The next two theorems are consequence of the compact embedding, Theorem 6.12, the a priori estimate (4.21) and standard arguments of Functional Analysis.

Theorem 4.7 ***(Semi Fredholm)*** *Assume the operator L satisfies hypotheses(H), $s \geq 2$ and $-\frac{3}{2} < \delta < -\frac{1}{2}$. Then $L : H_{s,\delta} \to H_{s-2,\delta+2}$ is semi Fredholm, that is,*

(i) $\mathbf{dim}(\mathbf{Ker}L) < \infty$*;*

(ii) *If L is injective, then there is a constant C such that*

$$\|u\|_{H_{s,\delta}} \leq C\|Lu\|_{H_{s-2,\delta+2}}; \tag{4.31}$$

(iii) *L has a closed range.*

Theorem 4.8 ***(A homotopy argument)*** *Let $s \geq 2$ and $-\frac{3}{2} < \delta < -\frac{1}{2}$. Assume L be an elliptic operator of the form (4.13) that fulfilled the hypotheses (H) and L_t is a continuous family of operators which satisfy hypotheses (H) for $t \in [0,1]$, $L_1 = L$ and*

$$L_t : H_{s,\delta} \to H_{s-2,\delta+2} \text{ is injective.}$$

If

$$L_0 : H_{s,\delta} \to H_{s-2,\delta+2} \text{ is an isomorphism,}$$

then the same is true for L.

The next Lemma shows that solutions to the homogeneous system have lower growth at infinity. We follow Christodoulou and O'Murchadha's proof [15].

Lemma 4.9 ***(Lower growth of homogeneous solutions)*** *Assume L satisfies hypotheses (H), $u \in H_{s,\delta}$, $s \geq 2$ and $-\frac{3}{2} < \delta < -\frac{1}{2}$. If $Lu = 0$, then $u \in H_{s,\delta'}$ for any $-\frac{3}{2} < \delta' < -\frac{1}{2}$.*

Proof (of Lemma 4.9) We recall that if $\delta' < \delta$, then it follows from Remark 5.4 that $H_{s,\delta} \hookrightarrow H_{s,\delta'}$. Hence it suffices to show the statement for $\delta' > \delta$. The conditions on δ_i imply that we may find $\delta' > \delta$ so that $\delta_i + \delta + i > \delta' + 2 - \frac{3}{2}$. Then applying the Proposition 6.7 to

$$f := A_\infty u - Lu = (A_\infty - A(x))\left(D^2u\right) - B(x)(Du) - C(x)u,$$

we obtain that f belongs to $H_{s-2,\delta'+2}$. Now $Lu = 0$, so $A_\infty u = f$ and since $A_\infty : H_{s,\delta'} \to H_{s-2,\delta'+2}$ is isomorphism by Lemma 4.5, we conclude that hence $u \in H_{s,\delta'}$. We now replace δ by δ' repeat the above arguments. ∎

4.3 Semi Linear Elliptic Equations on Asymptotically Flat Manifolds

A Riemannian 3-manifold (M,h) is asymptotically flat (AF) if there is a compact subset K such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus B_1(0)$ and the metric h tends to the identity I at infinity. A natural definition of the last statement in our case is $h - I \in H_{s',\delta'}$. We will hence assume here that $h - I \in H_{s',\delta'}$, $s' > \frac{3}{2}$ and $\delta' > -\frac{3}{2}$.

We denote by Δ_h be the Laplace-Beltrami operator on (M,h). In the coordinates (x^1, x^2, x^3) it takes the form

$$\Delta_h = \frac{1}{\sqrt{|h|}} \partial_j \left(\sqrt{|h|} h^{ij} \partial_i \right), \tag{4.32}$$

where $|h| = \det(h_{ij})$ and $h^{ij} = (h_{ij})^{-1}$. Inserting the identity $\partial_j |h| = |h| \mathrm{tr}(h^{ij}(\partial_j(h_{ij}))$ into (4.32), we have

$$\Delta_h = h^{ij}\partial_j\partial_i + \partial_j(h^{ij})\partial_i + \frac{1}{2}\mathrm{tr}(h^{ij}(\partial_j(h_{ij}))h^{ij}\partial_i. \tag{4.33}$$

Hence, by means of Proposition 6.7, Theorem 6.10 and Remark 6.11, the elliptic operator (4.33) satisfies hypothesis (H) of Section 4.2 provided that $s \leq s'$.

Let us introduce some more notations. We denote by $\mu_h = \sqrt{|h|}dx$ the Lebesgue measure on the manifold (M,h), $(Du \cdot Dv)_h = h^{ij}\partial_i u \partial_j v$, and $\|Du\|_h^2 = (Du \cdot Du)_h$. Integration by parts yields

$$\begin{aligned} \int (\Delta_h u)\, v d\mu_h &= \int \partial_j \left(\sqrt{|h|} h^{ij} \partial_i u \right) v dx \\ = - \int h^{ij}\partial_i u \partial_j v \sqrt{|h|} dx &= - \int (Du \cdot Dv)_h d\mu_h. \end{aligned} \tag{4.34}$$

Formula (4.34) holds whenever $v \in H_0^1(\mathbb{R}^3)$, $u \in H_{s,\delta}$ and $s \geq 1$. Therefore it enables us to define weak solutions on the manifold (M,h).

Definition 4.10 ***(Weak solutions)*** *A function u in $H_{s,\delta}$ is a weak solution of*

$$-\Delta_h u + c(x) u = f \in H_{s-2,\delta+2}$$

on (M,h), if

$$\int \left((Du \cdot Dv)_h + cuv \right) d\mu_h = \int f v d\mu_h, \tag{4.35}$$

for all $v \in H_0^1(\mathbb{R}^3)$.

Remark 4.11 *In case $u, v \in H_{s,\delta}$, $s \geq 2$ and $\delta \geq -1$, then by algebra $h^{ij}\partial_i u, \sqrt{|h|}\partial_j v \in H_{s-1,0}$ (see subsection 6.1). Applying the Cauchy Schwarz inequality*

$$\begin{aligned} &\int |(Du \cdot Dv)_h| d\mu_h = \int |h^{ij}\partial_i u \partial_j v| \sqrt{|h|} dx \\ &\leq \left(\int (h^{ij}\partial_i u)^2 \right)^{\frac{1}{2}} \left(\int \sqrt{|h|}\partial_j v)^2 \right)^{\frac{1}{2}} \leq \|h^{ij}\partial_i u\|_{H_{s-1,0}} \|\sqrt{|h|}\partial_j v\|_{H_{s-1,0}}, \end{aligned}$$

we see that $h^{ij}\partial_i u \partial_j v|\sqrt{|h|} \in L^1(\mathbb{R}^3)$*. Similarly, the integrand of the left hand side of (4.34) belongs to* $L^1(\mathbb{R}^3)$*. Hence, approximating* u *and* v *by* C_0^∞ *functions and using Lebesgue's Dominated Convergence Theorem we have*

$$\int (\Delta_h u)\, v d\mu_h = -\int (Du \cdot Dv)_h d\mu_h, \quad u, v \in H_{s,\delta}, \text{ whenever } s \geq 2, \text{ and } \delta \geq -1. \tag{4.36}$$

In this section we will prove existence and uniqueness for the semi-linear equation

$$-\Delta_h u = F(x,u) := \sum_{i=1}^{N} m_i(x) h_i(u), \tag{4.37}$$

where $m_i \in H_{s_0,\delta_0}$, $m_i(x) \geq 0$, $s_0 \geq 0$, $\delta_0 > \frac{1}{2}$ and for $u > -1$ the functions h_i are decreasing, nonnegative and smooth. These conditions ensure $F(\cdot, u)$ and $\frac{\partial F}{\partial p}(\cdot, u)$ are in $H_{s-2,\delta+2}$ whenever $u \in H_{s,\delta}$ and $s \geq 2$.

Theorem 4.12 ***(Existence and uniqueness)*** *Let* $h - I \in H_{s',\delta'}$*,* $s' > \frac{3}{2}$*,* $\delta' > -\frac{3}{2}$*,* $2 \leq s \leq s'$ *and* $-\frac{3}{2} < \delta < -\frac{1}{2}$*. Then equation (4.37) has a unique solution* u *in* $H_{s,\delta}$*. Furthermore,* $0 \leq u \leq K$ *for a nonnegative constant* K*.*

In order to show Theorem 4.12 we need the weak maximal principle:

Proposition 4.13 ***(Weak maximal principle)*** *Assume* $c \in H_{s'-2,\delta'+2}$ *is nonnegative. If* $u \in H_{s,\delta}$ *satisfies*

$$-\Delta_h u + cu \leq 0, \tag{4.38}$$

then $u \leq 0$*.*

Proof (of Proposition 4.13) For $\epsilon > 0$ we put $w = \max(u - \epsilon, 0)$. It has compact support since $\lim_{x\to\infty} u(x) = 0$. Further, $Dw = Du$ a.e. in $\{u(x) > \epsilon\}$ (see e. g. [18] or [21]. Thus, $w \in H_0^1(\mathbb{R}^3)$ and $w \geq 0$, so by (4.35)

$$0 \geq \int ((Du, Dw)_h + cuw)\, d\mu_h = \int_{\{u \geq \epsilon\}} \left(\|Du\|_h^2 + cu^2 \right) d\mu_h.$$

Therefore $u \equiv \epsilon$ in $\{u(x) \geq \epsilon\}$. Since ϵ is arbitrary, we have $u \leq 0$. ∎

Proof (of Existence) The proof will be done in several steps. We define a map $\Phi : \{H_{s,\delta} \times [0,1], u(x) > -1\} \to H_{s-2,\delta+2}$ by

$$\Phi(u, \tau) = -\Delta_h u - \tau F(x,u),$$

let $u(\tau)$ denotes a solution of $\Phi(u,\tau) = 0$ and put $J = \{0 \leq s \leq 1 : \Phi(u(s), s) = 0\}$. We will show that J is both open and closed set. Since $0 \in J$, $J = [0,1]$ which yields the existence result.

Step 1. *The set J is open.*

Let

$$Lw := \left(\frac{\partial \Phi}{\partial u}(u,\tau)\right)(w) = -\Delta_h w - \tau \frac{\partial F}{\partial p}(\cdot, u)w$$

and

$$L_t w = -\Delta_{\{th+(1-t)I\}} w - t\tau \frac{\partial F}{\partial p}(\cdot, u)w.$$

If $L_t w = 0$, then by Lemma (4.9) $w \in H_{s,-1}$. So we may use (4.36) and get

$$\int (L_t w)\, w d\mu_{\{th+(1-t)I\}} = \int \left(\|Dw\|^2_{\{th+(1-t)I\}} - t\tau \frac{\partial F}{\partial p}(\cdot, u)w^2\right) d\mu_{\{th+(1-t)I\}}.$$

Since $\frac{\partial F}{\partial p} \leq 0$, the above yields that $L_t w = 0$ implies $w \equiv 0$ for each $t \in [0,1]$. In addition $L_0 = -\Delta_I = -\Delta$ is an isomorphism according to Lemma 4.5. Therefore Theorem 4.8 implies that $L_1 = L$ is an isomorphism too. Thus J is open by the Implicit Function Theorem.

Step 2. $\|u(\tau)\|_{H_{s,\delta}} \leq C$ *for a constant C independent of τ.*

We first establish the bound in $H_{2,\delta}$-norm. The weak maximum principle implies $u(\tau) \geq 0$ and since $F(x,p)$ is decreasing in p,

$$\|F(\cdot, u(\tau))\|_{H_{0,\delta+2}} \leq \|F(\cdot, 0)\|_{H_{0,\delta+2}} \leq \left(\sum_{i=1}^{N} h_i(0)^2 \|m_i\|^2_{H_{0,\delta+2}}\right)^{\frac{1}{2}} := K.$$

We showed in Step 1 that $\Delta_h : H_{s,\delta} \to H_{s-2,\delta+2}$ is injective, therefore from Theorem 4.7 (ii),

$$\|u(\tau)\|_{H_{2,\delta}} \leq C\| - \Delta_h u(\tau)\|_{H_{0,\delta+2}} \leq C\|F(\cdot, 0)\|_{H_{0,\delta+2}} \leq CK. \tag{4.39}$$

Now, Theorem 6.10 implies $\|h_i(u(\tau))\|_{H_{2,\delta}} \leq C\|(u(\tau))\|_{H_{2,\delta}}$ and by Proposition 6.7, $\|F(\cdot, u(\tau))\|_{H_{2,\delta}} \leq C\|u(\tau)\|_{H_{2,\delta}}$. In order to improve (4.39), we take s'' so that $s'' - 2 \leq 2$ and $s'' \leq s$. Then we may apply again (4.31) and combine it with (4.39) and Remark 5.4, we have

$$\begin{aligned} \|u(\tau)\|_{H_{s'',\delta}} &\leq C\| - \Delta_h u(\tau)\|_{H_{s''-2,\delta+2}} \leq C\| - \Delta_h u(\tau)\|_{H_{2,\delta+2}} \\ &= C\|F(\cdot, u(\tau))\|_{H_{2,\delta}} \leq C\|u(\tau)\|_{H_{2,\delta}} \leq CK. \end{aligned} \tag{4.40}$$

We have proved the boundedness in case $s'' = s$, otherwise we can repeat the same procedure as above to improve regularity until we would reach the desired regularity. It is obvious that the bound on $\|u(\tau)\|_{H_{s,\delta}}$ does not depend on τ.

Step 3. *Lipschitz continuity with respect to τ:*

Differentiation of the equation $\Phi(u(\tau), \tau) = 0$ with respect to τ gives

$$-\Delta_h u_\tau(\tau) - \tau \frac{\partial F}{\partial p} F(x, u(\tau)) u_\tau(\tau) = F(x, u(\tau)).$$

Now $\frac{\partial F}{\partial p} F(x,p) \leq 0$, so in the same way as we did in Step 1 we obtain that the operator $L = -\Delta_h - \tau \frac{\partial F}{\partial p} F(x, u(\tau)) : H_{s,\delta} \to H_{s-2,\delta+2}$ is injective. Hence, by Theorem 4.7 (ii),

$$\|u_\tau\|_{H_{s,\delta}} \leq C \|L(u_\tau)\|_{H_{s-2,\delta+2}} = C \|F(x, u(\tau))\|_{H_{s-2,\delta+2}}. \tag{4.41}$$

Next, Step 2 implies

$$\|F(x, u(\tau))\|_{H_{s-2,\delta+2}} \leq C \|u(\tau)\|_{H_{s,\delta}} \left(\sum_{i=1}^{N} \|m_i\|_{H_{s_0,\delta_0}} \right) \leq C. \tag{4.42}$$

Thus, combining (4.41) with (4.42) we get

$$\|u(\tau_1) - u(\tau_2)\|_{H_{s,\delta}} \leq C |\tau_1 - \tau_2|. \tag{4.43}$$

Step 4. *The set J is closed:*

Take a sequence $\{\tau_n\} \subset J$ such that $\tau_n \to \tau_0$. By (4.43), $\{u(\tau_n)\}$ is Cauchy in $H_{s,\delta}$ and therefore it converges to $u_0 \in H_{s,\delta}$. Since the map Φ is continuous, it follows that $\Phi(u_0, \tau_0) = 0$, that is $\tau_0 \in J$. This completes the proof of the existence. ∎

Proof (of Uniqueness) Assume u_1 and u_2 are solutions to (4.37). We conduct the proof by showing that $\Omega := \{x : u_1(x) > u_2(x)\}$ is an empty set. Note that Ω is open since u_1 and u_2 are continuous. Put $w = u_1 - u_2$, then $-\Delta_h w = F(x, u_1) - F(x, u_2) \leq 0$ in Ω. So $w \leq 0$ in Ω by Proposition (4.13). That obviously leads to a contradiction unless Ω is empty. ∎

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Appendix

5 Construction of the Spaces $H_{s,\delta}$

The weighted Sobolev spaces of integer order below were introduced by Cantor [8] and independently by Nirenberg and Walker [27]. Nirenberg and Walker initiate the study of elliptic operators in these spaces, while Cantor used them to solve the constraint equations on asymptotically flat manifolds. For an nonnegative integer m and a real δ we define a norm

$$\left(\|u\|^*_{m,\delta}\right)^2 = \sum_{|\alpha|\le m} \int \left(\langle x\rangle^{\delta+|\alpha|}|\partial^\alpha u|\right)^2 dx, \tag{5.1}$$

where $\langle x\rangle = 1 + |x|$. The space $H_{s,m}$ is the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm (5.1). Note that the weight varies with the derivatives.

Here we will repeat Triebel's extension of these spaces into a fractional order, [32],[33]. Let $s = m + \lambda$, where m is a nonnegative integer and $0 < \lambda < 1$. One possibility of extending the ordinary integer order Sobolev spaces is the *Lipschitz-Sobolevskij Spaces*, having a norm

$$\|u\|^2_{m+\lambda,2} = \sum_{|\alpha|\le m} \int |\partial^\alpha u|^2 dx + \sum_{|\alpha|=m} \int\int \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x-y|^{3+\lambda 2}} dxdy. \tag{5.2}$$

Hence, a reasonable definition of *weighted fractional Sobolev norm* is a combination of the norm (5.1) with (5.2):

$$\left(\|u\|^*_{s,\delta}\right)^2 = \begin{cases} \displaystyle\sum_{|\alpha|\le m} \int |\langle x\rangle^{\delta+|\alpha|}\partial^\alpha u|^2 dx, & s = m \\ \left.\begin{array}{l} \displaystyle\sum_{|\alpha|\le m} \int |\langle x\rangle^{\delta+|\alpha|}\partial^\alpha u|^2 dx \\ \displaystyle + \sum_{|\alpha|=m} \int\int \frac{|\langle x\rangle^{m+\lambda+\delta}\partial^\alpha u(x) - \langle y\rangle^{m+\lambda+\delta}\partial^\alpha u(y)|^2}{|x-y|^{3+2\lambda}} dxdy \end{array}\right\}, & s = m + \lambda. \end{cases} \tag{5.3}$$

here m is a nonnegative integer and $0 < \lambda < 1$. The space $H_{s,\delta}$ is the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm (5.3).

The norm (5.3) is essential for the understating of the connections between the integer and the fractional order. But it has a disadvantage, namely, the double integral makes it almost impossible to establish any property (embedding, a priori estimate, etc.) needed for PDEs. We are therefore looking for an equivalent definition of the norm (5.3).

Let $K_j = \{x : 2^{j-3} \le |x| \le 2^{j+2}\}$, $(j = 1, 2, ...)$ and $K_0 = \{x : |x| \le 4\}$. Let $\{\psi_j\}_{j=0}^\infty \subset C_0^\infty(\mathbb{R}^3)$ be a sequence such that $\psi_j(x) = 1$ on K_j, $\text{supp}(\psi_j) \subset \{x : 2^{j-4} \le |x| \le 2^{j+3}\}$, for $j \ge 1$, $\text{supp}(\psi_0) \subset \{x : |x| \le 2^3\}$ and

$$|\partial^\alpha \psi_j(x)| \le C_\alpha 2^{-|\alpha|j}, \tag{5.4}$$

where the constant C_α does not depend on j.

We define now,

$$\left(\|u\|^{\bigstar}_{s,\delta}\right)^2 = \begin{cases} \sum_{j=0}^{\infty}\left(2^{\delta 2j}\|\psi_j u\|^2_{L^2} + 2^{(\delta+m)2j}\sum_{|\alpha|=m}\|\partial^\alpha(\psi_j u)\|^2_{L^2}\right), & s=m \\ \left.\begin{array}{l} \sum_{j=0}^{\infty}\left(2^{\delta 2j}\|\psi_j u\|^2_{L^2} + 2^{(\delta+m)2j}\sum_{|\alpha|=m}\|\partial^\alpha(\psi_j u)\|^2_{L^2}\right) \\ +\sum_{j=0}^{\infty}2^{(\delta+m+\lambda)2j}\left(\sum_{|\alpha|=m}\int\int\frac{|\partial^\alpha(\psi_j u)(x)-\partial^\alpha(\psi_j u)(y)|^2}{|x-y|^{3+2\lambda}}dxdy\right), \end{array}\right\} & s=m+\lambda. \end{cases} \tag{5.5}$$

Proposition 5.1 ***(Equivalence of norms)*** *There are two positive constants* c_0 *and* c_1 *depending only on* s*,* δ *and the constants in (5.4) such that*

$$c_0\|u\|^{\bigstar}_{s,\delta} \leq \|u\|^*_{s,\delta} \leq c_1\|u\|^{\bigstar}_{s,\delta}. \tag{5.6}$$

This equivalence was proved in [32] (see also [4]).

We express these norms in terms of Fourier transform. Let

$$\hat{u}(\xi) = \mathcal{F}(u)(\xi) = \frac{1}{(2\pi)^3}\int u(x)e^{-ix\cdot\xi}dx$$

denotes the Fourier transform, put

$$\Lambda^s u = \mathcal{F}^{-1}(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}u), \tag{5.7}$$

and let H^s denotes the Bessel Potentials space having the norm

$$\|u\|^2_{H^s} = \|\Lambda^s u\|^2_{L^2} = \int(1+|\xi|^2)^s|\hat{u}(\xi)|^2 d\xi. \tag{5.8}$$

We also set

$$\|u\|^2_{h^s} = \|\mathcal{F}^{-1}(|\xi|^s\mathcal{F}u)\|^2_{L^2} = \int(|\xi|^s|\hat{u}(\xi)|)^2 d\xi.$$

It is well known that (see e. g. [19]; p. 240-241)

$$\|u\|^2_{h^s} \simeq \begin{cases} \sum_{|\alpha|=m}\int|\partial^\alpha u|^2 dx & s=m \\ \sum_{|\alpha|=m}\int\int\frac{|\partial^\alpha u(x)-\partial^\alpha u(y)|^2}{|x-y|^{3+2\lambda}}dx & s=m+\lambda \end{cases} \tag{5.9}$$

and since $(1+|\xi|^2)^s \simeq (1+|\xi|^s)$,

$$\|u\|^2_{H^s} \simeq \left(\|u\|^2_{L^2} + \|u\|^2_{h^s}\right). \tag{5.10}$$

Hence, by (5.5),

$$\left(\|u\|^{\star}_{s,\delta}\right)^2 \simeq \sum_{j=0}^{\infty}\left(2^{\delta 2j}\|\psi_j u\|^2_{L^2} + 2^{(\delta+s)2j}\|\psi_j u\|^2_{\dot{h}^s}\right) \tag{5.11}$$

We invoke now the scaling $u_\epsilon(x) := u(\epsilon x)$ $(\epsilon > 0)$, then simple calculations yields $\|u_\epsilon\|^2_{L^2} = \epsilon^{-3}\|u\|^2_{L^2}$ and $\|u_\epsilon\|^2_{\dot{h}^s} = \epsilon^{2s-3}\|u\|^2_{\dot{h}^s}$. Combining the later one with (5.10), we have

$$\|u_\epsilon\|^2_{H^s} \simeq \epsilon^{-3}\left(\|u\|_{L^2} + \epsilon^{2s}\|u\|^2_{\dot{h}^s}\right). \tag{5.12}$$

Setting $\epsilon = 2^j$, multiplying (5.12) by 2^{3j} and inserting it in (5.11), we conclude

$$\left(\|u\|^{\star}_{s,\delta}\right)^2 \simeq \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j}\|(\psi_j u)_{2^j}\|^2_{H^s}. \tag{5.13}$$

The last one is the most convenience form of norm for applications and therefore the right hand side of (5.13) defines the norm of $H_{s,\delta}$ space (see Definition 2.1).

Combining Proposition 5.1 with (5.11) and (5.13) we get:

Theorem 5.2 ***(Equivalence of norms, Triebel)*** *There are two positive constant c_0 and c_1 depending only on s, δ and the constants in (5.4) such that*

$$c_0\|u\|_{Hs,\delta} \leq \|u\|^*_{s,\delta} \leq c_1\|u\|_{Hs,\delta}. \tag{5.14}$$

Remark 5.3 *Theorem 5.2 enables us to use both sorts of the norms (5.3) and (2.1), and for each application we will use the suitable type of norm.*

Remark 5.4 *Let $s' \leq s$ and $\delta' \leq \delta$, then the inclusion $H_{s,\delta} \hookrightarrow H_{s',\delta'}$ follows easily from the representations (5.8) and (2.1) of the norms.*

Remark 5.5 *The functions $\{\psi_j\}$ are constructed by means of a composition of exponential functions. Hence, for any positive γ there holds*

$$c_1(\gamma,\alpha)|\partial^\alpha\psi_j^\gamma(x)| \leq |\partial^\alpha\psi_j(x)| \leq c_2(\gamma,\alpha)|\partial^\alpha\psi_j^\gamma(x)|. \tag{5.15}$$

Therefore the equivalence (5.6) remains valid with ψ_j^γ replacing ψ_j and hence

$$\sum_j 2^{(\frac{3}{2}+\delta)2j}\|(\psi_j^\gamma u)_{(2^j)}\|^2_{H^s} \simeq \left(\|u\|^{\star}_{s,\delta}\right)^2 \simeq \sum_j 2^{(\frac{3}{2}+\delta)2j}\|(\psi_j u)_{(2^j)}\|^2_{H^s}. \tag{5.16}$$

Corollary 5.6 ***(Equivalence of norms)*** *For any positive γ, there are two positive constants c_0 and c_1 depending on s, δ and γ such that*

$$c_0\|u\|^2_{H_{s,\delta}} \leq \sum_j 2^{(\frac{3}{2}+\delta)2j}\|(\psi_j^\gamma u)_{(2^j)}\|^2_{H^s} \leq c_1\|u\|^2_{H_{s,\delta}}. \tag{5.17}$$

6 Some Properties of $H_{s,\delta}$

Theorem 6.1 ***(Complex interpolation, Triebel)***
Let $0 < \theta < 1$, $0 \le s_0 < s_1$ *and* $s_\theta = \theta s_0 + (1-\theta)s_1$, *then*

$$[H_{s_0,\delta}, H_{s_1,\delta}]_\theta = H_{s_\theta,\delta}, \tag{6.1}$$

where (6.1) is a complex interpolation.

As a consequence of the interpolation Theorem 6.1 we get

Corollary 6.2 ***($H_{s,\delta}$-norm of a derivative)***

$$\|\partial_i u\|_{H_{s-1,\delta+1}} \le \|u\|_{H_{s,\delta}} \tag{6.2}$$

Proof (of Corollary 6.2) Let m be a positive integer and define $T : H_{m,\delta} \to H_{m-1,\delta+1}$ by $T(u) = \partial_i u$. Using the norm (5.1) we see that $\|T(u)\|_{H_{m-1,\delta+1}} \le \|u\|_{H_{m,\delta}}$. So (6.2) follows from Theorem 6.1. ∎

Remark 6.3 *If* $\operatorname{supp} u \subset \{|x| \le R\}$, *then for any* δ

$$c_1(R)\|u\|_{H^s} \le \|u\|_{H_{s,\delta}} \le c_2(R)\|u\|_{H^s}. \tag{6.3}$$

This follows from the integral representation of the norm (5.1) and the interpolation (6.1).

Proposition 6.4 ***(Multiplication by smooth functions)*** *Let* $N \ge s$ *be an integer. Assume* $f \in C^N(\mathbb{R}^3)$ *satisfies* $\sup |D^k f| \le K$ *for* $k = 0, 1, ...N$, *then*

$$\|fu\|_{H_{s,\delta}} \le C_s K \|u\|_{H_{s,\delta}}. \tag{6.4}$$

Proof (of Proposition 6.4) By the well known property of H^s

$$\|fu\|_{H^s} \le C_s K \|u\|_{H^s} \tag{6.5}$$

we have

$$\begin{aligned} \|fu\|^2_{H_{s,\delta}} &= \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \left\|(\psi_j f u)_{2^j}\right\|^2_{H^s} \\ &\le (C_s K)^2 \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \left\|(\psi_j u)_{2^j}\right\|^2_{H^s} = (C_s K)^2 \|u\|^2_{H_{s,\delta}}. \end{aligned} \tag{6.6}$$

∎

Proposition 6.5 ***(Multiplication by cutoff function)*** *Let $\chi_R \in C^\infty(\mathbb{R}^3)$ satisfies $\chi_R(x) = 1$ for $|x| \le R$, $\chi_R(x) = 0$ for $|x| \ge 2R$ and $|\partial^\alpha \chi_R| \le c_\alpha R^{-|\alpha|}$. Then for $\delta' < \delta$ there holds*

$$\|(1-\chi_R)u\|_{H_{s,\delta'}} \le \frac{C(\delta,\delta')}{R^{\delta-\delta'}}\|u\|_{H_{s,\delta}}. \tag{6.7}$$

Proof (of Proposition 6.5) Let J_0 be the smallest integer such that $R \le 2^{J_0-3}$. Then $(1-\chi_R)\psi_j = 0$ for $j = 0, 1, ..., J_0 - 1$. Hence

$$\begin{aligned} &\|(1-\chi_R)u\|^2_{H_{s,\delta'}} = \sum_{j=J_0}^{\infty} 2^{\left(\frac{3}{2}+\delta'\right)2j} \left\|(\psi_j(1-\chi_R)u)_{2^j}\right\|^2_{H^s} \\ &\le C^2 \sum_{j=J_0}^{\infty} 2^{\left(\frac{3}{2}+\delta'\right)2j} \left\|(\psi_j u)_{2^j}\right\|^2_{H^s} = C^2 \sum_{j=J_0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} 2^{(\delta'-\delta)2j} \left\|(\psi_j u)_{2^j}\right\|^2_{H^s} \\ &\le C^2 2^{(\delta'-\delta)2J_0} \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \left\|(\psi_j u)_{2^j}\right\|^2_{H^s} \le \frac{C^2}{(8R)^{(\delta-\delta')2}}\|u\|^2_{H_{s,\delta}}. \end{aligned} \tag{6.8}$$

■

Proposition 6.6 ***(An intermediate estimate)*** *Let $0 \le s_0 < s < s_1$ and $\varepsilon > 0$, then there is a constant $C = C(\varepsilon)$ such that*

$$\|u\|_{H_{s,\delta}} \le \sqrt{2}\varepsilon\|u\|_{H_{s_1,\delta}} + C\|u\|_{H_{s_0,\delta}}, \tag{6.9}$$

holds for all $u \in H_{s_1,\delta}$.

Proof (of Proposition 6.6) Inequality (6.9) is well known in H^s spaces. We apply it to each term of the norm (2.1) and get

$$\begin{aligned} \|u\|^2_{H_{s,\delta}} &= \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \left\|(\psi_j u)_{2^j}\right\|^2_{H^s} \\ &\le 2\epsilon^2 \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \left\|(\psi_j u)_{2^j}\right\|^2_{H^{s_1}} + 2C^2(\epsilon) \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \left\|(\psi_j u)_{2^j}\right\|^2_{H^{s_0}} \\ &= 2\epsilon^2\|u\|^2_{H_{s_1,\delta}} + 2C^2(\epsilon)\|u\|^2_{H_{s_0,\delta}}, \end{aligned}$$

■

6.1 Algebra

Proposition 6.7 ***(Algebra in $H_{s,\delta}$)*** *If $s_1, s_2 \ge s$, $s_1 + s_2 > s + \frac{3}{2}$ and $\delta_1 + \delta_2 \ge \delta - \frac{3}{2}$, then*

$$\|uv\|_{H_{s,\delta}} \le C\|u\|_{H_{s_1,\delta_1}}\|v\|_{H_{s_2,\delta_2}}. \tag{6.10}$$

Proof (of Proposition 6.7) By Corollary 5.6,

$$\|uv\|^2_{H_{s,\delta}} \simeq \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \left\| \left(\psi_j^2 uv\right)_{2^j} \right\|^2_{H^s}. \tag{6.11}$$

We apply the classic algebra property $\|uv\|_{H^s} \le C\|u\|_{H^{s_1}}\|v\|_{H^{s_2}}$ (see e. g. [30] Ch. 3, Section 5), to each term of the norm (6.11) and then we use Cauchy Schwarz inequality,

$$\begin{aligned}
\|uv\|^2_{H_{s,\delta}} &\le C\sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \left\| \left(\psi_j^2 uv\right)_{2^j} \right\|^2_{H^s} \\
&\le C^2 \sum_{j=0}^{\infty} 2^{\left(\frac{3}{2}+\delta\right)2j} \left\| (\psi_j u)_{2^j} \right\|^2_{H^{s_1}} \left\| (\psi_j v)_{2^j} \right\|^2_{H^{s_2}} \\
&\le C^2 \sum_{j=0}^{\infty} \left(2^{\left(\frac{3}{2}+\delta_1\right)2j} \left\| (\psi_j u)_{2^j} \right\|^2_{H^{s_1}} \right) \left(2^{\left(\frac{3}{2}+\delta_2\right)2j} \left\| (\psi_j v)_{2^j} \right\|^2_{H^{s_2}} \right) \\
&\le C^2 \left(\sum_{j=0}^{\infty} \left(2^{\left(\frac{3}{2}+\delta_1\right)2j} \left\| (\psi_j u)_{2^j} \right\|^2_{H^{s_1}} \right)^2 \right)^{\frac{1}{2}} \left(\sum_{j=0}^{\infty} \left(2^{\left(\frac{3}{2}+\delta_2\right)2j} \left\| (\psi_j v)_{2^j} \right\|^2_{H^{s_2}} \right)^2 \right)^{\frac{1}{2}} \\
&\le C^2 \left(\sum_{j=0}^{\infty} \left(2^{\left(\frac{3}{2}+\delta_1\right)2j} \left\| (\psi_j u)_{2^j} \right\|^2_{H^{s_1}} \right) \right) \left(\sum_{j=0}^{\infty} \left(2^{\left(\frac{3}{2}+\delta_2\right)2j} \left\| (\psi_j v)_{2^j} \right\|^2_{H^{s_2}} \right) \right) \\
&\le C^2 \|u\|^2_{H_{s_1,\delta_1}} \|v\|^2_{H_{s_2,\delta_2}}.
\end{aligned}$$

∎

6.2 Fractional power $|u|^\gamma$

In [20] Kateb showed that if $u \in H^s \cap L^\infty$, $1 < \gamma$ and $0 < s < \gamma + \frac{1}{2}$, then

$$\||u|^\gamma\|_{H^s} \le C(\|u\|_{L^\infty})\|u\|_{H^s}. \tag{6.12}$$

Proposition 6.8 ***(Fractional power in*** $H_{s,\delta}$***)*** *Let* $u \in H_{s,\delta} \cap L^\infty$, $1 < \gamma$, $0 < s < \gamma + \frac{1}{2}$ *and* $\delta \in \mathbb{R}$, *then*

$$\||u|^\gamma\|_{H_{s,\delta}} \le C(\|u\|_{L^\infty})\|u\|_{H_{s,\delta}}. \tag{6.13}$$

Proof (of Proposition 6.8) Inequality (6.13) is a direct consequence of the Corollary 5.6 and (6.12). Because

$$\begin{aligned}
\||u|^\gamma\|^2_{Hs,\delta} &\simeq \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j^\gamma |u|^\gamma)_{(2^j)}\|^2_{H^s} \\
&\le \left(C(\|u\|_{L^\infty})\right)^2 \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j u)_{(2^j)}\|^2_{H^s} \le \left(C(\|u\|_{L^\infty})\right)^2 \|u\|^2_{H_{s,\delta}}.
\end{aligned} \tag{6.14}$$

∎

6.3 Moser type estimates

Y. Meyer proved the below Moser type estimate [25].See also Taylor [30].

Theorem 6.9 ***(Third Moser inequality for Bessel potentials spaces)*** *Let $F : \mathbb{R}^m \to \mathbb{R}^l$ be C^{N+1} function such that $F(0) = 0$. Let $s > 0$ and $u \in H^s \cap L^\infty$. Then*

$$\|F(u)\|_{H^s} \leq K\|u\|_{H^s}, \tag{6.15}$$

where

$$K = K_N(F, \|u\|_{L^\infty}) \leq C\|F\|_{C^{N+1}} \left(1 + \|u\|_{L^\infty}^N\right), \tag{6.16}$$

here N is a positive integer such that $N \geq [s] + 1$.

We generalize this important inequality to the $H_{s,\delta}$ spaces.

Theorem 6.10 ***(Third Moser inequality in $H_{s,\delta}$)*** *Let $F : \mathbb{R}^m \to \mathbb{R}^l$ be C^{N+1} function such that $F(0) = 0$. Let $s > 0$, $\delta \in \mathbb{R}$ and $u \in H_{s,\delta} \cap L^\infty$. Then*

$$\|F(u)\|_{H_{s,\delta}} \leq K\|u\|_{H_{s,\delta}}, \tag{6.17}$$

The constant K in (6.17) depends on one in (6.16) and in addition on δ.

Proof (of Theorem 6.10) Let $\{\psi_j\}$ be the sequence satisfying (5.4) and $\Psi_j(x) = \frac{1}{\varphi(x)}\psi_j(x)$, where $\varphi(x) = \sum_{j=0}^{\infty} \psi_j(x)$. From the properties of the sequence $\{\psi_j\}$, it follows that $1 \leq \varphi(x) \leq 7$. So the sequence $\{\Psi_j\} \subset C_0^\infty(\mathbb{R}^3)$ and $\sum_{j=0}^{\infty} \Psi_j(x) = 1$. From (5.12) we conclude that

$$\|u_\epsilon\|_{H^s}^2 \leq C \max\{\epsilon^{2s-3}, \epsilon^{-3}\}\|u\|_{H^s}^2 \tag{6.18}$$

and with the combination of (6.5) and Meyer's Theorem6.9 we have,

$$
\begin{aligned}
\|F(u)\|_{H_{s,\delta}}^2 &= \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j(F(u))_{(2^j)}\|_{H^s}^2 \\
&= \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \left\| \left(\psi_j F\left(\sum_{k=0}^{\infty} \Psi_k(x) u \right) \right)_{(2^j)} \right\|_{H^s}^2 \\
&= \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \left\| \left(\psi_j F\left(\sum_{k=j-4}^{j+3} \Psi_k(x) u \right) \right)_{(2^j)} \right\|_{H^s}^2 \\
&\leq CK^2 \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \sum_{k=j-4}^{j+3} \| (\Psi_k u)_{(2^j)} \|_{H^s}^2 \\
&\leq CK^2 \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \sum_{k=j-4}^{j+3} \| ((\Psi_k u)_{2^{j-k}})_{(2^k)} \|_{H^s}^2 \\
&\leq CK^2 \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \sum_{k=j-4}^{j+3} \max\{2^{(2s-3)(j-k)}, 2^{-3(j-k)}\} \| (\Psi_k u)_{(2^k)} \|_{H^s}^2 \\
&\leq C(s)K^2 \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \sum_{k=j-4}^{j+3} \| (\psi_k u)_{(2^k)} \|_{H^s}^2 \\
&\leq C(s,\delta)K^2 \sum_{j=0}^{\infty} \sum_{k=j-4}^{j+3} 2^{(\frac{3}{2}+\delta)2k} \| (\psi_k u)_{(2^k)} \|_{H^s}^2 \\
&\leq 7C(s,\delta)K^2 \sum_{k=0}^{\infty} 2^{(\frac{3}{2}+\delta)2k} \| (\psi_k u)_{(2^k)} \|_{H^s}^2 \leq 7C(s,\delta)K^2 \|u\|_{H_{s,\delta}}^2.
\end{aligned}
\tag{6.19}
$$

■

Remark 6.11 *If $F(0) \neq 0$ and $F(0) \in H_{s,\delta}$, then we can apply Theorem 6.10 to $\widetilde{F}(u) := F(u) - F(0)$ and get*

$$
\|F(u)\|_{H_{s,\delta}} \leq \|\widetilde{F}(u)\|_{H_{s,\delta}} + \|F(0)\|_{H_{s,\delta}} \leq K\|u\|_{H_{s,\delta}} + \|F(0)\|_{H_{s,\delta}}. \tag{6.20}
$$

6.4 Compact embedding

Theorem 6.12 ***(Compact embedding)*** *Let $0 \leq s' < s$ and $\delta' < \delta$, then the embedding*

$$
H_{s,\delta} \hookrightarrow H_{s',\delta'}. \tag{6.21}
$$

is compact.

Proof (of Theorem 6.12) Let $\{u_n\} \subset H_{s,\delta}$ be a sequence with $\|u_n\|_{H_{s,\delta}} \leq 1$. Since $H_{s,\delta}$ is a Hilbert space there is a subsequence, denoted by $\{u_n\}$, which converges weakly to u_0. We will complete the proof by showing that $u_n \to u_0$ strongly in $H_{s',\delta'}$.

Let $\chi_R \in C_0^\infty$ such that $\chi_R(x) = 1$ for $|x| \leq R$ and $\operatorname{supp}(\chi_R) \subset B_{2R}$. For a given $\epsilon > 0$, we take R such that $2\frac{C(\delta,\delta')}{R^{\delta-\delta'}} < \epsilon$, where $C(\delta,\delta')$ is the constant of inequality (6.7) of Proposition 6.5. For a bounded domain Ω, it is known that the embedding $H^s(\Omega) \hookrightarrow H^{s'}(\Omega)$ is compact and from Remark 6.3 it follows that $\|\chi_R u_n\|_{H^s} \leq C$, where C does not depend on n . Hence $\chi_R u_n$ converges strongly to $\hat{u}_0$ in $H^{s'}$. In addition, we have that $\chi_R u_n \to \chi_R u_0$ weakly in H^s and hence $\chi_R u_n \to \chi_R u_0$ weakly in $H^{s'}$. Thus the sequence $\{\chi_R u_n\}$ converges both strongly to $\hat{u}_0$ and weakly to $\chi_R u_0$ in $H^{s'}$, hence $\hat{u}_0 = \chi_R u_0$ (because $\lim_n\langle(\chi_R u_n - \chi_R u_0), (\hat{u}_0 - \chi_R u_0)\rangle_{s'} = \langle(\hat{u}_0 - \chi_R u_0), (\hat{u}_0 - \chi_R u_0)\rangle_{s'} = \|\hat{u}_0 - \chi_R u_0\|^2_{H^{s'}} = 0$). By Remark 6.3 $\lim_n \|\chi_R u_n - \chi_R u_0\|_{H_{s',\delta'}} = 0$, hence we may take n sufficiently large so that $\|\chi_R u_n - \chi_R u_0\|_{H_{s',\delta'}} < \epsilon$. Therefore

$$
\begin{aligned}
&\|u_n - u_0\|_{H_{s',\delta'}} = \|(\chi_R u_n - \chi_R u_0) + (1-\chi_R)(u_n - u_0)\|_{H_{s',\delta'}} \\
&\leq \|(\chi_R u_n - \chi_R u_0)\|_{H_{s',\delta'}} + \|(1-\chi_R)(u_n - u_0)\|_{H_{s',\delta'}} \\
&< \epsilon + \frac{C}{R^{\delta-\delta'}}\|(u_n - u_0)\|_{H_{s,\delta}} \leq \epsilon + \frac{C}{R^{\delta-\delta'}}\left(\|u_n\|_{H_{s,\delta}} + \|u_0\|_{H_{s,\delta}}\right) \\
&\leq \epsilon + 2\frac{C(\delta,\delta')}{R^{\delta-\delta'}} < 2\epsilon
\end{aligned}
\tag{6.22}
$$

and that completes the proof. ∎

6.5 Embedding into the continuous

We introduce the following notations. For a nonnegative integer m and $\beta \in \mathbb{R}$, we set

$$\|u\|_{C^m_\beta} = \textstyle\sum_{|\alpha|\leq m} \sup_x \left((1+|x|)^{\beta+|\alpha|}|\partial^\alpha u(x)|\right)$$

Let C^m_β be the functions spaces corresponding to the above norms.

Theorem 6.13 ***(Embedding into the continuous)*** *If $s > \frac{3}{2} + m$ and $\delta + \frac{3}{2} \geq \beta$, then any $u \in H_{s,\delta}$ has a representative $\tilde{u} \in C^m_\beta$ satisfying*

$$\|\tilde{u}\|_{C^m_\beta} \leq C\|u\|_{H_{s,\delta}}. \tag{6.23}$$

Proof (of Theorem 6.13) We first show (6.23) when $m = 0$. In order to make notations simpler we will use the convention $2^k = 0$ if $k < 0$. Recall that $\psi_j(x) = 1$ on $K_j := \{2^{j-3} \leq |x| \leq 2^{j+2}\}$. Using the known embedding $\sup_x |u(x)| \leq C\|u\|_{H^s}$ (see e. g. [22]),we have

$$
\begin{aligned}
&\sup_x(1+|x|)^\beta|u(x)| \leq 2^\beta \sup_{j\geq -1}\left(2^{\beta j}\sup_{\{2^j\leq|x|\leq 2^{j+1}\}}|u(x)|\right) \\
\leq &2^\beta \sup_{j\geq -1}\left(2^{\beta j}\sup|\psi_j(x)u(x)|\right) = 2^\beta \sup_{j\geq -1}\left(2^{\beta j}\sup|\psi_j(2^j x)u(2^j x)|\right) \\
\leq &2^\beta C \sup_{j\geq -1}\left(2^{\beta j}\|(\psi_j u)_{2^j}\|_{H^s}\right) \leq 2^\beta C \sup_{j\geq -1}\left(2^{(\frac{3}{2}+\delta)j}\|(\psi_j u)_{2^j}\|_{H^s}\right) \leq 2^\beta C\|u\|_{H_{s,\delta}}.
\end{aligned}
\tag{6.24}
$$

If $m > 1$, $s > \frac{3}{2} + m$ and $\delta + \frac{3}{2} \geq \beta$, then $\partial^\alpha u \in H_{s-|\alpha|,\delta+|\alpha|}$ for $1 \leq |\alpha| \leq m$. So we may apply (6.24) to $\partial^\alpha u$ and obtain $\|\partial^\alpha u\|_{C_{\beta+k}} \leq C\|\partial^\alpha u\|_{H_{s-|\alpha|,\delta+|\alpha|}}$. ∎

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